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Optimal Trajectories For Drones With Limited Observability Constraints

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Abstract. The dynamic output stabilization problem is a central and important topic in control theory. A compelling example is the control of a drone where the only measurable output is its distance to a target. This scenario presents unique challenges, as the drone's coordinates are not always observable, even after symmetry reductions. From a practical point of view, minimizing the time required to reach the target—closely tied to fuel efficiency—is critical, particularly in military applications. Interestingly, the optimal trajectories that achieve minimum time, typically straight-line paths, coincide with trajectories where the system is unobservable. This paper introduces a novel methodology to reconcile these seemingly conflicting objectives: ensuring observability while minimizing travel time.

2020 Mathematics Subject Classifications: 49J15, 93C95, 93B07

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1. Introduction

Dynamic output stabilization is a classical problem in control theory. When the system is uniformly observable, the stabilization problem turns out to be effective. However, nonlinear systems observability, is a property depending upon the control as a function of time. In the unobservable situation, the dynamic output feedback stabilization becomes a complicated problem. The work presented throughout this paper is a continuation of earlier work [1], in the spirit of the series of articles [2], [3] and the review paper [4].

The subject of this long-term study is dynamic output stabilization by coupling state estimators and static feedback. Our ideas rely on the seminal work [5].

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Most of the other studies assume a kind of strong observability, see [6], [7], [8], [9],[10] and [11] . Here, and in the aforementioned series of papers, the situation where the system is not strongly observable is considered.

It turns out that the academic drone model used in the following (the classical Dubbins model, see [12], [13]), even after some reduction eliminating the symmetries, still remains non-observable.

In view of military applications and other real situations, one of the most important criteria to be minimized is fuel consumption, which is as a first approximation, equivalent to the minimum time problem. After eliminating symmetries, the minimum time problem for the drone model can be exactly solved, as it is done in [14], [15], using the general theory of [16]. The interesting point is that most of the time (which means out of two small sequences of nonzero controls close to the origin and close to the target), the minimum time strategy is the zero control as expected, i.e., straight-line trajectories. However, precisely, these zero controls correspond to the trajectories along which an observability defect appears.

This paper aims to overcome this difficulty (the observability vs minimum time strategy problem) and to show what kind of a surprising phenomenon may appear. In fact, the idea is to introduce an additional perturbation input to the controlled system in regions where the system is unobservable to reduce convergence time. It is well known that bilinear systems are more easily tractable from the observability point of view, and general systems can be at least approximated by bilinear ones. Here, the presented case is a very pleasant case where the system can be exactly embedded into a bilinear one (see [17]). This is crucial for the proposed strategy.

This paper is organized as follows: the next section presents the system model, its properties, and the successive reduction and extension that are used. Then, Section 3 recalls the previous results about the minimum time strategy, the observer, and the result regarding dynamic output stabilization. The last section, Section 4, presents the proposed strategy to overcome the conflict between the lack of observability and a minimum time strategy.

2. The Model, Reduction, and Extension

This section presents the system model, its properties, and the successive reduction and extension that are used. The drone kinematic equations are presented in the following:

$$\dot{x} = \cos(\theta),\ \ \dot{y} = \sin(\theta), \dot{\theta} = u, \qquad -u\max \leq u \leq u\max, \tag{1}$$

which describes the movement of the drone on the (x, y) plane, with a constant velocity equal to 1 and with minimum possible curvature radius r of trajectories being $r = \frac{1}{u\max}$.

This model is inspired by the Dubbins model [12], and it has been discussed in many papers, see for instance [12], [13].

We consider here the situation in which we have minimum information, i.e., the only available continuous observation (measurement) is the distance to the target. It is an interesting case from the theoretical point of view since it makes the system non-observable. Let us mention that this case also corresponds to a certain practical situation in the military context.

Moreover, in this mission, the target is not a point, but a circle of minimum curvature radius r (staying above the real target), since the drone has constant velocity (or at least minimum velocity). At the end of the mission, while the drone is hovering the target, after some signal, the charge is dropped over the target. To express this requirement, we define the target (traveled counter-clockwise) $\mathcal{T}$ by:

$$\mathcal{T} = \{(x, y)|\, x = r\sin(\omega),\, y = -r\cos(\omega),\, \omega \in [0, 2\pi]\}$$

The first step is eliminating the rotational symmetry, which is done via a "moving frame" strategy, i.e., set:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In these new coordinates, the system 1 can be rewritten as:

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} u\,\tilde{y} + 1 \\ -u\,\tilde{x} \end{pmatrix}. \tag{2}$$

This system 2 possesses two equilibria for $u = \pm u_{\max}$, namely $e_1 = (0, r)$, $e_2 = (0, -r)$. They correspond to the target $\mathcal{T}$ being browsed counter-clockwise and clockwise, respectively. If u is changed for $-u$, the two equilibria are exchanged so that the set of equilibria is unchanged. It means that we can indifferently consider one among the two equilibria positions.

Changing the $(\tilde{x}, \tilde{y})$ coordinates for $(\bar{x}, \bar{y}) = (\tilde{x}, \tilde{y} + r)$, the stabilization problem to $\mathcal{T}$ is reformulated in these variables as the stabilization problem to the submanifold $\{(\bar{x}, \bar{y}, \theta)|\bar{x} = \bar{y} = 0\}$. Equivalently, we will refer to the convergence of $(\bar{x}, \bar{y})$ to the point $(0, 0)$ of the reduced state space. The reader can easily check that the $(\bar{x}, \bar{y})$ obey the following equations:

$$\begin{cases} \frac{d\bar{x}}{dt} & = & u\,\bar{y} + 1 - r\,u \\ \frac{d\bar{y}}{dt} & = & -u\,\bar{x} \end{cases}$$

Now, the "minimum information output", i.e. the square distance to the target becomes: $\rho^2 = x^2 + y^2 = \tilde{x}^2 + \tilde{y}^2 = \bar{x}^2 + (\bar{y} - r)^2$.

Let us now make some observability analysis: The system 1 is not even weakly observable in the sense of [18]: changing θ for $\theta + \theta_0$ leaves both the system and the observation invariant.

On its side, the system 2 is weakly observable (it is just the quotient system 1 by the "weak indistinguishability" relation from [18]), but it is not strongly observable in the sense of [8]. That is, for the constant control $u \equiv 0$, the variable $\tilde{x}$ (resp. $\bar{x}$) only can be reconstructed from the observation ρ^2. That is $\tilde{y}$ can be reconstructed up to sign only.

Therefore, even after this first reduction, in which the stabilization concept (at a point) makes sense, and the system becomes weakly observable, observability problems are remaining, and the general theory of dynamic output stabilization does not apply.

Besides [18], one can also consult [19], [20], [21] for the general theory of quotienting (modding out) through unobservability.

At this point, another important fact appears: this system 2 can be embedded into a state-affine one: following [17], its "observation space" is finite-dimensional. Actually, setting $z = (z_1, z_2, z_3)$, $z_1 = x^2 + y^2 = \tilde{x}^2 + \tilde{y}^2$, $z_2 = \tilde{x}$, $z_3 = \tilde{y}$, and denoting the output by s, we get the following (state affine in the sense of [17]) system:

$$\begin{aligned} \dot{z} &= Az + uBz + b, \\ s &= Cz, \ u \in [-u_{\max}, u_{\max}] \end{aligned} \tag{3}$$

with $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$.

3. Previous results and minimum time

Although since the system described by equation 2 is not strongly observable, it follows that the system described in 3, which has a higher dimensional state space and includes the states of the original system, is also not observable. However, in our paper [1] and [22], we were able to prove a dynamic output stabilization result by coupling an observer with static stabilizing feedback. The observer is as follows:

It is the following standard Luenberger-type observer for System 3:

$$\frac{d\hat{x}}{dt} = A\hat{x} + u \ B\hat{x} + b - K(C\hat{x} - s) \tag{4}$$

with $K^{'} = (\alpha, 2, 0)$ for some arbitrary $\alpha > 0$. Here $\hat{x}$ stands for the estimate of z in 3. Therefore, the coupled system feedback-observer is (for system 3):

$$\begin{aligned} \frac{d\hat{x}}{dt} &= A\hat{x} + u(\hat{x}) \ B\hat{x} + b - K(C\hat{x} - z_2^2 - z_3^2), \\ \frac{dz_2}{dt} &= z_3 u(\hat{x}) + 1, \\ \frac{dz_3}{dt} &= -z_2 u(\hat{x}). \end{aligned} \tag{5}$$

We were able (in [1]) to prove the following theorem:

Theorem 1. *For any smooth stabilizing feedback at the target, for systems 1, 5, our Luenberger-type observer is such that the coupled closed-loop system is asymptotically stable at the target, with an arbitrarily large basin of attraction (depending on the parameter α).*

Remark 1. *(i) In the paper [14], we have exhausted a smooth stabilizing feedback control law for the system 1 that may be used in the applications of the theorem.*

(ii) Note that there is no local stabilization of the linearized system (due to effective constraints on the inputs at the equilibrium). Center manifold theory cannot be avoided in the proof [23].

(iii) See [24] for a similar example of a dynamic-output stabilization problem, where additionally, the target point is not observable.

(iv) It is well known (see, for instance, [9]) that global feedback stabilization plus strong observability does not in general imply the possibility of global dynamic output stabilization. Here, the situation is even worse: the system 2 is not even strongly observable. However, we obtain semi-global output stabilization.

Computing the minimum time strategy for the reduced system 2 is not that easy ([14] in the case of constant velocity and [15] in the varying velocity case). However, it is less difficult than computing the point-to-point minimum-time strategy for system 1, a beautiful classical result, see [12] and [13] for instance.

In fact, the problem of constructing the time-optimal synthesis for 2-dimensional systems is completely solved (at least via some formal iterative algorithm) in [16]. We used this method to get our result. It is a fascinating exercise since more or less all possible generic singularities occur in the computations. We present the phase portrait of the optimal synthesis in Figure 1. See [14] for details.

This optimal strategy being nonsmooth (even not continuous, as we said) makes Theorem 1 unapplicable. The first step is to approximate the time-optimal strategy with a smooth one.

It is not hard to understand that there is a real problem when, close to the target, one has to choose between turning clockwise (resp. counter-clockwise) around the target $\mathcal{T}$. Equivalently, for the reduced system 2, we have to choose between one of the equilibria e_1, e_2.

Once this choice is made, it is not difficult to approximate the optimal method by a smooth one. The following control algorithm was presented in [25] and [26], this method approximates the minimum time controller in a smooth and computationally efficient manner. It is proven in [25] that the following control law stabilizes the system (this control algorithm forces the system to satisfy a Lyapunov function):

$$u = \begin{cases} a(\bar{x}, \bar{y}) & \text{if } \bar{x} \leq -\varepsilon \\ \frac{u_{\max} - a(\bar{x},\bar{y})}{1+e^{1/(\bar{x}+\varepsilon)+1/\bar{x}}} + a(\bar{x}, \bar{y}) & \text{if } \bar{x} \in (-\varepsilon, 0) \\ u_{\max} & \text{if } \bar{x} \geq 0, \end{cases} \tag{6}$$

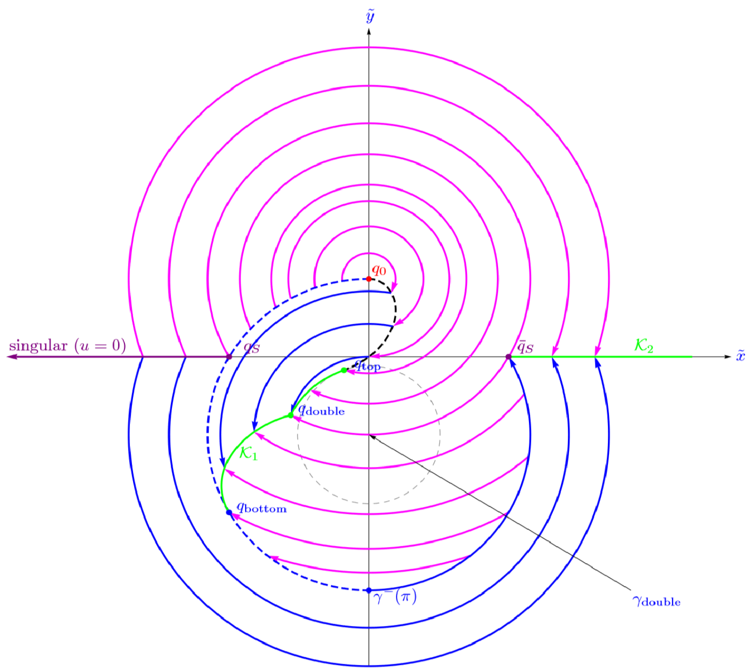

Figure 1: The optimal synthesis. All optimal trajectories start at q0 . The dashed black curve is the switching curve, the purple curve is the singular trajectory and the green curves are cut loci. Notice that the minimum-time function is not continuous along the blue dashed curve.

Where ϵ is a constant. Considering $\bar{x} < 0$, let $\phi = \arctan(\bar{y}/\bar{x})$ be the angular coordinate of the drone in the $(\bar{x}, \bar{y})$-plane. Let $a(\bar{x}, \bar{y}) = \beta \sin(\phi)$ with β such that $0 \leq \beta < u_{max}$. Parameter β controls the convergence rate of the drone towards the target. By choosing a high value of β (close to u_{max}) the drone will converge to its circular hovering pattern using the tangent to this circle which is a near optimal convergence rate. By decreasing the value of β the convergence rate is decreased, however, the trajectory used by the drone is smoother but longer. As verified in [25],the control law is smooth.

As mentioned previously, this method allows nearly time minimum performances. More specifically, this algorithm allows the drone to slide onto the circular hovering pattern using the tangent to the circle. The parameter β control the speed in which the drone converges toward the tangent. In fact, by increasing β the rate of convergence will increase as well and vice versa.

As it is possible to see, the proposed controller presented in 6 requires the knowledge of the system states namely the coordinates x, y and θ, However in the proposed scenario the drone in capable of observing only the distance between itself and the target. Thus, the state observer presented in 5 should be used along with the controller in 6 to stabilize the system and thus track the target.

In the following, the performances of the nearly time minimum controller along with the state observer are evaluated. Then a strategy to enhance the performances of the overall system is presented and verified using software simulations.

4. The strategy to overcome the conflict

Given the smooth feedback law that approximates the minimum time strategy, obtained from the previous section, Theorem 1 can be used. This theorem tells that the full closed-loop system 5 must be asymptotically stable, with an arbitrarily large basin of attraction, depending on the parameter α in (4) above.

Figure 2 displays a typical result, which is quite intriguing. In this first simulation the drone is starting from an initial position x=-50, y=40 and an initial angle θ=0. The target is considered to be fixed located at x=0 and y=0. The speed of the drone is constant equal to $1m/s$, parameter ϵ is equal to 0.5, $\beta = 0.2$ and $\alpha = 5$. The radius of the circular hovering pattern is chosen to be $r = 1m$. Figures 2 shows that the drone is able to converge toward the target at the end, however, the path that is being used is weird. It is possible to explain what is happening. In fact, at the beginning of the simulation the drone starts turning in order to face the target (or in other words the drone will turn in order to intercept the tangent to the hovering pattern). During that phase and since the control input u is not zero the system coordinates (states) are observable for a short period of time. Thus, the observer makes a (poor) initial assessment of which direction to move in, consequently an error still exists between the estimated and the real drone coordinates. In the second phase the controller thinks that it is intercepting the tangent to the hovering pattern thus it is moving in a straight line toward the target. In that phase the system is not observable thus the proposed state observer is not reducing the estimation error. This scenario continues until the observer/controller realizes that it is diverging from the target consequently the controller makes adjustment to the control input during which the system is observable (the control input different from zero) thus the estimation error is reduced. Then the system goes through a cycle of observable and non-observable phases repeatedly until it reaches its hovering pattern. During the hovering phase the system is observable (the control input is not zero) thus the system estimation error goes to zero.

Similar to figure 2, figure 3 presents the simulation results of a drone starting from an initial position (x_0, y_0, θ_0) and converging toward a target located at the origin (in both simulation the system and simulation parameters are the same). The control algorithm is the one previously described in 6 coupled to the observer of 5. These figures presents the drone real and estimated path on the same figure. In conclusion, although the proposed method converges towards the target (hovering pattern), the convergence time is prolonged because the drone frequently misses the target on multiple occasions due to a lack of observability whenever the chosen path is a straight line (resulting in a control input of zero). Indeed, in that special case a nearly minimum time strategy is playing a negative role in optimizing the system performances.

The question now is how to improve on this bad behavior. The strategy is to choose a periodic perturbation of the control input to improve the observability. This strategy is in accordance with the general theory of [5]. The perturbation is added to the input whenever the system is lacking observability or in other words whenever the system is sliding toward the tangent to the circle (control input is close to zero). The selected perturbation is a sine-wave function having an magnitude of γu_{max} where γ is a constant $\in [0; 1[$ and a

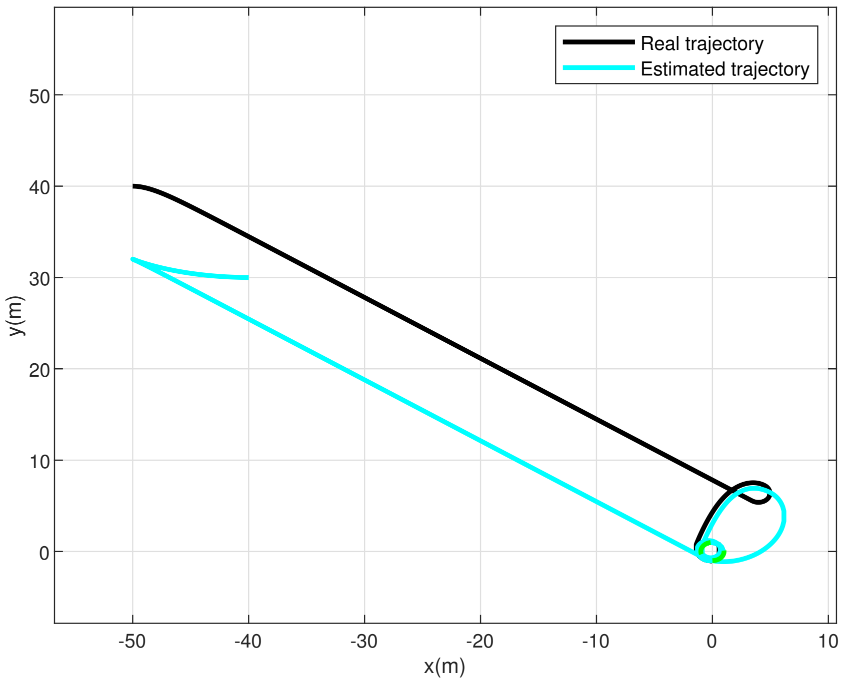

Figure 2: Coupling the proposed observer with the nearly minimum time control strategy, the drone is starting from (x_0, y_0, θ_0) equal to $(-50\ m, 40\ m, 0\ rad/s)$

frequency $f = 0.1Hz$. the expression of this input perturbation u_p is given by:

$$u_p = \gamma u_{max} \sin(2\pi f t) \tag{7}$$

where t is the time variable. The perturbation will be introduced into the control input in situations where the system lacks observability, specifically when the absolute value of the control input is less than 0.1.

As mentioned previously γ is a parameter within the interval $[0; 1[$. γ could not exceed 1 since the input of the system should be always within the interval $[U_{min}, U_{max}]$. It should be mentioned that u_p is applied when the system is not observable meaning that u_p is applied when the input is close to zero (0.1 in this case). Choosing a high value of γ increase the magnitude of the perturbation and thus accelerating the convergence rate of the observer, conversely, decreasing the value of γ decreases the convergence rate. The parameter f represents the frequency of the perturbation. A high value of f (high-frequency perturbation) has a limited effect on the drone's position (the angular position in the integral of the angular speed; thus, a high-frequency speed has minimal influence on the position). Consequently, a high value of f does not considerably improve observability. Conversely, low-frequency input perturbations improve observability; however, they also increase the drone's convergence time to the target (resulting in a longer path to the target).

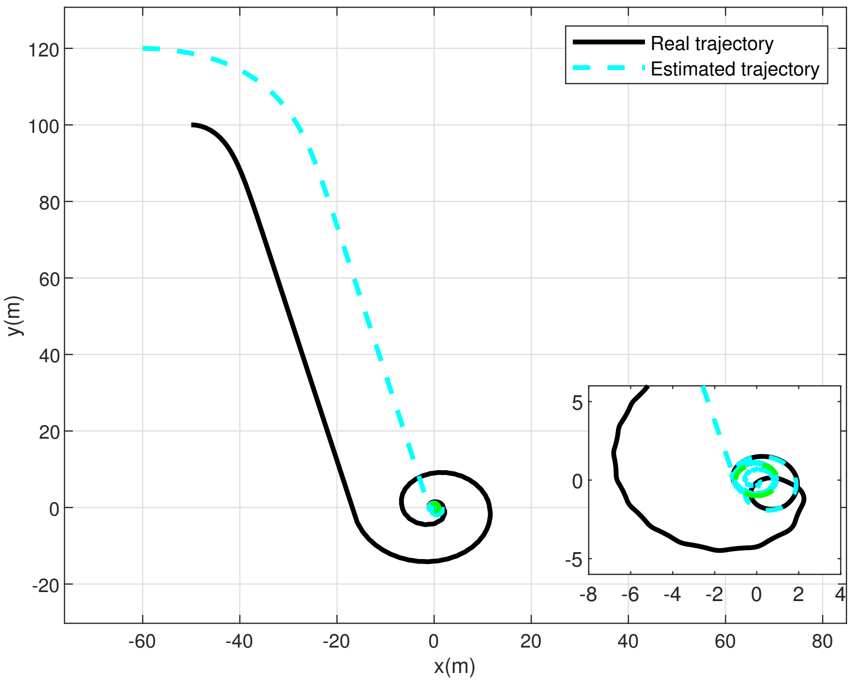

Figure 3: Coupling the proposed observer with the nearly minimum time control strategy, the drone is starting from (x_0, y_0, θ_0) equal to $(-50\ m, 100\ m, 0\ rad/s)$

The full control algorithm is summarized in Figure 4. The proposed controller starts by measuring the distance between the drone and the target. Then, the observer presented in Eq. (5) is computed. The output of the observer represents the system state in the stationary reference frame. With this information, the drone's coordinates in the new rotating reference frame $(\bar{x}, \bar{y})$ are determined. This is followed by computing the control law presented in Eq. 6. Finally, the periodic perturbation is added to the output of the control law and applied to the system.

Figures 5 and 6 present the real and estimated path of the drone. In Figure 5, the drone is starting from the same initial position as in Figure 2. The initial conditions in Figure 6 are also the same as in Figure 3, and the simulation and system parameters are the same in all simulations. The introduction of perturbation in both of these tests exhibits a pronounced effect. In fact, it is clear that the drone is no more traveling in straight lines thus the system is nearly always observable. Consequently, the estimated and real coordinates (states) of the drone converge rapidly and the performances of the controller/observer in this scenario are superior.

It should be noted that the control expression in Eq. 6 contains parameter β, which affects the convergence time of the controller. Increasing this parameter causes the drone to converge more quickly towards the tangent to the hovering pattern, while decreasing it has the opposite effect. This parameter appears to affect the observer convergence time

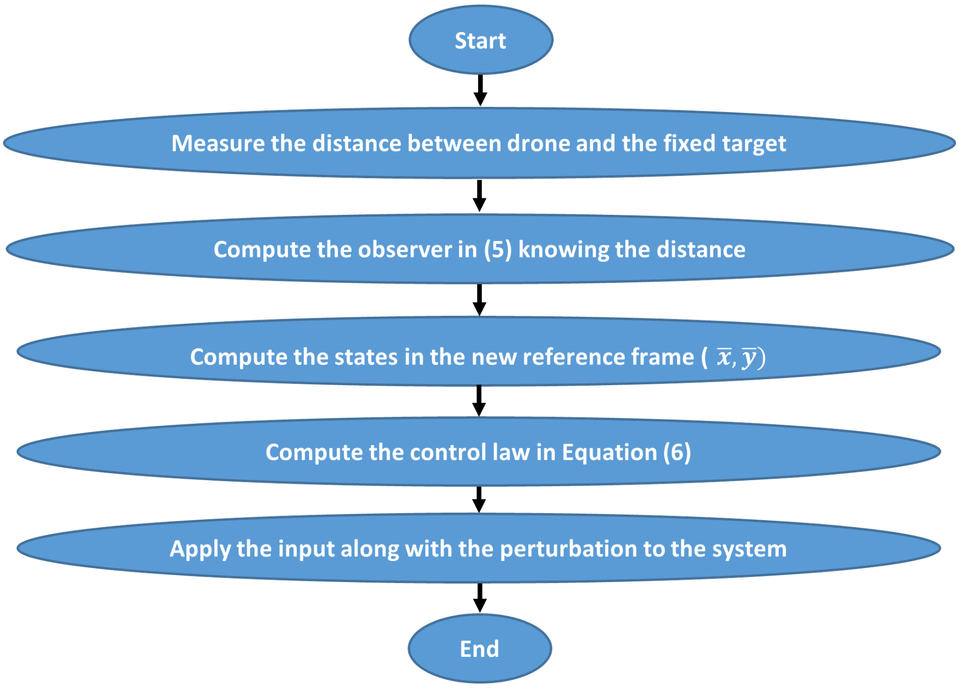

Figure 4: Drone Full Control Algorithm Flowchart

but this time in a negative way. In fact, in the case where parameter β has a high value (the controller is quick), the perturbation that are added to the control input are quickly rectified by the controller causing the observer convergence time to increase. Conversely, decreasing β slows down the controller's response, making the effect of perturbations more pronounced and reducing the observer convergence time. On one hand, it is not possible to decrease β to very small values since the proposed controller should behave like the minimum time controller (or in other words it should nearly resemble to the minimum time controller). On the other hand, the observer should converge in a reasonable amount of time (quickly). Consequently, parameter β should be carefully chosen to compromise between the observer and the controller convergence time.

To prove the benefits of adding the perturbation to the controller/observer, the time needed for the drone to travel from its initial position to intercept the hovering pattern is calculated for both of the previously presented scenarios. In the first scenario (Figures 2 and 5), the time needed to intercept the circle is 77.3 seconds with perturbation and 92.9 seconds without. In the second scenario (Figures 3 and 6), the time needed to intercept the circle is 124.5 seconds with perturbation and 196.9 seconds without. Thus by adding the proposed perturbation the convergence time toward the target was reduced by 25% in the first scenario and 58% in the second one. This is considered a good improvement in the convergence time and the trajectories look much more reasonable. These results prove the superiority of adding a perturbation to the control input.

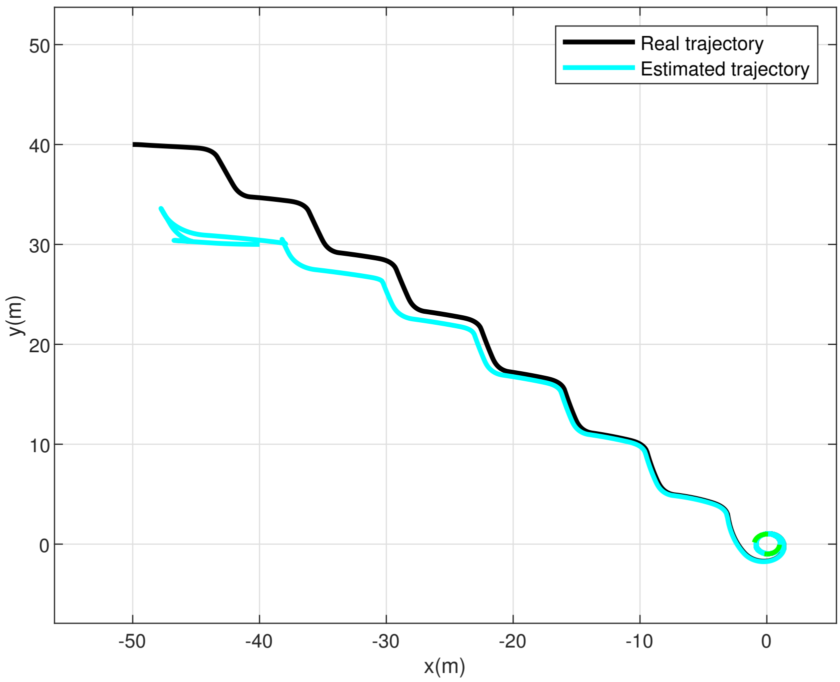

Figure 5: Coupling the proposed observer plus perturbation with the nearly minimum time control strategy, the drone is starting from (x_0, y_0, θ_0) equal to $(-50\ m, 40\ m, 0\ rad/s)$

5. Conclusion

In this paper, a tracking control algorithm for fixed-wing drones is presented. This algorithm enables the drone to intercept and hover around a target in a circular motion, knowing only the distance separating the drone from the fixed target. The proposed approach combines a nearly optimal minimum-time control with a state observer. The results show that this controller is convergent, ensuring that the drone intercepts the target. However, a major limitation was observed: whenever the drone moves in a straight line, the system loses observability, which affects its performance and convergence time. To overcome this issue, a sinusoidal perturbation was added to the control input, significantly improving observability. As a result, the system's convergence time was reduced, and the trajectory became smoother and shorter. These findings open the door to further improvements. Future work could focus on applying the same technique to moving target applications. In summary, the proposed method offers a promising step forward in drone control with limited observability enabling a shorter path decreasing fuel consumption.

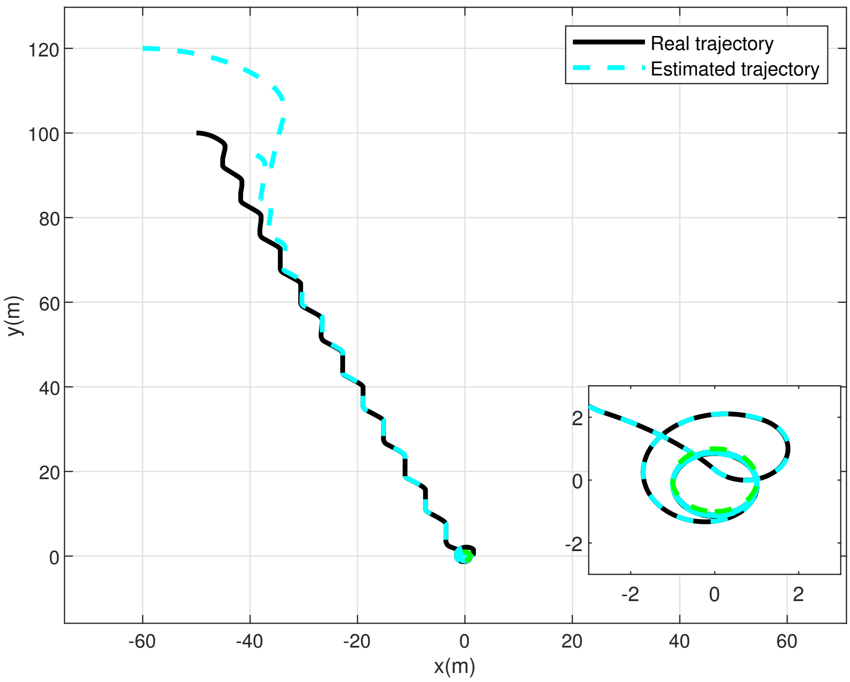

Figure 6: Coupling the proposed observer plus perturbation with the nearly minimum time control strategy, the drone is starting from (x_0, y_0, θ_0) equal to $(-50\ m, 100\ m, 0\ rad/s)$

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