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Finite Rank Solution for Conformable Second-Order Abstract Cauchy Problem in Hilbert Space

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Abstract. This paper presents a comprehensive analytical framework for constructing finite-rank solution to second-order conformable fractional abstract Cauchy problem. We examine the mathematical structure:

$$Eu^{(2\alpha)}(t) + Au^{(\alpha)}(t) + Bu(t) = f(t)$$

subject to prescribed initial conditions $u(0) = u_0$ and $u^{(\alpha)}(0) = u_0^{(\alpha)}$, where A, B, and E represent closed linear operators acting on a Banach space X, $f : [0, \infty) \to X$ is continuous, and u is continuously differentiable on $[0, \infty)$. Our analytical methodology exploits tensor product decomposition techniques to transform the problem into finite-dimensional systems. This work proves solution existence and uniqueness under specific conditions, and provides computational methods for many types of this problem.

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1. Introduction

For a Banach space X and $I = [0, 1]$ or $[0, \infty)$, $C(I)$ is the Banach space of all real-valued continuous functions defined on I with the supremum norm, $C(I, X)$ is the space of all continuous functions defined on I taking values in X, and $C^{(2\alpha)}(I, X)$ is the space of functions on I taking values in X with continuous conformable derivatives up to order 2α.

The abstract Cauchy problem represents one of the most fundamental classes of differential equations in applied mathematics, with applications spanning from heat conduction and wave propagation to population dynamics and financial modeling. The

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general form of the second-order conformable fractional abstract Cauchy problem under investigation is:

$$
\begin{gathered}
Eu^{(2\alpha)}(t) + Au^{(\alpha)}(t) + Bu(t) = f(t)z \\
u(0) = u_0 \\
u^{(\alpha)}(0) = u_0^{(\alpha)}
\end{gathered}
\tag{1}
$$

where A, B and E are closed linear operators on a Banach space X, $u \in C^{(2\alpha)}(I, X)$ is the unknown function, $f \in C(I)$ and $z, u_0, u_0^{(\alpha)} \in X$.

If $f = 0$ or $z = 0$, then the equation is homogeneous otherwise it is called non-homogeneous. If $f \neq 0$ and $z \neq 0$, then we have two cases. If f is given, then the problem is called a direct problem, otherwise the problem is called an inverse problem.

In this paper, we find a finite-rank solution of the second-order fractional type of the abstract Cauchy problem with some conditions on A, B and E.

Among various fractional derivative definitions available in the literature [12,15], this paper employs the conformable fractional derivative introduced by Khalil et al. [10] due to its advantageous properties.

Definition 1. *Let $f : [0, \infty) \to \mathbb{R}$ be a function. The conformable fractional derivative of f of order α, where $0 < \alpha \leq 1$ is defined by:*

$$
f^{(\alpha)}(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}
$$

for all $t > 0$. If f is $\alpha-$differentiable on $(0, c)$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then we define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

The power of this definition is that it satisfies the most of the properties of the usual derivatives such as product rule, quotient rule and chain rule, etc.

To read more about the conformable fractional derivatives see [1,2,9].

First of all, let us define what the tensor product and the finite-rank function are.

Definition 2. *Let X and Y be Banach spaces, and $T \in X^*$ (the dual of X). For $x \in X$ and $y \in Y$, the tensor product of x and y is the map $x \otimes y : X^* \to Y$ as $x \otimes y(T) = T(x)y$ for all $T \in X^*$.*

The operator $x \otimes y$ is bounded and linear with $\|x \otimes y\| = \|x\| \, \|y\|$ (see [13]). Such operators are called atoms, and every atom has rank 1.

The span of all atoms forms a subspace of $L(X^*, Y)$, denoted by $X \otimes Y$. A finite sum of atoms: ${}_{i=1}^{n} x_i \otimes y_i$ constitutes a finite-rank function, which forms the basis of our solution approach.

There are many norms on $X \otimes Y$, but the most important one is that called the injective norm. Furthermore, for any $T ={}_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, the injective norm is defined as

$$
\|T\|_{\vee} = \sup \{{}_{i=1}^{n} x^*(x_i).y^*(y_i) : x^* \in X^*, y^* \in Y^*, \|x^*\| = \|y^*\| = 1\}
$$

It should be noted that the space $(X \otimes Y, \|.\|_{\vee})$ does not necessarily exhibit completeness. We denote by $X \overset{\vee}{\otimes} Y$ the completion of $(X \otimes Y, \|.\|_{\vee})$, which is referred to as the completed injective tensor product of X with Y.

A fundamental result with significant applications in differential equation theory can be formulated as follows:

Theorem 1. *[16] For any compact hausdorff space K and any Banach space X, the space $C(K, X)$ is isometrically isomorphic to $C(K) \overset{\vee}{\otimes} X^*$.*

This theorem yields the important corollary that for any two compact metric spaces I and J, we have $C(I \times J) \cong C(I) \overset{\vee}{\otimes} C(J)$.

For more on tensor product we refer to [6,7,13,16].

2. Direct Problem

Let u be an $2\alpha-$differentiable on $I = [0, 1]$ into the Hilbert space $X = \ell^2$, where $\ell^2 = \{(x_n) :_{n=1}^{\infty} |x_n|^2 < \infty\}$. The standard basis of ℓ^2 is denoted by $\{\delta_1, \delta_2, ...\}$. Let A, B be two closed operators on ℓ^2 such that domains of A and B contain the elements of the standard basis of ℓ^2.

In this section we look for a solution to the direct problem (1) among finite-rank functions of the form $u(t) =_{i=1}^{n} u_i(t)\delta_i$, where $u_i^{(2\alpha)}(t) \in C(I)$, $i = 1, 2, ..., n$.

Before analyzing the main problem, we establish the theoretical foundation using fundamental matrices.

Definition 3. *The conformable fractional fundamental matrix $\phi_\alpha(t)$ is the unique $n \times n$ matrix-valued function that satisfies the fractional differential equation $\phi_\alpha^{(\alpha)}(t) = A\phi_\alpha(t)$, $t > 0$ and $\alpha \in (0, 1]$ with initial condition $\phi_\alpha(0) = I$, where I is the $n \times n$ identity matrix.*

For the conformable fractional derivative, the fundamental matrix can be expressed as:

$$\phi_\alpha(t) = \exp\left(\frac{At^\alpha}{\alpha}\right)$$

The essential properties of the fundamental matrix are:

1- $\phi_\alpha(t)$ is continuously differentiable in the conformable sense for $t > 0$.
2- $\phi_\alpha(t)$ is continuous at $t = 0$.
3- $\phi_\alpha(t)$ is invertible for all $t \geq 0$.
4- The inverse satisfies $\phi_\alpha^{-1}(t) = \phi_\alpha(-t)$.
5- $\frac{d}{dt}\phi_\alpha(t) = \frac{1}{t^{1-\alpha}}A\phi_\alpha(t)$.
To read more about $\phi_\alpha(t)$ see [10].
We now present the main theorem.

Theorem 2. *In problem (1), let $E = I$ (the identity operator) and $u(t) =_{i=1}^{n} u_i(t)\delta_i$, where $u_i^{(2\alpha)}(t) \in C(I)$, $i = 1, 2, ..., n$. Then, the problem has a unique solution.*

Proof. *Since $u(t) =_{i=1}^{n} u_i(t)\delta_i$, the derivatives are $u^{(\alpha)}(t) =_{i=1}^{n} u_i^{(\alpha)}(t)\delta_i$, $u^{(2\alpha)}(t) =_{i=1}^{n} u_i^{(2\alpha)}(t)\delta_i$. Substituting into problem (1):*

$$_{i=1}^{n} u_i^{(2\alpha)}(t)\delta_i +_{i=1}^{n} u_i^{(\alpha)}(t)A(\delta_i) +_{i=1}^{n} u_i(t)B(\delta_i) = f(t)z \tag{2}$$

Taking the inner product with δ_j (2):

$$_{i=1}^{n} u_i^{(2\alpha)}(t)\langle \delta_i, \delta_j\rangle +_{i=1}^{n} u_i^{(\alpha)}(t)\langle A(\delta_i), \delta_j\rangle +_{i=1}^{n} u_i(t)\langle B(\delta_i), \delta_j\rangle = f(t)\langle z, \delta_j\rangle \tag{3}$$

Using orthonormality of the standard basis:

$$u_j^{(2\alpha)}(t) +_{i=1}^{n} u_i^{(\alpha)}(t)\langle A(\delta_i), \delta_j\rangle +_{i=1}^{n} u_i(t)\langle B(\delta_i), \delta_j\rangle = f(t)\langle z, \delta_j\rangle \tag{4}$$

This yields a system of second-order conformable fractional differential equations. Converting to first-order form by introducing new variables:

$$v_i = u_i, v_{n+i} = u_i^{(\alpha)}, i = 1, 2, ..., n$$

Then the system can be written as:

$$v_i^{(\alpha)} = v_{n+i}, \ i = 1, 2, ..., n$$

$$v_{n+j}^{(\alpha)}(t) = -\sum_{i=1}^{n} v_{n+i}(t)\langle A(\delta_i), \delta_j\rangle - \sum_{i=1}^{n} v_i(t)\langle B(\delta_i), \delta_j\rangle + f(t)\langle z, \delta_j\rangle \tag{5}$$

In matrix form:

$$\boldsymbol{V}^{(\alpha)}(t) = \boldsymbol{G}\boldsymbol{V}(t) + \mathbf{F}(t) \tag{6}$$

where:

$$V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \\ v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{2n} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ u_1^{(\alpha)} \\ u_2^{(\alpha)} \\ \vdots \\ u_n^{(\alpha)} \end{pmatrix}, \ \boldsymbol{G} = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\mathcal{B} & -\mathcal{A} \end{pmatrix}, \ \mathbf{F}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(t)\langle z, \delta_1\rangle \\ \vdots \\ f(t)\langle z, \delta_n\rangle \end{pmatrix}$$

Here, $\mathbf{0}$ is a zero matrix, $\boldsymbol{I}$ is the identity matrix, $\mathcal{A}$ and $\mathcal{B}$ are $n \times n$ coefficient matrices:

$$\mathcal{A} = (a_{ij})_{n\times n}\,,\ \mathcal{B} = (b_{ij})_{n\times n}$$

where $a_{ij} = \langle A(\delta_i), \delta_j\rangle\,,\, b_{ij} = \langle B(\delta_i), \delta_j\rangle$ *for all* $i, j = 1, 2, ..., n$.
This system has a unique solution of the form:

$$\boldsymbol{V}(t) = \phi_\alpha(t)\,\boldsymbol{V}(0) + \phi_\alpha(t)\int_0^t \frac{\phi_\alpha^{-1}(s)\,\mathbf{F}(s)}{s^{1-\alpha}}ds \tag{7}$$

where $\phi_\alpha(t)$ *is the* $n \times n$ *conformable fundamental matrix, and* $\boldsymbol{V}(0)$ *is the initial condition vector containing the initial value of* u_0 *and* $u_0^{(\alpha)}$.

Now, the following theorem takes a special case on the operators A and B.

Theorem 3. *Consider problem (1) with* $E = I$ *and* $u(t) =_{i=1}^{n} u_i(t)\delta_i$, *where* $u_i^{(2\alpha)}(t) \in C(I)$, $i = 1, 2, ..., n$. *If* $A(\delta_i) = \lambda_i\delta_i$ *and* $B(\delta_i) = \beta_i\delta_i$, *then the problem has a unique solution.*

Proof. We have $u(t) =_{i=1}^{n} u_i(t)\delta_i$, then $u^{(\alpha)}(t) =_{i=1}^{n} u_i^{(\alpha)}(t)\delta_i$, and $u^{(2\alpha)}(t) =_{i=1}^{n} u_i^{(2\alpha)}(t)\delta_i$. Thus

$$_{i=1}^{n}u_i^{(2\alpha)}(t)\delta_i +_{i=1}^{n} u_i^{(\alpha)}(t)A(\delta_i) +_{i=1}^{n} u_i(t)B(\delta_i) = f(t)z \tag{8}$$

If we take the inner product of δ_j with both sides of equation (8), then the equation becomes

$$_{i=1}^{n}u_i^{(2\alpha)}(t)\,\langle\delta_i, \delta_j\rangle +_{i=1}^{n} u_i^{(\alpha)}(t)\,\langle A(\delta_i), \delta_j\rangle +_{i=1}^{n} u_i(t)\,\langle B(\delta_i), \delta_j\rangle = f(t)\,\langle z, \delta_j\rangle \tag{9}$$

But since the standard basis is orthonormal and $A(\delta_i) = \lambda_i\delta_i$, $B(\delta_i) = \beta_i\delta_i$, then

$$u_j^{(2\alpha)}(t) + \lambda_j u_j^{(\alpha)}(t) + \beta_j u_j(t) = f(t)\,\langle z, \delta_j\rangle \tag{10}$$

Each equation represents a second-order linear conformable fractional differential equation with constant coefficients. Given appropriate initial conditions $u_j(0)$ and $u_j^{(\alpha)}(0)$ for all $j = 1, 2, ..., n$, each equation admits a unique solution.

Now, in the following theorem we take the case where $E \neq I$.

Theorem 4. *Consider problem (1) where* E_n *is orthogonally diagonalizable with* $A_n|_{\ker(E_n)}$ *invertible. If* $u(t) =_{i=1}^{n} u_i(t)\delta_i$ *with* $u_i^{(2\alpha)}(t) \in C(I)$ *for* $i = 1, 2, ..., n$, *then problem (1) has a unique solution.*

Proof. Let $\{\theta_1, \theta_2, ..., \theta_n\}$ be an orthonormal basis such that the matrix representation of E_n with respect to this basis is $D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ with $\lambda_1, \lambda_2, ..., \lambda_n$ corresponding eigenvalues. Now, if $\lambda_i \neq 0$, for all $i = 1, 2, ..., n$, then problem (1) reduces to:

$$u^{(2\alpha)}(t) + E_n^{-1}A_n u^{(\alpha)}(t) + E_n^{-1}B_n u(t) = E_n^{-1}f(t)$$

This has a unique solution by Theorem 3.

Suppose $\lambda_i \neq 0$, for $i = 1, 2, ..., r$, and $\lambda_i = 0$, for $i = r+1, r+2, ..., n$. Let $u(t) = \sum_{i=1}^{n} v_i(t)\,\theta_i$. The system becomes:

$$\sum_{i=1}^{n} v_i^{(2\alpha)}(t)\,E_n(\theta_i) + \sum_{i=1}^{n} v_i^{(\alpha)}(t)\,A_n(\theta_i) + \sum_{i=1}^{n} v_i(t)\,B_n(\theta_i) = f(t)\,z \tag{11}$$

Taking the inner product of θ_j with both sides of equation (11), we obtain

$$\sum_{i=1}^{n} v_i^{(2\alpha)}(t)\,\langle E_n(\theta_i), \theta_j\rangle + \sum_{i=1}^{n} v_i^{(\alpha)}(t)\,\langle A_n(\theta_i), \theta_j\rangle + \sum_{i=1}^{n} v_i(t)\,\langle B_n(\theta_i), \theta_j\rangle = f(t)\,\langle z, \theta_j\rangle \tag{12}$$

Since θ_i is an eigenvector of E_n with corresponding eigenvalue λ_i for every i and $\{\theta_1, \theta_2, ..., \theta_n\}$ is an orthonormal basis, we get

$$\lambda_j v_j^{(2\alpha)}(t) + \sum_{i=1}^{n} v_i^{(\alpha)}(t)\,\langle A_n(\theta_i), \theta_j\rangle + \sum_{i=1}^{n} v_i(t)\,\langle B_n(\theta_i), \theta_j\rangle = f(t)\,\langle z, \theta_j\rangle \tag{13}$$

Introducing new variables to represents the first derivatives of v_i :

$$w_i = v_i, w_{n+i} = v_i^{(\alpha)}, i = 1, 2, ..., n$$
$$w_i^{(\alpha)} = w_{n+i}, i = 1, 2, ..., n$$

Then the system can be written as:

$$\lambda_j w_{n+j}^{(\alpha)}(t) = -\sum_{i=1}^{n} w_{n+i}(t)\,\langle A_n(\theta_i), \theta_j\rangle - \sum_{i=1}^{n} w_i(t)\,\langle B_n(\theta_i), \theta_j\rangle + f(t)\,\langle z, \theta_j\rangle \tag{14}$$

So, we get the following system

$$\begin{pmatrix} \boldsymbol{I}_n & \mathbf{0} \\ 0 & \widetilde{D} \end{pmatrix} \mathbf{W}^{(\alpha)}(t) = \begin{pmatrix} \mathbf{0} & \boldsymbol{I}_n \\ -\mathcal{B} & -\mathcal{A} \end{pmatrix} W(\mathbf{t}) + \mathbf{F}(\mathbf{t}) \tag{15}$$

where $\mathbf{W}(\mathbf{t}) = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ w_{n+1} \\ w_{n+2} \\ \vdots \\ w_{2n} \end{pmatrix}$, $\widetilde{D} = \begin{pmatrix} D & \mathbf{0} \\ 0 & 0 \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} G_1 & G_2 \\ G_3 & \widehat{G} \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$

where $\widehat{G} = A_n|_{\ker(E_n)} = [\langle A_n(\theta_j), \theta_i\rangle]_{i,j=r+1,...,n}$.

Now, multiplying (15) by $\begin{pmatrix} \boldsymbol{I}_n & \mathbf{0} \\ 0 & K \end{pmatrix}_{2n\times 2n}$ where $K = \begin{pmatrix} D^{-1} & \mathbf{0} \\ 0 & \widehat{G}^{-1} \end{pmatrix}$, we get

$$\begin{pmatrix} \boldsymbol{I}_n & \mathbf{0} \\ 0 & K\widetilde{D} \end{pmatrix} \mathbf{W}^{(\alpha)}(t) = \begin{pmatrix} \mathbf{0} & \boldsymbol{I}_n \\ -K\mathcal{B} & -K\mathcal{A} \end{pmatrix} \mathbf{W}(\mathbf{t}) + \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ 0 & K \end{pmatrix} \mathbf{F}(\mathbf{t}) \tag{16}$$

where $K\widetilde{D} = \begin{pmatrix} \boldsymbol{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, $K\mathcal{A} = \begin{pmatrix} D^{-1}G_1 & D^{-1}G_2 \\ \widehat{G}^{-1}G_3 & I_{n-r} \end{pmatrix}$ and $K\mathcal{B} = \begin{pmatrix} D^{-1}H_1 & D^{-1}H_2 \\ \widehat{G}^{-1}H_3 & \widehat{A}^{-1}H_4 \end{pmatrix}$

Let $U_1 = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{pmatrix}$, $U_2 = \begin{pmatrix} w_{r+1} \\ w_{r+2} \\ \vdots \\ w_n \end{pmatrix}$, $U_3 = \begin{pmatrix} w_{n+1} \\ w_{n+2} \\ \vdots \\ w_{n+r} \end{pmatrix}$, $U_4 = \begin{pmatrix} w_{n+r+1} \\ w_{n+r+2} \\ \vdots \\ w_{2n} \end{pmatrix}$, $F_1(t) = f(t) \begin{pmatrix} \langle z,\theta_1\rangle \\ \langle z,\theta_2\rangle \\ \vdots \\ \langle z,\theta_r\rangle \end{pmatrix}$ and $F_2(t) = f(t) \begin{pmatrix} \langle z,\theta_{r+1}\rangle \\ \langle z,\theta_{r+2}\rangle \\ \vdots \\ \langle z,\theta_n\rangle \end{pmatrix}$. Then, we have

$$U_1^{(\alpha)}(t) = U_3(t) \tag{17}$$

$$U_2^{(\alpha)}(t) = U_4(t) \tag{18}$$

$$U_3^{(\alpha)}(t) = -D^{-1}H_1U_1(t) - D^{-1}H_2U_2(t) - D^{-1}G_1U_3(t) - D^{-1}G_2U_4(t) + D^{-1}F_1(t) \tag{19}$$

and

$$\mathbf{0} = -\widehat{G}^{-1}H_3U_1(t) - \widehat{G}^{-1}H_4U_2(t) - \widehat{G}^{-1}G_3U_3(t) - U_4(t) + \widehat{G}^{-1}F_2(t) \tag{20}$$

From the last equation, we get

$$U_4(t) = -\widehat{G}^{-1}H_3U_1(t) - \widehat{G}^{-1}H_4U_2(t) - \widehat{G}^{-1}G_3U_3(t) + \widehat{G}^{-1}F_2(t) \tag{21}$$

Substituting (21) in (18) and (19), we get

$$U_2^{(\alpha)}(t) = -\widehat{G}^{-1}H_3U_1(t) - \widehat{G}^{-1}H_4U_2(t) - \widehat{G}^{-1}G_3U_3(t) + \widehat{G}^{-1}F_2(t) \tag{22}$$

$$\begin{aligned} U_3^{(\alpha)}(t) \;&=\; D^{-1}\left(G_2\widehat{G}^{-1}H_3 - H_1\right)U_1(t) + D^{-1}\left(G_2\widehat{G}^{-1}H_4 - H_2\right)U_2(t) + D^{-1}\left(G_2\widehat{G}^{-1}G_3 - G_1\right)U_3(t) \\ &\quad + D^{-1}F_1(t) - D^{-1}G_2\widehat{G}^{-1}F_2(t) \end{aligned} \tag{23}$$

If we define a combined state vector $U(t) = \begin{pmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \end{pmatrix}$ that includes all variables except U_4 (which has been eliminated), then the merged system can be written as:

$$U^{(\alpha)}(t) = MU(t) + G(t) \tag{24}$$

where $M = \begin{pmatrix} \mathbf{0} & \mathbf{0} & I_r \\ -\widehat{G}^{-1}H_3 & -\widehat{G}^{-1}H_4 & -\widehat{G}^{-1}G_3 \\ D^{-1}\left(G_2\widehat{G}^{-1}H_3 - H_1\right) & D^{-1}\left(G_2\widehat{G}^{-1}H_4 - H_2\right) & D^{-1}\left(G_2\widehat{G}^{-1}G_3 - G_1\right) \end{pmatrix}$

and

$$G(t) = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ D^{-1} \end{pmatrix} F_1(t) + \begin{pmatrix} \mathbf{0} \\ \widehat{G}^{-1} \\ -D^{-1}G_2\widehat{G}^{-1} \end{pmatrix} F_2(t).$$

Thus, the system (24) has a unique solution and then the problem (1) has a unique solution as required.

3. Inverse Problem Case

In this section, we consider the inverse problem where both solution $u(t)$ and function $f(t)$ have finite-rank representations: $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$, $f(t) = \sum_{i=1}^{n} f_i(t)\delta_i$, with $u_i^{(2\alpha)}, f_i \in C(I)$ for $i = 1, 2, ..., n$.

Theorem 5. *Consider problem (1) with* $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$ *and* $f(t) = \sum_{i=1}^{n} f_i(t)\delta_i$, *where* $u_i^{(2\alpha)}, f_i \in C(I)$ *for* $i = 1, 2, ..., n$. *If the following conditions hold:*

1) There exists $x \in \ell^2$ *such that* $\langle u_i(t)\delta_i, x\rangle = h_i(t)$ *where* $h_i^{(2\alpha)} \in C(I)$ *and* $\langle \delta_i, x\rangle \neq 0$

2) $A\delta_i = \lambda_i\delta_i$ *and* $B\delta_i = \beta_i\delta_i$ *for all* $i = 1, 2, ..., n$.

Then the problem has a unique solution.

Proof. Under the given representations, problem (1) becomes

$$\sum_{i=1}^{n} u_i^{(2\alpha)}(t)\delta_i + \sum_{i=1}^{n} u_i^{(\alpha)}(t)A\delta_i + \sum_{i=1}^{n} u_i(t)B\delta_i = \sum_{i=1}^{n} f_i(t)\delta_i \tag{25}$$

But $A\delta_i = \lambda_i\delta_i$ and $B\delta_i = \beta_i\delta_i$ for all $i = 1, 2, ..., n$ by condition 2, so we get

$$\sum_{i=1}^{n} u_i^{(2\alpha)}(t)\delta_i + \sum_{i=1}^{n} \lambda_i u_i^{(\alpha)}(t)\delta_i + \sum_{i=1}^{n} \beta_i u_i(t)\delta_i = \sum_{i=1}^{n} f_i(t)\delta_i \tag{26}$$

Taking inner products with δ_j:

$$\sum_{i=1}^{n} u_i^{(2\alpha)}(t)\langle\delta_i, \delta_j\rangle + \sum_{i=1}^{n} \lambda_i u_i^{(\alpha)}(t)\langle\delta_i, \delta_j\rangle + \sum_{i=1}^{n} \beta_i u_i(t)\langle\delta_i, \delta_j\rangle = \sum_{i=1}^{n} f_i(t)\langle\delta_i, \delta_j\rangle \tag{27}$$

Moreover, the basis $\{\delta_i\}_{i=1}^{n}$ is orthonormal so then the equation (27) becomes

$$u_j^{(2\alpha)}(t) + \lambda_j u_j^{(\alpha)}(t) + \beta_j u_j(t) = f_j(t) \tag{28}$$

Multiplying (28) by δ_j and taking inner product x :

$$g_j^{(2\alpha)}(t) + \lambda_j g_j^{(\alpha)}(t) + \beta_j g_j(t) = f_j(t)\langle\delta_j, x\rangle \tag{29}$$

Therefore, $f_j(t)$ is uniquely determined by

$$f_j(t) = \frac{g_j^{(2\alpha)}(t) + \lambda_j g_j^{(\alpha)}(t) + \beta_j g_j(t)}{\langle \delta_j, x \rangle} \tag{30}$$

since $\langle \delta_j, x \rangle \neq 0$ by assumption. Hence, $f(t)$ has been completely determined. Each differential equation admits a unique solution given initial conditions $u_j(0)$ and $u_j^{(\alpha)}(0)$.

4. Physical Applications and Engineering Relevance

1- **Viscoelastic Material Modeling:**

In viscoelastic material analysis, stress-stain relationships naturally incorporate memory effects through fractional derivative formulations:

$$E\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + A\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + Bu = f(t)$$

where u represents displacement field distribution, E captures internal effects (potentially degenerate in quasi-static scenarios), A models viscous damping mechanisms, and B represents elastic restoring forces.

2- Anomalous Diffusion Phenomena:

In systems exhibiting anomalous diffusion characteristics, finite-rank approaches naturally capture dominant transport modes:

$$E\frac{\partial^{2\alpha} c}{\partial t^{2\alpha}} - D\frac{\partial^{\alpha} c}{\partial t^{\alpha}} + Rc = S(x,t)$$

where c denotes concentration distribution, D represents diffusion operator, and R models reaction mechanisms.

5. Conclusion

This study presents a comprehensive framework for analyzing conformable fractional abstract Cauchy problems through finite-rank solution techniques. The main contributions include:

1- Theoretical Achievements

Existence and Uniqueness Theorems: Established under various operator conditions including degenerate cases.

Solution Methodology: Developed systematic approach using tensor product decomposition.

Computational Framework: Provided constructive algorithms for solution computation.

2- Practical Impact

The finite-rank approach offers significant computational advantages by reducing infinite-dimensional problems to finite-dimensional systems. Applications in viscoelastic materials and anomalous diffusion demonstrate practical relevance.

3- Novel Contributions

Extension of tensor product techniques to conformable fractional derivatives.

Treatment of degenerate operators through spectral decomposition.

Unified approach handling both direct and inverse problems.

4- Future Research Directions

Higher-Order Problems: Extension to conformable fractional problems of order greater than 2.

The framework established here provides a solid foundation for further research in fractional differential equations and their applications to real-world phenomena exhibiting memory effects and anomalous behavior.

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