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New Multi-Step Collocation Methods for Non-Standard Volterra Integral Equations

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Abstract. Non-standard Volterra integral equations are in the category of non-linear Volterra integral equations which appear in certain mathematical modelling processes. In the current study, new multi-step collocation techniques have been employed to numerically address non-standard Volterra integral equations. Several established outcomes concerning convergence of the numerical solution for this problem are investigated. In conclusion, two test problems are thoroughly examined to substantiate theoretical accomplishments in practical terms. Also, we apply one-step and multi-step collocation methods to solve Non-standard Volterra integral equations and compare numerical results to show the efficiency and high accuracy of the proposed method.

2020 Mathematics Subject Classifications: 65R20, 45G10

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1. Introduction

In general, the kernels of integral equations are in the form $K(t, s, x(s))$. But in some scientific and engineering applications, especially in physical sciences, a special case of nonlinear integral equations appears, where the kernels of the equation is dependent on time t as well as space s. This dependence of the kernels is shown in the general form as $K(t, s, x(t), x(s))$. Therefore, integral equations including these types of kernels are called non-standard integral equations.

A non-linear VIE as

$$y(\sigma) = f(\sigma) + \int_0^\sigma K(\sigma, \tau, y(\sigma), y(\tau))\mathrm{d}\tau, \quad \sigma \in [0, T] = I, \tag{1}$$

is called a NVIEs. Different kind of physical and biological phenomena that lead to NVIEs (1) can be found in [1–5]. Some researchers [6–12] have investigated a number of these non-standard integral equations such as auto convolution equations and implicit VIEs.

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Certain established outcomes rooted in regularity, existence, uniqueness, as well as the convergence of the solution for the general auto-convolution Volterra integral equations studied by Zhang et al [12]. Guan et al. in [13], studied NVIEs (1) and obtained several positive outcomes concerning regularity, existence, uniqueness, and convergence of the one-step collocation method for this equation. In [14], the authors by adding conditions for interpolation in the previous r step points, applied multi-step collocation scheme to solve the equation (1) and increased the order of convergence for the m points and r steps to $m+r$ compared with the one-step collocation method (Order of convergence is m [13]).

Actually, nonlinear VIEs originate in various scientific realms and this is the primary reason why they were a subject of refined investigation from the both theoretical and numerical perspectives so far. Specifically, they span a wide range of applied science such as engineering, mathematical physics, economics, etc[15–19]. One of the particular kind of non-linear VIEs is Volterra-Hammerstein IEs which have been studied in [20]. Usually, finding an analytical solution for the non-linear VIEs is not possible, then researchers try to provide numerical methods to avoid this complicated situation(See [16, 21, 22]). Moreover, the great contributions of recent developments in the conceptually promising and challenging areas of the nonlinear VIEs are all gathered in the book due to Brunner [21].

Here, we consider new multi-step collocation scheme to NVIEs (1) and try to increase the order of convergence compared with the one-step and multi-step collocation methods in [13, 14].

The paper is structured in the following manner: Section 2 is dedicated to introducing a new multi-step polynomial collocation method for discretizing NVIEs using widely recognized interpolation polynomials. In the sequel, we analyze the convergence of the numerical solutions, considering Peano theorems for interpolation, in section 3. Section 4 then delves into the examination of two test problems to validate the theoretical results.

2. Numerical methods

The multi-step method in the context of integral equations serves a similar purpose as in ordinary differential equations improving accuracy and computational efficiency but is tailored to handle integral operators. Its primary aim is to solve integral equations numerically by leveraging past computed values to approximate the solution at subsequent steps more effectively.

The fundamental purposes of multi-step techniques in integral equations include:

1. Higher-Order Approximation;
   By using information from multiple previous steps (e.g., past values of the unknown function and its integrals), multi-step methods achieve better accuracy than single-step quadrature-based approaches.

2. Reduced Computational Cost;
   Reusing previously computed integral evaluations (e.g., in Volterra or Fredholm

equations) avoids redundant calculations, making the method more efficient than naive quadrature rules.

3. Handling Non-Local Dependence;
Integral equations involve global dependence (due to the integral term), and multi-step methods help approximate the history integral more efficiently by exploiting smoothness and past data.

4. Improved Stability for Volterra Equations;
Multi-step discretizations can stabilize solutions for weakly singular or nonlinear integral equations.

We will now introduce and examine two numerical methods: the standard multi-step collocation method and an advanced version referred to as the new multi-step collocation method.

2.1. Multi-step collocation scheme for NVIE

The examination conducted in [14], we consider the multi-step collocation scheme for the equation (1). Consider $\{t_n = nh, \quad n = 0, 1, \ldots, N \quad (t_N = T)\}$, as an equidistant grid on I. Suppose that S_h be given by

$$S_h := \{t_{n,i} = t_n + c_i h : 0 < c_1 < \cdots < c_m \leq 1, \quad 0 \leq n \leq N-1\},$$

where $\{t_{n,i}\}$ are collocation points with $\{c_i\}$ as collocation parameters. Take the collocation solution on the interval $\epsilon_n = (t_n, t_{n+1}]$ for $s \in [0, 1]$ by

$$y(t_n + sh) = \sum_{k=0}^{r-1} L_k(s) x_{n-k} + \sum_{j=1}^{m} \hat{L}_j(s) Y_{n,j}, \quad n = r, \cdots, N-1, \tag{2}$$

with

$$Y_{n,j} := y(t_{n,j}), \quad x_{n-k} := y(t_{n-k}),$$

$$L_k(s) = \prod_{i=1}^{m} \frac{s - c_i}{-k - c_i} \cdot \prod_{i=0, i\neq k}^{r-1} \frac{s + i}{-k + i}, \quad \hat{L}_j(s) = \prod_{i=0}^{r-1} \frac{s + i}{c_j + i} \cdot \prod_{i=1, i\neq j}^{m} \frac{s - c_i}{c_j - c_i}.$$

Note that the starting values $x_1, \ldots, x_r$ can be obtained by a suitable one-step method. In the sequel, we find y so that the collocation equation is hold for $t \in S_h$,

$$y(t) = f(t) + \int_0^t K\Big(t, s, y(t), y(s)\Big) \mathrm{d}s.$$

Hence, on every subinterval ϵ_n, we have the system of algebraic equations for the unknowns $Y_{n,j},\ j = 1, \ldots, m,\ n = r, \ldots, N-1,$

$$\begin{aligned} Y_{n,j} &= f(t_{n,j}) + h\sum_{k=0}^{r-1}\int_0^1 K\Big(t_{n,j}, t_k + sh, Y_{n,j}, y(t_k + sh)\Big)\mathrm{d}s \\ &+ h\sum_{k=r}^{n-1}\int_0^1 K\Big(t_{n,j}, t_k + sh, Y_{n,j}, \sum_{j=0}^{r-1} L_j(s)x_{k-j} + \sum_{i=1}^{m}\hat{L}_i(s)Y_{k,i}\Big)\mathrm{d}s \\ &+ h\int_0^{c_j} K\Big(t_{n,j}, t_n + sh, Y_{n,j}, \sum_{k=0}^{r-1} L_k(s)x_{n-k} + \sum_{i=1}^{m}\hat{L}_i(s)Y_{n,i}\Big)\mathrm{d}s. \end{aligned} \tag{3}$$

Also, the convergence theorem results can be presented as:

Theorem 1. *[14] Consider*
i) $f \in C^{m+r}(I)$ *and for* $D := \{(t,s) : 0 \leq s, t \leq T\}$, $K \in C^{m+r}(D \times \mathbb{R} \times \mathbb{R})$,

ii) $|K(t,s,r_1,s_1) - K(t,s,r_2,s_2)| \leq Q(t,s)|r_1 - r_2| + M|s_1 - s_2|$ *for all* $(t,s) \in D, r_i, s_i \in \mathbb{R}$ *with* $0 < Q(s,s) \leq M$ *and* $\int_0^t Q(t,s)\mathrm{d}s \leq 1 - \Delta$ *for* $\Delta \in (0,1)$,

iii) The order of primary error is $m + r$, *and the spectral radius of the matrix* A *is less than 1,* $(\rho(A) < 1)$.

Then the NIVE (1) *has a unique solution* $x \in C^{m+r}(I)$ *and for the multi-step collocation method, we have:*

$$\|x - y\|_\infty \leq Ch^{m+r},$$

such that C *is a constant not depending on* N.

2.2. New multi-step collocation methods

To construct the new multi-step methods, we seek a collocation polynomial of the form

$$y(t_n + sh) = \sum_{k=0}^{r-1} L_k(s)x_{n-k} + \sum_{j=1}^{m}\hat{L}_j(s)Y_{n,j} + \sum_{j=1}^{m}\tilde{L}_j(s)Y_{n+1,j},\ n = r-1, \ldots, N-2, \tag{4}$$

where

$$Y_{n,j} = y(t_{n,j}), \quad Y_{n+1,j} = y(t_{n+1,j}),$$

$$L_k(s) = \prod_{\substack{i=0 \\ i \neq k}}^{r-1}\frac{s+i}{-k+i} \cdot \prod_{i=1}^{m}\frac{s - c_i}{-k - c_i} \cdot \prod_{i=1}^{m}\frac{s - c_i - 1}{-k - c_i - 1}, \tag{5}$$

$$\hat{L}_j(s) = \prod_{i=0}^{r-1}\frac{s+i}{c_j+i} \cdot \prod_{\substack{i=1 \\ i \neq j}}^{m}\frac{s - c_i}{c_j - c_i} \cdot \prod_{i=1}^{m}\frac{s - c_i - 1}{c_j - c_i - 1}, \tag{6}$$

$$\tilde{L}_j(s) = \prod_{i=0}^{r-1} \frac{s+i}{1+c_j+i} \cdot \prod_{i=1}^{m} \frac{s-c_i}{1+c_j-c_i} \cdot \prod_{\substack{i=1\\ i\neq j}}^{m} \frac{s-c_i-1}{c_j-c_i}, \tag{7}$$

also the collocation parameters are assumed to satisfy $c_1 \neq 0$ and $c_i \neq c_j$ for $i \neq j$. In the equations (1) by substituting $t = t_{n+1,i}$ and approximate solution (4), we will have

$$\begin{aligned} Y_{n+1,j} = \;& f(t_{n+1,j}) + h\sum_{z=0}^{r-2}\int_0^1 K\Big(t_{n+1,j}, t_z+sh, Y_{n+1,j}, y(t_z+sh)\Big)\mathrm{d}s \\ & + \sum_{z=r-1}^{n-1} h\int_0^1 \Bigg(K\Big(t_{n+1,j}, t_z+sh, Y_{n+1,j}, \sum_{k=0}^{r-1} L_k(s)x_{z-k} + \sum_{i=1}^{m}\hat{L}_i(s)Y_{z,i} \\ & + \sum_{i=1}^{m}\tilde{L}_i(s)Y_{z+1,i}\Big)\Bigg)\mathrm{d}s + h\int_0^1 \Bigg(K\Big(t_{n+1,j}, t_n+sh, Y_{n+1,j}, \sum_{k=0}^{r-1} L_k(s)x_{n-k} \\ & + \sum_{i=1}^{m}\hat{L}_i(s)Y_{n,i} + \sum_{i=1}^{m}\tilde{L}_i(s)Y_{n+1,i}\Big)\Bigg)\mathrm{d}s + h\int_0^{c_j}\Bigg(K\Big(t_{n+1,j}, t_{n+1}+sh, \\ & Y_{n+1,j}, \sum_{k=0}^{r-1} L_k(1+s)x_{n-k} + \sum_{i=1}^{m}\hat{L}_i(1+s)Y_{n,i} + \sum_{i=1}^{m}\tilde{L}_i(1+s)Y_{n+1,i}\Big)\Bigg)\mathrm{d}s, \end{aligned} \tag{8}$$

with

$$y_{n+1} = \sum_{k=0}^{r-1} L_k(1)x_{n-k} + \sum_{j=1}^{m}\hat{L}_j(1)Y_{n,j} + \sum_{j=1}^{m}\tilde{L}_j(1)Y_{n+1,j}.$$

In general, the integrals in the system (8), can be computed approximately by using quadrature formulas to yield the discretized version of this non-linear system.

3. Convergence analysis

Consider the NVIE (1), and it's corresponding collocation equation

$$y(t) = f(t) + \int_0^t K(t,s,y(t),y(s))\mathrm{d}s. \tag{9}$$

By subtracting equation (9) from (1), we will get

$$e(t) = \int_0^t \Big(K(t,s,x(t),x(s)) - K(t,s,y(t),y(s))\Big)\mathrm{d}s, \tag{10}$$

where $e(t) = x(t) - y(t)$, is the error function.

Before investigating the convergence analysis of the new multi-step collocation solutions, Taylor's theorem to multivariate functions [16, 23, 24] is considered as:

Lemma 1. *Assume that the function $g : \mathbb{R}^n \to \mathbb{R}$, for some $r > 0$ in a closed ball $\mathcal{A} = \{y \in \mathbb{R}^n : ||x_0 - y|| \leq r\}$, $g \in C^{k+1}(\mathcal{A})$. Then we can derive the following formula for the remainder in $\mathcal{A}$, Namely,*

$$g(x) = \sum_{|\alpha| \leq k} \frac{D^\alpha g(x_0)}{\alpha!}(x - x_0)^k + \sum_{|\beta|=k+1} R_\beta(x)(x - x_0)^\beta,$$

where

$$R_\beta(x) = \frac{|\beta|}{\beta!} \int_0^1 (1 - t)^{|\beta|-1} D^\beta g(x_0 + t(x - x_0))\mathrm{d}t.$$

Theorem 2. *Let the second condition of Theorem 1 is hold and for the known functions in* (1)*:*

A1. $K \in C^p(D \times \mathbb{R} \times \mathbb{R})$, *for some* $m \geq 1$, *and* $f \in C^p(I)$.

A2. Order of primary error is p and the spectral radius of the matrix $\tilde{A}$ is less than 1 ($\rho(\tilde{A}) < 1$) where

$$\tilde{A} = \left[\begin{array}{c|c} \mathbf{0}_{r-1,1} & \mathbf{I}_{r-1} \\ \hline L_{r-1}(1) & L_{r-2}(1), \ldots, L_0(1), \end{array} \right]. \tag{11}$$

Then, there exist constant C not depending on N such that

$$\|x - y\|_\infty \leq Ch^p,$$

where $p = 2m + r$.

Proof. Now, by Peano's theorem [16, 25], for $s \in [0, 1]$, the error $e(t) := x(t) - y(t)$ is written as:

$$e(t_\lambda + sh) = \sum_{k=0}^{r-1} L_k(s)\varepsilon_{\lambda-k} + \sum_{j=1}^{m} \hat{L}_j(s)\varepsilon_{\lambda,j} + \sum_{j=1}^{m} \tilde{L}_j(s)\varepsilon_{\lambda+1,j} + h^{2m+r}R_{m,r,\lambda}(s), \tag{12}$$

where $R_{m,r,n}(s)$ is the Peano remainder and $\varepsilon_{\lambda,j} := X_{\lambda,j} - Y_{\lambda,j}$. From (8) and (10), for $\lambda = n + 1$, we have

$$
\begin{aligned}
\varepsilon_{n+1,j} = & \int_0^{t_{n+1,j}} K(t_{n+1,j}, s, x_{n+1,j}, x(s))\mathrm{d}s - \int_0^{t_{n+1,j}} K(t_{n+1,j}, s, y_{n+1,j}, y(s))\mathrm{d}s \\
= & \int_0^{t_{n+1,j}} \Big(K(t_{n+1,j}, s, X_{n+1,j}, x(s)) - K(t_{n+1,j}, s, Y_{n+1,j}, x(s)) \Big)\mathrm{d}s \\
& + h\sum_{l=0}^{n} \int_0^1 \Big(K(t_{n+1,j}, t_l + sh, Y_{n+1,j}, x(t_l + sh)) \\
& - K(t_{n+1,j}, t_l + sh, Y_{n+1,j}, y(t_l + sh)) \Big)\mathrm{d}s \\
& + h\int_0^{c_j} \Big(K(t_{n+1,j}, t_{n+1} + sh, Y_{n+1,j}, x(t_{n+1} + sh)) \\
& - K(t_{n+1,j}, t_{n+1} + sh, Y_{n+1,j}, y(t_{n+1} + sh)) \Big)\mathrm{d}s.
\end{aligned} \tag{13}
$$

Now, assume that

$$
\hat{K}_l(t_{\eta,j}, s) := \int_0^1 \frac{\partial K}{\partial v}(t_{\eta,j}, t_l + sh, U_{\eta,j}, u(t_l + sh) + \zeta(y(t_l + sh) - u(t_l + sh)))\mathrm{d}\zeta, \tag{14}
$$

where for $l = 0, 1, ..., n-1$ the function $\hat{K}_l(t_{\eta,j}, s) \in C[0,1]$. Likewise, for $l = n$, $\hat{K}_n(t_{\eta,j}, s) \in C[0, c_j]$.
Also,

$$
\hat{K}(t_{\eta,j}, s) := \int_0^1 \frac{\partial K}{\partial u}(t_{\eta,j}, s, U_{\eta,j} + \zeta(Y_{\eta,j} - U_{\eta,j}), y(s))d\zeta, \tag{15}
$$

then $\hat{K}(t_{\eta,j}, s) \in C[0, t_{\eta,j}]$. Using $A1$, possess

$$
|\hat{K}_l(t_{\eta,j}, s)| \leq M,
$$

and

$$
|\hat{K}(t_{\eta,j}, s)| \leq H(t_{\eta,j}, s).
$$

For further detail see [13].

By using of this assumptions, for $\eta = n+1$, equation (13) can be written

$$\begin{aligned}\varepsilon_{n+1,j} = & \int_0^{t_{n+1,j}} \hat{K}(t_{n+1,j}, s)\mathrm{d}s\varepsilon_{n+1,j} \\ & + h\sum_{z=0}^{n}\int_0^1 \Big(\hat{K}_z(t_{n+1,j}, s)(x(t_z + sh) - y(t_z + sh))\Big)\mathrm{d}s \\ & + h\int_0^{c_j} \Big(\hat{K}_{n+1}(t_{n+1,j}, s)(x(t_{n+1} + sh) - y(t_{n+1} + sh))\Big)\mathrm{d}s \\ = & \int_0^{t_{n+1,j}} \hat{K}(t_{n+1,j}, s)\mathrm{d}s\varepsilon_{n+1,j} + h\sum_{z=0}^{n}\int_0^1 \hat{K}_z(t_{n+1,j}, s)e(t_z + sh)\mathrm{d}s \\ & + h\int_0^{c_j} \Big(\hat{K}_{n+1}(t_{n+1,j}, s)e(t_{n+1} + sh)\Big)\mathrm{d}s.\end{aligned} \tag{16}$$

From $A2$, for $z = 0, \ldots, r-1$, and $s \in [0, 1)$, we have

$$e(t_z + sh) = h^p q_z(s), \quad ||q_z||_\infty \leq C_1, \tag{17}$$

and then using the (17), equation (16) reduced to the following form

$$
\begin{aligned}
&\varepsilon_{n+1,j} - \int_0^{t_n+1,j} \hat{K}(t_{n+1,j}, s)\mathrm{d}s\varepsilon_{n+1,j} \\
&-h\sum_{i=1}^{m}\int_0^{c_j} \hat{K}_{n+1}(t_{n+1,j}, s)\tilde{L}_i(s+1)\mathrm{d}s\varepsilon_{n+1,j} \\
&-h\sum_{i=1}^{m}\int_0^{1} \hat{K}_{n}(t_{n+1,j}, s)\tilde{L}_i(s)\mathrm{d}s\varepsilon_{n+1,j} \\
&-h\sum_{i=1}^{m}\int_0^{c_j} \hat{K}_{n+1}(t_{n+1,j}, s)\hat{L}_i(s+1)\mathrm{d}s\varepsilon_{n,j} \\
&-h\sum_{i=1}^{m}\int_0^{1} \hat{K}_{n-1}(t_{n+1,j}, s)\tilde{L}_i(s)\mathrm{d}s\varepsilon_{n,j} \\
&-h\sum_{i=1}^{m}\int_0^{1} \hat{K}_{n}(t_{n+1,j}, s)\hat{L}_i(s)\mathrm{d}s\varepsilon_{n,j} \\
&= h^{p+1}\sum_{z=0}^{r-1}\int_0^{1} \hat{K}_{z}(t_{n+1,j}, s)q_z(s)\mathrm{d}s \qquad (18)\\
&+h^{p+1}\sum_{z=r}^{n}\int_0^{1} \hat{K}_{z}(t_{n+1,j}, s)R_{m,r,z}(s)\mathrm{d}s \\
&+h^{p+1}\int_0^{c_j} \hat{K}_{n}(t_{n+1,j}, s)R_{m,r,n}(1+s)\mathrm{d}s \\
&+h\sum_{z=r}^{n}\sum_{k=0}^{r-1}\int_0^{1} \hat{K}_{z}(t_{n+1,j}, s)L_k(s)\mathrm{d}s\varepsilon_{z-k} \\
&+h\sum_{z=r}^{n-1}\sum_{i=1}^{m}\int_0^{1} \hat{K}_{z}(t_{n+1,j}, s)\hat{L}_i(s)\mathrm{d}s\varepsilon_{z,j} \\
&+h\sum_{z=r}^{n-2}\sum_{i=1}^{m}\int_0^{1} \hat{K}_{z}(t_{n+1,j}, s)\tilde{L}_i(s)\mathrm{d}s\varepsilon_{z+1,j} \\
&+h\sum_{k=0}^{r-1}\int_0^{c_j} \hat{K}_{n+1}(t_{n+1,j}, s)L_k(s+1)\mathrm{d}s\varepsilon_{n-k}.
\end{aligned}
$$

Now, to get the matrix form of equation (18), we define the following vectors

$$\bar{a}_{n+1}^{j} = 1 - \int_{0}^{t_n+1,j} \hat{K}(t_{n+1,j}, s)\mathrm{d}s, \quad j = 1, 2, ..., m,$$
$$E_{1,z} = [\varepsilon_{z-r+1}, \varepsilon_{z-r}, ..., \varepsilon_z]^T, \qquad E_{2,z} = [\varepsilon_{z,1}, \varepsilon_{z,2}, ..., \varepsilon_{z,m}]^T, \tag{19}$$

and matrices

$$\bar{A}_{n+1} = \quad \mathrm{diag}(\bar{a}_{n+1}^{1}, ..., \bar{a}_{n+1}^{m}), \tag{20}$$

$$(B_{n+1}^{(z)})_{j,k} = \begin{cases} \displaystyle\int_0^1 \hat{K}_z(t_{n+1,j}, s)L_k(s)\mathrm{d}s, & z = r, ..., n, \\ \displaystyle\int_0^{c_j} \hat{K}_z(t_{n+1,j}, s)L_k(1+s)\mathrm{d}s, & z = n+1, \end{cases} \tag{21}$$

$$(C_{n+1}^{(z)})_{j,i} = \begin{cases} \displaystyle\int_0^1 \hat{K}_z(t_{n+1,j}, s)\hat{L}_i(s)\mathrm{d}s, & z = r, ..., n, \\ \displaystyle\int_0^{c_j} \hat{K}_z(t_{n+1,j}, s)\hat{L}_i(1+s)\mathrm{d}s, & z = n+1, \end{cases} \tag{22}$$

and

$$(D_{n+1}^{(z)})_{j,i} = \begin{cases} \displaystyle\int_0^1 \hat{K}_z(t_{n+1,j}, s)\tilde{L}_i(s)\mathrm{d}s, & z = r, ..., n, \\ \displaystyle\int_0^{c_j} \hat{K}_z(t_{n+1,j}, s)\tilde{L}_i(1+s)\mathrm{d}s, & z = n+1. \end{cases} \tag{23}$$

Therefore, by replacing these vectors and matrices in equation (18) and after some computations, we will have

$$\begin{aligned} &\left(\bar{A}_{n+1} - h\Big(D_{n+1}^{(n+1)} + D_{n+1}^{(n)}\Big)\right)E_{2,n+1} - h\left(C_{n+1}^{(n+1)} + C_{n+1}^{(n)} + D_{n+1}^{(n-1)}\right)E_{2,n} \\ &= h\sum_{z=r}^{n-1} B_{n+1}^{(z)}E_{1,z} + hB_{n+1}^{(z)}E_{1,n+1} + hB_{n+1}^{(z)}E_{1,n} + h\sum_{z=r}^{n-1} C_{n+1}^{(z)}E_{2,z} \\ &\quad + h\sum_{z=r}^{n-2} D_{n+1}^{(z)}E_{2,z} + \mathcal{O}(h^{p+1}). \end{aligned} \tag{24}$$

Similar to what we did before for equation (13) at point $t = t_{n+1,j}$, the matrix form of this equation at point $t = t_{n,j}$ can be expressed as

$$\begin{aligned} &-hD_n^{(n)}E_{2,n+1} + \left(\bar{A}_n - h\Big(C_n^{(n)} + D_n^{(n-1)}\Big)\right)E_{2,n} \\ &= h\sum_{z=r}^{n} B_n^{(z)}E_{1,z} + h\sum_{z=r}^{n-2} D_n^{(z)}E_{2,z+1} + h\sum_{z=r}^{n-2} C_n^{(z)}E_{2,z} + C_n^{(n-1)}E_{2,n-1} + \mathcal{O}(h^{p+1}). \end{aligned} \tag{25}$$

Now, in the equation (12), we set $n = l - 1$, then we get the difference equation

$$E_{1,l} = \tilde{A}E_{1,l-1} + \tilde{S}E_{2,l-1} + \tilde{T}E_{2,l} + \mathcal{O}(h^p), \tag{26}$$

with exact solution

$$E_{1,l} = \tilde{A}^{l-r+1}E_{1,r-1} + \sum_{j=r-1}^{l-1} \tilde{A}^{l-j-1}(\tilde{S}E_{2,j} + \tilde{T}E_{2,j+1}) + \mathcal{O}(h^p), \tag{27}$$

where

$$\tilde{S} = \left[\begin{array}{c} \mathbf{0}_{r-1,m} \\ \hline \bar{L}(1), \dots, \bar{L}_m(1) \end{array}\right], \quad \tilde{T} = \left[\begin{array}{c} \mathbf{0}_{r-1,m} \\ \hline \hat{L}_1(1), \dots, \hat{L}_m(1) \end{array}\right]. \tag{28}$$

Inserting (27) in equations (24) and (25), we will get the general form of the system of their equations in the form of the following system

$$\begin{cases} \left(\bar{A}_{n+1} - h\left(D_{n+1}^{(n+1)} + D_{n+1}^{(n)} + B_{n+1}^{(z)}\tilde{T}\right)\right)E_{2,n+1} \\ -h\left(C_{n+1}^{(n+1)} + C_{n+1}^{(n)} + D_{n+1}^{(n-1)} + B_{n+1}^{(z)}\tilde{S} + B_{n+1}^{(z)}\tilde{A}\tilde{T} + B_{n+1}^{(z)}\tilde{T}\right)E_{2,n} \\ = h\sum\limits_{z=r}^{n-1} B_{n+1}^{(z)}E_{1,z} + hB_{n+1}^{(z)}\tilde{A}^{n-r+2}E_{1,r-1} + h\sum\limits_{j=r-1}^{n-1} B_{n+1}^{(z)}\tilde{A}^{n-j}\tilde{S}E_{2,j} \\ \quad +h\sum\limits_{j=r-1}^{n-2} B_{n+1}^{(z)}\tilde{A}^{n-j}\tilde{T}E_{2,j+1} + hB_{n+1}^{(z)}\tilde{A}^{n-r+1}E_{1,r-1} \\ \quad +h\sum\limits_{j=r-1}^{n-1} B_{n+1}^{(z)}\tilde{A}^{n-j-1}\tilde{S}E_{2,j} + h\sum\limits_{j=r-1}^{n-2} B_{n+1}^{(z)}\tilde{A}^{n-j-1}\tilde{T}E_{2,j+1} \\ \quad +h\sum\limits_{z=r}^{n-1} C_{n+1}^{(z)}E_{2,z} + h\sum\limits_{z=r}^{n-2} D_{n+1}^{(z)}E_{2,z} + \mathcal{O}(h^{p+1}), \\ \\ -hD_n^{(n)}E_{2,n+1} + \left(\bar{A}_n - h\left(C_n^{(n)} + D_n^{(n-1)} - B_n^{(z)}\tilde{T}\right)\right)E_{2,n} \\ = h\sum\limits_{z=r}^{n-1} B_n^{(z)}E_{1,z} + hB_n^{(z)}\tilde{A}^{n-r+1}E_{1,r-1} + h\sum\limits_{j=r-1}^{n-1} B_n^{(z)}\tilde{A}^{n-j-1}\tilde{S}E_{2,j} \\ \quad +h\sum\limits_{j=r-1}^{n-2} B_n^{(z)}\tilde{A}^{n-j-1}\tilde{T}E_{2,j+1} + h\sum\limits_{z=r}^{n-2} D_n^{(z)}E_{2,z+1} + h\sum\limits_{z=r}^{n-2} C_n^{(z)}E_{2,z} \\ \quad +C_n^{(n-1)}E_{2,n-1} + \mathcal{O}(h^{p+1}). \end{cases} \tag{29}$$

Note that, the coefficients matrix on the left hand of (29) has the form:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{30}$$

where

$$\begin{aligned}
a_{11} &= \bar{A}_{n+1} - h\Big(D_{n+1}^{(n+1)} + D_{n+1}^{(n)} + B_{n+1}^{(z)}\tilde{T}\Big),\\
a_{12} &= -h\left(C_{n+1}^{(n+1)} + C_{n+1}^{(n)} + D_{n+1}^{(n-1)} + B_{n+1}^{(z)}\tilde{S} + B_{n+1}^{(z)}\tilde{A}\tilde{T} + B_{n+1}^{(z)}\tilde{T}\right),\\
a_{21} &= -hD_{n}^{(n)},\\
a_{22} &= \bar{A}_{n} - h\Big(C_{n}^{(n)} + D_{n}^{(n-1)} - B_{n}^{(z)}\tilde{T}\Big).
\end{aligned} \tag{31}$$

Now, we set

$$\boldsymbol{\zeta} = \begin{bmatrix} E_{2,n+1} \\ E_{2,n} \end{bmatrix}, \tag{32}$$

and $\boldsymbol{A}_h = \boldsymbol{A} + \mathcal{O}(h)$, then we have

$$\boldsymbol{A}_h\boldsymbol{\zeta} = \boldsymbol{b}, \tag{33}$$

where vector $\boldsymbol{b}$, is clear and

$$\boldsymbol{A}_h = \begin{bmatrix} \bar{A}_{n+1} & 0 \\ 0 & \bar{A}_n \end{bmatrix}. \tag{34}$$

Then, matrix $\boldsymbol{A}_h$ is invertible if for $\eta = n, n+1$, the matrices $\bar{A}_\eta$, are invertible. According to theorem 3.6 from reference [13], matrices $\bar{A}_n$, and $\bar{A}_{n+1}$, are invertible. Therefore, the bound for $\boldsymbol{\zeta}$ can be obtained by the same way as described in [13] (Theorem 3.6), then we obtain

$$||\boldsymbol{\zeta}|| \leq Ch^p,$$

and the proof has completes here.

4. Numerical results

Now, we aim to validate the previously established analytical findings by employing numerical solutions to address two test problems. At the same time, we will tabulate the attained satisfactory outcomes concerning L_∞ errors and optimal convergence orders for some values of N with $m, r = 2$. This is done for the purpose of numerically verifying the underlying theorem 2. We consider two following cases:

1) $c_1 = \dfrac{1}{2}, c_2 = 1 \Rightarrow \rho(\tilde{A}) = 0 < 1.$

2) $c_1 = \dfrac{1}{3}, c_2 = \dfrac{7}{8} \Rightarrow \rho(\tilde{A}) = 0.505485 < 1.$

To ensure clarity, it's important to mention that all implementations have been carried out in the Mathematica ® software. Continuing in this section, the primary values have been gotten from well-established exact solutions. The comparison of the obtained numerical results in Tables 1- 4, for $m, r = 2$, shows that new multi-step method (NMSM) is more accurate than the one-step method (1-SM) and multi-step methods (MSMs) used in [13, 14]. Also in Fig. 1 and Fig. 2, we plot error functions for the one-step, multi-step and

N	$(c_1, c_2) = (\frac{1}{2}, 1)$ 1-SM [13]	$(c_1, c_2) = (\frac{1}{2}, 1)$ MSM [14]	$(c_1, c_2) = (\frac{1}{2}, 1)$ NMSM
4	5.20×10^{-6}	5.58×10^{-10}	1.30×10^{-11}
8	1.30×10^{-6}	6.52×10^{-11}	2.29×10^{-13}
16	3.25×10^{-7}	5.34×10^{-12}	4.21×10^{-15}
Order	2.00	3.60	5.76

Table 1: Numerical report for example 1.

N	$(c_1, c_2) = (\frac{1}{2}, 1)$ 1-SM [13]	$(c_1, c_2) = (\frac{1}{2}, 1)$ MSM [14]	$(c_1, c_2) = (\frac{1}{2}, 1)$ NMSM
4	1.70×10^{-5}	2.04×10^{-9}	1.77×10^{-10}
8	4.18×10^{-6}	4.21×10^{-10}	9.04×10^{-12}
16	1.03×10^{-6}	4.82×10^{-11}	1.82×10^{-13}
Order	2.01	3.12	5.62

Table 2: Numerical report for example 2.

new multi-step scheme with $N = 16$. As can be seen, the accuracy of the numerical results obtained by the new multi-step method is higher than the other ones.

Example 1. *[13] Take for the equation* (1) *with* $T = 1$, *as*

$$x(t) = f(t) + \int_0^t \frac{s+t+5}{(2s+4t+10)(x^2(t)+1+x^2(s))} \mathrm{d}s,$$

whose exact solution is $x(t) = \sqrt{2+t}$ *and*

$$f(t) = (2+t)^{\frac{1}{2}} - \ln(\frac{5+3t}{5+2t})^{\frac{1}{2}}.$$

Example 2. *[13]Take for the equation* (1) *for* $0 \leq t \leq 1$, *as*

$$x(t) = (1+t)^{\frac{1}{2}} - \frac{1}{2}\exp(-(t+2t)) + \frac{1}{2}\exp(-2(t+1)) + \int_0^t \frac{1}{2}\exp\Big(-(y^2(t)+y^2(s))\Big)\mathrm{d}s,$$

whose exact solution is $x(t) = (1+t)^{\frac{1}{2}}$.

5. Conclusion

In this paper, we studied NMSM to solve the nonstandard Volterra integral equation, numerically. We have displayed that this collocation approach yields an effective and very

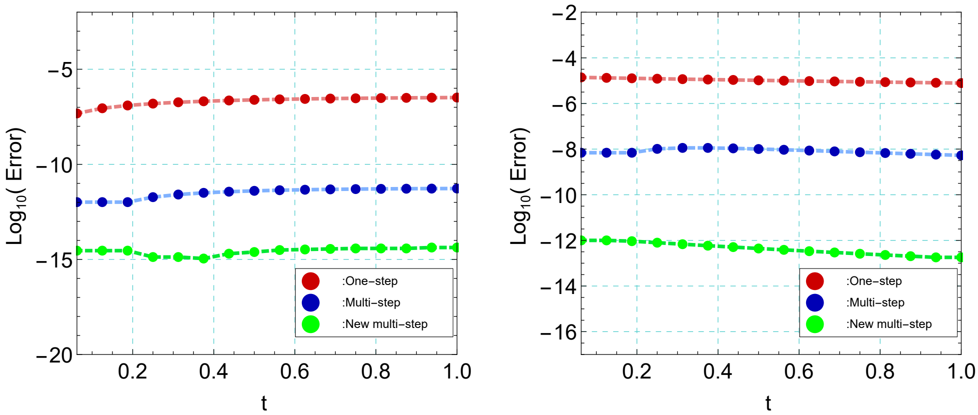

Figure 1: Plot of error functions for the one-step, muti-step and new multi-step scheme with $N = 16, c_1 = .5, c_2 = 1$ in Example 1 (left). Plot of error functions related for the one-step, muti-step and new multi-step scheme with $N = 16, c_1 = \frac{1}{3}, c_2 = \frac{7}{8}$ in Example 1 (right).

N	$(c_1, c_2) = (\frac{1}{3}, \frac{7}{8})$ 1-SM [13]	$(c_1, c_2) = (\frac{1}{3}, \frac{7}{8})$ MSM [14]	$(c_1, c_2) = (\frac{1}{3}, \frac{7}{8})$ NMSM
4	2.01×10^{-4}	1.23×10^{-6}	1.97×10^{-9}
8	5.37×10^{-5}	1.37×10^{-7}	4.95×10^{-11}
16	1.38×10^{-5}	1.13×10^{-8}	1.00×10^{-12}
Order	1.95	3.52	5.62

Table 3: Numerical report for example 1.

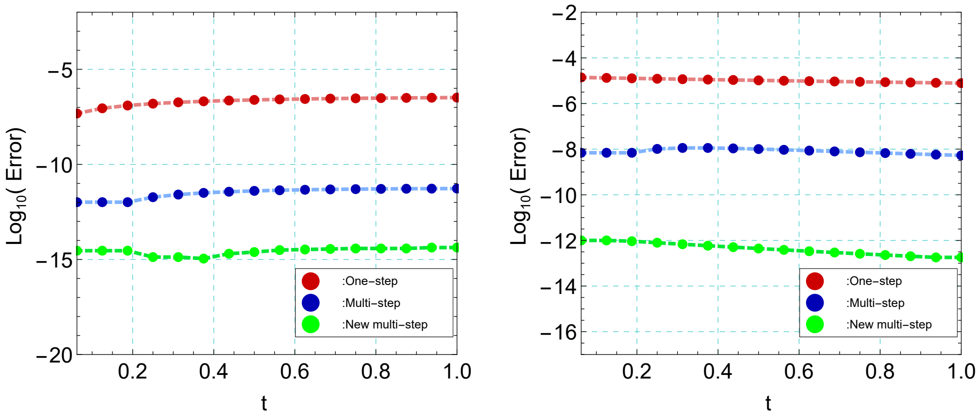

Figure 2: Plot of error functions for the one-step, muti-step and new multi-step scheme with $N = 16, c_1 = 0.5$ and $c_2 = 1$ in Example 2 (left). Plot of error functions related for the one-step, muti-step and new multi-step scheme with $N = 16, c_1 = \frac{1}{3}$ and $c_2 = \frac{7}{8}$ in Example 2 (right).

N	$(c_1, c_2) = (\frac{1}{3}, \frac{7}{8})$ 1-SM [13]	$(c_1, c_2) = (\frac{1}{3}, \frac{7}{8})$ MSM [14]	$(c_1, c_2) = (\frac{1}{3}, \frac{7}{8})$ NMSM
4	5.04×10^{-4}	6.36×10^{-6}	3.92×10^{-8}
8	1.42×10^{-4}	8.73×10^{-7}	1.39×10^{-9}
16	3.80×10^{-5}	9.21×10^{-8}	1.39×10^{-11}
Order	1.90	3.25	5.31

Table 4: Numerical report for example 2.

accurate numerical method for the approximation of solutions to NVIE. The proposed scheme exhibited satisfactory global convergence, with the observation that the convergence orders were primarily influenced by the collocation parameter and the conditions for $\rho(\tilde{A})$.

References

[1] M. Bohner and A. Peterson. *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston, MA, 2003.

[2] F. Brauer and C. Castillo-Chavez. *Mathematical Models in Population Biology and Epidemiology*, volume 40 of *Texts in Applied Mathematics*. Springer, New York, 2001.

[3] D. B. Pachpatte. Properties of solutions to nonlinear dynamic integral equations on time scales. *Electronic Journal of Differential Equations*, 136:1–8, 2008.

[4] D. B. Pachpatte. On a nonstandard volterra type dynamic integral equation on time scales. *Electronic Journal of Qualitative Theory of Differential Equations*, 72:1–14, 2009.

[5] C. C. Tisdell and A. Zaidi. Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling. *Nonlinear Analysis*, 68:3504–3524, 2008.

[6] P. Darania, S. Pishbin, and A. Ebadi. Convergence analysis of multi-step collocation methods to solve generalized auto-convolution volterra integral equations. *Numerical Analysis and Applications*, 16(2):123–134, 2023.

[7] U. Kosel and L. von Wolfersdorf. Nichtlineare integralgleichungen. In *Seminar Analysis*, pages 93–128, Berlin, Akademie der Wissenschaften der DDR, 1986. Karl-Weierstrass-Institut für Mathematik.

[8] L. Liu and J. Ma. Collocation boundary value methods for auto-convolution volterra integral equations. *Applied Numerical Mathematics*, 177:1–17, 2022.

[9] L. von Wolfersdorf. Einige klassen quadratischer integralgleichungen. *Sitzungsberichte der Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Naturwissenschaftliche Klasse*, 128(2):1–34, 2000.

[10] L. von Wolfersdorf. Autoconvolution equations and special functions. *Integral Transforms and Special Functions*, 19:677–686, 2008.

[11] L. von Wolfersdorf. Autoconvolution equations of the third kind with abel integral. *Journal of Integral Equations and Applications*, 23:113–136, 2011.

[12] R. Zhang, H. Liang, and H. Brunner. Analysis of collocation methods for generalized auto-convolution volterra integral equations. *SIAM Journal on Numerical Analysis*, 54(2):899–920, 2016.

[13] Q. Guan, R. Zhang, and Y. Zou. Analysis of collocation methods for nonstandard volterra integral equations. *IMA Journal of Numerical Analysis*, 32:1755–1785, 2012.

[14] S. Pishbin and A. Ebadi. High-order convergence of multistep collocation methods for nonstandard volterra integral equations. *International Journal of Computer Mathematics*, 100(4):824–837, 2023.

[15] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong. *Positive Solutions of Differential, Difference and Integral Equations.* Kluwer Academic, Dordrecht, 1999.

[16] H. Brunner. *Collocation Methods for Volterra Integral and Related Functional Differential Equations.* Cambridge University Press, Cambridge, 2004.

[17] T. A. Burton. *Volterra Integral and Differential Equations.* Academic Press, New York, 1983.

[18] A. J. Jerri. *Introduction to Integral Equations with Applications.* Wiley-Interscience, New York, 1999.

[19] D. O'Regan and M. Mehhan. *Existence Theory for Nonlinear Integral and Integro-Differential Equations.* Kluwer Academic, Dordrecht, 1998.

[20] G. Gripenberg, S. O. Londen, and O. Staffans. *Volterra Integral and Functional Equations.* Cambridge University Press, Cambridge, 1990.

[21] H. Brunner. *Volterra Integral Equations: An Introduction to Theory and Applications.* Cambridge University Press, Cambridge, UK, 2017.

[22] F. Ghoreishi and M. Hadizadeh. Numerical computation of the tau approximation for the volterra-hammerstein integral equations. *Numerical Algorithms*, 52:541–559, 2009.

[23] T. Apostol. *Mathematical Analysis.* Addison-Wesley, Reading, MA, 1974.

[24] D. Conte, Z. Jackiewicz, and B. Paternoster. Two-step almost collocation methods for volterra integral equations. *Applied Mathematics and Computation*, 204:839–853, 2008.

[25] S. Fazeli, G. Hojjati, and S. Shahmorad. Super implicit multistep collocation methods for nonlinear volterra integral equations. *Mathematical and Computer Modelling*, 55:590–607, 2015.