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Numerical Solution of the Intuitionistic Fuzzy Complex Heat Equation with Intuitionistic Complex Dirichlet Boundary Conditions via the Explicit Finite Difference Method

Abd Ulazeez Alkouri[1,*], Sadeq Damrah[2], Hamzeh Zureigat[3], Osama Ogilat[4], Eman Hussein[5]

[1] *Department of Mathematics, Science college, Ajloun National University, P.O.43, Ajloun-26810, Jordan*
[2] *Department of Mathematics and Physics, College of Engineering, Australian University, West Mishref, Safat 13015, Kuwait*
[3] *Department of Mathematics, Faculty of Science, Jadara University, Irbid, Jordan*
[4] *Department of Basic Sciences, Faculty of Arts and Science, Al-Ahliyya Amman University, Amman 19328, Jordan*
[5] *Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman, Jordan*

Abstract. Recent developments in complex fuzzy (CF) sets have extended the classical fuzzy framework from the unit interval [0,1] to the unit disk in the complex plane C, allowing for the modeling of uncertainties in both magnitude and phase. Building upon this foundation, this study introduces-for the first time-the use of complex intuitionistic fuzzy (CIF) numbers to solve partial differential equations, specifically focusing on the CIF heat equation. The CIF framework integrates both membership and non-membership functions with a complex-valued representation, enabling a more expressive treatment of uncertainty, including hesitation. An explicit finite difference method, namely the Forward Time Central Space (FTCS) scheme, is employed to discretize and solve the CIF heat equation. The model considers fuzziness in the initial and boundary conditions, where uncertainty impacts amplitude and phase terms. To represent this uncertainty, triangular fuzzy numbers are used for the real and imaginary parts within the complex unit disk. The proposed approach demonstrates numerical stability and achieves second-order spatial and first-order temporal accuracy, validating its reliability and effectiveness. A numerical example confirms the feasibility of the method, showing strong alignment with theoretical predictions. This work generalizes existing CF heat equation models and provides a foundation for solving more complex systems involving higher-order and bipolar uncertainties in future studies.

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*Corresponding author.
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Email addresses: alkouriabdulazeez@anu.edu.jo (A. U. Alkouri), s.damrah@au.edu.kw (S. Damrah), hamzeh.zu@jadara.edu.jo (H. Zureigat), o.oqilat@ammanu.edu.jo (O.Ogilat), e.hussein@aau.edu.jo (E. Hussein)

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1. Introduction

Zadeh founded the notion of fuzzy sets in 1965, and since then, numerous mathematicians have written articles extending classical mathematical theories to fuzzy mathematics. Subsequently, Atanassov developed and published the notion of intuitionistic fuzzy sets in 1986. Atanassov extended the notion of a fuzzy set to include a second degree, namely degree of non-membership, and investigated the characteristics of sets with both degrees. Several applications of interest [1],[2], [3], [4], [5] in the field of numerical solutions have involved in both fuzzy and intuitionistic fuzzy sets. Recently, several fields—such as environmental science, fixed point theory, and algebra—have seen successful applications of fuzzy set theory and its generalizations, including CF sets [6], [7], [8], [9], [10], [11], [12], [13], [14], [15].

Recent advancements in fuzzy and intuitionistic fuzzy partial differential equations (FPDEs, IFPDEs) enhance their applicability in solving complex heat equations under uncertainty. Arshad et al. [16] proposed the Laplace-Residual Power Series Method (LRPSM) to derive fast-converging analytical solutions for FPDEs, demonstrating its effectiveness in handling boundary conditions using Laurent series and fractional calculus techniques. This approach aligns with the need for precise numerical solutions in heat transfer models with fuzzy uncertainty. Similarly, Rahman et al. [17] extended intuitionistic fuzzy calculus to model uncertainty in decision-making, emphasizing acceptance-rejection dynamics. Their use of intuitionistic fuzzy differential equations (IFDEs) in economic systems highlights the broader applicability of intuitionistic complex Dirichlet conditions in modeling imprecise systems. The combination of these methodologies supports the explicit finite difference method used in this study for solving CIF heat equations, ensuring improved computational accuracy under fuzzy and intuitionistic fuzzy environments. Additionally, Payal et al. [6] investigated a fuzzy prey–predator model incorporating the Holling Type II functional response, with a focus on mathematical analysis and qualitative behaviors relevant to biological systems such as pest control. In 2025, Rahaman [7]constructed the Riemann–Liouville fractional integral within this framework and extended the analysis to include both Riemann–Liouville and Caputo fractional derivatives. A Type 2 interval-valued Laplace transform was also developed. The theoretical results were illustrated through an application to an economic lot maintenance model. Al-Qudah et al. [8] proposed an efficient analytic-numeric method using fuzzy logic and generalized Taylor expansion to solve fractional-order differential equations with uncertain initial data. The method showed high accuracy and reliability in obtaining fuzzy approximate solutions, suitable for physics and engineering applications.

Buckley introduced a novel definition of fuzzy complex numbers (FCNs) [18], [19], [20], [21] by incorporating complex-valued fuzzy numbers to extend classical fuzzy set theory. In this formulation, FCNs are represented as ordinary fuzzy sets with a membership function ranging between [0,1]. Subsequently, a new idea emerged in 2002 to extend the membership function's image from [0, 1] to the unit disk in the complex plane C. This gave rise to CF sets (CFSs) [22]. Unlike FCNs, CFSs feature complex-valued membership grades, which represent a significant advancement in fuzzy set theory. The concept of CFS has since garnered extensive attention with widespread applications and research contributions [23], [24], [25] Incorporating complex numbers into the fuzzy theory and intuitionistic fuzzy theory, Ramote et al. [26]and Alkouri and Salleh [27], introduced the useful notions of the CFS and Complex intuitionistic fuzzy set (CIFS). These notions extended the range of membership and (non-membership) functions from [0, 1] to unit disk in the C. This extension allows mathematicians to convey and represent a special type of information to mathematical formulation and vice versa without losing the interpretability provided by human experts at various levels or stages. Moreover, several real-life applications of CFS and CIFS have been reported in the literature [28], [29], [30], [31], [32]. The first combination of CFS with numerical methods for solving FPDE involving fuzzy heat equation (FHE) has been introduced by Zureigat et al. in 2023 [33]

The research gaps and motivations for extending the CF heat equation to the novel notion of a CIF heat equation come from the inability of CF heat equation to account for hesitation degrees, which are often present in real-world physical or engineering systems. These systems may involve not only uncertainty in measurement but also incomplete trust in that uncertainty due to imprecise, conflicting, or missing information. The CIFS framework enriches the model by incorporating both the degree of membership and non-membership in complex-valued form, along with a residual hesitation margin. This provides a more expressive and reliable modeling framework for environments where data vagueness and conflict coexist with complex-valued parameters. Therefore, the novelty of our work is that it is among the first to systematically extend the complex fuzzy heat equation to a CIF setting, combining the strengths of intuitionistic fuzzy theory and complex representation. Moreover, the proposed formulation introduces new solution procedures under CIF boundary conditions, which have not been widely investigated in the literature.

This paper is considered an extension of the work presented in [25] which combined CIFSs with numerical methods to solve FPDEs involving the FHE. In this study, we address the case involving two complex-valued functions: membership and nonmembership. The key contributions of this work are threefold. First, it introduces a novel extension from the classical heat equation and the CF heat equation to the CIF heat equation. To the best of our knowledge, this is the first attempt to integrate complex-valued and intuitionistic fuzzy uncertainty into a heat conduction modeling framework. Second, the paper formulates the CIF heat equation under CIF boundary conditions and proposes a numerical solution using a modified Crank–Nicolson scheme adapted to this new setting. It also generalizes the concept of Hukuhara differentiability to CIF-valued functions. Third, the proposed model effectively captures multiple layers of uncertainty-namely magnitude uncertainty, phase variation, and hesitation (incompleteness)-offering a richer and more expressive framework for modeling real-world thermal systems under deep uncertainty.

The implications of the current research can be viewed from both theoretical and methodological perspectives. From a theoretical standpoint, the study contributes to the advancement of fuzzy differential equations by introducing the CIF heat equation, which integrates the strengths of CFS with intuitionistic fuzzy logic. Furthermore, it presents a new class of solutions involving CIF-valued functions and extends the concept of Hukuhara differentiability to a more expressive and generalized setting. Additionally, it opens new research directions for addressing higher-order uncertainties, where amplitude, phase, and hesitation coexist—offering a mathematically richer framework for future developments. From a methodological perspective, the study demonstrates how classical numerical schemes can be adapted to solve partial differential equations under complex intuitionistic fuzzy conditions, thereby broadening the applicability of traditional solvers in uncertain environments. Moreover, this work lays the foundation for integrating the proposed framework with time-fractional derivatives, stochastic modeling, and hybrid intelligent systems in future research.

It is well known that values in the interval [0, 1] represent the fuzzification used in fuzzy heat or diffusion equations. The objective of the current research is to extend this concept by focusing on the fuzzification of the CIF heat equation, which is characterized by two distinct components: the amplitude and phase terms within the unit disk in the complex plane C. This approach aims to provide a generalization and improved accuracy for solving intuitionistic fuzzy heat equations, considering the novel semantics of periodicity embedded within CIF information, as discussed in [27].The following flowchart Figuer 1 illustrates the modeling framework proposed in this research.

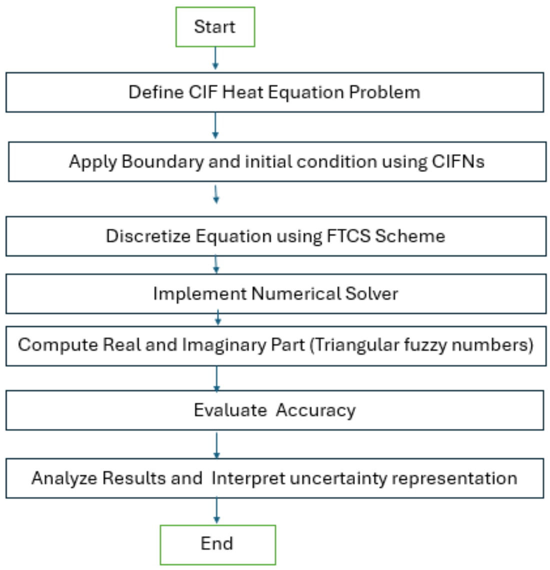

Figure 1: Modeling framework

2. Complex Intuitionistic Fuzzy Heat Equation

At this stage, we consider the general form of the one-dimensional fuzzy heat equation [34]:

$$\frac{\partial \widetilde{u}(\xi,t)}{\partial t} = \widetilde{D}(\xi,t)\frac{\partial^2 \widetilde{u}(\xi,t)}{\partial \xi^2} + \widetilde{b}(\xi,t), \quad 0<\xi<l,\ t>0$$

$$\widetilde{u}(\xi,0) = \widetilde{f}(\xi),\ \widetilde{u}(0,t) = \widetilde{g}(t),\ \widetilde{u}(l,t) = \widetilde{z}(t) \tag{1}$$

Such that:

$\widetilde{u}(\xi,t)$: is the unknown fuzzy function at ξ *and* t variables.

$\frac{\partial \widetilde{u}(\xi,t)}{\partial t}$, $\frac{\partial^2 \widetilde{u}(\xi,t)}{\partial \xi^2}$: are the first order and the second order fuzzy partial spatial derivatives with involving $\widetilde{D}(\xi,t)$ and $\widetilde{b}(\xi,t)$ are known fuzzy functions. Also, $\widetilde{u}(0,t)$ and $\widetilde{u}(l,t)$ are the fuzzy known boundary conditions and the $\widetilde{u}(0,\xi)$ is the fuzzy known initial condition, where the range of the given functions is confined to the unit disk in $\mathbb{C}$.

The fuzzy heat equation in Eq. 1 is transform to CF heat equation: [33]:

$$\begin{gathered}\frac{\partial \widetilde{u}(\xi,t;r,\theta)}{\partial t} = \left[\widetilde{q_1}(r)\ e^{i\alpha\ \widetilde{w_1}(\theta)}\ \ s_1(\xi,t)\right]\frac{\partial^2 \widetilde{u}(\xi,t;r,\theta)}{\partial \xi^2} + \widetilde{q_2}(r)\ e^{i\alpha\ \widetilde{w_2}(\theta)}\ \ s_2(\xi,t)\\ \widetilde{u}(\xi,0;r,\theta) = \widetilde{q_3}(r)\ e^{i\alpha\ \widetilde{w_3}(\theta)}\ \ s_3(\xi)\\ \widetilde{u}(0,t;r,\theta) = \widetilde{q_4}(r)\ e^{i\alpha\ \widetilde{w_4}(\theta)}\ \ s_4(t)\ ,\ \ \widetilde{u}(l,t;r,\theta) = \widetilde{q_5}(r)\ e^{i\alpha\ \widetilde{w_5}(\theta)}\ \ s_5(t)\end{gathered} \tag{2}$$

where in Eq. 1 the CF functions or fuzzy parameters $\widetilde{D}(\xi)$, $\widetilde{b}(\xi)$, $\widetilde{f}(\xi)$, $\widetilde{g}(t)$ and $\widetilde{z}(t)$ are being CF convex normalized numbers that defined as following [33].

$$\begin{cases}\widetilde{D}(\xi,t) = \widetilde{q_1}\ e^{i\alpha\ \widetilde{w_1}}\ \ s_1(\xi,t)\\ \widetilde{b}(\xi,t) = \widetilde{q_2}\ \ e^{i\alpha\ \widetilde{w_2}}\ \ s_2(\xi,t)\\ \widetilde{f}(\xi) = \widetilde{q_3}\ e^{i\alpha\ \widetilde{w_3}}\ \ s_3(\xi)\\ \widetilde{g}(t) = \widetilde{q_4}\ e^{i\alpha\ \widetilde{w_4}}\ s_4(t)\\ \widetilde{z}(t) = \widetilde{q_5}\ e^{i\alpha\ \widetilde{w_5}}\ \ s_5(t)\end{cases}$$

where $s_1(\xi,t)$, $s_2(\xi,t)$, $s_3(\xi)$, $s_4(t)$and $s_5(t)$ are known crisp functions of x and t variables with $\widetilde{q}_1$, $\widetilde{q}_2$, $\widetilde{q}_3$, $\widetilde{q}_4$, $\widetilde{q}_5$, $\widetilde{w}_1$, $\widetilde{w}_2$, $\widetilde{w}_3$, $\widetilde{w}_4$, and $\widetilde{w}_5$ are fuzzy normalized convex numbers.

Now, the CF heat equation in Eq. 2 is generalized to represent the general form of the CIF heat equation using the single parametric form based on the r,β-cut approach for all $r,\beta,\theta \in [0,1]$ as follows:

$$[\widetilde{u}(\xi,t)]_{r,\ \ \theta,\beta} = [\ \underline{u}(\xi,t;r,\theta), \overline{u}(\xi,t;r,\theta)]\,, \left[\ \underline{u}^{'}(\xi,t;\beta,\theta),\ \ \overline{u}^{'}(\xi,t;\beta,\theta)\right] \tag{3}$$

$$\left[\frac{\partial \widetilde{u}(\xi,t)}{\partial t}\right]_{r,\theta,\beta} = \left[\frac{\partial \underline{u}(\xi,t;r,\theta)}{\partial t}, \frac{\partial \overline{u}(\xi,t;r,\theta)}{\partial t}\right],\ \ \left[\frac{\partial \underline{u}^{'}(\xi,t;\beta,\theta)}{\partial t},\ \ \frac{\partial \overline{u^{'}}(\xi,t;\beta,\theta)}{\partial t}\right] \tag{4}$$

$$\left[\frac{\partial^2\widetilde{u}(\xi,t)}{\partial\xi^2}\right]_{r,\theta,\beta}=\left[\frac{\partial^2\underline{u}(\xi,t;r,\theta)}{\partial\xi^2},\frac{\partial^2\overline{u}(\xi,t;r,\theta)}{\partial\xi^2}\right],\left[\frac{\partial^2\underline{u'}(\xi,t;\beta,\theta)}{\partial\xi^2},\quad\frac{\partial^2\overline{u'}(\xi,t;\beta,\theta)}{\partial\xi^2}\right]\tag{5}$$

$$\left[\widetilde{D}(\xi,t)\right]_{r,\theta,\beta}=\left[\underline{D}(\xi,t;r,\theta),\overline{D}(\xi,t;r,\theta)\right],\left[\underline{D'}(\xi,t;\beta,\theta),\quad\overline{D'}(\xi,t;\beta,\theta)\right]\tag{6}$$

$$\left[\widetilde{b}(\xi,t)\right]_{r,\theta,\beta}=\left[\underline{b}(\xi,t;r,\theta),\overline{b}(\xi,t;r,\theta)\right],\left[\underline{b'}(\xi,t;\beta,\theta),\quad\overline{b'}(\xi,t;\beta,\theta)\right]\tag{7}$$

$$\left[\widetilde{u}(\xi,0)\right]_{r,\theta,\beta}=\left[\underline{u}(\xi,0;r,\theta),\overline{u}(\xi,0;r,\theta)\right],\left[\underline{u'}(\xi,0;\beta,\theta),\quad\overline{u'}(\xi,0;\beta,\theta)\right]\tag{8}$$

$$\left[\widetilde{u}(0,t)\right]_{r,\theta,\beta}=\left[\underline{u}(0,t;r,\theta),\overline{u}(0,t;r,\theta)\right],\quad\left[\underline{u'}(0,t;\beta,\theta),\overline{u'}(0,t;\beta,\theta)\right]\tag{9}$$

$$\left[\widetilde{u}(l,t)\right]_{r,\theta,\beta}=\left[\underline{u}(l,t;r,\theta),\overline{u}(l,t;r,\theta)\right],\left[\underline{u'}(l,t;\beta,\theta),\quad\overline{u'}(l,t;\beta,\theta)\right]\tag{10}$$

$$\left[\widetilde{f}(\xi)\right]_{r,\theta,\beta}=\left[\underline{f}(x;r,\theta),\overline{f}(x;r,\theta)\right],\quad\left[\underline{f'}(x;r,\theta),\ \overline{f'}(x;r,\theta)\right]\tag{11}$$

$$\begin{cases}\left[\widetilde{g}\ (t)\right]_{r,\theta,\beta}=\left[\underline{g}(t;r,\theta),\overline{g}(t;r,\theta)\right],\left[\underline{g'}(t;r,\theta),\overline{g'}(t;\beta,\theta)\right]\\ \left[\widetilde{z}(t)\right]_{r,\theta,\beta}=\left[\underline{z}(t;r,\theta),\overline{z}(t;r,\theta)\right],\left[\underline{z'}(t;r,\theta),\overline{z'}(t;\beta,\theta)\right]\end{cases}\tag{12}$$

$$\begin{cases}\left[\widetilde{D}(\xi,t)\right]_{r,\theta,\beta}=\left[\left[\underline{q}_1(r),\overline{q}_1(r)\right]\ e^{i\alpha\ [\underline{w}_1(\theta),\overline{w}_1(\theta)]}s_1(\xi,t)\right],\left[\left[\underline{q'}_1(\beta),\overline{q'}_1(\beta)\right]\ e^{i\alpha\ [\underline{w}_1(\theta),\overline{w}_1(\theta)]}s_1(\xi,t)\right]\\ \left[\widetilde{b}(\xi,t)\right]_{r,\theta,\beta}=\left[\left[\underline{q}_2(r),\overline{q}_2(r)\right]\ e^{i\alpha[\underline{w}_2(\theta),\overline{w}_2(\theta)]}\ s_2(\xi,t)\right],\left[\left[\underline{q'}_2(\beta),\overline{q'}_2(\beta)\right]\ e^{i\alpha[\underline{w}_2(\theta),\overline{w}_2(\theta)]}\ s_2(\xi,t)\right]\\ \left[\widetilde{f}(\xi)\right]_{r,\theta,\beta}=\left[\left[\underline{q}_3(r),\overline{q}_3(r)\right]\ e^{i\alpha[\underline{w}_3(\theta),\overline{w}_3(\theta)]}\ s_3(\xi)\right],\left[\left[\underline{q'}_3(\beta),\overline{q'}_3(\beta)\right]\ e^{i\alpha[\underline{w}_3(\theta),\overline{w}_3(\theta)]}\ s_3(\xi)\right]\\ \left[\widetilde{g}(t)\right]_{r,\theta,\beta}=\left[\left[\underline{q}_4(r),\overline{q}_4(r)\right]\ e^{i\alpha\ [\underline{w}_4(\theta),\overline{w}_4(\theta)]}\ s_4(t)\right],\left[\left[\underline{q'}_4(\beta),\overline{q'}_4(\beta)\right]\ e^{i\alpha\ [\underline{w}_4(\theta),\overline{w}_4(\theta)]}\ s_4(t)\right]\\ \left[\widetilde{z}(t)\right]_{r,\theta,\beta}=\left[\left[\underline{q}_5(r),\overline{q}_5(r)\right]\ e^{i\alpha\ [\underline{w}_5(\theta),\overline{w}_5(\theta)]}\ s_5(t)\right],\left[\left[\underline{q'}_5(\beta),\overline{q'}_5(\beta)\right]\ e^{i\alpha\ [\underline{w}_5(\theta),\overline{w}_5(\theta)]}\ s_5(t)\right]\end{cases}\tag{13}$$

The complex membership and non-membership functions are defined by using the fuzzy extension principle [33]as follows:

$$\begin{cases}\underline{u}(\xi,t;r,\theta)=\left[\min\left\{\widetilde{u}\left(\widetilde{\mu}(r,\theta),t\right)\right)|\widetilde{\mu}(r,\theta)\in\widetilde{u}(\xi,t;r,\theta)\right\}\right]\\ \overline{u}(\xi,t;r,\theta)=\left[\max\left\{\widetilde{u}\left(\widetilde{\mu}(r,\theta),t\right)|\widetilde{\mu}(r,\theta)\in\widetilde{u}(\xi,t;r,\theta)\right\}\right]\end{cases}\tag{14}$$

$$\begin{cases}\underline{u'}(\xi,t;\beta,\theta)=\left[\min\left\{\widetilde{u}\left(\widetilde{\mu}(\beta,\theta),t\right)\right)|\widetilde{\mu}(r,\theta)\in\widetilde{u}(\xi,t;\beta,\theta)\right\}\right]\\ \overline{u'}(\xi,t;\beta,\theta)=\left[\max\left\{\widetilde{u}\left(\widetilde{\mu}(\beta,\theta),t\right)|\widetilde{\mu}(r,\theta)\in\widetilde{u}(\xi,t;\beta,\theta)\right\}\right]\end{cases}\tag{15}$$

Here, substituting all Equations 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and 13 into equation 2, the following is obtained:

$$\begin{cases} \frac{\partial \underline{u}(\xi,t;r,\theta)}{\partial t} = \left[\underline{q}_1(r)\ e^{i\alpha\ \underline{w}_1(\theta)}\ \ s_1(\xi,t)\right]\ \frac{\partial^2 \underline{u}(\xi,t;r,\theta)}{\partial \xi^2} + \underline{q}_2(r)\ e^{i\alpha\ \underline{w}_2(\theta)}\ \ s_2(\xi,t) \\ \underline{u}(\xi,0;r,\theta) = \underline{q}_3(r)\ e^{i\alpha\ \underline{w}_3(\theta)}\ \ s_3(\xi) \\ \underline{u}(0,t;r,\theta) = \underline{q}_4(r)\ e^{i\alpha\ \underline{w}_4(\theta)}\ \ s_4(t)\ ,\ \ \underline{u}(l,t;r,\theta) = \underline{q}_5(r)\ e^{i\alpha\ \underline{w}_5(\theta)}\ \ s_5(t) \end{cases} \tag{16}$$

$$\begin{cases} \frac{\partial \overline{u}(\xi,t;r,\theta)}{\partial t} = \left[\overline{q}_1(r)\ e^{i\alpha\ \overline{w}_1(\theta)}\ \ s_1(\xi,t)\right]\ \frac{\partial^2 \overline{u}(\xi,t;r,\theta)}{\partial \xi^2} + \overline{q}_2(r)\ e^{i\alpha\ \overline{w}_2(\theta)}\ \ s_2(\xi,t) \\ \overline{u}(\xi,0;r,\theta) = \overline{q}_3(r)\ e^{i\alpha\ \overline{w}_3(\theta)}\ \ s_3(\xi) \\ \overline{u}(0,t;r,\theta) = \overline{q}_4(r)\ e^{i\alpha\ \overline{w}_4(\theta)}\ \ s_4(t)\ ,\ \ \overline{u}(l,t;r,\theta) = \overline{q}_5(r)\ e^{i\alpha\ \overline{w}_5(\theta)}\ \ s_5(t) \end{cases} \tag{17}$$

$$\begin{cases} \frac{\partial \underline{u}'(\xi,t;\beta,\theta)}{\partial t} = \left[\underline{q}'_1(\beta)\ e^{i\alpha\ \underline{w}_1(\theta)}\ \ s_1(\xi,t)\right]\ \frac{\partial^2 \underline{u}'(\xi,t;\beta,\theta)}{\partial \xi^2} + \underline{q}'_2(\beta)\ e^{i\alpha\ \underline{w}_2(\theta)}\ \ s_2(\xi,t) \\ \underline{u}'(\xi,0;\beta,\theta) = \underline{q}'_3(\beta)\ e^{i\alpha\ \underline{w}_3(\theta)}\ \ s_3(\xi) \\ \underline{u}'(0,t;\beta,\theta) = \underline{q}'_4(\beta)\ e^{i\alpha\ \underline{w}_4(\theta)}\ \ s_4(t)\ ,\ \ \underline{u}'(l,t;\beta,\theta) = \underline{q}'_5(\beta)\ e^{i\alpha\ \underline{w}_5(\theta)}\ \ s_5(t) \end{cases} \tag{18}$$

$$\begin{cases} \frac{\partial \overline{u'}(\xi,t;\beta,\theta)}{\partial t} = \left[\overline{q'}_1(\beta)\ e^{i\alpha\ \overline{w}_1(\theta)}\ \ s_1(\xi,t)\right]\ \frac{\partial^2 \overline{u'}(\xi,t;\beta,\theta)}{\partial \xi^2} + \overline{q'}_2(\beta)\ e^{i\alpha\ \overline{w}_2(\theta)}\ \ s_2(\xi,t) \\ \overline{u'}(\xi,0;\beta,\theta) = \overline{q'}_3(\beta)\ e^{i\alpha\ \overline{w}_3(\theta)}\ \ s_3(\xi) \\ \overline{u'}(0,t;\beta,\theta) = \overline{q'}_4(\beta)\ e^{i\alpha\ \overline{w}_4(\theta)}\ \ s_4(t)\ ,\ \ \overline{u'}(l,t;\beta,\theta) = \overline{q'}_5(\beta)\ e^{i\alpha\ \overline{w}_5(\theta)}\ \ s_5(t) \end{cases} \tag{19}$$

The Eq. 16 and Eq. 17 display the lower and upper bounds of membership function respectively of the general form of CIF heat equation while the Eq. 18 and Eq. 19 present the lower and upper bounds of non-membership function respectively of the general form of CIF heat equation.

3. Forward Time Centered Space Scheme for solution of complex intuitionistic fuzzy heat equation

This section highlights the adaptation and utilization of a forward difference for the first order derivative in time with centre difference for the second-order derivative in space to solve the CIF heat equation,

The partial time derivative $\dfrac{\partial \underline{u}(\xi,t;r,\theta)}{\partial t}, \dfrac{\partial \overline{u}(\xi,t;r,\theta)}{\partial t}, \dfrac{\partial \underline{u}'(\xi,t;\beta,\theta)}{\partial t}, \dfrac{\partial \overline{u'}(\xi,t;\beta,\theta)}{\partial t}$ is discretised as follows:

$$\frac{\partial \underline{u}_{i,j}(\xi,t;r,\theta)}{\partial t} = \frac{\underline{u}_{i,j+1}(\xi,t;r,\theta) - \underline{u}_{i,j}(\xi,t;r,\theta)}{\Delta t} \tag{20}$$

$$\frac{\partial \overline{u}_{i,j}(\xi,t;r,\theta)}{\partial t} = \frac{\overline{u}_{i,j+1}(\xi,t;r,\theta) - \overline{u}_{i,j}(\xi,t;r,\theta)}{\Delta t} \tag{21}$$

$$\frac{\partial \underline{u}'_{i,j}(\xi,t;\beta,\theta)}{\partial t} = \frac{\underline{u}'_{i,j+1}(\xi,t;\beta,\theta) - \underline{u}'_{i,j}(\xi,t;\beta,\theta)}{\Delta t} \tag{22}$$

$$\frac{\partial \overline{u'}_{i,j}(\xi,t;\beta,\theta)}{\partial t} = \frac{\overline{u'}_{i,j+1}(\xi,t;\beta,\theta) - \overline{u'}_{i,j}(\xi,t;\beta,\theta)}{\Delta t} \tag{23}$$

Furthermore, the second fuzzy partial derivatives

$$\frac{\partial^2 \underline{u}_{i,j}(\xi,t;r,\theta)}{d\xi^2}, \frac{\partial^2 \overline{u}_{i,j}(\xi,t;r,\theta)}{d\xi^2}, \frac{\partial^2 \underline{u'}_{i,j}(\xi,t;\beta,\theta)}{d\xi^2}, \frac{\partial^2 \overline{u'}_{i,j}(\xi,t;\beta,\theta)}{d\xi^2}$$

can be formulated as:

$$\frac{\partial^2 \underline{u}_{i,j}(\xi,t;r,\theta)}{d\xi^2} = \frac{\underline{u}_{i+1,j}(\xi,t;r,\theta) - 2\underline{u}_{i,j}(\xi,t;r,\theta) + \underline{u}_{i-1,j}(\xi,t;r,\theta)}{\Delta\xi^2} \tag{24}$$

$$\frac{\partial^2 \overline{u}_{i,j}(\xi,t;r,\theta)}{d\xi^2} = \frac{\overline{u}_{i+1,j}(\xi,t;r,\theta) - 2\overline{u}_{i,j}(\xi,t;r,\theta) + \overline{u}_{i-1,j}(\xi,t;r,\theta)}{\Delta\xi^2} \tag{25}$$

$$\frac{\partial^2 \underline{u'}_{i,j}(\xi,t;\beta,\theta)}{d\xi^2} = \frac{\underline{u'}_{i+1,j}(\xi,t;\beta,\theta) - 2\underline{u'}_{i,j}(\xi,t;\beta,\theta) + \underline{u'}_{i-1,j}(\xi,t;\beta,\theta)}{\Delta\xi^2} \tag{26}$$

$$\frac{\partial^2 \overline{u'}_{i,j}(\xi,t;\beta,\theta)}{d\xi^2} = \frac{\overline{u'}_{i+1,j}(\xi,t;\beta,\theta) - 2\overline{u'}_{i,j}(\xi,t;\beta,\theta) + \overline{u'}_{i-1,j}(\xi,t;\beta,\theta)}{\Delta\xi^2} \tag{27}$$

Now substitute Eq. 20, 21, 22, 23, 24, 25, 26, and 27 in Eq. 16, 17, 18, and 19 respectively in order to obtain the following:

$$\frac{\underline{u}_{i,j+1}(\xi,t;r,\theta) - \underline{u}_{i,j}(\xi,t;r,\theta)}{\Delta t} = \underline{D}(\xi,t;r,\theta)\frac{\underline{u}_{i+1,j}(\xi,t;r,\theta) - 2\underline{u}_{i,j}(\xi,t;r,\theta) + \underline{u}_{i-1,j}(\xi,t;r,\theta)}{\Delta\xi^2} + \underline{b}(\xi,t;r,\theta) \tag{28}$$

$$\frac{\overline{u}_{i,j+1}(\xi,t;r,\theta) - \overline{u}_{i,j}(\xi,t;r,\theta)}{\Delta t} = \overline{D}(\xi,t;r,\theta)\ \frac{\overline{u}_{i+1,j}(\xi,t;r,\theta) - 2\overline{u}_{i,j}(\xi,t;r,\theta) + \overline{u}_{i-1,j}(\xi,t;r,\theta)}{\Delta\xi^2} + \overline{b}(\xi,t;r,\theta) \tag{29}$$

$$\frac{\underline{u'}_{i,j+1}(\xi,t;\beta,\theta) - \underline{u'}_{i,j}(\xi,t;\beta,\theta)}{\Delta t} = \underline{D'}(\xi,t;\beta,\theta)\ \frac{\underline{u'}_{i+1,j}(\xi,t;\beta,\theta) - 2\underline{u'}_{i,j}(\xi,t;\beta,\theta) + \underline{u'}_{i-1,j}(\xi,t;\beta,\theta)}{\Delta\xi^2} + \underline{b'}(\xi,t;\beta,\theta) \tag{30}$$

$$\frac{\overline{u'}_{i,j+1}(\xi,t;\beta,\theta) - \overline{u'}_{i,j}(\xi,t;\beta,\theta)}{\Delta t} = \overline{D'}(\xi,t;\beta,\theta)\ \frac{\overline{u'}_{i+1,j}(\xi,t;\beta,\theta) - 2\overline{u'}_{i,j}(\xi,t;\beta,\theta) + \overline{u'}_{i-1,j}(\xi,t;\beta,\theta)}{\Delta\xi^2} + \overline{b'}(\xi,t;\beta,\theta) \tag{31}$$

By assume that $\widetilde{s}(r,\theta) = \frac{\widetilde{D}(\xi,t;r,\theta)\Delta t}{\Delta\xi^2}$ and $\widetilde{s}(\beta,\theta) = \frac{\widetilde{D}(\xi,t;\beta,\theta)\Delta t}{\Delta\xi^2}$ then the Eq. 28, 29, 30, and 31 are simplified to get lower and upper bounds of membership function and non-membership function respectively of the general form of CIF heat equation. for all $r,\theta,\beta \in [0,1]$ as follows:

$$\underline{u}_{i,j+1}(\xi,t;r,\theta) = (1-2s)\underline{u}_{i,j}(\xi,t;r,\theta) + s\ (\underline{u}_{i+1,j}(\xi,t;r,\theta) + \underline{u}_{i-1,j}(\xi,t;r,\theta) + \Delta t\ \underline{b}(\xi,t;r,\theta) \tag{32}$$

$$\overline{u}_{i,j+1}(\xi,t;r,\theta) = (1-2s)\overline{u}_{i,j}(\xi,t;r,\theta) + s(\overline{u}_{i+1,j}(\xi,t;r,\theta) + \overline{u}_{i-1,j}(\xi,t;r,\theta) + \Delta t\ \overline{b}(\xi,t;r,\theta) \tag{33}$$

$$\underline{u}'_{i,j+1}(\xi,t;\beta,\theta) = (1-2s)\underline{u}'_{i,j}(\xi,t;\beta,\theta) + s\ (\underline{u}'_{i+1,j}(\xi,t;\beta,\theta) + \underline{u}'_{i-1,j}(\xi,t;\beta,\theta) + \Delta t\ \underline{b}'(\xi,t;\beta,\theta) \tag{34}$$

$$\overline{u'}_{i,j+1}(\xi,t;\beta,\theta) = (1-2s)\overline{u'}_{i,j}(\xi,t;\beta,\theta) + s(\overline{u'}_{i+1,j}(\xi,t;\beta,\theta) + \overline{u'}_{i-1,j}(\xi,t;\beta,\theta) + \Delta t\ \overline{b}'(\xi,t;\beta,\theta) \tag{35}$$

4. **Numerical Simulation**

The CIF heat equation is generalized by adding the complex non-membership term to CF heat equation as discussed in [33] as the following:

$$\frac{\partial\widetilde{U}(\xi,t)}{\partial t} - \frac{\partial^2\widetilde{U}(\xi,t)}{\partial\xi^2} = 0,\ 0<\xi<1,\ t>0 \tag{36}$$

where the boundary conditions $\widetilde{u}(0,t) = \widetilde{u}(1,t) = 0$ and initial condition $\widetilde{u}(\xi,0) = \widetilde{f}(\xi)$, with CF function $\widetilde{f}(\xi)$ is defined as follows

$$\widetilde{f}(\xi) = \widetilde{k}\ e^{i2\pi\ \widetilde{w}} Cos\left(\pi\xi - \frac{\pi}{2}\right) \tag{37}$$

In r-cut and β-cut formula of the intuitionistic fuzzy number is shown as below:

$$\left[\widetilde{k}\right]_{r,\beta} = \left\{\ \left[\underline{k}(r), \overline{k}(r)\right], \left[\underline{k}(\beta), \overline{k}(\beta)\right]\ \right\} = \{[r-1, 1-r], [-1.25\beta, 1.25\beta]\} \tag{38}$$

$$[\widetilde{w}]_{\theta,\beta} = \{\ [\underline{w}(\theta), \overline{w}(\theta)], [\underline{w}(\beta), \overline{w}(\beta)]\ \} = \{[\theta-1, 1-\theta], [-1.25\beta, 1.25\beta]\} \tag{39}$$

The exact complex intuitionistic solution of Eq. 36 is described as below:

$$\widetilde{u}(\xi,t;r) = \widetilde{k}\ e^{i2\pi\ \widetilde{w}\ -\pi t} Cos\left(\pi\xi - \frac{\pi}{2}\right) \tag{40}$$

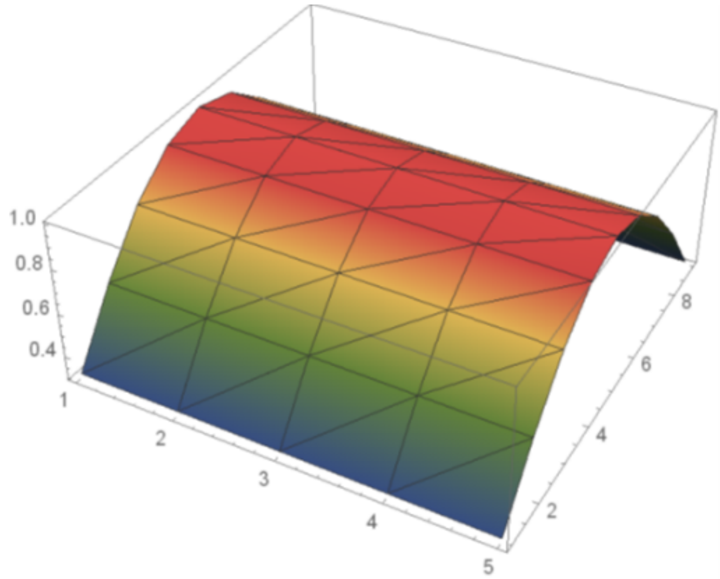

Figure 2: The exact solution of Eq. 33 at $\theta = 0.2$ *and for all* $r, \beta \in [0, 1]$

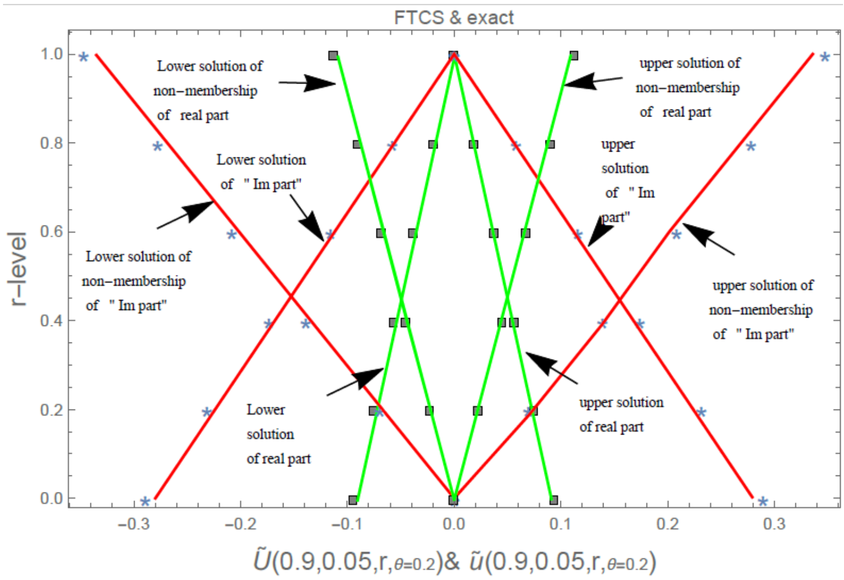

Figure 3: The numerical and exact solutions by FTCS of Equation 33 at $t = 0.05$, $\xi = 0.9$ and $\theta = 0.2$ for all $r,\ \beta \in [0, 1]$

Table 1: Numerical lower solutions of Eq. 33 by FTCS at $t = 0.05$, $\xi = 0.9$, for every $r, \beta, \theta \in [0, 1]$.

θ	r, β	$\tilde{u}(0.9, 0.5; r, \theta)$	$\tilde{E}(0.9, 0.5; r, \theta)$	$\tilde{u}(0.9, 0.5; \beta, \theta)$	$\tilde{E}(0.9, 0.5; \beta, \theta)$
0.2	0	$-0.09091 - 0.279796i$	$0.0030 + 0.0095i$	0	0
	0.2	$-0.07273 - 0.22383i$	$0.0025 + 0.0076i$	$-0.021818 - 0.067149i$	$0.00074276 + 0.002286i$
	0.4	$-0.05455 - 0.16787i$	$0.0019 + 0.0057i$	$-0.043636 - 0.1343i$	$0.0014855 + 0.0045719i$
	0.6	$-0.03636 - 0.11191i$	$0.0012 + 0.0038i$	$-0.065454 - 0.20145i$	$0.0022283 + 0.0068579i$
	0.8	$-0.01818 - 0.05596i$	$0.00061 + 0.0019i$	$-0.087272 - 0.2686i$	$0.002971 + 0.0091439i$
	1	0	0	$-0.10909 - 0.33574i$	$0.0037138 + 0.01143i$
0.4	0	$0.23800 - 0.17292i$	$0.0081 + 0.0059i$	0	0
	0.2	$0.19040 - 0.13833i$	$0.0065 + 0.0047i$	$0.057120 - 0.041500i$	$0.00194456 + 0.00141280i$
	0.4	$0.14280 - 0.10375i$	$0.0049 + 0.0035i$	$0.114241 - 0.083001i$	$0.00388912 + 0.00282562i$
	0.6	$0.09520 - 0.06917i$	$0.0032 + 0.0023i$	$0.171361 - 0.124501i$	$0.00583369 + 0.00423842i$
	0.8	$0.04760 - 0.03458i$	$0.0016 + 0.0012i$	$0.228481 - 0.166001i$	$0.00777825 + 0.00565123i$
	1	0	0	$0.285602 - 0.207502i$	$0.00972282 + 0.00706404i$

Table 2: Numerical lower solutions of Eq. 33 by FTCS at $t = 0.05$, $\xi = 0.9$, for every $r, \beta, \theta \in [0, 1]$.

θ	r, β	$\tilde{u}(0.9, 0.5; r, \theta)$	$\tilde{E}(0.9, 0.5; r, \theta)$	$\tilde{u}(0.9, 0.5; \beta, \theta)$	$\tilde{E}(0.9, 0.5; \beta, \theta)$
0.6	0	$0.23800 + 0.17292i$	$0.0081 + 0.0059i$	0	0
	0.2	$0.19040 + 0.13833i$	$0.0065 + 0.0047i$	$0.057120 + 0.041500i$	$0.00194456 + 0.00141280i$
	0.4	$0.14280 + 0.10375i$	$0.0049 + 0.0035i$	$0.114241 + 0.083001i$	$0.00388912 + 0.00282562i$
	0.6	$0.09520 + 0.06917i$	$0.0032 + 0.0023i$	$0.171361 + 0.124501i$	$0.00583369 + 0.00423842i$
	0.8	$0.04760 + 0.03458i$	$0.0016 + 0.0012i$	$0.228481 + 0.166001i$	$0.00777825 + 0.00565123i$
	1	0	0	$0.285602 + 0.207502i$	$0.00972282 + 0.00706404i$
0.8	0	$-0.09091 - 0.279796i$	$0.0030 + 0.0095i$	0	0
	0.2	$-0.07273 - 0.22383i$	$0.0025 + 0.0076i$	$-0.021818 + 0.067149i$	$0.00074276 + 0.002286i$
	0.4	$-0.05455 - 0.16787i$	$0.0019 + 0.0057i$	$-0.043636 + 0.1343i$	$0.0014855 + 0.0045719i$
	0.6	$-0.03636 - 0.11191i$	$0.0012 + 0.0038i$	$-0.065454 + 0.20145i$	$0.0022283 + 0.0068579i$
	0.8	$-0.01818 - 0.05596i$	$0.00061 + 0.0019i$	$-0.087272 + 0.2686i$	$0.002971 + 0.0091439i$
	1	0	0	$-0.10909 + 0.33574i$	$0.0037138 + 0.01143i$

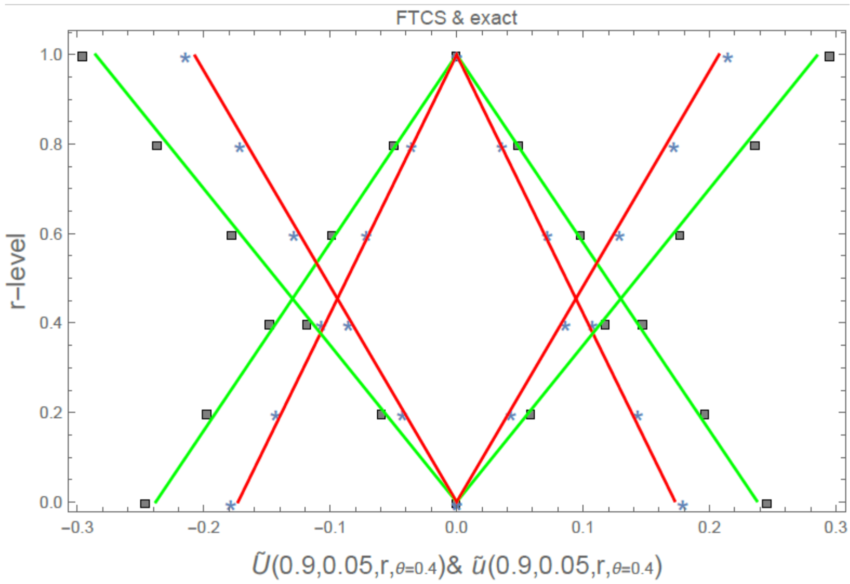

Figure 4: The numerical and exact solutions by FTCS of Equation 33 at $t = 0.05$, $\xi = 0.9$ and $\theta = 0.4$ for all $r,\ \beta \in [0, 1]$

The numerical solutions derived using the explicit FTCS method demonstrate strong agreement with the exact solution at $\xi = 0.9,\ t = 0.05\ \ \forall r, \beta, \theta \in [0, 1]$, as demonstrated in Figures 2, 3, and 4 and Tables 1, 2. Moreover, the numerical solution and exact solution obtained from our proposed schemes assume the structure of triangular fuzzy numbers in both the amplitude and phase parts, aligning with the characteristics of a C. In addition, as can be seen from tables 1, 2, The accuracy of the numerical outcomes is contingent upon the phase term value θ. This supports our theoretical framework and illustrates the important influence of including the complex non-membership function. It is assumed that the numerical solution of the intuitionistic fuzzy heat equation is derived from the CIF heat equation through substitution $\ \theta = 0\ and\ 1$. In more detail, Tables 1 and 2 shows the complex intuitionistic lower and complex intuitionistic upper solutions for membership and non-membership function for different values of $r, \theta, \beta \in [0, 1]$. The results show that the proposed scheme consistently maintains small errors. For instance, in Table 1 at $r = 0$ and $\theta = 0$, the approximate value is $-0.09091 - 0.279796i$, closely matching the exact value with an error of $0.0030 + 0.0095i$. Then, as r increases, the errors remain relatively small, indicating that the proposed scheme maintains good accuracy across different r values. Similarly, the upper solutions in Table 2 also show high accuracy. These results show that the proposed scheme accurately models the behavior of CIF heat equation. Fig. 3 and Fig. 4 illustrate the intuitionistic fuzzy lower and upper solutions for membership and non-membership functions, respectively. These visual representations support the numerical findings in Tables 1 and 2, showing good agreement between the proposed scheme results and the exact solutions.

In addition to its mathematical contribution, the proposed model based on CIFSs offer meaningful physical interpretations and practical relevance. CIFSs enable the simultaneous representation of uncertainty in amplitude, phase, and degree of hesitation, which are critical in many real-world systems where uncertainty is multi-faceted and not purely scalar. This makes the CIF heat equation framework particularly useful for modeling and solving problems in electromagnetic signal processing, where uncertain phase and intensity variations are common; biomedical imaging, such as MRI or EEG, where data uncertainty includes both measurement noise and interpretative ambiguity; and thermal analysis in composite or anisotropic materials, where heat conduction properties are not precisely known. Furthermore, the ability of CIFSs to capture membership, non-membership, and hesitation information allows for more nuanced modeling in decision-making systems, robotics, and intelligent control under uncertainty. These applications demonstrate the broader utility of the proposed CIF heat equation model beyond pure mathematical theory.

4. Conclusions

In this study, we proposed and analyzed a solution approach for the CIF heat equation using the FTCS finite difference scheme. The model incorporates CIF numbers to represent uncertainties in both the amplitude and phase of membership and non-membership functions. This dual representation captures a higher degree of uncertainty compared to classical or even complex fuzzy models. The results demonstrate that the FTCS scheme preserves the structural features of CIF numbers when triangular fuzzy numbers are used for both amplitude and phase parts. The scheme achieves a computational accuracy of order $O(\Delta t+\Delta\xi^2)$, confirming its numerical stability and convergence under complex intuitionistic fuzzy conditions. Moreover, the proposed method proves to be computationally efficient and generalizable for modeling periodic heat transfer and other dynamic systems involving complex uncertainty. The findings support the viability of CIF heat equation for a wide range of real-world applications, such as thermal systems, material modeling, and engineering processes under uncertainty. This work makes a significant contribution by extending the classical and complex fuzzy heat equation models to a more expressive CIF framework. The integration of both membership and non-membership uncertainties—along with phase information—offers a richer mathematical tool for handling high-order fuzzy uncertainty. For future research, we recommend exploring the use of alternative numerical schemes, such as explicit methods for time-fractional heat equations, to enhance modeling flexibility. Additionally, further generalization toward complex bipolar fuzzy and complex bipolar intuitionistic fuzzy heat equations is proposed to model systems.

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