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Behavior and Solution Representations of Fourth-Order Rational Systems of Difference Equations

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Abstract. Our aim in this paper is to obtain formulas expressions for solutions of the difference equations. The purpose of this article is to determine the expressions of solutions for the following rational difference systems

$$\Theta_{n+1} = \frac{\Theta_{n-2}\Omega_n}{\alpha \pm \Theta_{n-2} \pm \Omega_{n-3}}, \quad \Omega_{n+1} = \frac{\Theta_n\Omega_{n-2}}{\beta + \Theta_{n-3} + \Omega_{n-2}}, \quad n = 0, 1, 2, \ldots,$$

where the real numbers α and β are arbitrary. Additionally, the qualitative behavior of the solutions is analyzed, including their boundedness as well as their local and global stability. We will illustrate our findings with numerical examples.

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Key Words and Phrases: Difference equations, Fourth-Order, Rational systems, Local and global stability

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1. Introduction

The study of rational difference equations and their systems has garnered significant attention in recent years due to their essential role in analyzing dynamic systems. These equations serve as mathematical models for simulating various real-world phenomena and act as effective tools for approximating solutions to continuous problems. As the demand for a deeper understanding of these equations grows, an increasing number of researchers are investigating their qualitative properties. This interest arises from their capacity to accurately represent a wide range of physical, biological, and economic phenomena. Moreover, research in this field has led to notable advancements by providing numerical solutions and analytical techniques that establish links between difference and differential equations, ultimately contributing to the development of more precise and adaptable models for understanding complex systems.

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Extensive research has been carried out regarding the method of determining the general form of the solution for certain special cases of the problem. The systems and behavior of rational difference equations have been extensively studied (see, for instance, [1]-[2]).

The solutions of the following three-dimensional system of difference equations were examined by Tollu and Yalçınkaya [3].

$$s_{n+1} = \frac{a + t_n u_n}{t_n + u_n}, \quad t_{n+1} = \frac{a + s_n u_n}{u_n + s_n}, \quad u_{n+1} = \frac{a + s_n t_n}{s_n + t_n}.$$

Khaliq et al. [4]. conducted a dynamical analysis of the following discrete-time Lotka–Volterra system with two predators and one prey

$$\begin{aligned} s_{n+1} &= \frac{\alpha s_n - \beta s_n t_n - \gamma s_n u_n}{1 + \delta s_n}, \\ t_{n+1} &= \frac{\zeta y_n + \eta s_n t_n - \mu t_n u_n}{1 + \varepsilon t_n}, \\ u_{n+1} &= \frac{v z_n + \rho s_n u_n - \sigma t_n u_n}{1 + \omega u_n}. \end{aligned}$$

Tollu et al. [2]. examined the general solutions of the following system of second-order difference equations.

$$s_{n+1} = \frac{a}{+1 + t_n s_{n-1}}, \quad t_{n+1} = \frac{b}{+1 + t_{n-1} s_n}.$$

In [5]. Khaliq and Shoaib analyzed the global and local stability of the equilibrium point for the three-dimensional system of difference equations

$$\begin{aligned} s_{n+1} &= \frac{s_n}{a + s_{n-1} t_{n-1} u_{n-1}}, \\ t_{n+1} &= \frac{t_n}{b + t_{n-1} u_{n-1} s_{n-1}}, \\ u_{n+1} &= \frac{u_n}{-c + u_{n-1} s_{n-1} t_{n-1}}. \end{aligned}$$

Elsayed and Alharbi [6]. derived a closed-form expression for the solutions of the following systems.

$$s_{n+1} = \frac{s_n t_{n-1}}{s_n + t_n}, \quad t_{n+1} = \frac{t_n s_{n-1}}{s_n + t_n}.$$

The formula for the general solution of the system has been determined by Halim et al. [7].

$$s_{n+1} = \frac{t_{n-1} s_{n-2}}{t_n \left(\alpha + \beta t_{n-1} s_{n-2}\right)}, \quad t_{n+1} = \frac{s_{n-1} t_{n-2}}{s_n \left(\alpha + \beta s_{n-1} t_{n-2}\right)}.$$

[8]. Elsayed et al. discuss the structure of the system's solutions:

$$\begin{aligned} s_{n+1} &= \frac{\alpha_1 u_{n-1} t_{n-1}}{s_{n-1} + t_{n-1} + u_{n-1}}, \\ t_{n+1} &= \frac{\alpha_2 u_{n-1} s_{n-1}}{s_{n-1} + t_{n-1} + u_{n-1}}, \\ u_{n+1} &= \frac{\alpha_3 s_{n-1} t_{n-1}}{s_{n-1} + t_{n-1} + u_{n-1}}. \end{aligned}$$

Finding the solution expressions for the following rational difference systems is the primary objective of this article:

$$\Theta_{n+1} = \frac{\Theta_{n-2}\Omega_n}{\alpha \pm \Theta_{n-2} \pm \Omega_{n-3}}, \quad \Omega_{n+1} = \frac{\Theta_n \Omega_{n-2}}{\beta + \Theta_{n-3} + \Omega_{n-2}}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where α and β are arbitrary real values, and the initial conditions $\Theta_{-3}, \Theta_{-2}, \Theta_{-1}$, $\Theta_0, \Omega_{-3}, \Omega_{-2}, \Omega_{-1}$ and Ω_0 are nonzero real integers. Additionally, we examine the behavior of the solutions, including their boundedness and their local and global stability.

2. Main Results

Let I_Θ and I_Ω be any real number intervals, and let $f : I_\Theta^3 \times I_\Omega^3 \to I_\Theta, g : I_\Theta^3 \times I_\Omega^3 \to I_\Omega$ be continuously differentiable functions. Then for each initial condition $(\Theta_i, \Omega_i) \in I_\Theta \times I_\Omega$ for $i \in \{-3, -2, -1, 0\}$, the system of difference equations:

$$\begin{aligned} \Theta_{n+1} &= f\left(\Theta_n, \Theta_{n-2}, \Theta_{n-3}, \Omega_n, \Omega_{n-2}, \Omega_{n-3}\right), \\ \Omega_{n+1} &= g\left(\Theta_n, \Theta_{n-2}, \Theta_{n-3}, \Omega_n, \Omega_{n-2}, \Omega_{n-3}\right), \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{2}$$

has a unique solution $\{\Theta_n, \Omega_n\}_{n=-3}^{\infty}$.

Definition 1. *A point $(\bar{\Theta}, \bar{\Omega})$ is said to be an equilibrium point of (2) if*

$$\begin{aligned} \bar{\Theta} &= f(\bar{\Theta}, \bar{\Theta}, \bar{\Theta}, \bar{\Omega}, \bar{\Omega}, \bar{\Omega}), \\ \bar{\Omega} &= g(\bar{\Theta}, \bar{\Theta}, \bar{\Theta}, \bar{\Omega}, \bar{\Omega}, \bar{\Omega}) \end{aligned}$$

are satisfied.

Definition 2. *Assume that $(\bar{\Theta}, \bar{\Omega})$ is a fixed point of (2).*
(i): $(\bar{\Theta}, \bar{\Omega})$ is said to be stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every initial condition $(\Theta_i, \Omega_i) \in I_\Theta \times I_\Omega$ for $i \in \{-3, -2, -1, 0\}$ if

$$\left\| \sum_{i=-3}^{0} (\Theta_i, \Omega_i) - (\bar{\Theta}, \bar{\Omega}) \right\| < \delta \Rightarrow \left\| (\Theta_n, \Omega_n) - (\bar{\Theta}, \bar{\Omega}) \right\| < \varepsilon \text{ for all } n > 0.$$

(ii): $(\bar{\Theta}, \bar{\Omega})$ is said to be unstable if it is not stable.
(iii): $(\bar{\Theta}, \bar{\Omega})$ is called asymptotically stable if there exists $\gamma > 0$ such that

$$\left\| \sum_{i=-3}^{0} (\Theta_i, \Omega_i) - (\bar{\Theta}, \bar{\Omega}) \right\| < \gamma, (\Theta_n, \Omega_n) \to (\bar{\Theta}, \bar{\Omega}) \text{ as } n \to \infty.$$

(iv): $(\bar{\Theta}, \bar{\Omega})$ *is called global attractor if* $(\Theta_n, \Omega_n) \to (\bar{\Theta}, \bar{\Omega})$ *as* $n \to \infty$.
(v): $(\bar{\Theta}, \bar{\Omega})$ *is called globally asymptotically stable if it is a global attractor and stable.*

Theorem 1. *[9]. Assume that* $(\Theta_{n+1}, \Omega_{n+1}) = F(\Theta_n, \Omega_n), n = 0, 1, \ldots,$ *is a system of difference equations where* F *is continuously differentiable on open neighborhood* $H \subseteq \mathbb{R}^{n+1}$ *and* $(\bar{\Theta}, \bar{\Omega})$ *is a fixed point of* F *then*
(1) If all eigenvalues of the Jacobian matrix J_F *at equilibrium point* $(\bar{\Theta}, \bar{\Omega})$ *lie inside the unit disk i.e.* $|\lambda_i| < 1$ *then* $(\bar{\Theta}, \bar{\Omega})$ *is locally asymptotically stable.*
(2) If at least one eigenvalues at equilibrium point $(\bar{\Theta}, \bar{\Omega})$ *outside the unit disk, then (* $\bar{\Theta}, \bar{\Omega}$ *) is unstable.*

Theorem 2. *[10]. Let* $[a_1, a_2]$ *and* $[b_1, b_2]$ *be an interval of real numbers Moreover, suppose that* $f : [a_1, a_2] \times [b_1, b_2] \to [a_1, a_2]$ *and* $g : [a_1, a_2] \times [b_1, b_2] \to [b_1, b_2]$ *are continuous functions. Let*

$$\Theta_{n+1} = f(\Theta_n, \Omega_n) \text{ and } \Omega_{n+1} = g(\Theta_n, \Omega_n),$$

and assume (m_1, m_2, M_1, M_2 *) be a solution of the system*

$$\begin{aligned} m_1 &= f(m_1, m_2), \quad M_1 = f(M_1, M_2), \\ m_2 &= g(m_1, m_2), \quad M_2 = g(M_1, M_2). \end{aligned}$$

where

$$m_i = \begin{cases} m & \text{if } f \text{ or } g \text{ nondecreasing in } \Theta \text{ or } \Omega, \\ M & \text{if } f \text{ or } g \text{ nonincreasing in } \Theta \text{ or } \Omega, \end{cases}$$

and

$$M_i = \begin{cases} M & \text{if } f \text{ or } g \text{ nondecreasing in } \Theta \text{ or } \Omega, \\ m & \text{if } f \text{ or } g \text{ nonincreasing in } \Theta \text{ or } \Omega. \end{cases}$$

Thus, $m_1 = M_1$ *and* $m_2 = M_2$. *Then, the system of difference equations has a unique equilibrium point and it is a global attractor.*

3. On the System: $\boldsymbol{\Theta}_{n+1} = \frac{\boldsymbol{\Theta}_{n-2}\boldsymbol{\Omega}_n}{\alpha + \boldsymbol{\Theta}_{n-2} + \boldsymbol{\Omega}_{n-3}}, \boldsymbol{\Omega}_{n+1} = \frac{\boldsymbol{\Theta}_n \boldsymbol{\Omega}_{n-2}}{\beta + \boldsymbol{\Theta}_{n-3} + \boldsymbol{\Omega}_{n-2}}$

The behavior of the solutions of the following system of difference equations is analyzed in this section

$$\Theta_{n+1} = \frac{\Theta_{n-2}\Omega_n}{\alpha + \Theta_{n-2} + \Omega_{n-3}}, \quad \Omega_{n+1} = \frac{\Theta_n \Omega_{n-2}}{\beta + \Theta_{n-3} + \Omega_{n-2}}, \quad n = 0, 1, 2, \ldots, \tag{3}$$

Then, we derive a specific expression for the solutions of the following system of difference equations in (3) by considering the special case $\alpha = \beta = 0$

$$\Theta_{n+1} = \frac{\Theta_{n-2}\Omega_n}{\Theta_{n-2} + \Omega_{n-3}}, \quad \Omega_{n+1} = \frac{\Theta_n\Omega_{n-2}}{\Theta_{n-3} + \Omega_{n-2}}, \quad n = 0, 1, 2, \ldots, \tag{4}$$

where $\Theta_{-3}, \Theta_{-2}, \Theta_{-1}, \Theta_0, \Omega_{-3}, \Omega_{-2}, \Omega_{-1}$ and Ω_0 are nonzero real numbers that serve as initial conditions.

3.1. Local Stability of Equilibrium Point

The stability of the critical point $O = (0, 0)$ of system (3) is examined in this subsection.

Theorem 3. *The equilibrium point O is locally asymptotically stable.*

Proof. To investigate the stability of the critical point O, we assume

$$x_n = \Theta_{n-3}, y_n = \Theta_{n-2}, z_n = \Theta_{n-1},$$
$$u_n = \Omega_{n-3}, v_n = \Omega_{n-2}, w_n = \Omega_{n-1},$$

Consequently, system (3) can be expressed as follows:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ \Theta_{n+1} \\ u_{n+1} \\ v_{n+1} \\ w_{n+1} \\ \Omega_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ z_n \\ \Theta_n \\ \frac{y_n\Omega_n}{\alpha+y_n+u_n} \\ v_n \\ w_n \\ \Omega_n \\ \frac{\Theta_n v_n}{\beta+x_n+v_n} \end{pmatrix}. \tag{5}$$

The Jacobian matrix of (5) is given by:

$$J_F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha\Omega_n+u_n\Omega_n}{(\alpha+y_n+u_n)^2} & 0 & 0 & \frac{-y_n\Omega_n}{(\alpha+y_n+u_n)^2} & 0 & 0 & \frac{y_n}{\alpha+y_n+u_n} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-\Theta_n v_n}{(\beta+x_n+v_n)^2} & 0 & 0 & \frac{v_n}{\beta+x_n+v_n} & 0 & \frac{\beta\Theta_n+x_n\Theta_n}{(\beta+x_n+v_n)^2} & 0 & 0 \end{pmatrix}.$$

If we evaluate the Jacobian matrix about the equilibrium point, we get all eigenvalues $|\lambda_i| = 0, i = 1, 2, \ldots, 8$. So all eigenvalues are inside the unit disk. Therefore Theorem 1 ensures that the origin is locally asymptotically stable.

3.2. Global Attractor

We will show in this subsection that the global attractor is the critical point.

Theorem 4. *The equilibrium point O of the system (3) is the global attractor.*

Proof. To prove this, we will assume that

$$f(p,q) = \frac{pq}{\alpha + p + q}, \text{ and } g(p,q) = \frac{pq}{\beta + p + q},$$

from here,

$$\frac{\partial f}{\partial p} = \frac{\alpha q + q^2}{(\alpha + p + q)^2}, \quad \frac{\partial f}{\partial q} = \frac{\alpha p + p^2}{(\alpha + p + q)^2},$$
$$\frac{\partial g}{\partial p} = \frac{\beta q + q^2}{(\beta + p + q)^2}, \quad \frac{\partial g}{\partial q} = \frac{\beta q + q^2}{(\beta + p + q)^2}.$$

We observe that $f(p,q)$ and $g(p,q)$ are nondecreasing in p and q. Let (m_1, m_2, M_1, M_2) be a solution of system (3) such that:

$$m_1 = f(m_1, m_2), \quad M_1 = f(M_1, M_2),$$
$$m_2 = g(m_1, m_2), \quad M_2 = g(M_1, M_2).$$

So we have

$$m_1 = \frac{m_1 m_2}{\alpha + m_1 + m_2} \Rightarrow \alpha m_1 + m_1^2 + m_1 m_2 = m_1 m_2, \tag{6}$$

$$M_1 = \frac{M_1 M_2}{\alpha + M_1 + M_2} \Rightarrow \alpha M_1 + M_1^2 + M_1 M_2 = M_1 M_2, \tag{7}$$

$$m_2 = \frac{m_1 m_2}{\beta + m_1 + m_2} \Rightarrow \beta m_2 + m_1 m_2 + m_2^2 = m_1 m_2, \tag{8}$$

$$M_2 = \frac{M_1 M_2}{\beta + M_1 + M_2} \Rightarrow \beta M_2 + M_1 M_2 + M_2^2 = M_1 M_2. \tag{9}$$

By subtracting (6) from (7), we obtain:

$$\alpha(m_1 - M_1) + \left(m_1^2 - M_1^2\right) = 0,$$
$$\alpha(m_1 - M_1) + (m_1 - M_1)(m_1 + M_1) = 0,$$
$$(m_1 - M_1)[\alpha + m_1 + M_1] = 0,$$

since $\alpha + m_1 + M_1 \neq 0$, hence, we find

$$m_1 - M_1 = 0 \quad \text{tends to} \quad m_1 = M_1.$$

Likewise, we obtain $m_2 = M_2$. Thus, system (3) has a unique equilibrium point, which is a global attractor according to Theorem 2.

Theorem 5. *system (3) has a globally asymptotically stable equilibrium point O.*

Proof. The equilibrium point O is locally stable according to Theorem 3. Furthermore, it is a universal attractor according to Theorem 4. Therefore, the equilibrium point O is globally asymptotically stable according to Definition 2.

3.3. On Solution of System (4)

In this subsection, we find the solution to system (4).

Theorem 6. *Assume that system (4) has a solution $\{\Theta_n, \Omega_n\}_{n=-3}^{\infty}$. For $n = 0, 1, 2, \ldots,$*

$$\Theta_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\zeta\mu^n\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i)\kappa)\end{array}\right]},$$

$$\Theta_{6n-2} = \frac{\eta^n\lambda^n\sigma^{n+1}\mu^n\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i)\tau)\\((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Theta_{6n-1} = \frac{\eta^n\lambda^{n+1}\sigma^n\mu^n\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Theta_{6n} = \frac{\eta^{n+1}\lambda^n\sigma^n\mu^n\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Theta_{6n+1} = \frac{\eta^n\lambda^n\sigma^{n+1}\mu^{n+1}\tau^n\kappa^n}{\left[\begin{array}{c}(\sigma+\delta)\prod_{i=0}^{n-1}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)\\(\sigma+(2i+1)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Theta_{6n+2} = \frac{\eta^{n+1}\lambda^{n+1}\sigma^n\mu^n\tau^n\kappa^{n+1}}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-1}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]},$$

$$\Omega_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^n\delta}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i)\tau)\\((2i)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Omega_{6n-2} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^{n+1}}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Omega_{6n-1} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^{n+1}\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Omega_{6n} = \frac{\eta^n\lambda^n\sigma^n\mu^{n+1}\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Omega_{6n+1} = \frac{\eta^{n+1}\lambda^n\sigma^n\mu^n\tau^n\kappa^{n+1}}{\left[\begin{array}{c}(\zeta+\kappa)\prod_{i=0}^{n-1}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]},$$

$$\Omega_{6n+2} = \frac{\eta^n\lambda^n\sigma^{n+1}\mu^{n+1}\tau^{n+1}\kappa^n}{\left[\begin{array}{c}(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-1}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

where $\Theta_{-3}=\zeta, \Theta_{-2}=\sigma, \Theta_{-1}=\lambda, \Theta_0=\eta, \Omega_{-3}=\delta, \Omega_{-2}=\kappa, \Omega_{-1}=\tau$ *and* $\Omega_0=\mu$.

Proof. The outcome holds true for $n=0$. Assuming that $n>0$ and that our hypothesis is correct for $n-1$, we have:

$$\Theta_{6n-9} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\zeta\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i)\kappa)\end{array}\right]},$$

$$\Theta_{6n-8} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^n\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i)\tau)\\((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Theta_{6n-7} = \frac{\eta^{n-1}\lambda^n\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Theta_{6n-6} = \frac{\eta^n\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Theta_{6n-5} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^n\mu^n\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)\\(\sigma+(2i+1)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Theta_{6n-4} = \frac{\eta^n\lambda^n\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]},$$

$$\Omega_{6n-9} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}\delta}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i)\tau)\\((2i)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Omega_{6n-8} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Omega_{6n-7} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Omega_{6n-6} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Omega_{6n-5} = \frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]},$$

$$\Omega_{6n-4} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}.$$

Now, it can be inferred from system (4) that

$$\Theta_{6n-3} = \frac{\Theta_{6n-6}\,\Omega_{6n-4}}{\Theta_{6n-6}+\Omega_{6n-7}}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]}\right]\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]}\right]+\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]}\right]}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)\\((2i+2)\lambda+\kappa)(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{\left[\frac{\eta}{\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}\right]+\left[\frac{\tau}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{\eta+\left[\frac{\tau\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{\eta+((2n-2)\eta+\tau)}$$

$$= \frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma+\delta)((2n-1)\eta+\tau)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)\\((2i+2)\lambda+\kappa)(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}.$$

As a result, we obtain

$$\Theta_{6n-3} = \frac{\eta^{n}\lambda^{n}\sigma^{n}\zeta\mu^{n}\tau^{n}\kappa^{n}}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i)\kappa)\end{array}\right]}.$$

Similarly, we can infer from system (4) that

$$\Omega_{6n-3} = \frac{\Theta_{6n-4}\Omega_{6n-6}}{\Theta_{6n-7}+\Omega_{6n-6}}$$

$$= \frac{\left[\dfrac{\eta^n\lambda^n\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]\left[\dfrac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^n\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{\left[\dfrac{\eta^{n-1}\lambda^n\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]+\left[\dfrac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^n\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}$$

$$= \frac{\left[\dfrac{\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)\\((2i+3)\lambda+\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]}{\left[\frac{\lambda}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\left[\frac{\mu}{\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}\right]}$$

$$= \frac{\left[\dfrac{\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]}{\left[\frac{\lambda\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\mu}$$

$$= \frac{\left[\dfrac{\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]}{(\lambda+(2n-2)\mu)+\mu}$$

$$= \frac{\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta+\kappa)(\lambda+(2n-1)\mu)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)\\((2i+3)\lambda+\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}.$$

Hence, we obtain

$$\Omega_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^n\delta}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i)\tau)\\((2i)\sigma+\delta)(\zeta+(2i+1)\kappa)\end{array}\right]}.$$

A similar approach can be used to study other expressions.

To confirm the results presented in this section, a numerical example is provided to demonstrate a representative type of solution to system (4).

Example 1. *The initial conditions are taken as* $\Theta_{-3} = 0.15, \Theta_{-2} = -0.02, \Theta_{-1} = 0.08, \Theta_0 = -0.05, \Omega_{-3} = 0.1, \Omega_{-2} = -0.06, \Omega_{-1} = 0.05$ *and* $\Omega_0 = -0.02$. *The resulting solution and represented graphically and its behavior is illustrated in Figure 1.*

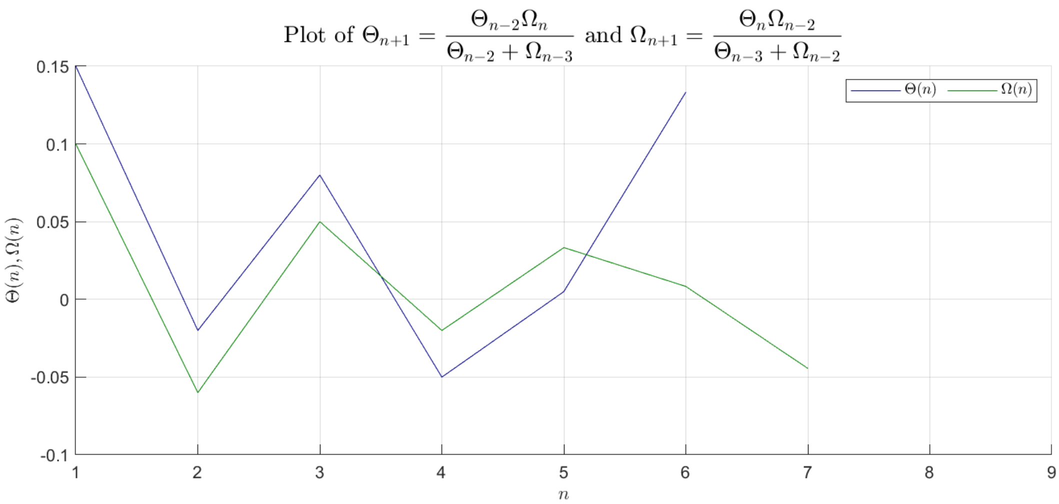

Figure 1

From Figure 1, we conclude that the solutions are bounded and that O is globally asymptotically stable and bounded. It is observed that the solutions has oscillatory harmonically in the all range. The results confirm Theorem 5 and Lemma 1. This result is in good agreement with the results obtained by Elsayed and Alshabi (2023)

3.4. Boundedness of The Solution

In this subsection, we show that the positive solutions of system (4) are bounded.

Lemma 1. *All positive solutions of system (4) are bounded and converge to zero.*

Proof. system (4) demonstrates that

$$\Theta_{n+1} = \frac{\Theta_{n-2}\Omega_n}{\Theta_{n-2} + \Omega_{n-3}} \leq \frac{\Theta_{n-2}\Omega_n}{\Theta_{n-2}} \leq \Omega_n,$$
$$\Omega_{n+1} = \frac{\Theta_n\Omega_{n-2}}{\Theta_{n-3} + \Omega_{n-2}} \leq \frac{\Theta_n\Omega_{n-2}}{\Omega_{n-2}} \leq \Theta_n.$$

We conclude that

$$\Theta_{n+1} \leq \Omega_n \text{ and } \Omega_{n+1} \leq \Theta_n.$$

Setting $n = n+1, n+2, \ldots$, yields

$$\Theta_{n+2} \leq \Omega_{n+1} \leq \Theta_n \text{ and } \Omega_{n+2} \leq \Theta_{n+1} \leq \Omega_n$$
$$\Theta_{n+3} \leq \Omega_{n+2} \leq \Theta_{n+1} \text{ and } \Omega_{n+3} \leq \Theta_{n+2} \leq \Omega_{n+1}, \ldots,$$

and so on. This suggests that the subsequences $\{\Theta_{6n-3}\}_{n=0}^{\infty}$, $\{\Theta_{6n-2}\}_{n=0}^{\infty}$, $\{\Theta_{6n-1}\}_{n=0}^{\infty}$, $\{\Theta_{6n}\}_{n=0}^{\infty}$, $\{\Theta_{6n+1}\}_{n=0}^{\infty}$, $\{\Theta_{6n+2}\}_{n=0}^{\infty}$ and $\{\Omega_{6n-3}\}_{n=0}^{\infty}$, $\{\Omega_{6n-2}\}_{n=0}^{\infty}$, $\{\Omega_{6n-1}\}_{n=0}^{\infty}$, $\{\Omega_{6n}\}_{n=0}^{\infty}$, $\{\Omega_{6n+1}\}_{n=0}^{\infty}$, $\{\Omega_{6n+2}\}_{n=0}^{\infty}$ are decreasing and are therefore bounded above by

$$\Theta_{\max} = \max\left\{\Theta_{-3}, \Theta_{-2}, \Theta_{-1}, \Theta_0\right\},$$

and

$$\Omega_{\max} = \max\left\{\Omega_{-3}, \Omega_{-2}, \Omega_{-1}, \Omega_0\right\}.$$

Setting $\alpha = \beta = 0$ in (1) yields the solution expression in the following cases.

4. On the System: $\boldsymbol{\Theta}_{n+1} = \frac{\boldsymbol{\Theta}_{n-2}\boldsymbol{\Omega}_n}{\boldsymbol{\Theta}_{n-2}-\boldsymbol{\Omega}_{n-3}}, \quad \boldsymbol{\Omega}_{n+1} = \frac{\boldsymbol{\Theta}_n\boldsymbol{\Omega}_{n-2}}{\boldsymbol{\Theta}_{n-3}+\boldsymbol{\Omega}_{n-2}}$

The solutions of the following system of difference equations will be explicitly expressed in this section

$$\Theta_{n+1} = \frac{\Theta_{n-2}\Omega_n}{\Theta_{n-2} - \Omega_{n-3}}, \quad \Omega_{n+1} = \frac{\Theta_n\Omega_{n-2}}{\Theta_{n-3} + \Omega_{n-2}}, \quad n = 0, 1, 2, \ldots, \tag{10}$$

where $\Theta_{-3}, \Theta_{-2}, \Theta_{-1}, \Theta_0, \Omega_{-3}, \Omega_{-2}, \Omega_{-1}$ and Ω_0 are nonzero real numbers that serve as initial conditions.

Theorem 7. *Assume that system (10) has a solution* $\{\Theta_n, \Omega_n\}_{n=-3}^{\infty}$. *For* $n = 0, 1, 2, \ldots,$

$$\Theta_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\zeta\mu^n\tau^n}{(\eta-\tau)^n(\sigma-\delta)^n\prod_{i=0}^{n-1}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i)\kappa)},$$
$$\Theta_{6n-2} = \frac{\eta^n\lambda^n\sigma^{n+1}\mu^n\kappa^n}{\delta^n(\lambda-\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)},$$
$$\Theta_{6n-1} = \frac{\eta^n\lambda^{n+1}\sigma^n\mu^n\tau^n}{(\eta-\tau)^n(\sigma-\delta)^n\prod_{i=0}^{n-1}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Theta_{6n} = \frac{\eta^{n+1}\lambda^n\sigma^n\mu^n\kappa^n}{\delta^n(\lambda-\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)},$$
$$\Theta_{6n+1} = \frac{\eta^n\lambda^n\sigma^{n+1}\mu^{n+1}\tau^n}{(\eta-\tau)^n(\sigma-\delta)^{n+1}\prod_{i=0}^{n-1}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$
$$\Theta_{6n+2} = \frac{\eta^{n+1}\lambda^{n+1}\sigma^n\mu^n\kappa^{n+1}}{(\zeta+\kappa)\delta^n(\lambda-\kappa)^{n+1}\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)},$$
$$\Omega_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\mu^n\kappa^n}{\delta^{n-1}(\lambda-\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)},$$
$$\Omega_{6n-2} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa}{(\eta-\tau)^n(\sigma-\delta)^n\prod_{i=0}^{n-1}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$
$$\Omega_{6n-1} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau\kappa^n}{\delta^n(\lambda-\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)},$$
$$\Omega_{6n} = \frac{\eta^n\lambda^n\sigma^n\mu^{n+1}\tau^n}{(\eta-\tau)^n(\sigma-\delta)^n\prod_{i=0}^{n-1}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$
$$\Omega_{6n+1} = \frac{\eta^{n+1}\lambda^n\sigma^n\mu^n\kappa^{n+1}}{(\zeta+\kappa)\delta^n(\lambda-\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)},$$
$$\Omega_{6n+2} = \frac{\eta^n\lambda^n\sigma^{n+1}\mu^{n+1}\tau^{n+1}}{(\sigma+\tau)(\eta-\tau)^n(\sigma-\delta)^{n+1}\prod_{i=0}^{n-1}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)},$$

where $\Theta_{-3}=\zeta, \Theta_{-2}=\sigma, \Theta_{-1}=\lambda, \Theta_0=\eta, \Omega_{-3}=\delta, \Omega_{-2}=\kappa, \Omega_{-1}=\tau$ *and* $\Omega_0=\mu$.

Proof. The outcome holds true for $n=0$. Assuming that $n>0$ and that our hypothesis is correct for $n-1$, we have:

$$\Theta_{6n-9} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\zeta\mu^{n-1}\tau^{n-1}}{(\eta-\tau)^{n-1}(\sigma-\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i)\kappa)},$$
$$\Theta_{6n-8} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^n\mu^{n-1}\kappa^{n-1}}{\delta^{n-1}(\lambda-\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)},$$
$$\Theta_{6n-7} = \frac{\eta^{n-1}\lambda^n\sigma^{n-1}\mu^{n-1}\tau^{n-1}}{(\eta-\tau)^{n-1}(\sigma-\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$
$$\Theta_{6n-6} = \frac{\eta^n\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\kappa^{n-1}}{\delta^{n-1}(\lambda-\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)},$$
$$\Theta_{6n-5} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^n\mu^n\tau^{n-1}}{(\eta-\tau)^{n-1}(\sigma-\delta)^n\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$
$$\Theta_{6n-4} = \frac{\eta^n\lambda^n\sigma^{n-1}\mu^{n-1}\kappa^n}{(\zeta+\kappa)\delta^{n-1}(\lambda-\kappa)^n\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)},$$
$$\Omega_{6n-9} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\kappa^{n-1}}{\delta^{n-2}(\lambda-\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)},$$
$$\Omega_{6n-8} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa}{(\eta-\tau)^{n-1}(\sigma-\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Omega_{6n-7} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau\kappa^{n-1}}{\delta^{n-1}(\lambda-\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)},$$
$$\Omega_{6n-6} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}}{(\eta-\tau)^{n-1}(\sigma-\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$
$$\Omega_{6n-5} = \frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\kappa^{n}}{(\zeta+\kappa)\delta^{n-1}(\lambda-\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)},$$
$$\Omega_{6n-4} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}}{(\sigma+\tau)(\eta-\tau)^{n-1}(\sigma-\delta)^{n}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)}.$$

We now derive from system (10) that

$$\Theta_{6n-3} = \frac{\Theta_{6n-6}\Omega_{6n-4}}{\Theta_{6n-6}-\Omega_{6n-7}}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\kappa^{n-1}}{(\delta^{n-1}(\lambda-\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)}\right]\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}}{(\sigma+\tau)(\eta-\tau)^{n-1}(\sigma-\delta)^{n}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)}\right]}{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\kappa^{n-1}}{(\delta^{n-1}(\lambda-\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)}\right] - \left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau\kappa^{n-1}}{\delta^{n-1}(\lambda-\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)}\right]}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}}{(\sigma+\tau)(\eta-\tau)^{n-1}(\sigma-\delta)^{n}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)}\right]}{(\eta-\tau)}$$

$$= \frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}}{(\sigma+\tau)(\eta-\tau)^{n}(\sigma-\delta)^{n}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)}.$$

Thus, we obtain

$$\Theta_{6n-3} = \frac{\eta^{n}\lambda^{n}\sigma^{n}\mu^{n}\tau^{n}\zeta}{(\eta-\tau)^{n}(\sigma-\delta)^{n}\prod_{i=0}^{n-1}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i)\kappa)}.$$

Likewise, we can analyze the relationships using the same approach

$$\Omega_{6n-3} = \frac{\Theta_{6n-4}\Omega_{6n-6}}{\Theta_{6n-7}+\Omega_{6n-6}}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n-1}\kappa^{n}}{(\zeta+\kappa)\delta^{n-1}(\lambda-\kappa)^{n}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)}\right]\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}}{(\eta-\tau)^{n-1}(\sigma-\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)}\right]}{\left[\frac{\eta^{n-1}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}}{(\eta-\tau)^{n-1}(\sigma-\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)}\right]+\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}}{(\eta-\tau)^{n-1}(\sigma-\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)}\right]}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n}\kappa^{n}}{\left[\begin{array}{c}(\zeta+\kappa)\delta^{n-1}(\lambda-\kappa)^{n}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)\\(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]}{\left[\frac{\lambda}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\left[\frac{\mu}{\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}\right]}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n}\kappa^{n}}{\left[(\zeta+\kappa)\delta^{n-1}(\lambda-\kappa)^{n}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)\right]}\right]}{\left[\frac{\lambda\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\mu}$$

$$= \frac{\left[\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n}\kappa^{n}}{\left[(\zeta+\kappa)\delta^{n-1}(\lambda-\kappa)^{n}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)\right]}\right]}{(\lambda+(2n-2)\mu)+\mu}$$

$$= \frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n}\kappa^{n}}{\left[\begin{array}{c}(\zeta+\kappa)\delta^{n-1}(\lambda-\kappa)^{n}(\lambda+(2n-1)\mu)\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)\\(\zeta+(2i+3)\kappa)\end{array}\right]}.$$

Then

$$\Omega_{6n-3} = \frac{\eta^{n}\lambda^{n}\sigma^{n}\mu^{n}\kappa^{n}}{\delta^{n-1}(\lambda-\kappa)^{n}\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)}.$$

The same method can be used to prove other relations.

Example 2. *Let the initial values be* $\Theta_{-3} = -0.1, \Theta_{-2} = 0.09, \Theta_{-1} = -0.03, \Theta_0 = 0.05, \Omega_{-3} = 0.03, \Omega_{-2} = -0.04, \Omega_{-1} = 0.1$ *and* $\Omega_0 = -0.02$. *See Figure 2.*

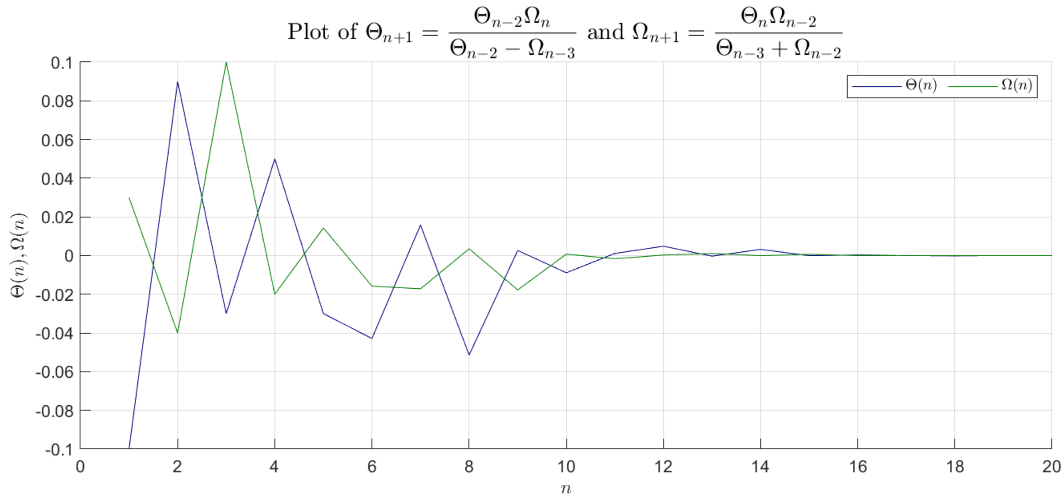

Figure 2

Figure 2, explained that the solutions are bounded and that O is globally asymptotically stable and bounded. It is observed that the solutions has oscillatory harmonically in the all range and its converges to zero for $n > 10$. The results confirm Theorem 5 and Lemma 1. This result is in good agreement with the results obtained by Khan et al. (2019).

5. On the System: $\boldsymbol{\Theta}_{n+1} = \frac{\boldsymbol{\Theta}_{n-2}\boldsymbol{\Omega}_n}{-\boldsymbol{\Theta}_{n-2}+\boldsymbol{\Omega}_{n-3}}, \quad \boldsymbol{\Omega}_{n+1} = \frac{\boldsymbol{\Theta}_n\boldsymbol{\Omega}_{n-2}}{\boldsymbol{\Theta}_{n-3}+\boldsymbol{\Omega}_{n-2}}$

The solutions of the following system of difference equations will be explicitly expressed in this section

$$\Theta_{n+1} = \frac{\Theta_{n-2}\Omega_n}{-\Theta_{n-2}+\Omega_{n-3}}, \quad \Omega_{n+1} = \frac{\Theta_n\Omega_{n-2}}{\Theta_{n-3}+\Omega_{n-2}}, \quad n = 0, 1, 2, \ldots, \tag{11}$$

where $\Theta_{-3}, \Theta_{-2}, \Theta_{-1}, \Theta_0, \Omega_{-3}, \Omega_{-2}, \Omega_{-1}$ and Ω_0 are nonzero real numbers that serve as initial conditions.

Theorem 8. *Assume that system (11) has a solution $\{\Theta_n, \Omega_n\}_{n=-3}^{\infty}$. For $n = 0, 1, 2, \ldots,$*

$$\Theta_{6n-3} = \frac{(-1)^n\eta^n\lambda^n\sigma^n\zeta\mu^n\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta-\tau)(\lambda+(2i)\mu)((2i)\lambda-\kappa)(\sigma+(2i+1)\tau)\\ ((2i+1)\sigma-\delta)(\zeta+(2i)\kappa)\end{array}\right]},$$

$$\Theta_{6n-2} = \frac{(-1)^n\eta^n\lambda^n\sigma^{n+1}\mu^n\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i)\tau)\\ ((2i+2)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Theta_{6n-1} = \frac{(-1)^n\eta^n\lambda^{n+1}\sigma^n\mu^n\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta-\tau)(\lambda+(2i)\mu)((2i+2)\lambda-\kappa)(\sigma+(2i+1)\tau)\\ ((2i+1)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Theta_{6n} = \frac{(-1)^n\eta^{n+1}\lambda^n\sigma^n\mu^n\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Theta_{6n+1} = \frac{(-1)^{n+1}\eta^n\lambda^n\sigma^{n+1}\mu^{n+1}\tau^n\kappa^n}{\left[\begin{array}{c}(\sigma-\delta)\prod_{i=0}^{n-1}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)\\(\sigma+(2i+1)\tau)((2i+3)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Theta_{6n+2} = \frac{(-1)^{n+1}\eta^{n+1}\lambda^{n+1}\sigma^n\mu^n\tau^n\kappa^{n+1}}{\left[\begin{array}{c}(\lambda-\kappa)(\zeta+\kappa)\prod_{i=0}^{n-1}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda-\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma-\delta)(\zeta+(2i+3)\kappa)\end{array}\right]},$$

$$\Omega_{6n-3} = \frac{(-1)^n\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^n\delta}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i)\tau)\\((2i)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Omega_{6n-2} = \frac{(-1)^n\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^{n+1}}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta-\tau)(\lambda+(2i)\mu)((2i)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Omega_{6n-1} = \frac{(-1)^n\eta^n\lambda^n\sigma^n\mu^n\tau^{n+1}\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Omega_{6n} = \frac{(-1)^n\eta^n\lambda^n\sigma^n\mu^{n+1}\tau^n\kappa^n}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Omega_{6n+1} = \frac{(-1)^n\eta^{n+1}\lambda^n\sigma^n\mu^n\tau^n\kappa^{n+1}}{\left[\begin{array}{c}(\zeta+\kappa)\prod_{i=0}^{n-1}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma-\delta)(\zeta+(2i+3)\kappa)\end{array}\right]},$$

$$\Omega_{6n+2} = \frac{(-1)^{n+1}\eta^n\lambda^n\sigma^{n+1}\mu^{n+1}\tau^{n+1}\kappa^n}{\left[\begin{array}{c}(\sigma+\tau)(\sigma-\delta)\prod_{i=0}^{n-1}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

where $\Theta_{-3}=\zeta, \Theta_{-2}=\sigma, \Theta_{-1}=\lambda, \Theta_0=\eta, \Omega_{-3}=\delta, \Omega_{-2}=\kappa, \Omega_{-1}=\tau$ *and* $\Omega_0=\mu$.

Proof. The outcome holds true for $n=0$. Assuming that $n>0$ and that our hypothesis is correct for $n-1$, we have:

$$\Theta_{6n-9} = \frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\zeta\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i)\mu)((2i)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i)\kappa)\end{array}\right]},$$

$$\Theta_{6n-8} = \frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^n\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i)\tau)\\((2i+2)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Theta_{6n-7}=\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i)\mu)((2i+2)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Theta_{6n-6}=\frac{(-1)^{n-1}\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Theta_{6n-5}=\frac{(-1)^{n}\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma-\delta)\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)\\(\sigma+(2i+1)\tau)((2i+3)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Theta_{6n-4}=\frac{(-1)^{n}\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\lambda-\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda-\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma-\delta)(\zeta+(2i+3)\kappa)\end{array}\right]},$$

$$\Omega_{6n-9}=\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}\delta}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i)\tau)\\((2i)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Omega_{6n-8}=\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i)\mu)((2i)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Omega_{6n-7}=\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]},$$

$$\Omega_{6n-6}=\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]},$$

$$\Omega_{6n-5}=\frac{(-1)^{n-1}\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma-\delta)(\zeta+(2i+3)\kappa)\end{array}\right]},$$

$$\Omega_{6n-4}=\frac{(-1)^{n}\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma-\delta)\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}.$$

Now, it can be inferred from system (11) that

$$\Theta_{6n-3}=\frac{\Theta_{6n-6}\Omega_{6n-4}}{-\Theta_{6n-6}+\Omega_{6n-7}}$$

$$= \frac{\left[\dfrac{(-1)^{n-1}\eta^n\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]}\right]\left[\dfrac{(-1)^{n}\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma-\delta)\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{-\left[\dfrac{(-1)^{n-1}\eta^n\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]}\right]+\left[\dfrac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i+2)\tau)\\((2i+2)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]}\right]}$$

$$= \frac{\left[\dfrac{(-1)^{n-1}\eta^n\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma-\delta)\prod_{i=0}^{n-2}((2i+2)\eta-\tau)\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)\\((2i+2)\lambda-\kappa)(\sigma+(2i+3)\tau)((2i+3)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{\left[\frac{\eta}{\prod_{i=0}^{n-2}((2i+2)\eta-\tau)}\right]-\left[\frac{\tau}{\prod_{i=0}^{n-2}((2i)\eta-\tau)}\right]}$$

$$= \frac{\left[\dfrac{(-1)^{n-1}\eta^n\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma-\delta)\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{\eta-\left[\frac{\tau\prod_{i=0}^{n-2}((2i+2)\eta-\tau)}{\prod_{i=0}^{n-2}((2i)\eta-\tau)}\right]}$$

$$= \frac{\left[\dfrac{(-1)^{n-1}\eta^n\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma-\delta)\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{\eta+((2n-2)\eta-\tau)}$$

$$= \frac{(-1)^{n-1}\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma-\delta)((2n-1)\eta-\tau)\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)\\((2i+2)\lambda-\kappa)((\sigma+(2i+3)\tau)((2i+3)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}.$$

Therefore, we get

$$\Theta_{6n-3} = \frac{(-1)^{n}\eta^{n}\lambda^{n}\sigma^{n}\zeta\mu^{n}\tau^{n}\kappa^{n}}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i+1)\eta-\tau)(\lambda+(2i)\mu)((2i)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i)\kappa)\end{array}\right]}.$$

Similarly, we can infer from system (11) that

$$\Omega_{6n-3} = \frac{\Theta_{6n-4}\Omega_{6n-6}}{\Theta_{6n-7}+\Omega_{6n-6}}$$

$$= \frac{\left[\frac{(-1)^{n}\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\lambda-\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda-\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma-\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]\left[\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}{\left[\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i)\mu)((2i+2)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]+\left[\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}\prod_{i=0}^{n-2}((2i+1)\eta-\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda-\kappa)(\sigma+(2i+1)\tau)\\((2i+1)\sigma-\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}\right]}$$

$$= \frac{\left[\frac{(-1)^{n}\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\lambda-\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)\\((2i+3)\lambda-\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma-\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]}{\left[\frac{\lambda}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\left[\frac{\mu}{\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}\right]}$$

$$= \frac{\left[\frac{(-1)^n\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda-\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda-\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma-\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]}{\left[\frac{\lambda\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\mu}$$

$$= \frac{\left[\frac{(-1)^n\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda-\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda-\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma-\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]}{(\lambda+(2n-2)\mu)+\mu}$$

$$= \frac{(-1)^n\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda-\kappa)(\zeta+\kappa)(\lambda+(2n-1)\mu)\prod_{i=0}^{n-2}((2i+2)\eta-\tau)(\lambda+(2i+1)\mu)\\((2i+3)\lambda-\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma-\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}.$$

Hence, we obtain

$$\Omega_{6n-3} = \frac{(-1)^n\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^n\delta}{\left[\begin{array}{c}\prod_{i=0}^{n-1}((2i)\eta-\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda-\kappa)(\sigma+(2i)\tau)\\((2i)\sigma-\delta)(\zeta+(2i+1)\kappa)\end{array}\right]}.$$

A similar method can be applied to prove the following cases.

Example 3. *Consider the initial values* $\Theta_{-3} = 0.07, \Theta_{-2} = -0.03, \Theta_{-1} = 0.1, \Theta_0 = -0.09, \Omega_{-3} = -0.02, \Omega_{-2} = 0.05, \Omega_{-1} = -0.08$ *and* $\Omega_0 = 0.04$. *See Figure 3.*

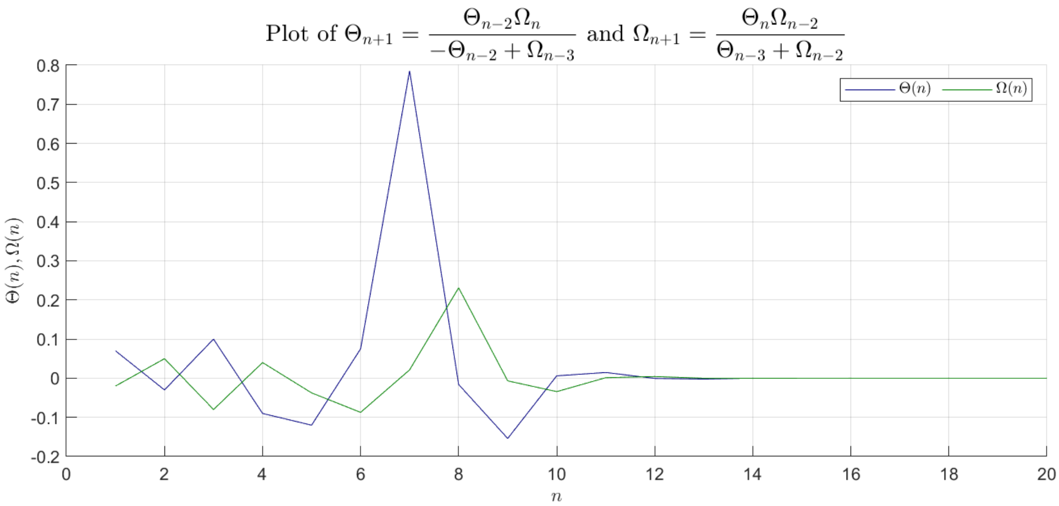

Figure 3

Figure 3, demonstrated that the solutions are bounded and that O is globally asymptotically stable and bounded. It is observed that the solutions has oscillatory harmonically in the all range and its converges to zero for $n > 12$. The results confirm Theorem 5 and Lemma 1. This result is in good agreement with the results obtained by Khaliq et al. (2022).

6. On the System: $\boldsymbol{\Theta}_{n+1} = \frac{\boldsymbol{\Theta}_{n-2}\boldsymbol{\Omega}_n}{-\boldsymbol{\Theta}_{n-2}-\boldsymbol{\Omega}_{n-3}}, \quad \boldsymbol{\Omega}_{n+1} = \frac{\boldsymbol{\Theta}_n\boldsymbol{\Omega}_{n-2}}{\boldsymbol{\Theta}_{n-3}+\boldsymbol{\Omega}_{n-2}}$

The solutions of the following system of difference equations will be explicitly expressed in this section

$$\Theta_{n+1} = \frac{\Theta_{n-2}\Omega_n}{-\Theta_{n-2} - \Omega_{n-3}}, \quad \Omega_{n+1} = \frac{\Theta_n\Omega_{n-2}}{\Theta_{n-3} + \Omega_{n-2}}, \quad n = 0, 1, 2, \ldots, \tag{12}$$

where $\Theta_{-3}, \Theta_{-2}, \Theta_{-1}, \Theta_0, \Omega_{-3}, \Omega_{-2}, \Omega_{-1}$ and Ω_0 are nonzero real numbers that serve as initial conditions.

Theorem 9. *Assume that system (12) has a solution $\{\Theta_n, \Omega_n\}_{n=-3}^{\infty}$. For $n = 0, 1, 2, \ldots,$*

$$\Theta_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\zeta\mu^n\tau^n}{(\eta+\tau)^n(\sigma+\delta)^n\prod_{i=0}^{n-1}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i)\kappa)},$$

$$\Theta_{6n-2} = \frac{(-1)^n\eta^n\lambda^n\sigma^{n+1}\mu^n\kappa^n}{\delta^n(\lambda+\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)},$$

$$\Theta_{6n-1} = \frac{\eta^n\lambda^{n+1}\sigma^n\mu^n\tau^n}{(\eta+\tau)^n(\sigma+\delta)^n\prod_{i=0}^{n-1}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Theta_{6n} = \frac{(-1)^n\eta^{n+1}\lambda^n\sigma^n\mu^n\kappa^n}{\delta^n(\lambda+\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)},$$

$$\Theta_{6n+1} = \frac{-\eta^n\lambda^n\sigma^{n+1}\mu^{n+1}\tau^n}{(\eta+\tau)^n(\sigma+\delta)^{n+1}\prod_{i=0}^{n-1}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Theta_{6n+2} = \frac{(-1)^{n+1}\eta^{n+1}\lambda^{n+1}\sigma^n\mu^n\kappa^{n+1}}{(\zeta+\kappa)\delta^n(\lambda+\kappa)^{n+1}\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)},$$

$$\Omega_{6n-3} = \frac{(-1)^n\eta^n\lambda^n\sigma^n\mu^n\kappa^n}{\delta^{n-1}(\lambda+\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)},$$

$$\Omega_{6n-2} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa}{(\eta+\tau)^n(\sigma+\delta)^n\prod_{i=0}^{n-1}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Omega_{6n-1} = \frac{(-1)^n\eta^n\lambda^n\sigma^n\mu^n\tau\kappa^n}{\delta^n(\lambda+\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)},$$

$$\Omega_{6n} = \frac{\eta^n\lambda^n\sigma^n\mu^{n+1}\tau^n}{(\eta+\tau)^n(\sigma+\delta)^n\prod_{i=0}^{n-1}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Omega_{6n+1} = \frac{(-1)^n \eta^{n+1} \lambda^n \sigma^n \mu^n \kappa^{n+1}}{(\zeta+\kappa)\delta^n(\lambda+\kappa)^n \prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)},$$

$$\Omega_{6n+2} = \frac{-\eta^n \lambda^n \sigma^{n+1} \mu^{n+1} \tau^{n+1}}{(\sigma+\tau)(\eta+\tau)^n(\sigma+\delta)^{n+1} \prod_{i=0}^{n-1}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)},$$

where $\Theta_{-3} = \zeta, \Theta_{-2} = \sigma, \Theta_{-1} = \lambda, \Theta_0 = \eta, \Omega_{-3} = \delta, \Omega_{-2} = \kappa, \Omega_{-1} = \tau$ *and* $\Omega_0 = \mu$.

Proof. The outcome holds true for $n = 0$. Assuming that $n > 0$ and that our hypothesis is correct for $n-1$, we have:

$$\Theta_{6n-9} = \frac{\eta^{n-1} \lambda^{n-1} \sigma^{n-1} \zeta \mu^{n-1} \tau^{n-1}}{(\eta+\tau)^{n-1}(\sigma+\delta)^{n-1} \prod_{i=0}^{n-2}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i)\kappa)},$$

$$\Theta_{6n-8} = \frac{(-1)^{n-1} \eta^{n-1} \lambda^{n-1} \sigma^n \mu^{n-1} \kappa^{n-1}}{\delta^{n-1}(\lambda+\kappa)^{n-1} \prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)},$$

$$\Theta_{6n-7} = \frac{\eta^{n-1} \lambda^n \sigma^{n-1} \mu^{n-1} \tau^{n-1}}{(\eta+\tau)^{n-1}(\sigma+\delta)^{n-1} \prod_{i=0}^{n-2}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Theta_{6n-6} = \frac{(-1)^{n-1} \eta^n \lambda^{n-1} \sigma^{n-1} \mu^{n-1} \kappa^{n-1}}{\delta^{n-1}(\lambda+\kappa)^{n-1} \prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)},$$

$$\Theta_{6n-5} = \frac{-\eta^{n-1} \lambda^{n-1} \sigma^n \mu^n \tau^{n-1}}{(\eta+\tau)^{n-1}(\sigma+\delta)^n \prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Theta_{6n-4} = \frac{(-1)^n \eta^n \lambda^n \sigma^{n-1} \mu^{n-1} \kappa^n}{(\zeta+\kappa)\delta^{n-1}(\lambda+\kappa)^n \prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)},$$

$$\Omega_{6n-9} = \frac{(-1)^{n-1} \eta^{n-1} \lambda^{n-1} \sigma^{n-1} \mu^{n-1} \kappa^{n-1}}{\delta^{n-2}(\lambda+\kappa)^{n-1} \prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)},$$

$$\Omega_{6n-8} = \frac{\eta^{n-1} \lambda^{n-1} \sigma^{n-1} \mu^{n-1} \tau^{n-1} \kappa}{(\eta+\tau)^{n-1}(\sigma+\delta)^{n-1} \prod_{i=0}^{n-2}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Omega_{6n-7} = \frac{(-1)^{n-1} \eta^{n-1} \lambda^{n-1} \sigma^{n-1} \mu^{n-1} \tau \kappa^{n-1}}{\delta^{n-1}(\lambda+\kappa)^{n-1} \prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)},$$

$$\Omega_{6n-6} = \frac{\eta^{n-1} \lambda^{n-1} \sigma^{n-1} \mu^n \tau^{n-1}}{(\eta+\tau)^{n-1}(\sigma+\delta)^{n-1} \prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)},$$

$$\Omega_{6n-5} = \frac{(-1)^{n-1} \eta^n \lambda^{n-1} \sigma^{n-1} \mu^{n-1} \kappa^n}{(\zeta+\kappa)\delta^{n-1}(\lambda+\kappa)^{n-1} \prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)},$$

$$\Omega_{6n-4} = \frac{-\eta^{n-1} \lambda^{n-1} \sigma^n \mu^n \tau^n}{(\sigma+\tau)(\eta+\tau)^{n-1}(\sigma+\delta)^n \prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)}.$$

We now demonstrate that the findings hold true for n. It can be inferred from system (12) that

$$\Theta_{6n-3} = \frac{\Theta_{6n-6}\Omega_{6n-4}}{-\Theta_{6n-6} - \Omega_{6n-7}}$$

$$= \frac{\left[\frac{(-1)^{n-1}\eta^n\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\kappa^{n-1}}{\delta^{n-1}(\lambda+\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)}\right]\left[\frac{-\eta^{n-1}\lambda^{n-1}\sigma^n\mu^n\tau^n}{(\sigma+\tau)(\eta+\tau)^{n-1}(\sigma+\delta)^n\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)}\right]}{-\left[\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau\kappa^{n-1}}{\delta^{n-1}(\lambda+\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)}\right] - \left[\frac{(-1)^{n-1}\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau\kappa^{n-1}}{\delta^{n-1}(\lambda+\kappa)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+1)\kappa)}\right]}$$

$$= \frac{\left[\frac{\eta^n\lambda^{n-1}\sigma^n\mu^n\tau^n}{(\sigma+\tau)(\eta+\tau)^{n-1}(\sigma+\delta)^n\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)}\right]}{(\eta+\tau)}$$

$$= \frac{\eta^n\lambda^{n-1}\sigma^n\mu^n\tau^n}{(\sigma+\tau)(\eta+\tau)^n(\sigma+\delta)^n\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+3)\tau)(\zeta+(2i+2)\kappa)}.$$

So, we have

$$\Theta_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\zeta\mu^n\tau^n}{(\eta+\tau)^n(\sigma+\delta)^n\prod_{i=0}^{n-1}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i)\kappa)}.$$

Additionally, we can see from system (12) that

$$\Omega_{6n-3} = \frac{\Theta_{6n-4}\Omega_{6n-6}}{\Theta_{6n-7} + \Omega_{6n-6}}$$

$$= \frac{\left[\frac{(-1)^n\eta^n\lambda^n\sigma^{n-1}\mu^{n-1}\kappa^n}{(\zeta+\kappa)\delta^{n-1}(\lambda+\kappa)^n\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)}\right]\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^n\tau^{n-1}}{(\eta+\tau)^{n-1}(\sigma+\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)}\right]}{\left[\frac{\eta^{n-1}\lambda^n\sigma^{n-1}\mu^{n-1}\tau^{n-1}}{(\eta+\tau)^{n-1}(\sigma+\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)}\right] + \left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^n\tau^{n-1}}{(\eta+\tau)^{n-1}(\sigma+\delta)^{n-1}\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)(\sigma+(2i+1)\tau)(\zeta+(2i+2)\kappa)}\right]}$$

$$= \frac{\left[\frac{(-1)^n\eta^n\lambda^n\sigma^{n-1}\mu^n\kappa^n}{\left[\begin{array}{c}(\zeta+\kappa)\delta^{n-1}(\lambda+\kappa)^n\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)\\(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)\end{array}\right]}\right]}{\left[\frac{\lambda}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\left[\frac{\mu}{\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}\right]}$$

$$= \frac{\left[\frac{(-1)^n\eta^n\lambda^n\sigma^{n-1}\mu^n\kappa^n}{\left[(\zeta+\kappa)\delta^{n-1}(\lambda+\kappa)^n\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)\right]}\right]}{\left[\frac{\lambda\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\mu}$$

$$= \frac{\left[\frac{(-1)^n\eta^n\lambda^n\sigma^{n-1}\mu^n\kappa^n}{\left[(\zeta+\kappa)\delta^{n-1}(\lambda+\kappa)^n\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)(\zeta+(2i+3)\kappa)\right]}\right]}{(\lambda+(2n-2)\mu)+\mu}$$

$$= \frac{(-1)^n\eta^n\lambda^n\sigma^{n-1}\mu^n\kappa^n}{\left[\begin{array}{c}(\zeta+\kappa)\delta^{n-1}(\lambda+\kappa)^n(\lambda+(2n-1)\mu)\prod_{i=0}^{n-2}(\lambda+(2i+1)\mu)(\sigma+(2i+2)\tau)\\(\zeta+(2i+3)\kappa)\end{array}\right]}.$$

Thus, we obtain

$$\Omega_{6n-3} = \frac{(-1)^n\eta^n\lambda^n\sigma^n\mu^n\kappa^n}{\delta^{n-1}(\lambda+\kappa)^n\prod_{i=0}^{n-1}(\lambda+(2i+1)\mu)(\sigma+(2i)\tau)(\zeta+(2i+1)\kappa)}.$$

Likewise, we can investigate additional relationships by applying the same methodology.

Example 4. *Take the initial values* $\Theta_{-3} = 0.1, \Theta_{-2} = -0.04, \Theta_{-1} = 0.06, \Theta_0 = -0.02, \Omega_{-3} = 0.05, \Omega_{-2} = -0.03, \Omega_{-1} = 0.09$ *and* $\Omega_0 = -0.1$. *See Figure 4.*

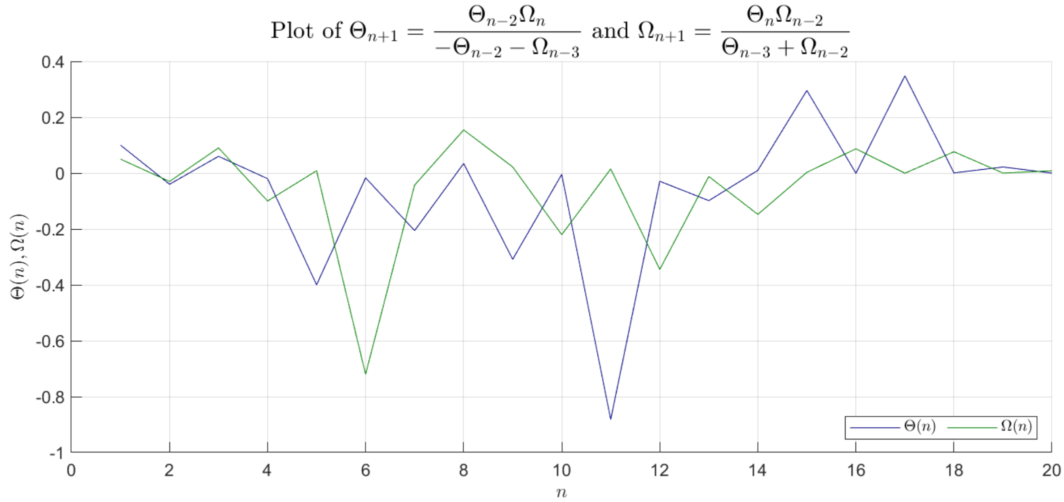

Figure 4

Figure 4, shows that the solutions are bounded and that O is globally asymptotically stable and bounded. It is observed that the solutions has oscillatory harmonically in the all range. The results confirm Theorem 5 and Lemma 1. This result is in good agreement with the results obtained by Elsayed and Ibrahim (2015).

7. Conclusion

The majority of studies on nonlinear rational difference equations analyze the behavior of solutions by presenting a general solution form. The initial conditions of the systems considered in this study are nonzero real integers. Since obtaining explicit solution formulations can be challenging, researchers investigate the stability characteristics of the equilibrium point.

This article presents the solution expressions for a few specific examples that serve as applications of fourth-order rational difference systems. In Section 3, we examined the qualitative behavior of the solutions, including their boundedness, as well as local and global stability. Additionally, we derived the general solution form of system 4.

Moreover, we obtained solutions for three specific cases of the studied systems, namely systems 10, 11, and 12, which are discussed in Sections 4, 5, and 6 respectively. Illustrative and numerical examples are presented within Sections 3, 4, 5, and 6 to support the theoretical discussions and verify the analytical results.

The expressions and results reported in this research might be beneficial for applications in seismology and materials characterization of difference equations.

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