EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6226
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Implicative and Positive Implicative INK-Ideals of Neutrosophic INK-Algebras

Remala Mounikalakshmi[1], Eswarlal Tamma[1], Venkata Kalyani Uppuluri[2], Aiyared Iampan[3,*], T. Srinivasa Rao[1]

[1] *Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Andhra Pradesh 522302, India*
[2] *Freshman Engineering Department, NRI Institute of Technology, Pothavarapadu, Nunna, Agiripalli, Vijayawada-521212, India*
[3] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

Abstract. This paper introduces and develops the concepts of implicative INK-ideals (MINK-Is) and positive implicative INK-ideals (PMINK-Is) within the framework of INK-algebras. These notions are subsequently extended to fuzzy and neutrosophic contexts, resulting in the definitions of fuzzy implicative INK-ideals (FMINK-Is), fuzzy positive implicative INK-ideals (FPMINK-Is), neutrosophic implicative INK-ideals (NMINK-Is), and neutrosophic positive implicative INK-ideals (NPMINK-Is). The structural properties of these ideals are investigated, including their behavior under intersection and union, where it is shown that intersection preserves the respective implicative properties, while union does not in general. Furthermore, we establish that the homomorphic pre-images of these ideals also preserve their respective fuzzy and neutrosophic implicative structures. The results contribute to a deeper understanding of ideal theory in INK-algebras under uncertainty and open pathways for applications in fuzzy logic and neutrosophic systems.

2020 Mathematics Subject Classifications: 03G25, 03E72

Key Words and Phrases: INK-algebra, fuzzy set, neutrosophic set, implicative INK-ideal, positive implicative INK-ideal, fuzzy implicative INK-ideal, fuzzy positive implicative INK-ideal, neutrosophic implicative INK-ideal, neutrosophic positive implicative INK-ideal

1. Introduction

Iséki and Tanaka [1, 2] have worked on the thought of BCK and BCI-algebras to pick up their characteristics and applications. There exists an immense area of empirical applications in fuzzy sets and their generalizations. The literature works on fuzzy

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6226
Email addresses: mouninaidu0521@gmail.com (R. Mounikalakshmi), eswarlal@kluniversity.in (E. Tamma), u.v.kalyani@gmail.com (V. Kalyani Uppuluri), aiyared.ia@up.ac.th (A. Iampan), tsr_2505@kluniversity.in (T. Srinivasa Rao)

subalgebras and fuzzy K-ideals in INK-algebras, fuzzy p-ideal in INK-algebras, and fuzzy translation of INK-ideals of INK-algebras. Kaviyarasu et al. [3, 4] proposed intuitionistic fuzzy INK-ideals of INK-algebras, direct product of intuitionistic fuzzy K-ideals of INK-algebras, intuitionistic fuzzy translation on INK-algebras, and they have discussed neutrosophic h-ideals in INK-algebras. Homomorphism and anti-homomorphism of neutrosophic INK-algebras have been done by Mounikalakshmi et al. [5, 6]. Moreover, the work of INK-algebras has been explored in different aspects of fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets.

Recent developments in the study of INK-algebras have increasingly embraced the framework of neutrosophic logic to handle indeterminacy and ambiguity in algebraic reasoning. Notably, Al-Masarwah et al. [7] introduced the concept of Fermatean neutrosophic INK-algebras, enriching the traditional neutrosophic structure by incorporating higher degrees of membership and indeterminacy for enhanced expressive power. Complementing this, Al-Omeri et al. [8] examined translation mechanisms within neutrosophic INK-algebras, providing insights into structural transformations and their algebraic consequences. Further foundational work by Kaviyarasu et al. [9] investigated the direct product of neutrosophic INK-algebras, establishing critical results on ideal preservation and operational behavior in product structures. Collectively, these contributions have expanded the theoretical landscape of INK-algebras, offering versatile tools for modeling logical uncertainty in abstract algebraic systems.

Zadeh [10] explored the idea of fuzzy sets (FSs) in 1965, an extension of classical set theory that deals with vagueness and uncertainty in the given data. Classical set theory asserts that an element is either a member of a set or not. FS theory discourses this by introducing membership values scaling from 0 to 1, which have been used to represent the degree to which an element is affiliated with a set. FS theory has innumerable implementations in diverse fields, including artificial intelligence, decision-making, and control systems that allow both modeling and handling of vague and imprecise information. Moreover, Jun et al. [11] studied the idea of fuzzy implicative ideals and constructed a fuzzy characteristic implicative ideal in BCK-algebras. Paad [12] introduced the concept of fuzzy implicative ideals in BL-algebras, and they state several properties of it. Also, Sowmia and Jeyalakshmi [13] have worked on the perception of fuzzy implicative ideals in Z-algebras.

Eventually, Atanassov [14] proposed the generalization of FSs, which is an intuitionistic fuzzy set (IFS) in 1980, which gives information about a fresh parameter known as nonmembership degree, where fuzzy tells us about the membership degree but in IFS gives information about uncertainty and vagueness regarding membership degrees and nonmembership degrees. IFSs have applications in decision-making where uncertainty plays a significant role. Various approaches to IFS include expert systems, risk assessment, and medical diagnosis, where precise decisions are difficult based on vague or uncertain data or information. Satyanarayana et al. [15, 16] have worked on the concept of intuitionistic fuzzy implicative hyper BCK-ideals of hyper BCK-algebras and interval-valued intuitionistic fuzzy (implicative and commutative) ideals of BCK-algebras. Also, Satynarayana et al. [17] have done with the concept of derivations of intuitionistic fuzzy implicative ideals of BCK-algebras. Also, Rasuli [18] has worked on intuitionistic fuzzy BCI-algebras (implicative ideals, closed implicative ideals, commutative ideals) under norms.

Later, the neutrosophic sets (NSs) were introduced by Smarandache [19] in 1999, which is the generalization of classical sets, FSs, and IFSs involving a new parameter called indeterminacy. Neutrosophic sets are particularly useful when uncertainty exists and the basic components of truth, falsehood, and indeterminacy are simultaneously considered. NSs have many approaches in many fields, like decision-making, expert systems, image processing, and fuzzy logic, thus enabling us to do effective modeling and analysis in situations where the classical set theory falls short. Some authors, Jun and Roh [20] have discussed a few results on MBJ-neutrosophic ideals of BCK/BCI-algebras. Borzooei et al. [21] have worked on the concept of positive implicative BMBJ-neutrosophic ideals in BCK-algebras. Bordbar et al. [22] have worked on the concept of positive implicative ideals of BCK-algebras based on NSs and falling shadows. Also, Satyanarayana and Baji [23] have worked on positive implicative, implicative, and commutative SB-neutrosophic ideals in BCK/BCI-algebras.

This article aims to introduce and investigate the notions of MINK-Is and PMINK-Is in the framework of INK-algebras. The study further extends these notions to fuzzy and neutrosophic environments, resulting in the definitions of FMINK-Is, FPMINK-Is, NMINK-Is, and NPMINK-Is. We examine their structural properties, including closure under intersection and behavior under homomorphisms. The results presented herein provide a comprehensive generalization of ideal theory within INK-algebras and contribute to the algebraic modeling of uncertainty and indeterminacy.

List of Abbreviations

To facilitate the understanding of frequently used terminology, we provide a list of abbreviations employed throughout the paper.

Abbreviation	Full Term
FS	Fuzzy Set
IFS	Intuitionistic Fuzzy Set
NS	Neutrosophic Set
MINK-I	Implicative INK-Ideal
PMINK-I	Positive Implicative INK-Ideal
FINK-S	Fuzzy INK-Subalgebra
FINK-I	Fuzzy INK-Ideal
FMINK-I	Fuzzy Implicative INK-Ideal
FPMINK-I	Fuzzy Positive Implicative INK-Ideal
NINK-S	Neutrosophic INK-Subalgebra
NINK-I	Neutrosophic INK-Ideal
NMINK-I	Neutrosophic Implicative INK-Ideal
NPMINK-I	Neutrosophic Positive Implicative INK-Ideal

Table 1: List of abbreviations used in the paper

2. Preliminaries

In this section, we present some fundamental definitions and properties that are essential for developing the main results of this paper. These include the basic structure and axioms of INK-algebras, the concepts of INK-subalgebras and INK-ideals, as well as their fuzzy and neutrosophic counterparts. The definitions and notations introduced herein will form the groundwork for the subsequent sections, where we extend the notions of implicative INK-ideals (MINK-Is) and positive implicative INK-ideals (PMINK-Is) within fuzzy and neutrosophic frameworks.

Definition 1. *[24] An algebra $(\mathcal{I}, \bullet, 0)$ is termed to be an INK-algebra where $\bullet$ is a binary and the 0 is a constant of $\mathcal{I}$ if it gratifies the following conditions:*
INK-1: $((\epsilon \bullet \xi) \bullet (\epsilon \bullet \varsigma)) \bullet (\varsigma \bullet \xi) = 0$
INK-2: $((\epsilon \bullet \varsigma) \bullet (\xi \bullet \varsigma)) \bullet (\epsilon \bullet \xi) = 0$
INK-3: $\epsilon \bullet 0 = \epsilon$
INK-4: $\epsilon \bullet \xi = 0, \xi \bullet \epsilon = 0 \Rightarrow \epsilon = \xi, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Definition 2. *[24] Let $\mathcal{S}$ be a non-empty subset of an INK-algebra $\mathcal{I}$. Then $\mathcal{S}$ is termed to be an INK-subalgebra of $\mathcal{I}$ if $\epsilon \bullet \xi \in \mathcal{S}, \forall \epsilon, \xi \in \mathcal{S}$.*

Definition 3. *[24] Let $\mathcal{S}$ be a non-empty subset of an INK-algebra $\mathcal{I}$. Then $\mathcal{S}$ is entitled as INK-ideal of $\mathcal{I}$ if*
(b1) $0 \in \mathcal{S}$
(b2) $(\varsigma \bullet \epsilon) \bullet (\varsigma \bullet \xi) \in \mathcal{S}, \xi \in \mathcal{S} \Rightarrow \epsilon \in \mathcal{S}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Definition 4. *[24] Let $\mathcal{I}$ be an INK-algebra with binary operation $\bullet$ and constant element 0. For any $\epsilon, \xi \in \mathcal{I}$, we define the partial order relation $\leq$ as follows:*

$$\epsilon \leq \xi \Leftrightarrow \epsilon \bullet \xi = 0.$$

Theorem 1. *Every INK-ideal $\mathcal{S}$ of an INK-algebra $\mathcal{I}$ has the following assertion:*

$$\epsilon \leq \xi \Rightarrow \epsilon \in \mathcal{S}, \forall \epsilon \in \mathcal{I}, \xi \in \mathcal{S}.$$

Proof. It is obtained immediately from Definition 3 by replacing ς with 0.

Definition 5. *[25] An FS g in an INK-algebra $\mathcal{I}$ is entitled as a fuzzy INK-subalgebra (FINK-S) of $\mathcal{I}$ if $g(\epsilon \bullet \xi) \geq \min\{g(\epsilon), g(\xi)\}, \forall \epsilon, \xi \in \mathcal{I}$.*

Definition 6. *[9] Let g and h be FSs of an INK-algebra $(\mathcal{I}, \bullet, 0)$. Then the direct product of $g \times h : \mathcal{I} \times \mathcal{I} \to [0, 1]$ is well entitled as $(g \times h)(\epsilon, \xi) = \min\{g(\epsilon), h(\xi)\}, \forall \epsilon, \xi \in \mathcal{I}$.*

Definition 7. *[25] Let an FS g in an INK-algebra $\mathcal{I}$ is known as a fuzzy INK-ideal (FINK-I) of $\mathcal{I}$ if it gratifies*
(c1) $g(0) \geq g(\epsilon)$
(c2) $g(\epsilon) \geq \min\{g(\epsilon \bullet \xi), g(\xi)\}, \forall \epsilon, \xi \in \mathcal{I}$.

Definition 8. *[26] An NS $g = (g_T, g_I, g_F)$ in an INK-algebra $\mathcal{I}$ is called a neutrosophic INK-subalgebra (NINK-S) of $\mathcal{I}$ if it gratifies the following condition:*

(d1) $g_T(\epsilon \bullet \xi) \geq \min\{g_T(\epsilon), g_T(\xi)\}$

(d2) $g_I(\epsilon \bullet \xi) \leq \max\{g_I(\epsilon), g_I(\xi)\}$

(d3) $g_F(\epsilon \bullet \xi) \leq \max\{g_F(\epsilon), g_F(\xi)\}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Definition 9. *[26] An NS $g = (g_T, g_I, g_F)$ in an INK-algebra $\mathcal{I}$ is termed as a neutrosophic INK-ideal (NINK-I) of $\mathcal{I}$ if it gratifies the following condition:*

(d4) $g_T(0) \geq g_T(\epsilon), g_I(0) \leq g_I(\epsilon), g_F(0) \leq g_F(\epsilon)$

(d5) $g_T(\epsilon) \geq \min\{g_T(\epsilon \bullet \xi), g_T(\xi)\}$

(d6) $g_I(\epsilon) \leq \max\{g_I(\epsilon \bullet \xi), g_I(\xi)\}$

(d7) $g_F(\epsilon) \leq \max\{g_F(\epsilon \bullet \xi), g_F(\xi)\}, \forall \epsilon, \xi \in \mathcal{I}$.

Theorem 2. *An NS $g = (g_T, g_I, g_F)$ in an INK-algebra $\mathcal{I}$ is termed as an NINK-I of $\mathcal{I}$ if and only if it gratifies the following conditions (d4) and*

(d8) $g_T(\epsilon) \geq \min\{g_T((\varsigma \bullet \epsilon) \bullet (\varsigma \bullet \xi)), g_T(\xi)\}$

(d9) $g_I(\epsilon) \leq \max\{g_I((\varsigma \bullet \epsilon) \bullet (\varsigma \bullet \xi)), g_I(\xi)\}$

(d10) $g_F(\epsilon) \leq \max\{g_F((\varsigma \bullet \epsilon) \bullet (\varsigma \bullet \xi)), g_F(\xi)\}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Proof. Suppose g satisfies (d4), (d8), (d9), and (d10). We prove that g is an NINK-I, i.e., it also satisfies (d5), (d6), and (d7). Let $\epsilon, \xi \in \mathcal{I}$, and choose $\varsigma = 0$, the constant of the INK-algebra. From axiom (INK-3), we know

$$0 \bullet \epsilon = \epsilon, 0 \bullet \xi = \xi \Rightarrow (0 \bullet \epsilon) \bullet (0 \bullet \xi) = \epsilon \bullet \xi.$$

Now substitute into (d8)-(d10):

From (d8), we have

$$g_T(\epsilon) \geq \min\{g_T((0 \bullet \epsilon) \bullet (0 \bullet \xi)),\ g_T(\xi)\} = \min\{g_T(\epsilon \bullet \xi),\ g_T(\xi)\}$$

which is exactly (d5).

From (d9), we have

$$g_I(\epsilon) \leq \max\{g_I((0 \bullet \epsilon) \bullet (0 \bullet \xi)),\ g_I(\xi)\} = \max\{g_I(\epsilon \bullet \xi),\ g_I(\xi)\}$$

which is (d6).

From (d10), we have

$$g_F(\epsilon) \leq \max\{g_F((0 \bullet \epsilon) \bullet (0 \bullet \xi)),\ g_F(\xi)\} = \max\{g_F(\epsilon \bullet \xi),\ g_F(\xi)\}$$

which is (d7).

Therefore, g is an NINK-I of $\mathcal{I}$.

Conversely, suppose g is an NINK-I of $\mathcal{I}$, i.e., it satisfies (d4)-(d7). We prove that it satisfies (d8)-(d10). Let $\epsilon, \xi, \varsigma \in \mathcal{I}$. Using (d5), we have

$$g_T(\epsilon) \geq \min\{g_T(\epsilon \bullet \xi),\ g_T(\xi)\}$$

which holds for all elements in $\mathcal{I}$, and in particular, for $(\varsigma \bullet \epsilon) \bullet (\varsigma \bullet \xi)$ and ξ. Therefore,

$$g_T(\epsilon) \geq \min\{g_T((\varsigma \bullet \epsilon) \bullet (\varsigma \bullet \xi)),\ g_T(\xi)\}.$$

Similarly, from (d6) and (d7), we have

$$g_I(\epsilon) \leq \max\{g_I((\varsigma \bullet \epsilon) \bullet (\varsigma \bullet \xi)),\ g_I(\xi)\}, g_F(\epsilon) \leq \max\{g_F((\varsigma \bullet \epsilon) \bullet (\varsigma \bullet \xi)),\ g_F(\xi)\}.$$

Thus, g satisfies (d8), (d9), and (d10) as required.

Definition 10. *[26] Let $g = (g_T, g_I, g_F)$ and $h = (h_T, h_I, h_F)$ be two NSs in an INK-algebra $\mathcal{I}$. Then, the union and the intersection are defined as follows:*

(i) $(g \cup h)(\epsilon) = \{< \epsilon, \max\{g_T(\epsilon), h_T(\epsilon)\}, \min\{g_I(\epsilon), h_I(\epsilon)\}, \min\{g_F(\epsilon), h_F(\epsilon)\} >\mid \epsilon \in \mathcal{I}\}$

(ii) $(g \cap h)(\epsilon) = \{< \epsilon, \min\{g_T(\epsilon), h_T(\epsilon)\}, \max\{g_I(\epsilon), h_I(\epsilon)\}, \max\{g_F(\epsilon), h_F(\epsilon)\} >\mid \epsilon \in \mathcal{I}\}$.

3. Main results

In this section, we develop and analyze various classes of FINK-Is in the context of INK-algebras. The discussion is organized into three key subsections. First, in Subsection 3.1, we introduce the concepts of fuzzy implicative INK-ideals (FMINK-I) and fuzzy positive implicative INK-ideals (FPMINK-I) and explore their defining properties with supporting examples. Subsection 3.2 investigates the structural behavior of these FINK-Is under intersection and union operations, showing that intersection preserves the implicative properties, whereas union may not. Finally, Subsection 3.3 addresses the behavior of fuzzy implicative and positive implicative INK-ideals under homomorphic pre-images, proving that these properties are preserved under INK-algebra homomorphisms. These results establish a robust algebraic foundation for the study of fuzzy logical structures in INK-algebras and contribute to their potential applications in uncertainty modeling and reasoning.

3.1. Fuzzy implicative and positive implicative INK-ideals of INK-algebras

Building upon the foundational notions of FINK-Is introduced in the previous section, we now define and explore two novel types of FINK-Is: FMINK-Is and FPMINK-Is in the setting of INK-algebras. These concepts are natural extensions of classical implicative structures and aim to incorporate reasoning under uncertainty within algebraic systems. We provide precise definitions, discuss their algebraic properties, and illustrate them through relevant examples. The aim is to deepen the understanding of logical implications in fuzzy settings and establish the groundwork for further results concerning intersection properties and homomorphic behavior of these ideals.

Definition 11. *An INK-algebra $\mathcal{I}$ is called an implicative INK-algebra if it satisfies the following condition:*

(a1) $\epsilon \bullet (\xi \bullet \epsilon) = \epsilon, \forall \epsilon, \xi \in \mathcal{I}$.

Definition 12. *An INK-algebra $\mathcal{I}$ is entitled as a positive implicative INK-algebra if it satisfies the following condition:*

(a2) $(\epsilon \bullet \varsigma) \bullet (\xi \bullet \varsigma) = (\epsilon \bullet \xi) \bullet \varsigma, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Example 1. *Consider an INK-algebra $\mathcal{I} = \{0, \breve{a}, \breve{b}, \breve{c}\}$ with $\bullet$ as the binary operation:*

$\bullet$	0	$\breve{a}$	$\breve{b}$	$\breve{c}$
0	0	0	0	0
$\breve{a}$	$\breve{a}$	0	$\breve{a}$	$\breve{a}$
$\breve{b}$	$\breve{b}$	$\breve{b}$	0	$\breve{b}$
$\breve{c}$	$\breve{c}$	$\breve{c}$	$\breve{c}$	0

Then by routine calculations, the INK-algebra $(\mathcal{I}, \bullet, 0)$ is an MINK-I and a PMINK-I.

Definition 13. *A subset $\mathcal{S}$ of an INK-algebra $\mathcal{I}$ is entitled as an implicative INK-ideal (MINK-I) of $\mathcal{I}$ if it gratifies (b1) and*

(b3) $((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma \in \mathcal{S}, \varsigma \in \mathcal{S} \Rightarrow \epsilon \in \mathcal{S}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Proposition 1. *Let I_1 and I_2 be INK-ideals of an INK-algebra $\mathcal{I}$ with $I_1 \subseteq I_2$. If I_1 is an MINK-I, then I_2 is also an MINK-I.*

Definition 14. *A subset $\mathcal{S}$ of an INK-algebra $\mathcal{I}$ is entitled as a positive implicative INK-ideal (PMINK-I) of $\mathcal{I}$ if it satisfies (b1) and*

(b4) $(\epsilon \bullet \xi) \bullet \varsigma \in \mathcal{S}, \xi \bullet \varsigma \in \mathcal{S} \Rightarrow \epsilon \bullet \varsigma \in \mathcal{S}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Example 2. *Consider a subset $\mathcal{S} = \{0, \breve{a}, \breve{b}\}$ of an INK-algebra $\mathcal{I}$ defined in Example 1. Then, the subset $(\mathcal{S}, \bullet, 0)$ is an MINK-I and a PMINK-I of $(\mathcal{I}, \bullet, 0)$.*

Definition 15. *An FS g in an INK-algebra $\mathcal{I}$ is entitled as a fuzzy implicative INK-ideal (FMINK-I) of $\mathcal{I}$ if it gratifies (c1) and*

(c3) $g(\epsilon) \geq \min\{g((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g(\varsigma)\}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Definition 16. *An FS g in an INK-algebra $\mathcal{I}$ is entitled as a fuzzy positive implicative INK-ideal (FPMINK-I) of $\mathcal{I}$ if it gratifies (c1) and*

(c4) $g(\epsilon \bullet \varsigma) \geq \min\{g((\epsilon \bullet \xi) \bullet \varsigma), g(\xi \bullet \varsigma)\}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Example 3. *Consider an INK-algebra $\mathcal{I} = \{0, \breve{a}, \breve{b}\}$ with the following Cayley table:*

$\bullet$	0	$\breve{a}$	$\breve{b}$
0	0	0	0
$\breve{a}$	$\breve{a}$	0	$\breve{a}$
$\breve{b}$	$\breve{b}$	$\breve{b}$	0

Define an FS $g : \mathcal{I} \to [0, 1]$ by

	0	$\breve{a}$	$\breve{b}$
g	0.6	0.4	0.4

Then, g is an FMINK-I and an FPMINK-I of $\mathcal{I}$.

Theorem 3. *Every FMINK-I of an INK-algebra $\mathcal{I}$ is an FINK-I of $\mathcal{I}$.*

Proof. Let g be an FMINK-I of an INK-algebra $\mathcal{I}$. Substituting $\xi = 0$ in (c3). Then $\forall \epsilon, \varsigma \in \mathcal{I}$,

$$\begin{aligned} g(\epsilon) &\geq \min\{g((\epsilon \bullet (0 \bullet \epsilon)) \bullet \varsigma), g(\varsigma)\} \\ &= \min\{g((\epsilon \bullet 0) \bullet \varsigma), g(\varsigma)\} \\ &= \min\{g(\epsilon \bullet \varsigma), g(\varsigma)\}. \end{aligned}$$

This shows that g gratifies (c2). Combining (c1), g is an FINK-I of $\mathcal{I}$.

Note: Every FMINK-I of $\mathcal{I}$ is an FINK-I of $\mathcal{I}$, but the converse is not true.

Example 4. *Consider an INK-algebra $\mathcal{I} = \{0, \breve{a}, \breve{b}\}$ with the following Cayley table:*

$\bullet$	0	$\breve{a}$	$\breve{b}$
0	0	$\breve{b}$	$\breve{a}$
$\breve{a}$	$\breve{a}$	0	$\breve{b}$
$\breve{b}$	$\breve{b}$	$\breve{a}$	0

Define an FS $g : \mathcal{I} \to [0, 1]$ by

	0	$\breve{a}$	$\breve{b}$
g	0.5	0.3	0.3

Then, the above table satisfies the FINK-I conditions but not FMINK-I of $\mathcal{I}$.

Theorem 4. *If $\mathcal{I}$ is an implicative INK-algebra, then every FINK-I of $\mathcal{I}$ is an FMINK-I of $\mathcal{I}$.*

Proof. Suppose $\mathcal{I}$ is an implicative INK-algebra and g is an FINK-I of $\mathcal{I}$. Then by (c2), $g(\epsilon) \geq \min\{g(\epsilon \bullet \varsigma), g(\varsigma)\} = \min\{g((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g(\varsigma)\}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$. It follows that g is an FMINK-I of $\mathcal{I}$.

Theorem 5. *Every FPMINK-I of an INK-algebra $\mathcal{I}$ is an FINK-I of $\mathcal{I}$.*

Proof. Let g be an FPMINK-I of an INK-algebra $\mathcal{I}$. Substitute $\varsigma = 0$ in (c4), $g(\epsilon \bullet 0) \geq \min\{g((\epsilon \bullet \xi) \bullet 0), g(\xi \bullet 0)\}, \forall \epsilon, \xi \in \mathcal{I}$ by INK-3. Thus, $g(\epsilon) \geq \min\{g(\epsilon \bullet \xi), g(\xi)\}$. This shows that g satisfies (c2). Combining (c1), g is an FINK-I of $\mathcal{I}$.

The following example shows that the converse of Theorem 5 does not hold.

Example 5. *Consider an INK-algebra $\mathcal{I} = \{0, \breve{a}, \breve{b}, \breve{c}\}$ with the following Cayley table:*

$\bullet$	0	$\breve{a}$	$\breve{b}$	$\breve{c}$
0	0	$\breve{a}$	$\breve{b}$	$\breve{c}$
$\breve{a}$	$\breve{a}$	0	$\breve{c}$	$\breve{b}$
$\breve{b}$	$\breve{b}$	$\breve{c}$	0	$\breve{a}$
$\breve{c}$	$\breve{c}$	$\breve{b}$	$\breve{a}$	0

Define an FS $g : \mathcal{I} \to [0,1]$ *by*

	0	$\breve{a}$	$\breve{b}$	$\breve{c}$
g	0.5	0.3	0.3	0.5

Then, the above table satisfies the FINK-I conditions but not the FPMINK-I of $\mathcal{I}$.

3.2. Intersection of fuzzy implicative and fuzzy positive implicative INK-ideals of INK-algebras

After introducing FMINK-Is and FPMINK-Is, it is natural to investigate their behavior under set-theoretic operations. In this subsection, we analyze the intersection and union of FMINK-Is and FPMINK-Is. We demonstrate that the intersection of any two FMINK-Is (respectively, FPMINK-Is) yields another ideal of the same type, thus proving the closure property under intersection. However, through counterexamples, we also show that such closure does not hold in general for union. These results highlight the structural stability of FMINK-I and FPMINK-I under intersection, while also identifying the limitations of union in preserving implicative properties.

Theorem 6. *Let* g *and* h *be two FMINK-Is of an INK-algebra* $\mathcal{I}$. *Then* $g \cap h$ *is also an FMINK-I of* $\mathcal{I}$.

Proof. Let $\epsilon, \xi, \varsigma \in \mathcal{I}$. Then

$$\begin{aligned}(g \cap h)(\epsilon) &= \min\{g(\epsilon), h(\epsilon)\}\\ &\geq \min\{\min\{g((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g(\varsigma)\}, \min\{h((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), h(\varsigma)\}\}\\ &= \min\{\min\{g((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), h((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma)\}, \min\{g(\varsigma), h(\varsigma)\}\}\\ &\geq \min\{(g \cap h)((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), (g \cap h)(\varsigma)\}.\end{aligned}$$

Hence, $g \cap h$ is an FMINK-I of $\mathcal{I}$.

Note: The union of two FMINK-Is of an INK-algebra need not be an FMINK-I.

Example 6. *Consider an INK-algebra* $\mathcal{I} = \{0, 2, 4\}$ *with the following Cayley table:*

$\bullet$	0	2	4
0	0	0	0
2	2	0	2
4	4	4	0

Define an FS $g : \mathcal{I} \to [0,1]$ *by*

	0	2	4
g	0.3	0.4	0.6

Define an FS $h : \mathcal{I} \to [0,1]$ *by*

	0	2	4
h	0.5	0.3	0.5

Clearly, g and h are two FMINK-Is of $\mathcal{I}$. Here $T_{g\cup h}(0) = 0.3$, but it is not greater than or equal to $0.5 = \min\{T_{g\cup h}((0 \bullet (2 \bullet 0)) \bullet 4), T_{g\cup h}(4)\}$. Thus, the union of FMINK-Is of an INK-algebra is not an FMINK-I.

Theorem 7. *Let g and h be two FMINK-Is of an INK-algebra $\mathcal{I}$. If $g \subseteq h$ or $h \subseteq g$, then $g \cup h$ is an FMINK-I of $\mathcal{I}$.*

Proof. Obvious.

Theorem 8. *Let g and h be two FPMINK-Is of an INK-algebra $\mathcal{I}$. Then $g \cap h$ is also an FPMINK-I of $\mathcal{I}$.*

Proof. The proof is similar to Theorem 6.

Note: The union of two FPMINK-Is of an INK-algebra need not be an FPMINK-I.

Example 7. *From Example 6, g and h are two FPMINK-Is of $\mathcal{I}$. Here, $T_{g\cup h}(0 \bullet 2) = 0.3$, but it is not greater than or equal to $0.4 = \min\{T_{g\cup h}((0 \bullet 2) \bullet 4), T_{g\cup h}(2 \bullet 4)\}$. Thus, the union of FPMINK-I of $\mathcal{I}$ is not an FPMINK-I.*

Theorem 9. *Let g and h be two FPMINK-Is of an INK-algebra $\mathcal{I}$. If $g \subseteq h$ or $h \subseteq g$, then $g \cup h$ is an FPMINK-I of $\mathcal{I}$.*

Proof. Obvious.

3.3. Homomorphism of fuzzy implicative and positive implicative INK-ideals of INK-algebras

Homomorphisms play a crucial role in algebra by preserving structural properties across different algebraic systems. In this subsection, we examine the behavior of fuzzy implicative INK-ideals (FMINK-I) and fuzzy positive implicative INK-ideals (FPMINK-I) under homomorphic mappings between INK-algebras. Specifically, we focus on the pre-images of these ideals and prove that such pre-images retain the implicative and positive implicative properties of the original fuzzy ideals. These results establish the invariance of FMINK-I and FPMINK-I under structure-preserving transformations, thereby confirming their robustness across algebraic homomorphic images.

Definition 17. *Let $\kappa : \mathcal{I} \to \breve{\mathcal{I}}$ be a homomorphism of INK-algebras and g be the FS in $\breve{\mathcal{I}}$. Then the FS $g[\kappa]$ in $\mathcal{I}$ is defined by $g[\kappa](\epsilon) = g(\kappa(\epsilon))$ for every $\epsilon \in \mathcal{I}$ is called the pre-image of g under κ.*

Theorem 10. *A homomorphic pre-image of an FMINK-I of an INK-algebra is an FMINK-I.*

Proof. Let $\kappa : \mathcal{I} \to \breve{\mathcal{I}}$ be a homomorphism of INK-algebras. If g is an FMINK-I of an INK-algebra $\breve{\mathcal{I}}$, then $g[\kappa](\epsilon) = g(\kappa(\epsilon)) \geq g(0) = g(\kappa(0)) = g[\kappa](0), \forall \epsilon \in \mathcal{I}$. Let $\epsilon, \xi, \varsigma \in \mathcal{I}$. Then

$$\begin{aligned}
\min\{g[\kappa]((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g[\kappa](\varsigma)\} &= \min\{g(\kappa((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma)), g(\kappa(\varsigma))\} \\
&= \min\{g(\kappa(\epsilon \bullet \varsigma)), g(\kappa(\varsigma))\} \\
&= g(\kappa(\epsilon)) \\
&= g[\kappa](\epsilon).
\end{aligned}$$

Hence, $g[\kappa]$ is an FMINK-I of $\breve{\mathcal{I}}$.

Theorem 11. *A homomorphic pre-image of an FPMINK-I of INK-algebras is an FPMINK-I.*

Proof. The proof is similar to Theorem 10.

4. Neutrosophic implicative and positive implicative INK-ideals of INK-algebras

Expanding on the fuzzy framework, we now extend the notions of MINK-Is and PMINK-Is to the setting of NSs, which provide a more expressive structure for capturing uncertainty, indeterminacy, and inconsistency. In this section, we introduce the concepts of neutrosophic implicative INK-ideals (NMINK-Is) and neutrosophic positive implicative INK-ideals (NPMINK-Is) in INK-algebras. These definitions are formulated in terms of the three-valued membership functions of neutrosophic sets—truth, indeterminacy, and falsity—and are motivated by their capacity to model more complex decision-making environments. We present key properties, establish their relationships with standard neutrosophic ideals, and explore the conditions under which they are preserved or extended. Examples are provided to illustrate the distinctions between NMINK-Is/NPMINK-Is and general neutrosophic ideals.

Definition 18. *An NS $g = (g_T, g_I, g_F)$ in an INK-algebra $\mathcal{I}$ is entitled as a neutrosophic implicative INK-ideal (NMINK-I) of $\mathcal{I}$ if it gratifies (d4) and*

(d11) $g_T(\epsilon) \geq \min\{g_T((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_T(\varsigma)\}$

(d12) $g_I(\epsilon) \leq \max\{g_I((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_I(\varsigma)\}$

(d13) $g_F(\epsilon) \leq \max\{g_F((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_F(\varsigma)\}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Definition 19. *An NS $g = (g_T, g_I, g_F)$ in an INK-algebra $\mathcal{I}$ is entitled as a neutrosophic positive implicative INK-ideal (NPMINK-I) of $\mathcal{I}$ if it gratifies (d4) and*

(d14) $g_T(\epsilon \bullet \varsigma) \geq \min\{g_T((\epsilon \bullet \xi) \bullet \varsigma), g_T(\xi \bullet \varsigma)\}$

(d15) $g_I(\epsilon \bullet \varsigma) \leq \max\{g_I((\epsilon \bullet \xi) \bullet \varsigma), g_I(\xi \bullet \varsigma)\}$

(d16) $g_F(\epsilon \bullet \varsigma) \leq \max\{g_F((\epsilon \bullet \xi) \bullet \varsigma), g_F(\xi \bullet \varsigma)\}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$.

Example 8. *Consider an INK-algebra* $(\mathcal{I}, \bullet, 0)$ *from Example 3. Define an NS* $g = (g_T, g_I, g_F)$ *of* $\mathcal{I}$ *by*

	0	$\breve{a}$	$\breve{b}$
g_T	0.4	0.3	0.3
g_I	0.2	0.4	0.4
g_F	0.3	0.3	0.5

Then, $g = (g_T, g_I, g_F)$ *is an NMINK-I and NPMINK-I of* $\mathcal{I}$.

Theorem 12. *Every NMINK-I of an INK-algebra* $\mathcal{I}$ *is an NINK-I.*

Proof. Let $g = (g_T, g_I, g_F)$ be an NMINK-I of an INK-algebra $\mathcal{I}$. Let $\epsilon, \xi, \varsigma \in \mathcal{I}$. Then by (d11), we have

$$\begin{aligned} g_T(\epsilon) &\geq \min\{g_T((\epsilon \bullet (0 \bullet \epsilon)) \bullet \varsigma), g_T(\varsigma)\} \\ &= \min\{g_T((\epsilon \bullet 0) \bullet \varsigma)), g_T(\varsigma)\} \\ &= \min\{g_T(\epsilon \bullet \varsigma), g_T(\varsigma)\}. \end{aligned}$$

This shows that g_T gratifies (d5). By (d12), we have

$$\begin{aligned} g_I(\epsilon) &\leq \max\{g_I((\epsilon \bullet (0 \bullet \epsilon)) \bullet \varsigma), g_I(\varsigma)\} \\ &= \max\{g_I((\epsilon \bullet 0) \bullet \varsigma)), g_I(\varsigma)\} \\ &= \max\{g_I(\epsilon \bullet \varsigma), g_I(\varsigma)\}. \end{aligned}$$

This shows that g_I gratifies (d6). By (d13), we have

$$\begin{aligned} g_F(\epsilon) &\leq \max\{g_F((\epsilon \bullet (0 \bullet \epsilon)) \bullet \varsigma), g_F(\varsigma)\} \\ &= \max\{g_F((\epsilon \bullet 0) \bullet \varsigma)), g_F(\varsigma)\} \\ &= \max\{g_F(\epsilon \bullet \varsigma), g_F(\varsigma)\}. \end{aligned}$$

This shows that g_F gratifies (d7). By combining (d4), g is an NINK-I of $\mathcal{I}$.

Note: Every NMINK-I of an INK-algebra is an NINK-I, but the converse is not valid.

Example 9. *Consider an INK-algebra* $\mathcal{I} = \{0, \breve{a}, \breve{b}\}$ *with the following Cayley table:*

$\bullet$	0	$\breve{a}$	$\breve{b}$
0	0	$\breve{b}$	$\breve{a}$
$\breve{a}$	$\breve{a}$	0	$\breve{b}$
$\breve{b}$	$\breve{b}$	$\breve{a}$	0

Define an NS $g = (g_T, g_I, g_F)$ *of* $\mathcal{I}$ *by*

	0	$\breve{a}$	$\breve{b}$
g_T	0.6	0.3	0.3
g_I	0.5	0.4	0.4
g_F	0.8	0.4	0.4

Then, the above table satisfies the NINK-I conditions but does not satisfy the conditions of NMINK-I of $\mathcal{I}$.

Theorem 13. *If $\mathcal{I}$ is an implicative INK-algebra, then every NINK-I of $\mathcal{I}$ is an NMINK-I.*

Proof. Since $\mathcal{I}$ is an implicative INK-algebra, it follows that $\epsilon = \epsilon \bullet (\xi \bullet \epsilon), \forall \epsilon, \xi \in \mathcal{I}$. Let $g = (g_T, g_I, g_F)$ be an NINK-I of $\mathcal{I}$. Then by (d5), $g_T(\epsilon) \geq \min\{g_T(\epsilon \bullet \varsigma), g_T(\varsigma)\}, \forall \epsilon, \xi, \varsigma \in \mathcal{I}$, so $g_T(\epsilon) \geq \min\{g_T((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_T(\varsigma)\}$. Similarly, by (d6), $g_I(\epsilon) \leq \max\{g_I(\epsilon \bullet \varsigma), g_I(\varsigma)\}$, so $g_I(\epsilon) \leq \max\{g_I((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_I(\varsigma)\}$. Also by (d7), $g_F(\epsilon) \leq \max\{g_F(\epsilon \bullet \varsigma), g_F(\varsigma)\}$, so $g_F(\epsilon) \leq \max\{g_F((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_F(\varsigma)\}$. Hence, g is an NMINK-I of $\mathcal{I}$.

Theorem 14. *Every NPMINK-I of an INK-algebra $\mathcal{I}$ is an NINK-I.*

Proof. The proof is similar to Theorem 12.

Note: The following example shows that the converse of Theorem 14 does not hold.

Example 10. *Consider an INK-algebra $\mathcal{I} = \{0, \breve{a}, \breve{b}, \breve{c}\}$ from Example 5. Define an NS $g = (g_T, g_I, g_F)$ of $\mathcal{I}$ by*

	0	$\breve{a}$	$\breve{b}$	$\breve{c}$
g_T	0.5	0.3	0.3	0.5
g_I	0.2	0.4	0.4	0.2
g_F	0.3	0.5	0.5	0.4

Then, the above table satisfies the NINK-I conditions but does not satisfy the conditions of NPMINK-I of $\mathcal{I}$.

4.1. Intersection of neutrosophic implicative and positive implicative INK-ideals of INK-algebras

In this subsection, we investigate the stability of NMINK-Is and NPMINK-Is under intersection operation. Understanding how these classes of ideals behave under set-theoretic intersection is crucial for exploring their algebraic structure and for constructing new ideals from existing ones. We prove that the intersection of any two NMINK-Is (respectively, NPMINK-Is) results in an ideal of the same type, thus demonstrating the closure property under intersection. We also provide illustrative examples to confirm the validity of the results and highlight that, unlike intersection, the union of such ideals does not necessarily preserve the implicative properties.

Definition 20. *Let $g = (g_T, g_I, g_F)$ and $h = (h_T, h_I, h_F)$ be two NSs in $\mathcal{I}$. Then the intersection $g \cap h = (T_{g\cap h}, I_{g\cap h}, F_{g\cap h})$ is defined as follows:*
(i) $T_{g\cap h}(\epsilon) \geq \min\{g_T(\epsilon), h_T(\epsilon)\}$
(ii) $I_{g\cap h}(\epsilon) \leq \max\{g_I(\epsilon), h_I(\epsilon)\}$
(iii) $F_{g\cap h}(\epsilon) \leq \max\{g_F(\epsilon), h_F(\epsilon)\}, \forall \epsilon \in \mathcal{I}$.

Definition 21. *Let $g = (g_T, g_I, g_F)$ and $h = (h_T, h_I, h_F)$ be two NSs in $\mathcal{I}$. Then the union $g \cup h = (T_{g\cup h}, I_{g\cup h}, F_{g\cup h})$ is defined as follows:*
(i) $T_{g\cup h}(\epsilon) \leq \max\{g_T(\epsilon), h_T(\epsilon)\}$
(ii) $I_{g\cup h}(\epsilon) \geq \min\{g_I(\epsilon), h_I(\epsilon)\}$
(iii) $H_{g\cup h}(\epsilon) \geq \min\{g_F(\epsilon), h_F(\epsilon)\}, \forall \epsilon \in \mathcal{I}$.

Theorem 15. *Let $g = (g_T, g_I, g_F)$ and $h = (h_T, h_I, h_F)$ be two NMINK-Is of an INK-algebra $\mathcal{I}$. Then $g \cap h = (T_{g\cap h}, I_{g\cap h}, F_{g\cap h})$ is an NMINK-I of $\mathcal{I}$.*

Proof. Let g and h be two NMINK-Is of an INK-algebra $\mathcal{I}$. Let $\epsilon, \xi, \varsigma \in \mathcal{I}$. Then

$$\begin{aligned}
T_{g\cap h}(\epsilon) &= \min\{g_T(\epsilon), h_T(\epsilon)\} \\
&\geq \min\{\min\{g_T((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_T(\varsigma)\}, \min\{h_T((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), h_T(\varsigma)\}\} \\
&= \min\{\min\{g_T((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), h_T((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma)\}, \min\{g_T(\varsigma), h_T(\varsigma)\}\} \\
&\geq \min\{T_{g\cap h}((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), T_{g\cap h}(\varsigma)\},
\end{aligned}$$

$$\begin{aligned}
I_{g\cap h}(\epsilon) &= \max\{g_I(\epsilon), h_I(\epsilon)\} \\
&\leq \max\{\max\{g_I((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_I(\varsigma)\}, \max\{h_I((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), h_I(\varsigma)\}\} \\
&= \max\{\max\{g_I((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), h_I((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma)\}, \max\{g_I(\varsigma), h_I(\varsigma)\}\} \\
&\leq \max\{I_{g\cap h}((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), I_{g\cap h}(\varsigma)\},
\end{aligned}$$

$$\begin{aligned}
F_{g\cap h}(\epsilon) &= \max\{g_F(\epsilon), h_F(\epsilon)\} \\
&\leq \max\{\max\{g_F((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_F(\varsigma)\}, \max\{h_F((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), h_F(\varsigma)\}\} \\
&= \max\{\max\{g_F((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), h_F((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma)\}, \max\{g_F(\varsigma), h_F(\varsigma)\}\} \\
&\leq \max\{F_{g\cap h}((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), F_{g\cap h}(\varsigma)\}.
\end{aligned}$$

Hence, $g \cap h$ is an NMINK-I of $\mathcal{I}$.

Note: The union of NMINK-Is of an INK-algebra need not be an NMINK-I.

Example 11. *Consider an INK-algebra $\mathcal{I} = \{0, 2, 4\}$ from Example 6. Define an NS $g = (g_T, g_I, g_F)$ of $\mathcal{I}$ by*

	0	2	4
g_T	0.3	0.4	0.6
g_I	0.5	0.4	0.4
g_F	0.4	0.5	0.6

Define an NS $h = (h_T, h_I, h_F)$ of $\mathcal{I}$ by

	0	2	4
h_T	0.5	0.3	0.5
h_I	0.4	0.6	0.6
h_F	0.6	0.5	0.5

Then g and h are two NMINK-Is of $\mathcal{I}$. Here $T_{g\cup h}(0) = 0.3$, but it is not greater than or equal to $0.5 = \min\{T_{g\cup h}((0 \bullet (2 \bullet 0)) \bullet 4), T_{g\cup h}(4)\}$. Similarly, for $I_{g\cup h}(0) = 0.5$, but it is not less than or equal to $0.4 = \max\{I_{g\cup h}((0 \bullet (2 \bullet 0)) \bullet 4), I_{g\cup h}(4)\}$. Also, $F_{g\cup h}(0) = 0.6$, but it is not less than or equal to $0.5 = \max\{F_{g\cup h}((0 \bullet (2 \bullet 0)) \bullet 4), F_{g\cup h}(4)\}$. Therefore, $g \cup h$ is not an NMINK-I of $\mathcal{I}$. Thus, the union of NMINK-Is of an INK-algebra is not an NMINK-I.

Theorem 16. *Let g and h be two NMINK-Is of an INK-algebra $\mathcal{I}$. If $g \subseteq h$ or $h \subseteq g$, then $g \cup h$ is an NMINK-I of $\mathcal{I}$.*

Proof. Obvious.

Theorem 17. *Let $g = (g_T, g_I, g_F)$ and $h = (h_T, h_I, h_F)$ be two NPMINK-Is of an INK-algebra $\mathcal{I}$. Then $g \cap h = (T_{g\cap h}, I_{g\cap h}, F_{g\cap h})$ is an NPMINK-I of $\mathcal{I}$.*

Proof. The proof is similar to Theorem 15.

Note: The union of NPMINK-Is of an INK-algebra need not be an NPMINK-I.

Example 12. *Consider an INK-algebra $\mathcal{I} = \{0, 2, 4\}$ defined in Example 11. Define an NS $g = (g_T, g_I, g_F)$ of $\mathcal{I}$ by*

	0	2	4
g_T	0.3	0.4	0.6
g_I	0.5	0.4	0.4
g_F	0.4	0.5	0.6

Define an NS $h = (h_T, h_I, h_F)$ of $\mathcal{I}$ by

	0	2	4
h_T	0.5	0.3	0.5
h_I	0.4	0.6	0.6
h_F	0.6	0.5	0.6

Then g and h are two NPMINK-Is of $\mathcal{I}$. Here $T_{g\cup h}(0 \bullet 2) = 0.3$, but it is not greater than or equal to $0.4 = \min\{T_{g\cup h}((0 \bullet 2) \bullet 4), T_{g\cup h}(2 \bullet 4)\}$. Similarly, for $I_{g\cup h}(0 \bullet 2) = 0.5$, but it is not less than or equal to $0.4 = \max\{I_{g\cup h}((0 \bullet 2) \bullet 4), I_{g\cup h}(2 \bullet 4)\}$. Also, $F_{g\cup h}(0 \bullet 2) = 0.6$, but it is not less than or equal to $0.5 = \max\{F_{g\cup h}((0 \bullet 2) \bullet 4), F_{g\cup h}(2 \bullet 4)\}$. Therefore, $g \cup h$ is not an NPMINK-I of $\mathcal{I}$. Thus, the union of NPMINK-Is of INK-algebras is not an NPMINK-I.

Theorem 18. *Let g and h be two NPMINK-Is of an INK-algebra $\mathcal{I}$. If $g \subseteq h$ or $h \subseteq g$, then $g \cup h$ is an NPMINK-I of $\mathcal{I}$.*

Proof. Obvious.

4.2. Homomorphism of neutrosophic implicative and positive implicative INK-ideals of INK-algebras

This subsection is devoted to the study of the behavior of NMINK-Is and NPMINK-Is under homomorphisms between INK-algebras. Specifically, we investigate whether the pre-image of an NMINK-I or NPMINK-I under a homomorphic mapping retains its respective neutrosophic implicative properties. Establishing such preservation results is essential for ensuring that these ideal structures are robust under algebraic transformations, thus reinforcing their theoretical and practical applicability. The theorems presented in this section confirm that both NMINK-I and NPMINK-I are preserved under homomorphic pre-images, further highlighting the structural soundness of these generalized ideals.

Definition 22. *[6] Let $\kappa : \mathcal{I} \to \breve{\mathcal{I}}$ be a homomorphism of INK-algebras and $g = (g_T, g_I, g_F)$ be an NS in $\breve{\mathcal{I}}$. Then the NS $g[\kappa] = (g_T[\kappa], g_I[\kappa], g_F[\kappa])$ in $\mathcal{I}$ is defined by $g_T[\kappa](\epsilon) = g_T(\kappa(\epsilon)), g_I[\kappa](\epsilon) = g_I(\kappa(\epsilon))$, and $g_F[\kappa](\epsilon) = g_F(\kappa(\epsilon))$ for every $\epsilon \in \mathcal{I}$, is called the pre-image of g under κ.*

Theorem 19. *A homomorphic pre-image of an NMINK-I of an INK-algebra is an NMINK-I.*

Proof. Let $\kappa : \mathcal{I} \to \breve{\mathcal{I}}$ be a homomorphism of INK-algebras. If $g = (g_T, g_I, g_F)$ is an NMINK-I of $\breve{\mathcal{I}}$, then
$g_T[\kappa](\epsilon) = g_T(\kappa(\epsilon)) \geq g_T(0) = g_T(\kappa(0)) = g_T[\kappa](0)$,
$g_I[\kappa](\epsilon) = g_I(\kappa(\epsilon)) \leq g_I(0) = g_I g_I(\kappa(0)) = g_I[\kappa](0)$,
$g_F[\kappa](\epsilon) = g_F(\kappa(\epsilon)) \leq g_F(0) = g_F(\kappa(0)) = g_F[\kappa](0), \forall \epsilon \in \mathcal{I}$.
Let $\epsilon, \xi, \varsigma \in \mathcal{I}$. Then

$$\begin{aligned}
\min\{g_T[\kappa]((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_T[\kappa](\varsigma)\} &= \min\{g_T(\kappa((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma)), g_T(\kappa(\varsigma))\} \\
&= \min\{g_T(\kappa(\epsilon \bullet \varsigma)), g_T(\kappa(\varsigma))\} \\
&= g_T(\kappa(\epsilon)) \\
&= g_T[\kappa](\epsilon),
\end{aligned}$$

$$\begin{aligned}
\max\{g_I[\kappa]((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_I[\kappa](\varsigma)\} &= \max\{g_I(\kappa((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma)), g_I(\kappa(\varsigma))\} \\
&= \max\{g_I(\kappa(\epsilon \bullet \varsigma)), g_I(\kappa(\varsigma))\} \\
&= g_I(\kappa(\epsilon)) \\
&= g_I[\kappa](\epsilon),
\end{aligned}$$

$$\begin{aligned}
\max\{g_F[\kappa]((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma), g_F[\kappa](\varsigma)\} &= \max\{g_F(\kappa((\epsilon \bullet (\xi \bullet \epsilon)) \bullet \varsigma)), g_F(\kappa(\varsigma))\} \\
&= \max\{g_F(\kappa(\epsilon \bullet \varsigma)), g_F(\kappa(\varsigma))\} \\
&= g_F(\kappa(\epsilon)) \\
&= g_F[\kappa](\epsilon).
\end{aligned}$$

Hence, $g[\kappa]$ is an NMINK-I of $\mathcal{I}$.

Theorem 20. *A homomorphic pre-image of an NPMINK-I of INK-algebras is an NPMINK-I.*

Proof. The proof is similar to Theorem 19.

5. Conclusion

This study introduces and formalizes the concepts of MINK-Is and PMINK-Is within the structure of INK-algebras, and extends these notions to fuzzy and neutrosophic contexts. The resulting classes of FMINK-Is, FPMINK-Is, NMINK-Is, and NPMINK-Is are investigated in terms of their structural properties, including closure under intersection and behavior under homomorphic pre-images. It is shown that while intersection preserves the implicative nature of these ideals, union does not necessarily do so. These findings enhance the theoretical foundation of INK-algebras and open pathways for their application in domains that require reasoning under uncertainty, such as decision support systems and artificial intelligence.

Future research may focus on the practical application of these ideals in areas such as medical diagnostics and intelligent systems, where managing ambiguity is essential. Moreover, generalizing these concepts to other algebraic structures and developing computational frameworks or algorithms based on their properties could further support complex decision-making processes in real-time environments. In particular, the exploration of neutrosophic hyperstructures, as discussed by Alsubie and Al-Masarwah [27], offers a compelling direction for extending the present work to more generalized systems such as hyper BCK-algebras. Furthermore, the integration of fuzzy soft computing paradigms, such as fuzzy soft graphs [28], may provide enriched algebraic modeling tools for handling complex uncertainty and flexible information structures in abstract systems.

Acknowledgements

This research was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5027/2567).

References

[1] K. Iséki. On BCI-algebras. *Mathematics Seminar Notes (Kobe University)*, 8(1):125–130, 1980.

[2] K. Iséki and S. Tanaka. An introduction to the theory of BCK-algebras. *Mathematica Japonica*, 23(1):1–25, 1978.

[3] M. Kaviyarasu and K. Indhira. On intuitionistic fuzzy INK-ideals of INK-algebras. *IOP Conference Series: Materials Science and Engineering*, 263:042142, 2017.

[4] M. Kaviyarasu, K. Indhira, V. M. Chandrasekaran, and K. Jacob. *Interval-valued fuzzy subalgebra and fuzzy INK-ideal in INK-algebra, Advances in Algebra and Analysis. Trends in Mathematics.* Birkhäuser, Cham, 2018.

[5] R. Mounikalakshmi, E. Tamma, and C. Jana. Bipolar fuzzy INK-subalgebras of INK-algebras. *AIMS Mathematics*, 9(10):27593–27606, 2024.

[6] R. Mounikalakshmi, E. Tamma, V. Kalyani Uppuluri, and A. Iampan. Homomorphisms and anti-homomorphisms of neutrosophic INK-algebras. *International Journal of Neutrosophic Science*, 23(1):335–340, 2024.

[7] A. Al-Masarwah, M. Kaviyarasu, K. Alnefaie, and M. Rajeshwari. Fermatean neutrosophic INK-algebras. *European Journal of Pure and Applied Mathematics*, 17(2):1113–1128, 2024.

[8] W. F. Al-Omeri, M. Kaviyarasu, and M. Rajeshwari. Translation of neutrosophic INK-algebras. *Neutrosophic Sets and Systems*, 66:119–135, 2024.

[9] M. Kaviyarasu, K. Indhira, and V. M. Chandrasekaran. Direct product of neutrosophic INK-algebras. *Neutrosophic Sets and Systems*, 38:227–234, 2020.

[10] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[11] Y. B. Jun, E. H. Roh, and S. M. Mostafa. On fuzzy implicative ideals of BCK-algebras. *Soochow Journal of Mathematics*, 25(1):57–70, 1999.

[12] A. Paad. On fuzzy implicative ideals in BL-algebras. *Journal of Algebraic Hyperstructures and Logical Algebras*, 1(4):83–94, 2020.

[13] S. Sowmiya and P. Jeyalakshmi. On fuzzy implicative ideals in Z-algebras. *Advances and Applications in Mathematical Sciences*, 21(10):5911–5922, 2022.

[14] K. T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1):87–96, 1986.

[15] B. Satyanarayana, L. Krishna, and R. D. Prasad. On intuitionistic fuzzy implicative hyper BCK-ideals of hyper BCK-algebras. *International Journal of Mathematics and Statistics Invention*, 2(2):77–85, 2014.

[16] R. D. Prasad, L. Krishna, and B. Satyanarayana. On interval-valued intuitionistic fuzzy (implicative and commutative) ideals of BCK-algebra. *Advances in Fuzzy Mathematics*, 12(3):371–380, 2017.

[17] B. Satyanarayana, V. J. Sree, R. D. Prasad, and U. B. Madhavi. Derivations of intuitionistic fuzzy implicative ideals of BCK-algebra. *Advances and Applications in Mathematical Sciences*, 20(6):1147–1166, 2021.

[18] R. Rasuli. Intuitionistic fuzzy BCI-algebras (implicative ideals, closed implicative ideals, commutative ideals) under norms. *Mathematical Analysis and its Contemporary Applications*, 4(3):17–34, 2022.

[19] F. Smarandache. *A unifying field in logics: neutrosophic logic, neutrosophy, neutrosophic set, neutrosophic probability.* American Research Press, Rehoboth, New Mexico, 1999.

[20] Y. B. Jun and E. H. Roh. MBJ-neutrosophic ideals of BCK/BCI-algebras. *Open Mathematics*, 17(1):588–601, 2019.

[21] R. A. Borzooei, M. M. Takallo, F. Smarandache, and Y. B. Jun. Positive implicative BMBJ-neutrosophic ideals in BCK-algebras. *Neutrosophic Sets and Systems*, 23:126–141, 2018.

[22] H. Bordbar, X. L. Xin, R. A. Borzooei, and Y. B. Jun. Positive implicative ideals of BCK-algebras based on neutrosophic sets and falling shadows. *Neutrosophic Sets and Systems*, 48:9–30, 2022.

[23] B. Satyanarayana and S. Baji. Positive implicative, implicative, and commutative SB-neutrosophic ideals in BCK/BCI-algebras. *IAENG International Journal of Applied Mathematics*, 54(5):815–830, 2024.

[24] M. Kaviyarasu, K. Indhira, and V. M. Chandrasekaran. Introduction on INK-algebras. *International Journal of Pure and Applied Mathematics*, 115(9):1–10, 2017.

[25] M. Kaviyarasu, K. Indhira, and V. M. Chandrasekaran. Fuzzy subalgebras and fuzzy K-ideals in INK-algebras. *International Journal of Pure and Applied Mathematics*, 113(6):47–55, 2017.

[26] M. Kaviyarasu, K. Indhira, and V. M. Chandrasekaran. Neutrosophic set in INK-algebra. *Advances in Mathematics: Scientific Journal*, 9(7):4345–4352, 2020.

[27] A. Alsubie and A. Al-Masarwah. MBJ-neutrosophic hyper BCK-ideals in hyper BCK-algebras. *AIMS Mathematics*, 6(6):6107–6121, 2021.

[28] A. Al-Masarwah and M. A. Qamar. Certain types of fuzzy soft graphs. *New Mathematics and Natural Computation*, 14(2):145–156, 2018.