EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6222
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Algebraic Operations on Complex Trigonometric Fuzzy Sets with Reciprocal Fractional Floor Functions

Murugan Palanikumar[1], Nasreen Kausar[2], Brikena Vrioni[3,*], Ervin Hoxha[4]

[1] *Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai-602105, India*
[2] *Department of Mathematics, Faculty of Arts and Science, Balikesir University, 10145 Balikesir, Turkey*
[3] *Institute: School of Arts and Sciences, American International University, Kuwait*
[4] *School of Arts and Sciences, American International University, Kuwait*

Abstract. A set with membership values that are vectors in the complex plane's unit circle is called a complex fuzzy set. The capacity of the complex fuzzy set to explain both contentment and discontent, as well as the lack of confusing information in two-dimensional situations, are some of its noteworthy benefits. For this study to offer a strong and adaptable tool for complexity and uncertain circumstances, a fuzzy set and a complex fuzzy set must be combined. A novel approach to generating complex sine trigonometric reciprocal fractional floor functions from fuzzy sets is presented in this work. This work explores the usage of fuzzy sets in averaging, geometric, generalized weighted averaging and generalized weighted geometric. An aggregating model is used to determine the weighted average and geometric. Some sets with important properties will be further analyzed using algebraic approaches.

2020 Mathematics Subject Classifications: 08A72, 94D05, 03B52

Key Words and Phrases: Reciprocal fractional floor functions, averaging, geometric, generalized weighted averaging and generalized weighted geometric

1. Introduction

Aggregation functions in fuzzy systems have applications in many different fields due to their capacity to combine data from several sources or criteria, which is essential for uncertainty modeling and decision-making. Data analysis, fuzzy control, and decision support are just a few of the fields in which fuzzy aggregation functions find use. These routines create a single, representative value by combining many pieces of ambiguous data. They allow for more sophisticated reasoning than traditional methods and are particularly

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6222

Email addresses: palanimaths86@gmail.com (M. Palanikumar), nasreen.kausar@balikesir.edu.tr (N. Kausar), brikena.vrioni@yahoo.com (B. Vrioni), ervin.hoxha1990@gmail.com (E. Hoxha)

useful when dealing with ambiguous or missing data. In this demanding technological age, the majority of information qualities are ambiguous. Classical set theory is unable to handle these ambiguous and unclear data. Zadeh [1] introduced fuzzy set (FS) theory as a solution to this shortcoming of traditional set theory. In this competitive setting, inaccurate information makes decision-making (DM) more challenging. Traditionally, there is a wealth of knowledge on real-world issues. Uncertain and imprecise information is making it harder for a single decision-maker to make appropriate judgments, which is why it is getting more complicated [2]. This innovative concept marked a sea change in a number of domains, including image processing, industrial control and human DM. Numerous complications were added throughout time, and academics came up with a number of other generalizations of FSs and IFSs. The inability of FSs and IFSs to handle the periodicity included in some ambiguous, imprecise, and partial data is one of the main drawbacks of their use. Moreover, FSs and IFSs are unable to handle a number of additional issues with a two-dimensional framework. In order to address this shortcoming, Ramot et al. [2] expanded the FS structure to a complex fuzzy set (CFS), adding a phase variable and expanding the range from [0,1] to the unit circle in the complex plane, making the information more widely applicable.
Later, Ramot et al. [3, 4] generalized the usual FS to CFS. In this extension, the range encompasses a unit circle in the complex plane as well as the interval $0 \leq y \leq 1$. Let $A = \{(\tau, r_A(\tau)e^{1\gamma_A(\tau)})|\tau \in y\}$ be the phase and amplitude term for CFS. In this case, the amplitude term is $i = \sqrt{-1}, r_A(\tau) \in [0,1]$, and the phase term is $\gamma_A(\tau) \in [0, 2\pi]$. If you set the phase term to zero, CFS can function as a regular FS. In addition to being a straightforward expansion of the conventional fuzzy set, CFS offers an obvious way to solve challenging and impossible issues. The range of a CFS, which is an expanded version of a FS, includes both real and complex integers with certain imaginary characteristics. A complex-valued membership grade with amplitude and phase terms is indicated by CFS. This suggests that compared to a traditional FS, the CFS is more broad. The decision makers in the hesitant fuzzy set (HFS) offer a range of favorable (multi-favorable) membership values for simultaneously expressing their preferences and evaluations. However, the CFS gives us the option to include a phase component, which helps us learn more about a specific higher dimensional periodic problem. within the aforementioned talks of the different applications and generalizations. With the potential enlarged range of unit disk containing a phase component, we provide a logical extension of the current set to a unique idea of a cohesive FS, which may explicitly focus on the set of the favorable scenarios for a specific uncertain higher dimensional problem. One benefit of the phase component is that it can help with periodic impreciseness. The concept of a generalized OWA operator was introduced by Yager [5]. The complex fuzzy arithmetic AO was first presented by Bi et al. [6]. The use of fuzzy information by AOs has increased within the last few years.

Numerous ideas have been put forth to explain uncertainty, including fuzzy sets (FS) [1], which have membership grades (MG) ranging from 0 to 1. Atanassov [7] started an intuitionistic FS (IFS) where each element has two MGs, like $0 \leq \varpi + \kappa \leq 1$, $\varpi, \kappa \in [0,1]$, and positive ϖ and negative κ. The Pythagorean FSs (PFS) idea was developed by Yager [8] and is distinguished by its MG and non-MG (NMG) with $\varpi + \kappa \geq 1$ to $\varpi^2 + \kappa^2 \leq 1$. Cuong et al. [9] identified three main ideas of the picture FS: positive MG (ϖ), neutral MG (γ), and negative MG (κ). Furthermore, because $\varpi, \gamma, \kappa \in [0,1]$, it provides more advantages than PFS and IFS with $0 \leq \varpi + \gamma + \kappa \leq 1$. According on the picture FS description, expert comments such as "yes," "abstain," "no," and "refusal" will be provided. Shahzaib et al. [10] designed the SFS for certain AOs using MADM. SFS needs $0 \leq \varpi^2 + \gamma^2 + \kappa^2 \leq 1$, not $0 \leq \varpi + \gamma + \kappa \leq 1$. Hussain et al. [11] introduced the idea of an intelligent decision support system for SFS. In the q-rung orthogonal pair FS (q-ROFS), both the MG and the NMG have power q, but their total can never be more than 1. Xu et al. [12] used IFSs to develop geometric operators, including weighted, ordered weighted, and hybrid operators. Li et al. [13] introduced generalized ordered weighted averaging operators (GOWs) in 2002. Zeng et al. [14] presented a method for computing ordered weighted distances with AOs and measurements. Peng et al. [15] analyzed a simple PFS based on AO features. Palanikumar et al. studied algebraic structures and aggregation techniques for applications [16–18]. I shall adhere to the format given here for the remainder of my work. Several techniques on RFFFNs are described in Section 2. The AOs based on CSRFFFN are covered in Section 3. Future research and conclusion are covered in Section 4.

The following are the main reasons behind this investigation:

(i) The initial motivation for this research is the limitations of existing FS and AO techniques. While these techniques have proven beneficial, they have certain limitations in describing CFS settings and coping with periodic unpredictability. This encourages the investigation of CFSs with complex fuzzy AOs as prospective strategies to overcome these constraints and improve the task.

(ii) The practical necessity of an efficacy model is the second factor motivating this study. The current challenging circumstances need applying these ideas and techniques to actual problems and offering workable answers. This encourages the development of new theoretical frameworks in the research.

In Figure 1, we provide an explanatory tree diagram to help you get a sense of the different extensions and generalizations that are currently in use.

2. Operations for CSRFFFN

Throughout the section $x = 1$. The AOs are described using WA, WG, GWA and GWG. If $\mathcal{M}$ is a fractional part function, then the fractional part of x, where x is a real number, may be written as follows: The formula $\mathcal{M}(x) = \lfloor x \rfloor = x - \lfloor x \rfloor$. As an alternative, a fractional part function may be used to specify the difference between a real number and its biggest integer value, which is found using the greatest integer function. The fractional component of $x = 0$ if x is an integer. A reciprocal fractional part function

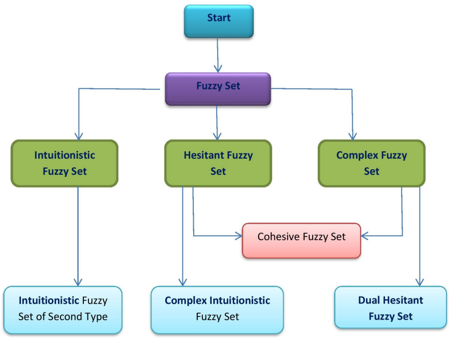

Figure 1: A generalizations of the fuzzy set.

$\mathcal{M}(x) = \frac{1}{x} = \propto$ is assumed. Every time x is an integer, its fractional component equals zero, as is commonly known. For $\mathcal{M}(x) = \frac{1}{x} = \propto$ to be defined, x must not be an integer. All real numbers, except integers, fall inside the domain of $\mathcal{M}(x) = \propto$. The concept of a complex sine trigonometric RFFFN is presented. This led to the establishment of $\sin \pi/2 = \lambda$, the CSRFFFN, and its functions.

Definition 1. *The CSRFFF set* $\Upsilon = \{\tau, \langle \lfloor \lfloor \lambda \mathcal{R} \rfloor (\tau) \cdot e^{\lfloor \lambda \mathcal{I} \rfloor (\tau), \lfloor \lambda \mho \rfloor (\tau) \cdot e^{\lfloor \lambda \Lambda \rfloor (\tau)}} \rfloor \rangle | \tau \in A\}$, *where* $\lfloor \lambda \mathcal{R} \rfloor, \lfloor \lambda \mho \rfloor : A \to \lfloor 0, 1 \rfloor$ *denote the MG and NMG of* $\tau \in A$ *to* Υ, *respectively and* $0 \leq \lfloor \lfloor \lambda \mathcal{R} \rfloor (\tau) \rfloor^{\propto} + \lfloor \lfloor \lambda \mho \rfloor (\tau) \rfloor^{\propto} \leq 1$ *and* $0 \leq \lfloor \lfloor \lambda \mathcal{I} \rfloor (\tau) \rfloor^{\propto} + \lfloor \lfloor \lambda \Lambda \rfloor (\tau) \rfloor^{\propto} \leq 1$. *For,* $\Upsilon = \left\langle \left\lfloor \lfloor \lambda \mathcal{R} \rfloor \cdot e^{\lfloor \lambda \mathcal{I} \rfloor}, \lfloor \lambda \mho \rfloor \cdot e^{\lfloor \lambda \Lambda \rfloor} \right\rfloor \right\rangle$ *is represent a CSRFFFN.*

Example 1. *Consider the CSRFFF set* $\Upsilon = \{y_1, y_2, y_3, y_4\}$ *and* $A = \{(y_1, 0.6e^{i\pi(0.45)}, 0.55e^{i\pi(0.65)}),$ $(y_2, 0.5e^{i\pi(0.4)}, 0.55e^{i\pi(0.7)}),\ (y_3, 0.65e^{i\pi(0.55)}, 0.45e^{i\pi(0.65)}), (y_4, 0.35e^{i\pi(0.6)}, 0.65e^{i\pi(0.7)})\}$.

Definition 2. *Let* $\Upsilon = \langle \lfloor \lfloor \lambda \mathcal{R} \rfloor \cdot e^{\lfloor \lambda \mathcal{I} \rfloor}, \lfloor \lfloor \lambda \mho \rfloor \rfloor \cdot e^{\lfloor \lambda \Lambda \rfloor} \rfloor \rangle, \Upsilon_1 = \langle \lfloor \lfloor \lambda \mathcal{R}_1 \rfloor \cdot e^{\lfloor \lambda \mathcal{I}_1 \rfloor}, \lfloor \lambda \mho_1 \rfloor \cdot e^{\lfloor \lambda \Lambda_1 \rfloor} \rfloor \rangle, \Upsilon_2 = \langle \lfloor \lfloor \lambda \mathcal{R}_2 \rfloor \cdot e^{\lfloor \lambda \mathcal{I}_2 \rfloor}, \lfloor \lambda \mho_2 \rfloor \cdot e^{\lfloor \lambda \Lambda_2 \rfloor} \rfloor \rangle$ *be any three CSRFFFNs. Then*

$$\textit{(i)}\ \ \Upsilon_1 \nabla \Upsilon_2 = \begin{bmatrix} \sqrt[\propto]{\begin{matrix} \lfloor \lfloor \lambda \mathcal{R}_1 \rfloor \rfloor^{\propto} + \lfloor \lfloor \lambda \mathcal{R}_2 \rfloor \rfloor^{\propto} \\ -\lfloor \lfloor \lambda \mathcal{R}_1 \rfloor \rfloor^{\propto} \cdot \lfloor \lfloor \lambda \mathcal{R}_2 \rfloor \rfloor^{\propto} \end{matrix}} \cdot e^{\sqrt[\propto]{\begin{matrix} \lfloor \lfloor \lambda \mathcal{I}_1 \rfloor \rfloor^{\propto} + \lfloor \lfloor \lambda \mathcal{I}_2 \rfloor \rfloor^{\propto} \\ -\lfloor \lfloor \lambda \mathcal{I}_1 \rfloor \rfloor^{\propto} \cdot \lfloor \lfloor \lambda \mathcal{I}_2 \rfloor \rfloor^{\propto} \end{matrix}}}, \\ \lfloor \lfloor \lambda \mho_1 \rfloor \rfloor^{\propto} \lfloor \lfloor \lambda \mho_2 \rfloor \rfloor^{\propto} \cdot e^{\lfloor \lfloor \lambda \Lambda_1 \rfloor \rfloor^{\propto} \lfloor \lfloor \lambda \Lambda_2 \rfloor \rfloor^{\propto}} \end{bmatrix},$$

$$\textit{(ii)}\ \ \Upsilon_1 \circ \Upsilon_2 = \begin{bmatrix} \lfloor \lfloor \lambda \mathcal{R}_1 \rfloor \rfloor^{\propto} \lfloor \lfloor \lambda \mathcal{R}_2 \rfloor \rfloor^{\propto} \cdot e^{\lfloor \lfloor \lambda \mathcal{I}_1 \rfloor \rfloor^{\propto} \lfloor \lfloor \lambda \mathcal{I}_2 \rfloor \rfloor^{\propto}}, \\ \sqrt[\propto]{\begin{matrix} \lfloor \lfloor \lambda \mho_1 \rfloor \rfloor^{\propto} + \lfloor \lfloor \lambda \mho_2 \rfloor \rfloor^{\propto} \\ -\lfloor \lfloor \lambda \mho_1 \rfloor \rfloor^{\propto} \cdot \lfloor \lfloor \lambda \mho_2 \rfloor \rfloor^{\propto} \end{matrix}} \cdot e^{\sqrt[\propto]{\begin{matrix} \lfloor \lfloor \lambda \mho_1 \rfloor \rfloor^{\propto} + \lfloor \lfloor \lambda \Lambda_2 \rfloor \rfloor^{\propto} \\ -\lfloor \lfloor \lambda \mho_1 \rfloor \rfloor^{\propto} \cdot \lfloor \lfloor \lambda \Lambda_2 \rfloor \rfloor^{\propto} \end{matrix}}} \end{bmatrix}$$

$$\textit{(iii)}\ \ \wp \cdot \Upsilon = \begin{bmatrix} \sqrt[\propto]{1 - \lfloor 1 - \lfloor \lambda \lfloor \mathcal{R} \rfloor^{\propto} \rfloor^{\wp}} \cdot e^{\sqrt[\propto]{1 - \lfloor 1 - \lfloor \lambda \lfloor \mathcal{I} \rfloor^{\propto} \rfloor^{\wp}}}, \\ \lfloor \lfloor \lambda \lfloor \mho \rfloor^{\propto} \rfloor^{\wp} \cdot e^{\lfloor \lfloor \lambda \lfloor \Lambda \rfloor^{\propto} \rfloor^{\wp}} \end{bmatrix}$$

(iv) $$\Upsilon^{\wp} = \left[\begin{array}{c} \lfloor\lfloor\lambda\lfloor\mathcal{R}\rfloor^{\propto}\rfloor^{\wp}\cdot e^{\lfloor\lfloor\lambda\lfloor\mathcal{I}\rfloor^{\propto}\rfloor^{\wp}}, \\ \sqrt[\propto]{1-\left\lfloor 1-\lfloor\lambda\lfloor\mho\rfloor^{\propto}\right\rfloor^{\wp}}\cdot e^{\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}\right\rfloor^{\wp}}} \end{array}\right].$$

Definition 3. *For any two CSRFFFNs* $\Upsilon_1 = \langle\lfloor\lfloor\lfloor\lambda\mathcal{R}_1\rfloor, \lfloor\lambda\mho_1\rfloor\rfloor\rfloor\rangle$ *and* $\Upsilon_2 = \langle\lfloor\lfloor\lfloor\lambda\mathcal{R}_2\rfloor, \lfloor\lambda\mho_2\rfloor\rfloor\rfloor\rangle$. *Then*

$$\mathbb{D}_E\lfloor\Upsilon_1,\Upsilon_2\rfloor = \sqrt{\frac{1}{2}\left[\left[\begin{array}{c} 1+\lfloor\lfloor\lambda\mathcal{R}_1\rfloor\rfloor^2-\lfloor\lfloor\lambda\mho_1\rfloor\rfloor^2 \\ -\lfloor 1+\lfloor\lfloor\lambda\mathcal{R}_2\rfloor\rfloor^2-\lfloor\lfloor\lambda\mho_2\rfloor\rfloor^2\rfloor \end{array}\right]^2 + \left[\begin{array}{c} \lfloor\lfloor\lambda\mathcal{I}_1\rfloor\rfloor^2-\lfloor\lfloor\lambda\Lambda_1\rfloor\rfloor^2 \\ -\lfloor\lfloor\lfloor\lambda\mathcal{I}_2\rfloor\rfloor^2-\lfloor\lfloor\lambda\Lambda_2\rfloor\rfloor^2\rfloor \end{array}\right]^2\right]}$$

where $\mathbb{D}_E\lfloor\Upsilon_1,\Upsilon_2\rfloor$ *is called the ED between* Υ_1 *and* Υ_2.

$$\mathbb{D}_H\lfloor\Upsilon_1,\Upsilon_2\rfloor = \frac{1}{2}\left[\left|\begin{array}{c} 1+\lfloor\lfloor\lambda\mathcal{R}_1\rfloor\rfloor^2-\lfloor\lfloor\lambda\mho_1\rfloor\rfloor^2 \\ -\lfloor 1+\lfloor\lfloor\lambda\mathcal{R}_2\rfloor\rfloor^2-\lfloor\lfloor\lambda\mho_2\rfloor\rfloor^2\rfloor \end{array}\right| + \left|\begin{array}{c} \lfloor\lfloor\lambda\mathcal{I}_1\rfloor\rfloor^2-\lfloor\lfloor\lambda\Lambda_1\rfloor\rfloor^2 \\ \lfloor -\lfloor\lfloor\lambda\mathcal{I}_2\rfloor\rfloor^2-\lfloor\lfloor\lambda\Lambda_2\rfloor\rfloor^2\rfloor \end{array}\right|\right]$$

where $\mathbb{D}_H\lfloor\Upsilon_1,\Upsilon_2\rfloor$ *is called the HD between* Υ_1 *and* Υ_2.

3. AOs based on CSRFFFN

Here, we discuss the concept of the CSRFFFNWA, CSRFFFNWG, GCSRFFFNWA and GCSRFFFNWG operator and its properties.

3.1. CTNWA

Definition 4. *Let* $\Upsilon_i = \langle\lfloor\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor}, \lfloor\lambda\mho_i\rfloor\cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor\rfloor\rangle$ *be the CSRFFFNs,* $W = \lfloor\zeta_1,\zeta_2,...,\zeta_z\rfloor$ *be the weight of* Υ_i, $\zeta_i \geq 0$ *and* $\nabla_{i\to1}^{z}\zeta_i = 1$. *Then the AO is defined as CSRFFFNWA* $\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor = \nabla_{i\to1}^{z}\zeta_i\Upsilon_i$.

Theorem 1. *Let* $\Upsilon_i = \left\langle\lfloor\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor}, \lfloor\lambda\mho_i\rfloor\cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor\rfloor\right\rangle$ *be the CSRFFFNs. Then the AO is determined as* $CTNWA\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor$

$$= \left[\begin{array}{c} \sqrt[\propto]{1-\triangle_{i\to1}^{z}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}}\cdot e^{\sqrt[\propto]{1-\triangle_{i\to1}^{z}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}}}, \\ \triangle_{i\to1}^{z}\lfloor\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}\cdot e^{\triangle_{i\to1}^{z}\lfloor\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}} \end{array}\right].$$

Proof. If $n=2$, then CSRFFFNWA$\lfloor\Upsilon_1,\Upsilon_2\rfloor = \zeta_1\Upsilon_1\nabla\zeta_2\Upsilon_2$, where

$$\zeta_1\Upsilon_1 = \left[\begin{array}{c} \sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_1\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_1}}\cdot e^{\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_1\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_1}}}, \\ \lfloor\lfloor\lfloor\lambda\mho_1\rfloor\rfloor^{\propto}\rfloor^{\zeta_1}\cdot e^{\lfloor\lfloor\lfloor\lambda\Lambda_1\rfloor\rfloor^{\propto}\rfloor^{\zeta_1}} \end{array}\right]$$

$$\zeta_2\Upsilon_2 = \begin{bmatrix} \sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_2\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_2}} \cdot e^{\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_2\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_2}}}, \\ \lfloor\lfloor\lfloor\lambda\mho_2\rfloor\rfloor^{\propto}\rfloor^{\zeta_2} \cdot e^{\lfloor\lfloor\lfloor\lambda\Lambda_2\rfloor\rfloor^{\propto}\rfloor^{\zeta_2}} \end{bmatrix}.$$

Now, $\zeta_1\Upsilon_1\nabla\zeta_2\Upsilon_2$

$$= \begin{bmatrix} \sqrt[\propto]{\begin{array}{c}\left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_1\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_1}\right\rfloor + \\ \left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_2\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_2}\right\rfloor \\ -\left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_1\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_1}\right\rfloor \cdot \\ \left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_2\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_2}\right\rfloor,\end{array}} \cdot e^{\sqrt[\propto]{\begin{array}{c}\left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_1\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_1}\right\rfloor + \\ \left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_2\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_2}\right\rfloor \\ -\left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_1\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_1}\right\rfloor \cdot \\ \left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_2\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_2}\right\rfloor,\end{array}}} \\ \lfloor\lfloor\lfloor\lambda\mho_1\rfloor\rfloor^{\propto}\rfloor^{\zeta_1} \cdot \lfloor\lfloor\lfloor\lambda\mho_2\rfloor\rfloor^{\propto}\rfloor^{\zeta_2} \cdot e^{\lfloor\lfloor\lfloor\lambda\Lambda_1\rfloor\rfloor^{\propto}\rfloor^{\zeta_1}\cdot\lfloor\lfloor\lfloor\lambda\Lambda_2\rfloor\rfloor^{\propto}\rfloor^{\zeta_2}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_1\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_1}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_2\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_2}} \cdot \\ e^{\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_1\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_1}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_2\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_2}}}, \\ \lfloor\lfloor\lfloor\lambda\mho_1\rfloor\rfloor^{\propto}\rfloor^{\zeta_1} \cdot \lfloor\lfloor\lfloor\lambda\mho_2\rfloor\rfloor^{\propto}\rfloor^{\zeta_2} \cdot \\ e^{\lfloor\lfloor\lfloor\lambda\Lambda_1\rfloor\rfloor^{\propto}\rfloor^{\zeta_1}\cdot\lfloor\lfloor\lfloor\lambda\Lambda_2\rfloor\rfloor^{\propto}\rfloor^{\zeta_2}} \end{bmatrix}$$

Hence, CT$NWA\lfloor\Upsilon_1,\Upsilon_2\rfloor$

$$= \begin{bmatrix} \sqrt[\propto]{1-\triangle_{i\to1}^{2}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}} \cdot e^{\sqrt[\propto]{1-\triangle_{i\to1}^{2}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}}}, \\ \triangle_{i\to1}^{2}\lfloor\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i} \cdot e^{\triangle_{i\to1}^{2}\lfloor\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}} \end{bmatrix}.$$

It valid for $z \geq 3$,
Thus, CT$NWA\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor$

$$= \begin{bmatrix} \sqrt[\propto]{1-\triangle_{i\to1}^{z}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}} \cdot e^{\sqrt[\propto]{1-\triangle_{i\to1}^{z}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}}}, \\ \triangle_{i\to1}^{z}\lfloor\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i} \cdot e^{\triangle_{i\to1}^{z}\lfloor\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}} \end{bmatrix}.$$

If $z = z + 1$, then CSRFFFNWA $\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z,\Upsilon_{z+1}\rfloor$

$$= \left[\begin{array}{c} \sqrt[\propto]{\begin{array}{l}\nabla_{i\to 1}^{z}\left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}\right\rfloor + \left\lfloor 1-\left\lfloor 1-\lfloor\mathcal{R}_{z+1}\rfloor^{\propto}\right\rfloor^{\zeta_{z+1}}\right\rfloor \\ -\triangle_{i\to 1}^{z}\left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}\right\rfloor \cdot \left\lfloor 1-\left\lfloor 1-\lfloor\mathcal{R}_{z+1}\rfloor^{\propto}\right\rfloor^{\zeta_{z+1}}\right\rfloor\end{array}} \cdot \\ e^{\sqrt[\propto]{\begin{array}{l}\nabla_{i\to 1}^{z}\left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}\right\rfloor + \left\lfloor 1-\left\lfloor 1-\lfloor\mathcal{I}_{z+1}\rfloor^{\propto}\right\rfloor^{\zeta_{z+1}}\right\rfloor \\ -\triangle_{i\to 1}^{z}\left\lfloor 1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}\right\rfloor \cdot \left\lfloor 1-\left\lfloor 1-\lfloor\mathcal{I}_{z+1}\rfloor^{\propto}\right\rfloor^{\zeta_{z+1}}\right\rfloor\end{array}}}, \\ \triangle_{i\to 1}^{z}\lfloor\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}\cdot\lfloor\lfloor\mho_{z+1}\rfloor^{\propto}\rfloor^{\zeta_{z+1}}\cdot e^{\triangle_{i\to 1}^{z}\lfloor\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}\cdot\lfloor\lfloor\Lambda_{z+1}\rfloor^{\propto}\rfloor^{\zeta_{z+1}}} \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[\propto]{1-\triangle_{i\to 1}^{z+1}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}} \cdot e^{\sqrt[\propto]{1-\triangle_{i\to 1}^{z+1}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}}}, \\ \triangle_{i\to 1}^{z+1}\lfloor\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}\cdot e^{\triangle_{i\to 1}^{z+1}\lfloor\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}} \end{array} \right].$$

Theorem 2. *Let* $\Upsilon_i = \left\langle \lfloor\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor},\ \lfloor\lambda\mho_i\rfloor\cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor\rfloor \right\rangle$ *be the CSRFFFNs. Then CSRFFFNWA* $\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor = \Upsilon$.

Proof. Since $\lfloor\lambda\mathcal{R}_i\rfloor = \lfloor\lambda\mathcal{R}\rfloor$, $\lfloor\lambda\mho_i\rfloor = \lfloor\lambda\mho\rfloor$ and $\lfloor\lambda\mathcal{I}_i\rfloor = \lfloor\lambda\mathcal{I}\rfloor$, $\lfloor\lambda\Lambda_i\rfloor = \lfloor\lambda\Lambda\rfloor$ and $\nabla_{i\to 1}^{z}\zeta_i = 1$. Now, $\mathrm{C}TNWA\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor$

$$= \left[\begin{array}{c} \sqrt[\propto]{1-\triangle_{i\to 1}^{z}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}} \cdot e^{\sqrt[\propto]{1-\triangle_{i\to 1}^{z}\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}}}, \\ \triangle_{i\to 1}^{z}\lfloor\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}\cdot e^{\triangle_{i\to 1}^{z}\lfloor\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}} \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[\propto]{1-\left\lfloor 1-\lfloor\lambda\lfloor\mathcal{R}\rfloor^{\propto}\right\rfloor^{\nabla_{i\to 1}^{z}\zeta_i}} \cdot e^{\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lambda\lfloor\mathcal{I}\rfloor^{\propto}\right\rfloor^{\nabla_{i\to 1}^{z}\zeta_i}}}, \\ \lfloor\lfloor\lambda\lfloor\mho\rfloor^{\propto}\rfloor^{\nabla_{i\to 1}^{z}\zeta_i}\cdot e^{\lfloor\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}\rfloor^{\nabla_{i\to 1}^{z}\zeta_i}} \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[\propto]{1-\left\lfloor 1-\lfloor\lambda\lfloor\mathcal{R}\rfloor^{\propto}\right\rfloor} \cdot e^{\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lambda\lfloor\mathcal{I}\rfloor^{\propto}\right\rfloor}}, \\ \lfloor\lambda\lfloor\mho\rfloor^{\propto}\cdot e^{\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}} \end{array} \right]$$

$$= \Upsilon.$$

Theorem 3. *Let* $\Upsilon_i = \left\langle \lfloor\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor},\ \lfloor\lambda\mho_i\rfloor\cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor\rfloor \right\rangle$ *be the CSRFFFNs. Then CSRFFFNWA*$\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor$, *where* $\underline{\lfloor\lambda\mathcal{R}\rfloor} = \min\lfloor\lambda\mathcal{R}_{ij}\rfloor$, $\overline{\lfloor\lambda\mathcal{R}\rfloor} = \max\lfloor\lambda\mathcal{R}_{ij}\rfloor$, $\underline{\lfloor\lambda\mho\rfloor} = \min\lfloor\lambda\mho_{ij}\rfloor$, $\overline{\lfloor\lambda\mho\rfloor} = \max\lfloor\lambda\mho_{ij}\rfloor$ *and* $\underline{\lfloor\lambda\mathcal{I}\rfloor} = \min\lfloor\lambda\mathcal{I}_{ij}\rfloor$, $\overline{\lfloor\lambda\mathcal{I}\rfloor} = \max\lfloor\lambda\mathcal{I}_{ij}\rfloor$, $\underline{\lfloor\lambda\Lambda\rfloor} = \min\lfloor\lambda\Lambda_{ij}\rfloor$, $\overline{\lfloor\lambda\Lambda\rfloor} = \max\lfloor\lambda\Lambda_{ij}\rfloor$ *and where* $1 \le i \le n,\ j = 1,2,...,i_j$. *Then,*
$\left\langle \underline{\lfloor\lambda\mathcal{R}\rfloor\cdot e^{\lfloor\lambda\mathcal{I}\rfloor}}, \overline{\lfloor\lambda\mho\rfloor\cdot e^{\lfloor\lambda\Lambda\rfloor}} \right\rangle \le CTNWA\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor \le \left\langle \overline{\lfloor\lambda\mathcal{R}\rfloor\cdot e^{\lfloor\lambda\mathcal{I}\rfloor}}, \underline{\lfloor\lambda\mho\rfloor\cdot e^{\lfloor\lambda\Lambda\rfloor}} \right\rangle.$

Proof. Since, $\underline{\lfloor\lambda\mathcal{R}\rfloor} = \min\lfloor\lambda\mathcal{R}_{ij}\rfloor$, $\overline{\lfloor\lambda\mathcal{R}\rfloor} = \max\lfloor\lambda\mathcal{R}_{ij}\rfloor$ and $\underline{\lfloor\lambda\mathcal{R}\rfloor} \leq \lfloor\lambda\mathcal{R}_{ij}\rfloor \leq \overline{\lfloor\lambda\mathcal{R}\rfloor}$ and $\underline{\lfloor\lambda\mathcal{I}\rfloor} = \min\lfloor\lambda\mathcal{I}_{ij}\rfloor$, $\overline{\lfloor\lambda\mathcal{I}\rfloor} = \max\lfloor\lambda\mathcal{I}_{ij}\rfloor$ and $\underline{\lfloor\lambda\mathcal{I}\rfloor} \leq \lfloor\lambda\mathcal{I}_{ij}\rfloor \leq \overline{\lfloor\lambda\mathcal{I}\rfloor}$.

Now

$$\begin{aligned}
\underline{\lfloor\lambda\mathcal{R}\rfloor}\cdot e^{\underline{\lfloor\lambda\mathcal{I}\rfloor}} &= \sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z}\left\lfloor \shortmid - \lfloor\underline{\lfloor\lambda\mathcal{R}\rfloor}\rfloor^{\propto}\right\rfloor^{\zeta_i}} \cdot e^{\sqrt[\propto]{\shortmid-\triangle_{i\to1}^{z}\left\lfloor\shortmid-\lfloor\underline{\lfloor\lambda\mathcal{I}\rfloor}\rfloor^{\propto}\right\rfloor^{\zeta_i}}} \\
&\leq \sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z}\left\lfloor \shortmid - \lfloor\lfloor\lfloor\lambda\mathcal{R}_{ij}\rfloor\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}} \cdot e^{\sqrt[\propto]{\shortmid-\triangle_{i\to1}^{z}\left\lfloor\shortmid-\lfloor\lfloor\lfloor\lambda\mathcal{I}_{ij}\rfloor\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}}} \\
&\leq \sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z}\left\lfloor \shortmid - \lfloor\overline{\lfloor\lambda\mathcal{R}\rfloor}\rfloor^{\propto}\right\rfloor^{\zeta_i}} \cdot e^{\sqrt[\propto]{\shortmid-\triangle_{i\to1}^{z}\left\lfloor\shortmid-\lfloor\overline{\lfloor\lambda\mathcal{I}\rfloor}\rfloor^{\propto}\right\rfloor^{\zeta_i}}} \\
&= \overline{\lfloor\lambda\mathcal{R}\rfloor}.
\end{aligned}$$

Since, $\underline{\lfloor\lambda\lfloor\mho\rfloor^{\propto}\rfloor} = \min\lfloor\lfloor\lambda\mho_{ij}\rfloor\rfloor^{\propto}$, $\overline{\lfloor\lambda\lfloor\mho\rfloor^{\propto}\rfloor} = \max\lfloor\lfloor\lambda\mho_{ij}\rfloor\rfloor^{\propto}$ and $\underline{\lfloor\lambda\lfloor\mho\rfloor^{\propto}\rfloor} \leq \lfloor\lfloor\lambda\mho_{ij}\rfloor\rfloor^{\propto} \leq \overline{\lfloor\lambda\lfloor\mho\rfloor^{\propto}\rfloor}$ and $\underline{\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}\rfloor} = \min\lfloor\lfloor\lambda\Lambda_{ij}\rfloor\rfloor^{\propto}$, $\overline{\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}\rfloor} = \max\lfloor\lfloor\lambda\Lambda_{ij}\rfloor\rfloor^{\propto}$ and $\underline{\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}\rfloor} \leq \lfloor\lfloor\lambda\Lambda_{ij}\rfloor\rfloor^{\propto} \leq \overline{\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}\rfloor}$.

We have,

$$\begin{aligned}
\underline{\lfloor\lambda\lfloor\mho\rfloor^{\propto}} &= \triangle_{i\to1}^{z}\underline{\lfloor\lambda\lfloor\mho\rfloor^{\propto}\rfloor}^{\zeta_i}\cdot e^{\triangle_{i\to1}^{z}\underline{\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}\rfloor}^{\zeta_i}} \\
&\leq \triangle_{i\to1}^{z}\lfloor\lfloor\lfloor\lambda\mho_{ij}\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}\cdot e^{\triangle_{i\to1}^{z}\lfloor\lfloor\lfloor\lambda\Lambda_{ij}\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}} \\
&\leq \triangle_{i\to1}^{z}\overline{\lfloor\lambda\lfloor\mho\rfloor^{\propto}\rfloor}^{\zeta_i}\cdot e^{\triangle_{i\to1}^{z}\overline{\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}\rfloor}^{\zeta_i}} \\
&= \overline{\lfloor\lambda\lfloor\mho\rfloor^{\propto}\rfloor}\cdot e^{\overline{\lfloor\lambda\lfloor\Lambda\rfloor^{\propto}\rfloor}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\triangle\left[\begin{array}{l}\left[\left\lfloor\sqrt[\propto]{\shortmid-\triangle_{i\to1}^{z}\left\lfloor\shortmid-\lfloor\underline{\lfloor\lambda\mathcal{R}\rfloor}\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{2}+\shortmid-\left\lfloor\triangle_{i\to1}^{z}\lfloor\overline{\lfloor\lambda\mho\rfloor}\rfloor^{\propto}\rfloor^{\zeta_i}\right\rfloor^{2}\right] \\ +\left[\left\lfloor\sqrt[\propto]{\shortmid-\triangle_{i\to1}^{z}\left\lfloor\shortmid-\lfloor\underline{\lfloor\lambda\mathcal{I}\rfloor}\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{2}-\left\lfloor\triangle_{i\to1}^{z}\lfloor\overline{\lfloor\lambda\Lambda\rfloor}\rfloor^{\propto}\rfloor^{\zeta_i}\right\rfloor^{2}\right]\end{array}\right] \\
\leq\ &\frac{1}{2}\left[\begin{array}{l}\left[\left\lfloor\sqrt[\propto]{\shortmid-\triangle_{i\to1}^{z}\left\lfloor\shortmid-\lfloor\lfloor\lambda\lfloor\lambda\mathcal{R}_{ij}\rfloor\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{2}+\shortmid-\left\lfloor\triangle_{i\to1}^{z}\lfloor\lfloor\lfloor\lambda\mho_{ij}\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}\right\rfloor^{2}\right] \\ +\left[\left\lfloor\sqrt[\propto]{\shortmid-\triangle_{i\to1}^{z}\left\lfloor\shortmid-\lfloor\lfloor\lambda\lfloor\lambda\mathcal{I}_{ij}\rfloor\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{2}-\left\lfloor\triangle_{i\to1}^{z}\lfloor\lfloor\lfloor\lambda\Lambda_{ij}\rfloor\rfloor^{\propto}\rfloor^{\zeta_i}\right\rfloor^{2}\right]\end{array}\right] \\
\leq\ &\frac{1}{2}\left[\begin{array}{l}\left[\left\lfloor\sqrt[\propto]{\shortmid-\triangle_{i\to1}^{z}\left\lfloor\shortmid-\lfloor\overline{\lfloor\lambda\mathcal{R}\rfloor}\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{2}+\shortmid-\left\lfloor\triangle_{i\to1}^{z}\lfloor\underline{\lfloor\lambda\mho\rfloor}\rfloor^{\propto}\rfloor^{\zeta_i}\right\rfloor^{2}\right] \\ +\left[\left\lfloor\sqrt[\propto]{\shortmid-\triangle_{i\to1}^{z}\left\lfloor\shortmid-\lfloor\overline{\lfloor\lambda\mathcal{I}\rfloor}\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{2}-\left\lfloor\triangle_{i\to1}^{z}\lfloor\underline{\lfloor\lambda\Lambda\rfloor}\rfloor^{\propto}\rfloor^{\zeta_i}\right\rfloor^{2}\right]\end{array}\right].
\end{aligned}$$

Hence, $\left\langle \underline{\lfloor\lambda\mathcal{R}\rfloor \cdot e^{\lfloor\lambda\mathcal{I}\rfloor}}, \overline{\lfloor\lambda\mho\rfloor \cdot e^{\lfloor\lambda\Lambda\rfloor}} \right\rangle \leq CTNWA\lfloor\Upsilon_1, \Upsilon_2, ..., \Upsilon_z\rfloor$
$\leq \langle \overline{\lfloor\lambda\mathcal{R}\rfloor \cdot e^{\lfloor\lambda\mathcal{I}\rfloor}}, \underline{\lfloor\lambda\mho\rfloor \cdot e^{\lfloor\lambda\Lambda\rfloor}} \rangle$.

Theorem 4. *Let* $\Upsilon_i = \langle\lfloor\lfloor\lambda\mathcal{R}_{t_{ij}}\rfloor \cdot e^{\lfloor\lambda\mathcal{I}_{t_{ij}}\rfloor}, \lfloor\lambda\mho_{t_{ij}}\rfloor \cdot e^{\lfloor\lambda\Lambda_{t_{ij}}\rfloor}\rfloor\rangle$ *and* $W_i = \langle\lfloor\lfloor\lambda\mathcal{R}_{h_{ij}}\rfloor \cdot e^{\lfloor\lambda\mathcal{I}_{h_{ij}}\rfloor}, \lfloor\lambda\mho_{h_{ij}}\rfloor \cdot e^{\lfloor\lambda\Lambda_{h_{ij}}\rfloor}\rfloor\rangle$, *be the CSRFFFNWAs. For any* i, *if there is* $\lfloor\lambda\mathcal{R}_{t_{ij}}\rfloor^2 \leq \lfloor\lambda\mathcal{R}_{h_{ij}}\rfloor^2$ *and* $\lfloor\lambda\mho_{t_{ij}}\rfloor^2 \geq \lfloor\lambda\mho_{h_{ij}}\rfloor^2$ *and* $\lfloor\lambda\mathcal{I}_{t_{ij}}\rfloor^2 \leq \lfloor\lambda\mathcal{I}_{h_{ij}}\rfloor^2$ *and* $\lfloor\lambda\Lambda_{t_{ij}}\rfloor^2 \geq \lfloor\lambda\Lambda_{h_{ij}}\rfloor^2$*or* $\Upsilon_i \leq W_i$. *Prove that* $CTNWA\lfloor\Upsilon_1, \Upsilon_2, ..., \Upsilon_z\rfloor \leq CTNWA(W_1, W_2, ..., W_z)$, *where* $\lfloor i = 1, 2, ..., z\rfloor; \lfloor j = 1, 2, ..., i_j\rfloor$.

Proof. For any i, $\lfloor\lambda\mathcal{R}_{t_{ij}}\rfloor^2 \leq \lfloor\lambda\mathcal{R}_{h_{ij}}\rfloor^2$ and $\shortmid - \lfloor\lfloor\lambda\mathcal{R}_{t_i}\rfloor\rfloor^2 \geq \shortmid - \lfloor\lfloor\lambda\mathcal{R}_{h_i}\rfloor\rfloor^2$.
Hence, $\triangle_{i\to1}^{z} \left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{R}_{t_i}\rfloor\rfloor^2 \right\rfloor^{\zeta_i} \geq \triangle_{i\to1}^{z} \left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{R}_{h_i}\rfloor\rfloor^2 \right\rfloor^{\zeta_i}$
and $\sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z} \left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{R}_{t_i}\rfloor\rfloor^{\propto} \right\rfloor^{\zeta_i}} \leq \sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z} \left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{R}_{h_i}\rfloor\rfloor^{\propto} \right\rfloor^{\zeta_i}}$.
Similarly, $\lfloor\lambda\mathcal{I}_{t_{ij}}\rfloor^2 \leq \lfloor\lambda\mathcal{I}_{h_{ij}}\rfloor^2$ and $\shortmid - \lfloor\lfloor\lambda\mathcal{I}_{t_i}\rfloor\rfloor^2 \geq \shortmid - \lfloor\lfloor\lambda\mathcal{I}_{h_i}\rfloor\rfloor^2$.
Hence, $\triangle_{i\to1}^{z} \left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{I}_{t_i}\rfloor\rfloor^2 \right\rfloor^{\zeta_i} \geq \triangle_{i\to1}^{z} \left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{I}_{h_i}\rfloor\rfloor^2 \right\rfloor^{\zeta_i}$
and $\sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z} \left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{I}_{t_i}\rfloor\rfloor^{\propto} \right\rfloor^{\zeta_i}} \leq \sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z} \left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{I}_{h_i}\rfloor\rfloor^{\propto} \right\rfloor^{\zeta_i}}$.
For any i, $\left\lfloor\lfloor\lambda\mho_{t_{ij}}\rfloor\right\rfloor^2 \geq \left\lfloor\lfloor\lambda\mho_{h_{ij}}\rfloor\right\rfloor^2$ and $\left\lfloor\lfloor\lambda\mho_{t_{ij}}\rfloor\right\rfloor^{\propto} \geq \left\lfloor\lfloor\lambda\mho_{h_{ij}}\rfloor\right\rfloor^{\propto}$.
Therefore, $\shortmid - \left\lfloor\triangle_{i\to1}^{z}\lfloor\lambda\mho_{t_{ij}}\rfloor\right\rfloor^{\propto} \leq \shortmid - \left\lfloor\triangle_{i\to1}^{z}\lfloor\lambda\mho_{h_{ij}}\rfloor\right\rfloor^{\propto}$.
Similarly, for any i,
$\left\lfloor\lfloor\lambda\Lambda_{t_{ij}}\rfloor\right\rfloor^2 \geq \left\lfloor\lfloor\lambda\Lambda_{h_{ij}}\rfloor\right\rfloor^2$ and $\left\lfloor\lfloor\lambda\Lambda_{t_{ij}}\rfloor\right\rfloor^{\propto} \geq \left\lfloor\lfloor\lambda\Lambda_{h_{ij}}\rfloor\right\rfloor^{\propto}$.
Therefore, $-\left\lfloor\triangle_{i\to1}^{z}\lfloor\lambda\Lambda_{t_{ij}}\rfloor\right\rfloor^{\propto} \leq -\left\lfloor\triangle_{i\to1}^{z}\lfloor\lambda\Lambda_{h_{ij}}\rfloor\right\rfloor^{\propto}$.
Hence,

$$\frac{1}{2}\left[\left[\begin{matrix}\left\lfloor \sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z}\left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{R}_{ti}\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}} \right\rfloor^2 \\ +1 - \left\lfloor\triangle_{i\to1}^{z}\lfloor\lfloor\lambda\mho_{t_i}\rfloor\rfloor^{\propto}\right\rfloor^2\end{matrix}\right] + \left[\begin{matrix}\left\lfloor \sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z}\left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{I}_{ti}\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}} \right\rfloor^2 \\ - \left\lfloor\triangle_{i\to1}^{z}\lfloor\lfloor\lambda\Lambda_{t_i}\rfloor\rfloor^{\propto}\right\rfloor^2\end{matrix}\right]\right]$$
$$\leq \frac{1}{2}\left[\left[\begin{matrix}\left\lfloor \sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z}\left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{R}_{hi}\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}} \right\rfloor^2 \\ +1 - \left\lfloor\triangle_{i\to1}^{z}\lfloor\lfloor\lambda\mho_{h_i}\rfloor\rfloor^{\propto}\right\rfloor^2\end{matrix}\right] + \left[\begin{matrix}\left\lfloor \sqrt[\propto]{\shortmid - \triangle_{i\to1}^{z}\left\lfloor \shortmid - \lfloor\lfloor\lambda\mathcal{I}_{hi}\rfloor\rfloor^{\propto}\right\rfloor^{\zeta_i}} \right\rfloor^2 \\ - \left\lfloor\triangle_{i\to1}^{z}\lfloor\lfloor\lambda\Lambda_{h_i}\rfloor\rfloor^{\propto}\right\rfloor^2\end{matrix}\right]\right].$$

Hence, CT$NWA\lfloor\Upsilon_1, \Upsilon_2, ..., \Upsilon_z\rfloor \leq CTNWA\lfloor W_1, W_2, ..., W_z\rfloor$.

3.2. CSRFFFNWG

Definition 5. *Let* $\Upsilon_i = \left\langle \left\lfloor \lfloor\lfloor\lambda\mathcal{R}_i\rfloor \cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor}, \lfloor\lambda\mho_i\rfloor \cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor \right\rfloor \right\rangle$ *be the CSRFFFNs. Then the AO RFFFNWG is defined as* $\lfloor\Upsilon_1, \Upsilon_2, ..., \Upsilon_z\rfloor = \triangle_{i\to1}^{z}\Upsilon_i^{\zeta_i}$.

Corollary 1. *Let* $\Upsilon_i = \left\langle \left\lfloor \lfloor\lfloor\lambda\mathcal{R}_i\rfloor \cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor}, \lfloor\lambda\mho_i\rfloor \cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor \right\rfloor \right\rangle$ *be the CSRFFFNs. Then the AO CSRFFFNWG is determined as* $\lfloor\Upsilon_1, \Upsilon_2, ..., \Upsilon_z\rfloor$

$$= \left[\frac{\triangle_{i\to 1}^{z} \lfloor\lfloor\lfloor \lambda\mathcal{R}_i \rfloor\rfloor^{\propto} \rfloor^{\zeta_i} \cdot e^{\triangle_{i\to 1}^{z} \lfloor\lfloor\lfloor \lambda\mathcal{I}_i \rfloor\rfloor^{\propto} \rfloor^{\zeta_i}}}{\sqrt[\propto]{\shortmid - \triangle_{i\to 1}^{z} \left\lfloor \shortmid - \lfloor\lfloor \lambda\mho_i \rfloor\rfloor^{\propto} \right\rfloor^{\zeta_i}} \cdot e^{\sqrt[\propto]{\shortmid - \triangle_{i\to 1}^{z} \left\lfloor \shortmid - \lfloor\lfloor \lambda\Lambda_i \rfloor\rfloor^{\propto} \right\rfloor^{\zeta_i}}}} \right].$$

Corollary 2. *(i) Let* $\Upsilon_i = \left\langle \left\lfloor \lfloor\lfloor \lambda\mathcal{R}_i \rfloor \cdot e^{\lfloor \lambda\mathcal{I}_i \rfloor}, \lfloor \lambda\mho_i \rfloor \cdot e^{\lfloor \lambda\Lambda_i \rfloor} \rfloor \right\rfloor \right\rangle$ *be the CSRFFFNs and all are equal. Then RFFFNWG*$\lfloor \Upsilon_1, \Upsilon_2, ..., \Upsilon_z \rfloor = \Upsilon$.

3.3. Generalized CSRFFFNWA (GCSRFFFNWA)

Definition 6. *Let* $\Upsilon_i = \left\langle \lfloor\lfloor\lfloor \lambda\mathcal{R}_i \rfloor, \lfloor \lambda\mho_i \rfloor\rfloor\rfloor \right\rangle$ *be the CSRFFFN. Then GCSRFFFNWA* $\lfloor \Upsilon_1, \Upsilon_2, ..., \Upsilon_z \rfloor = \left\lfloor \nabla_{i\to 1}^{z} \zeta_i \Upsilon_i^{\wp} \right\rfloor^{1/\wp}$.

Theorem 5. *Let* $\Upsilon_i = \left\langle \lfloor\lfloor\lfloor \lambda\mathcal{R}_i \rfloor, \lfloor \lambda\mho_i \rfloor\rfloor\rfloor \right\rangle$ *be the CSRFFFNs. Then GCSRFFFNWA* $\lfloor \Upsilon_1, \Upsilon_2, ..., \Upsilon_z \rfloor$

$$= \left[\begin{array}{c} \left\lfloor \sqrt[\propto]{\shortmid - \triangle_{i\to 1}^{z} \left\lfloor \shortmid - \left\lfloor \lfloor\lfloor \lambda\mathcal{R}_i \rfloor\rfloor^{\propto} \right\rfloor^{\propto} \right\rfloor^{\zeta_i}} \right\rfloor^{1/\propto} \cdot \\ e^{\left\lfloor \sqrt[\propto]{\shortmid - \triangle_{i\to 1}^{z} \left\lfloor \shortmid - \left\lfloor \lfloor\lfloor \lambda\mathcal{I}_i \rfloor\rfloor^{\propto} \right\rfloor^{\propto} \right\rfloor^{\zeta_i}} \right\rfloor^{1/\propto}}, \\ \sqrt[\propto]{\shortmid - \left\lfloor \shortmid - \left\lfloor \triangle_{i\to 1}^{z} \left\lfloor \sqrt[\propto]{\shortmid - \left\lfloor \shortmid - \lfloor\lfloor \lambda\mho_i \rfloor\rfloor^{\propto} \right\rfloor^{\propto}} \right\rfloor^{\zeta_i} \right\rfloor^{\propto} \right\rfloor^{1/\propto}} \cdot \\ e^{\sqrt[\propto]{\shortmid - \left\lfloor \shortmid - \left\lfloor \triangle_{i\to 1}^{z} \left\lfloor \sqrt[\propto]{\shortmid - \left\lfloor \shortmid - \lfloor\lfloor \lambda\Lambda_i \rfloor\rfloor^{\propto} \right\rfloor^{\propto}} \right\rfloor^{\zeta_i} \right\rfloor^{\propto} \right\rfloor^{1/\propto}}} \end{array} \right].$$

Proof. To illustrate this, we may first show that,

$$\nabla_{i\to 1}^{z} \zeta_i \Upsilon_i^{\propto} = \left[\begin{array}{c} \sqrt[\propto]{\shortmid - \triangle_{i\to 1}^{z} \left\lfloor \shortmid - \left\lfloor \lfloor\lfloor \lambda\mathcal{R}_i \rfloor\rfloor^{\propto} \right\rfloor^{\propto} \right\rfloor^{\zeta_i}} \cdot e^{\sqrt[\propto]{\shortmid - \triangle_{i\to 1}^{z} \left\lfloor \shortmid - \left\lfloor \lfloor\lfloor \lambda\mathcal{I}_i \rfloor\rfloor^{\propto} \right\rfloor^{\propto} \right\rfloor^{\zeta_i}}} \\ \triangle_{i\to 1}^{z} \left\lfloor \sqrt[\propto]{\shortmid - \left\lfloor \shortmid - \lfloor\lfloor \lambda\mho_i \rfloor\rfloor^{\propto} \right\rfloor^{\propto}} \right\rfloor^{\zeta_i} \cdot e^{\triangle_{i\to 1}^{z} \left\lfloor \sqrt[\propto]{\shortmid - \left\lfloor \shortmid - \lfloor\lfloor \lambda\Lambda_i \rfloor\rfloor^{\propto} \right\rfloor^{\propto}} \right\rfloor^{\zeta_i}} \end{array} \right].$$

Put $z = 2$, $\zeta_1 \Upsilon_1 \nabla \zeta_2 \Upsilon_2$

$$= \begin{bmatrix} \sqrt[\propto]{1-\triangle^2_{i\to 1}\left\lfloor 1-\left\lfloor\lfloor\lfloor\lambda\mathcal{R}_1\rfloor\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_i}} \cdot e^{\sqrt[\propto]{1-\triangle^2_{i\to 1}\left\lfloor 1-\left\lfloor\lfloor\lfloor\lambda\mathcal{I}_1\rfloor\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_i}}} \\ \triangle^2_{i\to 1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}}\right\rceil^{\zeta_i} \cdot \triangle^2_{i\to 1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}}\right\rceil^{\zeta_i} \end{bmatrix}.$$

Hence,

$$\nabla^{z}_{i\to 1}\zeta_i\Upsilon^{\wp}_i = \begin{bmatrix} \sqrt[\propto]{1-\triangle^z_{i\to 1}\left\lfloor 1-\left\lfloor\lfloor\lfloor\lambda\mathcal{R}_1\rfloor\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_i}} \cdot e^{\sqrt[\propto]{1-\triangle^z_{i\to 1}\left\lfloor 1-\left\lfloor\lfloor\lfloor\lambda\mathcal{I}_1\rfloor\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_i}}} \\ \triangle^z_{i\to 1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}}\right\rceil^{\zeta_i} \cdot e^{\triangle^z_{i\to 1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}}\right\rceil^{\zeta_i}} \end{bmatrix}.$$

If $z = z+1$, then $\nabla^{z}_{i\to 1}\zeta_i\Upsilon^{\wp}_i + \zeta_{z+1}\Upsilon^{\wp}_{z+1} = \nabla^{z+1}_{i\to 1}\zeta_i\Upsilon^{\wp}_i$.
Now,$\nabla^{z}_{i\to 1}\zeta_i\Upsilon^{\wp}_i + \zeta_{z+1}\Upsilon^{\wp}_{z+1} = \zeta_1\Upsilon^{\wp}_1\nabla\zeta_2\Upsilon^{\wp}_2\nabla ...\nabla\zeta_z\Upsilon^{\wp}_z\nabla\zeta_{z+1}\Upsilon^{\wp}_{z+1}$

$$= \begin{bmatrix} \sqrt[\propto]{\begin{array}{l} \left\lfloor\sqrt[\propto]{1-\triangle^z_{i\to 1}\left\lfloor 1-\left\lfloor\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_i}}\right\rceil^{\propto} + \left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\left\lfloor\lfloor\mathcal{R}_{z+1}\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_1}}\right\rceil^{\propto} \\ -\left\lfloor\sqrt[\propto]{1-\triangle^z_{i\to 1}\left\lfloor 1-\left\lfloor\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_i}}\right\rceil^{\propto}\cdot\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\left\lfloor\lfloor\mathcal{R}_{z+1}\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_1}}\right\rceil^{\propto} \end{array}} \\ \cdot e^{\sqrt[\propto]{\begin{array}{l} \left\lfloor\sqrt[\propto]{1-\triangle^z_{i\to 1}\left\lfloor 1-\left\lfloor\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_i}}\right\rceil^{\propto} + \left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\left\lfloor\lfloor\mathcal{I}_{z+1}\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_1}}\right\rceil^{\propto} \\ -\left\lfloor\sqrt[\propto]{1-\triangle^z_{i\to 1}\left\lfloor 1-\left\lfloor\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_i}}\right\rceil^{\propto}\cdot\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\left\lfloor\lfloor\mathcal{I}_{z+1}\rfloor^{\propto}\right\rceil^{\propto}\right\rceil^{\zeta_1}}\right\rceil^{\propto} \end{array}}} \\ \triangle^z_{i\to 1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}}\right\rceil^{\zeta_i}\cdot\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\mho_{z+1}\rfloor^{\propto}\right\rceil^{\propto}}\right\rceil^{\zeta_1} \\ \cdot e^{\triangle^z_{i\to 1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\right\rceil^{\propto}}\right\rceil^{\zeta_i}\cdot\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\Lambda_{z+1}\rfloor^{\propto}\right\rceil^{\propto}}\right\rceil^{\zeta_1}} \end{bmatrix}$$

$$\nabla_{i\to 1}^{z+1}\zeta_i\Upsilon_i^{\propto} = \begin{bmatrix} \sqrt[\propto]{1-\triangle_{i\to 1}^{z+1}\left\lfloor 1-\left\lfloor\lfloor\lambda\mathcal{R}_1\rfloor\right\rfloor^{\propto}\right\rfloor^{\zeta_i}}\cdot e^{\sqrt[\propto]{1-\triangle_{i\to 1}^{z+1}\left\lfloor 1-\left\lfloor\lfloor\lambda\mathcal{I}_1\rfloor\right\rfloor^{\propto}\right\rfloor^{\zeta_i}}} \\ \triangle_{i\to 1}^{z+1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\right\rfloor}\right\rfloor^{\zeta_i}\cdot e^{\triangle_{i\to 1}^{z+1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\right\rfloor}\right\rfloor^{\zeta_i}} \end{bmatrix}.$$

$$\left\lfloor\nabla_{i\to 1}^{z+1}\zeta_i\Upsilon_i^{\wp}\right\rfloor^{1/\wp} = \begin{bmatrix} \left\lfloor\sqrt[\propto]{1-\triangle_{i\to 1}^{z+1}\left\lfloor 1-\left\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\right\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{1/\propto}\cdot \\ e^{\left\lfloor\sqrt[\propto]{1-\triangle_{i\to 1}^{z+1}\left\lfloor 1-\left\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\right\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{1/\propto}} \\ \sqrt[\propto]{1-\left\lfloor 1-\left\lfloor\triangle_{i\to 1}^{z+1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mho_i\rfloor\rfloor^{\propto}\right\rfloor}\right\rfloor^{\zeta_i}\right\rfloor^{2}\right\rfloor^{1/\propto}}\cdot \\ e^{\sqrt[\propto]{1-\left\lfloor 1-\left\lfloor\triangle_{i\to 1}^{z+1}\left\lfloor\sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\Lambda_i\rfloor\rfloor^{\propto}\right\rfloor}\right\rfloor^{\zeta_i}\right\rfloor^{2}\right\rfloor^{1/\propto}}} \end{bmatrix}$$

Corollary 3. *(i) If $\lfloor\propto,\propto\rfloor = \lfloor 1,1\rfloor$, then CSRFFFNWA operator is convert to the GC-SRFFFNWA operator.*
(ii) If all $\Upsilon_i = \left\langle\left\lfloor\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor}, \lfloor\lambda\mho_i\rfloor\cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor\right\rfloor\right\rangle$ and all are equal. Then GCSRFFFNWA $\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor = \Upsilon$.

3.4. Generalized CSRFFFNWG(GCSRFFFNWG)

Definition 7. *Let $\Upsilon_i = \left\langle\left\lfloor\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor}, \lfloor\lambda\mho_i\rfloor\cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor\right\rfloor\right\rangle$ be the CSRFFFNs. Then the AO GCSRFFFNWG is determined as $\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor = \frac{1}{\wp}\left\lfloor\triangle_{i\to 1}^{z}\lfloor\wp\Upsilon_i\rfloor^{\zeta_i}\right\rfloor$.*

Corollary 4. *Let $\Upsilon_i = \left\langle\left\lfloor\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor}, \lfloor\lambda\mho_i\rfloor\cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor\right\rfloor\right\rangle$ be the CSRFFFNs. Then the AO GCSRFFFNWG is determined as $\lfloor\Upsilon_1,\Upsilon_2,...,\Upsilon_z\rfloor$*

$$= \left[\begin{array}{c} \sqrt[\propto]{1-\left\lfloor 1-\left\lfloor \triangle_{i\to 1}^{z}\left\lfloor \sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{R}_i\rfloor\rfloor^{\propto}\right\rfloor^{\propto}}\right\rfloor^{\zeta_i}\right\rfloor^{\propto}\right\rfloor}^{1/\propto} \cdot \\ e^{\sqrt[\propto]{1-\left\lfloor 1-\left\lfloor \triangle_{i\to 1}^{z}\left\lfloor \sqrt[\propto]{1-\left\lfloor 1-\lfloor\lfloor\lambda\mathcal{I}_i\rfloor\rfloor^{\propto}\right\rfloor^{\propto}}\right\rfloor^{\zeta_i}\right\rfloor^{\propto}\right\rfloor}^{1/\propto}} \\ \left\lfloor \sqrt[\propto]{1-\triangle_{i\to 1}^{z}\left\lfloor 1-\left\lfloor\lfloor\lambda\mho_i\rfloor\right\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{1/\propto} \cdot \\ e^{\left\lfloor \sqrt[\propto]{1-\triangle_{i\to 1}^{z}\left\lfloor 1-\left\lfloor\lfloor\lambda\Lambda_i\rfloor\right\rfloor^{\propto}\right\rfloor^{\zeta_i}}\right\rfloor^{1/\propto}} \end{array}\right].$$

Corollary 5. *(i) When $\wp = 1$, the GCSRFFFNWG is converted to the RFFFNWG.*
(ii) If all $\Upsilon_i = \left\langle \left\lfloor \lfloor\lfloor\lambda\mathcal{R}_i\rfloor \cdot e^{\lfloor\lambda\mathcal{I}_i\rfloor}, \lfloor\lambda\mho_i\rfloor \cdot e^{\lfloor\lambda\Lambda_i\rfloor}\rfloor \right\rfloor \right\rangle$ are equal.
Then GCSRFFFNWG$\lfloor\Upsilon_1, \Upsilon_2, ..., \Upsilon_z\rfloor = \Upsilon$.

4. Conclusion:

In some circumstances where existing approaches fail to produce appropriate results, the suggested method outperforms them in terms of CSRFFF discrimination. Furthermore, this technique employs the selection of operations with the aggregation operator property. However, our proposed solution isn't flawless. It is stuck due to a lack of theoretical justification. Additionally, we are working to find a better approach (or distance metric) for much better exploration and exploitation on CSRFFFs. Instead of being regarded as scalar numbers, CSRFFFs are described as vectors in the complex plane. From the perspective of vector theory, we ought to quantify the difference of CSRFFFs. The novel attributes of CSRFFFs that are induced by this concept of membership vectors differ significantly from those of FS. In this study, new weighted operators including WA and WG operators are introduced. These operators have several properties, such as monotonicity, idempotency, commutativity, associativity and boundedness. We examined several characteristics in order to characterize the weighted vector. We could expand the developed approach in subsequent work to incorporate a range of other AOs. These might include power AOs, Einstein operations, Hamacher operations, Dombi operations and advanced fuzzy ordered weighted quadratic averaging operators using CSRFFF information. Furthermore, we may look at using our recommended method in a number of domains.

Acknowledgements

We greatly appreciate the editors-in-chief and the reviewers for their valuable comments, which have improved the quality of the work.

References

[1] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[2] M. Zeleny. *Multiple Criteria Decision Making Kyoto 1975.* Springer-Verlag, Berlin, Germany, 1976.

[3] D. Ramot, R. Milo, M. Friedman, and A. Kandel. Complex fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 10:171–186, 2022.

[4] D. Ramot, M. Friedman, G. Langholz, and A. Kandel. Complex fuzzy logic. *IEEE Transactions on Fuzzy Systems*, 11:450–461, 2003.

[5] R. R. Yager. Generalized owa aggregation operators. *Fuzzy Optimization and Decision Making*, 3:93–107, 2004.

[6] L. Bi, S. Dai, B. Hu, and S. Li. Complex fuzzy arithmetic aggregation operators. *Journal of Intelligent & Fuzzy Systems*, 36:2765–2771, 2019.

[7] K. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1):87–96, 1986.

[8] R. R. Yager. Pythagorean membership grades in multi criteria decision-making. *IEEE Transactions on Fuzzy Systems*, 22:958–965, 2014.

[9] B. C. Cuong and V. Kreinovich. Picture fuzzy sets: A new concept for computational intelligence problems. In *Proceedings of 2013 Third World Congress on Information and Communication Technologies (WICT2013)*, pages 1–16, 2013.

[10] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, and T. Mahmood. Spherical fuzzy sets and their applications in multi-attribute decision making problems. *Journal of Intelligent and Fuzzy Systems*, 36:282–284, 2019.

[11] A. Hussain and K. Ullah. An intelligent decision support system for spherical fuzzy sugeno-weber aggregation operators and real-life applications. *Spectrum of Mechanical Engineering and Operational Research*, 1(1):177–188, 2024.

[12] Z. Xu and R. R. Yager. Some geometric aggregation operators based on intuitionistic fuzzy sets. *International Journal of General Systems*, 35:417–433, 2006.

[13] D. F. Li. Multi-attribute decision making method based on generalized owa operators with intuitionistic fuzzy sets. *Expert Systems with Applications*, 37:8673–8678, 2010.

[14] S. Zeng and W. Sua. Intuitionistic fuzzy ordered weighted distance operator. *Knowledge-Based Systems*, 24:1224–1232, 2011.

[15] X. Peng and H. Yuan. Fundamental properties of pythagorean fuzzy aggregation operators. *Fundamenta Informaticae*, 147:415–446, 2016.

[16] M. Palanikumar, K. Arulmozhi, and A. Iampan. Multi criteria group decision making based on vikor and topsis methods for fermatean fuzzy soft with aggregation operators. *ICIC Express Letters*, 16(10):1129–1138, 2022.

[17] M. Palanikumar and K. Arulmozhi. Mcgdm based on topsis and vikor using pythagorean neutrosophic soft with aggregation operators. *Neutrosophic Sets and Systems*, pages 538–555, 2022.

[18] M. Palanikumar, N. Kausar, H. Garg, A. Iampan, S. Kadry, and M. Sharaf. Medical robotic engineering selection based on square root neutrosophic normal interval-valued sets and their aggregated operators. *AIMS Mathematics*, 8(8):17402–17432, 2023.