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T-Ordering on Generalized Regular Intuitionistic Fuzzy Matrices

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Abstract. In this paper, we study T-ordering on generalized regular intuitionistic fuzzy matrices (IFM) named as $k-T$-ordering, as a generalization of the T-ordering on intuitionistic fuzzy matrices. Some equivalent conditions for this ordering using generalized inverses are derived. Further, we prove that $k-T$-ordering is not a partial ordering.

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Key Words and Phrases: Fuzzy Matrix, Intuitionistic Fuzzy Matrices, Partial Ordering, k-T-ordering

1. Introduction

Atanassov first introduced the concept of intuitionistic fuzzy sets [1], building on the foundation of fuzzy set theory. Meanwhile, Ben-Israel and Greville [2] explored the idea of generalized inverses for complex matrices. In fuzzy algebra, defined over the interval $F = [0, 1]$, matrix operations are carried out using the max-min operations, where addition is defined as $a + b = \max\{a, b\}$, and multiplication as $a \cdot b = \min\{a, b\}$ for all $a, b \in F$. The set $F_{m\times n}$ consists of all $m \times n$ fuzzy matrices under this algebra.

A fuzzy matrix $A \in F_{m\times n}$ is said to be regular if there exists a matrix X such that $AXA = A$, in which case X is termed a generalized (g-) inverse of A. Kim and Roush [3] extended fuzzy matrix theory by drawing analogies to Boolean matrices and studying their inverses.

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Further developments include Cho's analysis of the consistency of fuzzy matrix equations [4] and the introduction of k-regular fuzzy matrices by Meenakshi and Jenita [5], a generalization of regular fuzzy matrices. Khan and Paul [6] introduced the concept of generalized inverses for intuitionistic fuzzy matrices, while Pradhan and Pal [7] proposed a method to compute such inverses using block-wise decompositions.

Pal and Khan [8] further developed the fundamental properties of intuitionistic fuzzy matrices, and Meenakshi and Gandhimathi [9] examined their regularity. Ordering concepts in fuzzy matrices have also been explored, with Sriram and Murugadas investigating general ordering [10], Cen [11] proposing the idea of T-ordering, along with its relation to generalized inverses. Platil and Tanaka [12] proposed a multi-criteria evaluation framework based on set-relations for intuitionistic fuzzy sets, which may offer broader interpretive foundations for such ordering concepts.

Expanding on this, Poongodi et al. [13] discussed ordering in k-regular interval-valued fuzzy matrices, extending the minus ordering concept previously studied in [14]. Additionally, [15] explored special types of inverses for regular intuitionistic fuzzy matrices. In another significant contribution, Jenita, Karuppusamy, and Thangamani introduced k-regular intuitionistic fuzzy matrices in [16], extending the notion of regular intuitionistic fuzzy matrices and analyzing various types of inverses for them.

Meenakshi and Inbam [17] defined minus ordering on matrices using generalized inverses. As a continuation of this line of research, two further studies were conducted [18], [19], focusing on minus ordering and sharp ordering in the context of generalized regular intuitionistic fuzzy matrices. In [20] Radio fuzzy graphs and assignment of frequency in radio stations is discussed. Applications of edge colouring of fuzzy graphs was discussed in [21]. In [22] and [23] Rupkumar Mahapatra , Sovan Samanta,Madhumangal Palhave discussed about the link prediction in social networks by neutrosophic graph and generalized neutrosophic planar graphs. Detecting influential node in a network using neutrosophic graph, Edge colouring of neutrosophic graphs, A study on linguistic z-graph and its application in social networks, Centrality measure using linguistic Z-graph and its application was also discussed in [24–27].

2. Preliminaries

The matrix operations on IFM as stated in [9] will be followed. For $A, B \in (IFM)_{m\times n}$, the operations are defined as follows:

$$A + B = (\langle \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\vartheta}, b_{ij\vartheta}\}\rangle),$$
$$AB = \left(\left\langle \max_k \min\{a_{ik\mu}, b_{kj\mu}\}, \min_k \max\{a_{ik\vartheta}, b_{kj\vartheta}\}\right\rangle\right).$$

The order relation on $(IFM)_{m\times n}$ defined as:

$$A \leq B \Leftrightarrow a_{ij\mu} \leq b_{ij\mu} \text{ and } a_{ij\vartheta} \geq b_{ij\vartheta}, \quad \text{for all } i, j.$$

Throughout this paper we denoted right $k-$regular as $_{\text{right}}$k-reg, left $k-$regular as $_{\text{left}}$k-reg, right $k-g-$ inverse as $_{\text{right}}$k-g-inv, left $k-g-$ inverse as $_{\text{left}}$k-g-inv, right $k-$Moore-Penrose inverse as $_{\text{right}}$k-Moore -Penrose inv ,left $k-$Moore-Penrose inverse as $_{\text{left}}$k-Moore -Penrose inv ,$k-$regular as $k-$ reg, $k-g-$inverse as $k-g-$inv, and $k-$Moore-Penrose inverse as $k-$Moore-Penrose inv..

Definition 1. *[16] If there exists a matrix $X \in (IFM)_n$ such that $u^kXu = u^k$, for some positive integer k, then the matrix $u \in (IFM)_n$ is said to be $_{right}$k-reg . X is called a $_{right}$k-g-inv of u.*
Let, $u\{1_r^k\} = \{X \mid u^kXu = u^k\}$.

Definition 2. *[16] If there exists a matrix $Y \in (IFM)_n$ such that $uYu^k = u^k$, for some integer k, then the matrix $u \in (IFM)_n$ is said to be $_{left}$k-reg. Y is called a $_{left}$k-g-inv of u.*
Let, $u\{1^k\} = u\{1_r^k\} \cup u\{1_l^k\}$ and $u\{1_k\} = u\{1_r^k\} \cap u\{1_l^k\}$.

Theorem 1. *[28] Let $u \in (IFM)_n$ and k be a positive integer. Then,*

$$X \in u\{1_r^k\} \Leftrightarrow X^T \in u^T\{1_l^k\}.$$

Definition 3. *[15] A matrix $u \in (IFM)_n$ is said to have a $_{right}$k-Moore -Penrose inv if there exists a matrix $X \in (IFM)_n$ satisfying the four equations*

$$u^kXu = u^k, - - -\{1_r^k\}$$
$$XuX^k = X^k, - - -\{2_l^k\}$$
$$(u^kX)^T = u^kX - - - \{3^k\}$$
$$(Xu^k)^T = Xu^k - - - \{4^k\}.$$

This inverse is denoted as u_{rk}^+.

Definition 4. *[15] A matrix $u \in (IFM)_n$ is said to have a $_{left}$k-Moore -Penrose inv if there exists a matrix $Y \in (IFM)_n$ satisfying the four equations:*

$$uYu^k = u^k - - - \{1_l^k\}$$
$$Y^kuY = Y^k - - - \{2_r^k\}$$
$$(u^kY)^T = u^kY - - - \{3^k\}$$
$$(Yu^k)^T = Yu^k - - - \{4^k\}.$$

This inverse is denoted as u_{lk}^+.

3. k - T Ordering on IFM

Theorem 2. *Let $u \in (IFM)_{mn}$. The following are equivalent:*
(i) u^+ exists and $u^+ = u^T$.
(ii) u^T is a g-inverse of u.

Proof. (i) $\Rightarrow$ (ii)

$$u^{+} = u^{T} \Rightarrow u^{T} \text{ is a g-inverse of } u,$$

(ii) $\Rightarrow$ (i)

$$u^{T} \text{ is a g-inverse of } u \quad \Rightarrow uu^{T}u = u \tag{1}$$

By taking transpose on both sides in (1), we get:

$$u^{T} = u^{T}uu^{T}$$

$$(uu^{T})^{T} = uu^{T} \quad \text{and} \quad (u^{T}u)^{T} = u^{T}u,$$

Hence u^{+} exists and $u^{+} = u^{T}$.

Remark 1. *In general, for a k-regular IFM, the* ${}_{right}k$*-Moore -Penrose inv* u^{+}_{rk} *is different from* ${}_{left}k$*-Moore -Penrose inv* u^{+}_{lk} *and it is not unique.If* $u^{+}_{rk} = u^{+}_{lk}$*, let us call it as the k-Moore–Penrose inv, and it is denoted by* u^{+}_{k}*. Thus,*

$$u^{+}_{k} = u^{+}_{rk} = u^{+}_{lk}$$

This is shown in the following example.

Example 1. *Let us consider the matrix u as follows:*

$$u = \begin{bmatrix} \langle 0.5, 0\rangle & \langle 0.2, 0.5\rangle & \langle 0, 0\rangle \\ \langle 0.2, 0\rangle & \langle 0.5, 0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.2, 0.5\rangle & \langle 0.2, 0\rangle & \langle 0.5, 0\rangle \end{bmatrix}.$$

For the permutation matrices:

$$P_1 = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 1, 0\rangle & \langle 0, 0\rangle \\ \langle 0, 0\rangle & \langle 0, 1\rangle & \langle 1, 0\rangle \end{bmatrix},$$

$$P_2 = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle & \langle 0, 0\rangle \\ \langle 0, 1\rangle & \langle 0, 0\rangle & \langle 1, 0\rangle \\ \langle 0, 0\rangle & \langle 1, 0\rangle & \langle 0, 1\rangle \end{bmatrix},$$

$$P_3 = \begin{bmatrix} \langle 0, 0\rangle & \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 1, 0\rangle & \langle 0, 1\rangle & \langle 0, 0\rangle \\ \langle 0, 1\rangle & \langle 0, 0\rangle & \langle 1, 0\rangle \end{bmatrix},$$

$$P_4 = \begin{bmatrix} \langle 0, 1\rangle & \langle 1, 0\rangle & \langle 0, 0\rangle \\ \langle 0, 0\rangle & \langle 0, 1\rangle & \langle 1, 0\rangle \\ \langle 1, 0\rangle & \langle 0, 0\rangle & \langle 0, 1\rangle \end{bmatrix},$$

$$P_5 = \begin{bmatrix} \langle 0,0\rangle & \langle 0,1\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 0,0\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle & \langle 0,0\rangle \end{bmatrix},$$

$$P_6 = \begin{bmatrix} \langle 0,1\rangle & \langle 0,0\rangle & \langle 1,0\rangle \\ \langle 0,0\rangle & \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle & \langle 0,0\rangle \end{bmatrix}.$$

$$uP_1u = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \neq u,$$

$$uP_2u = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \end{bmatrix} \neq u.$$

$$uP_3u = \begin{bmatrix} \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \neq u,$$

$$uP_4u = \begin{bmatrix} \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \\ \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \end{bmatrix} \neq u,$$

$$uP_5u = \begin{bmatrix} \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \\ \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \end{bmatrix} \neq u$$

and

$$uP_6u = \begin{bmatrix} \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \end{bmatrix} \neq u.$$

Therefore, u is not regular.
For this u,

$$u^2 = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix}.$$

For

$$X = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix}$$
$$u^2Xu = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0.5\rangle & \langle 0,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0.5\rangle \\ \langle 0.2,0.5\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0.5\rangle & \langle 0,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0.5\rangle \\ \langle 0.2,0.5\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = u^2.$$

$$uXu^2 = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0.5\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0.5\rangle \\ \langle 0.2,0.5\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0.5\rangle \\ \langle 0.2,0.5\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = u^2.$$

$u^2Xu = u^2 = uXu^2$ *holds.*

Thus u is 2-reg.
For $k = 2$,

$$X^2uX = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0.5\rangle \\ \langle 0.2,0.5\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0.5\rangle & \langle 0,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0.5\rangle \\ \langle 0.2,0.5\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = X^2.$$

$$XuX^2 = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0.5\rangle & \langle 0,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0.5\rangle \\ \langle 0.2,0.5\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = X^2.$$

$$(u^2X)^T = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = u^2X$$

$$(Xu^2)^T = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = Xu^2.$$

Hence, X is a ${}_{right}2$-Moore -Penrose inverse as well as a ${}_{left}2$-Moore -Penrose inverse.
Therefore, $X = u_2^+ = u_{r2}^+ = u_{l2}^+$ exists.

$$u^T = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0,0.5\rangle \\ \langle 0.2,0.5\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0,0\rangle & \langle 0.2,0.5\rangle & \langle 0.5,0\rangle \end{bmatrix}.$$

$$u^2u^Tu = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = u^2.$$

$$uu^Tu^2 = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = u^2.$$

$$(u^T)^2uu^T = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = (u^T)^2,$$

$$u^Tu(u^T)^2 = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = (u^T)^2,$$

$$(u^2u^T)^T = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = u^2u^T$$

and

$$(u^Tu^2)^T = \begin{bmatrix} \langle 0.5,0\rangle & \langle 0.2,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.5,0\rangle & \langle 0.2,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle & \langle 0.5,0\rangle \end{bmatrix} = u^Tu^2.$$

Therefore, X and u^T are 2-Moore-Penrose inv of u.
Hence u_2^+ exists, but it is not unique.

Definition 5. *Let $u \in (IFM)_{mn}^-$ and $v \in (IFM)_{mn}$, the minus ordering denoted as $u \leq v$ is defined as:*

$$u \leq v \Leftrightarrow uX = vX \text{ and } Xu = Xv, \quad \text{for some } X \in u\{1\}.$$

$u\{1\} - set\ of\ generalized\ inverses$

Remark 2. *Let $u \in (IFM)_{mn}^-$ and $v \in (IFM)_{mn}$, if u^+ exists, then u^+ is unique and u^+=u^T.Then we have the following definition.*

Definition 6. *The T-ordering $u <^T v$ in $(IFM)_{mn}$ is defined as:*

$$u <^T v \iff uu^T = vu^T \quad and \quad u^Tu = u^Tv$$

Here, u^T is a g-inverse of u.

Lemma 1. *Let $u = \langle u_\mu, u_\nu\rangle \in (IFM)_{mn}$ and $v \in (IFM)_{mn}$. Then:*

$$u <^T v \Leftrightarrow u_\mu <^T v_\mu \text{ and } u_\nu <^T v_\nu.$$

Proof.

$$\begin{aligned} u <^T v &\iff uu^T = vu^T \text{ and } u^Tu = u^Tv \\ &\iff \langle u_\mu, u_\nu\rangle\langle u_\mu^T, u_\nu^T\rangle = \langle v_\mu, v_\nu\rangle\langle u_\mu^T, u_\nu^T\rangle \\ &\iff \langle u_\mu u_\mu^T, u_\nu u_\nu^T\rangle = \langle v_\mu u_\mu^T, v_\nu u_\nu^T\rangle \\ &\iff u_\mu u_\mu^T = v_\mu u_\mu^T \text{ and } u_\nu u_\nu^T = v_\nu u_\nu^T \end{aligned}$$

Similarly,

$$u^T u = u^T v \iff u_\mu^T u_\mu = u_\mu^T v_\mu \text{ and}$$

$$u_\nu^T u_\nu = u_\nu^T v_\nu$$

Remark 3. *$u <^T v \iff u \leq v$ with respect to u^+*
$\iff$

$$u^T u = u^T v \quad \textit{and} \quad uu^T = vu^T$$

where u^T is a g-inverse of u ,which is the definition of T-ordering

$$u <^T v \implies u \leq v$$

but the converse

$$u \leq v \implies u <^T v$$

need not be true. This is illustrated in the following example.

Example 2. *Let $u = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix}$ and $v = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle \end{bmatrix}$.*

$$u_\mu = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad u_\nu = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad v_\mu = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad v_\nu = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$u^T = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle \end{bmatrix}$$

$$u_\mu^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad u_\nu^T = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Here,

$$u_\mu u_\mu^T u_\mu = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = u_\mu$$

$$u_\nu u_\nu^T u_\nu = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = u_\nu$$

Hence,

$$u^T = \langle u_\mu^T, u_\nu^T \rangle$$

is a g-inverse of $u = \langle u_\mu, u_\nu \rangle$, u^+ exists and $u^+ = u^T$, also u is idempotent.

Since $u^2 = u$, u itself is a g-inverse of u,

$$uv = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix} = u$$

$$vu = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix} = u$$

Hence $u \leq v$.

But,

$$u_\mu^T u_\mu = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$u_\mu^T v_\mu = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad u_\mu^T u_\mu \neq u_\mu^T v_\mu$$

$$u_\nu^T u_\nu = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$u_\nu^T v_\nu = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad u_\nu^T u_\nu \neq u_\nu^T v_\nu$$

Also,

$$u_\mu u_\mu^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$v_\mu u_\mu^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad u_\mu u_\mu^T \neq v_\mu u_\mu^T$$

$$u_\nu u_\nu^T = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$v_\nu u_\nu^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow u_v u_v^T \neq v_v u_v^T$$

$$\Rightarrow u^T u \neq u^T v \text{ and } uu^T \neq vu^T$$

Hence $u \leq v$ *need not imply* $u <^T v$.

Definition 7. *The k-T ordering* $u <_k^T v$ *in* $(IFM)_n$ *is defined as:*

$$u <_k^T v \iff u^k u^T = v^k u^T \text{ and } u^T u^k = u^T v^k$$

where, $u^T \in u\{1_k\}$ *and* $u^T \in (u\{3^k\} \text{ or } u\{4^k\})$
(i.e.) $u^T \in u\{1_r^k\} \cap u\{1_l^k\}$ *and* $u^T \in \big(u\{3^k\} or u\{4^k\}\big)$.

Remark 4. *For $k = 1$ Definition 7, reduces to the definition of T-ordering for IFM: Also, from Definition 6 and Definition 7, it is to be noted that* $u^k <^T v^k \Leftrightarrow u <_k^T v$

Definition 8. *[18] For $u \in (IFM)_n^-$ and $v \in (IFM)_n$, the k-minus ordering, denoted as $u <_k^- v$, and is defined by*

$$u <_k^- v \Leftrightarrow u^k X = v^k X \text{ for some } X \in u\{1_r^k\} \text{ and } Y u^k = Y v^k \text{ for some } Y \in \{1_l^k\}.$$

Remark 5.

$$u <_k^T v \Leftrightarrow u <_k^- v \text{ with respect to } A_k^+.$$

Thus,

$$u <_k^T v \Rightarrow u <_k^- v.$$

But the converse,

$$u <_k^- v \Rightarrow u <_k^T v$$

need not be true. This is given in the following example.

Example 3.

$$u = \begin{bmatrix} <0.5, 0.1> & <0.2, 0.3> \\ <0.3, 0.2> & <0.1, 0.5> \end{bmatrix}, \quad v = \begin{bmatrix} <0.6, 0.1> & <0.2, 0.3> \\ <0.3, 0.2> & <0.5, 0.3> \end{bmatrix}$$

$$u_\mu = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}, \quad u_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix}$$

$$u_\mu^2 = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} \neq u_\mu$$

$$u_\nu^2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \neq u_\nu$$

$$P_1 = \begin{bmatrix} <1, 0> & <0, 1> \\ <0, 1> & <1, 0> \end{bmatrix}$$

$$P_2 = \begin{bmatrix} <0, 1> & <1, 0> \\ <1, 0> & <0, 1> \end{bmatrix}$$

$$u_\mu P_{1\mu} u_\mu \neq u_\mu$$

$$u_\mu P_{2\mu} u_\mu \neq u_\mu$$

$$u_\nu P_{1\nu} u_\nu \neq u_\nu$$

$$u_\nu P_{2\nu} u_\nu \neq u_\nu$$

Therefore, u is not regular.
For,

$$X = \begin{bmatrix} <0.5, 0.1> & <0.1, 0.5> \\ <0.1, 0.2> & <0.2, 0.3> \end{bmatrix}$$

$$X_\mu = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \quad X_\nu = \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix}$$

$$u_\mu^2 X_\mu u_\mu = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} = u_\mu^2$$

$$u_\nu^2 X_\nu u_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

$$\therefore u^2 X u = u^2$$

Hence, u is 2-reg and X is a 2-g-inv of u.
For,

$$v_\mu = \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}, \quad v_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

$$v_\mu^2 = \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = v_\mu$$

$$v_\nu^2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = v_\nu$$

Therefore, $v = v^2$.

$$u_\mu^2 X_\mu = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix}$$

$$v_\mu^2 X_\mu = \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix}$$

$$u_\nu^2 X_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

$$v_\nu^2 X_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

Therefore, $u^2 X = v^2 X$.

$$Y = \begin{bmatrix} \langle 0.5, 0.1 \rangle & \langle 0.2, 0.5 \rangle \\ \langle 0.1, 0.2 \rangle & \langle 0.2, 0.3 \rangle \end{bmatrix}$$

$$Y_\mu = \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, \quad Y_\nu = \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix}$$

$$u_\mu Y_\mu u_\mu^2 = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} = u_\mu^2$$

$$u_\nu Y_\nu u_\nu^2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

Therefore $uYu^2 = u^2$, Y *is a* $_{left}$ *2-g-inv. of* u.

$$Y_\mu u_\mu^2 = \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$Y_\nu u_\nu^2 = \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix}$$

$$Y_\mu v_\mu^2 = \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$Y_\nu v_\nu^2 = \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

Therefore $Yu^2 = Yv^2$. *Hence* $u <_k^- v$.

$$\text{Here } u_\mu^T = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix}, \quad u_\nu^T = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}$$

$$u_\mu^2 u_\mu^T u_\mu = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} = u_\mu^2$$

$$u_\nu u_\nu^T u_\nu^2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

$$u_\mu u_\mu^T u_\mu^2 = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} = u_\mu^2$$

$$u_\nu^2 u_\nu^T u_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

Therefore $u^2 u^T u = u^2$ *and* $uu^T u^2 = u^2$.

$$u_\mu^2 u_\mu^T = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$v_\mu^2 u_\mu^T = \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$u_\nu^2 u_\nu^T = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$v_\nu^2 u_\nu^T = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

Therefore $u^2 u^T = v^2 u^T$.

$$u_\mu^T u_\mu^2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$u_\mu^T v_\mu^2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}$$

$$u_\nu^T u_\nu^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$u_\nu^T v_\nu^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

Therefore $u^T u^2 \neq u^T v^2$.

$$u_\mu^2 u_\mu^T = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$(u_\mu^2 u_\mu^T)^T = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$u_\nu^2 u_\nu^T = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$(u_\nu^2 u_\nu^T)^T = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

Therefore $(u^2 u^T)^T = u^2 u^T$.
Therefore u^T *is a 2-Moore-Penrose inv of* u.

$$\textit{But } u^2 u^T = v^2 u^T \textit{ and } u^T u^2 \neq u^T v^2$$

Hence $u \not<_k^T v$.

Lemma 2. *Let* $< u_\mu, u_\nu > \in (IFM)_n$ *and* $v =< v_\mu, v_\nu > \in (IFM)_n$. $u <_k^T v \iff u_\mu <_k^T v_\mu$ *and* $u_\nu <_k^T v_\nu$.

Proof.

$$\begin{aligned} u <_k^T v \quad &\Leftrightarrow \quad u^k u^T = v^k u^T \text{ and } u^T u^k = u^T v^k \\ u^T u^k = u^T v^k \quad &\Rightarrow \quad \langle u_\mu^T, u_\nu^T \rangle \langle u_\mu^k, u_\nu^k \rangle \\ &= \langle u_\mu^T, u_\nu^T \rangle \langle v_\mu^k, v_\nu^k \rangle \\ &\Leftrightarrow \langle u_\mu^T u_\mu^k, u_\nu^T u_\nu^k \rangle = \langle u_\mu^T u_\mu^k, u_\nu^T v_\nu^k \rangle \\ &\iff u_\mu^T u_\mu^k = u_\mu^T v_\mu^k \text{ and } u_\nu^T u_\nu^k = u_\nu^T v_\nu^k \end{aligned}$$

Similarly,

$$u^k u^T = v^k u^T \Leftrightarrow u_\mu^k u_\mu^T = v_\mu^k u_\mu^T \text{ and } u_\nu^k u_\nu^T = v_\nu^k u_\nu^T$$

Hence, $u <_k^T v \iff u_\mu <_k^T v_\mu$ and $u_\nu <_k^T v_\nu$.

Theorem 3. *For $u \in (IFM)_n$, we have the following:*

$$u_k^+ = u^T \Leftrightarrow u^T \in u\{1_r{}^k\} \cap u\{1_l{}^k\} \quad and \quad [u^T \in u\{3^k\} \quad or \quad u^T \in u\{4^k\}].$$

Proof. Since $u_k^+ = u^T$,
the result directly follows from the Definition 3 and 4.
Conversely, let $u^T \in u\left\{1_r^k\right\} \cap u\left\{1_l^k\right\}$.
Then, by Definition 1,

$$u^k u^T u = u^k \tag{2}$$

and by Definition 2,

$$u^T u u^k = u^k \tag{3}$$

By taking the transpose on both sides in Equation (2) and (3), we have

$$u^T u (u^T)^k = (u^T)^k \tag{4}$$

and

$$(u^T)^k u u^T = (u^T)^k \tag{5}$$

$$\Rightarrow u^T \in u\left\{2_r^k\right\} \text{ and } u^T \in u\left\{2_l^k\right\}.$$

Let $u^T \in u\left\{3^k\right\}$.
We claim that u^T is a solution of equation $\{4^k\}$.
Since $u^T \in u\{3^k\}$,

$$(u^k u^T)^T = u^k u^T$$

$$\Rightarrow u(u^T)^k = u^k u^T.$$

Pre-multiplying by u^T and post-multiplying by u, we get

$$u^T u (u^T)^k u = u^T u^k u^T u.$$

By using Equations (2) and (4), we have

$$(u^T)^k u = u^T u^k$$

$$\Rightarrow (u^T u^k)^T = u^T u^k$$

$$\Rightarrow u^T \in u\{4^k\}.$$

Hence u_r^+ and u_ℓ^+ exist and they are equal.
By Remark 1 $u_k^+ = u_{rk}^+ = u_{\ell k}^+ = u^T$.
Hence the Theorem.

Theorem 4. *Let $u, v \in (IFM)_n$ and u_k^+ exists. Then the following conditions are equivalent:*

(i) $u <_k^T v$.

(ii) $u_k^+ u^k = u_k^+ v^k$ *and* $u^k u_k^+ = v^k u_k^+$.

Proof. The proof of this theorem directly follows from Definition 7 and Theorem 3

Theorem 5. *For $u \in (IFM)_n^+$ and $v \in (IFM)_n$, we have:*

$$u <_k^T v \Rightarrow u^k = uu^T v^k = v^k u^T u.$$

Proof.

$$\begin{aligned} &u <_k^T v \Leftrightarrow u^k u^T = v^k u^T \text{and} \\ &u^T u^k = u^T v^k, \\ &u^k u^T = v^k u^T \Rightarrow u^k = v^k u^T u, \\ &u^T u^k = u^T v^k \Rightarrow u^k = uu^T v^k. \end{aligned}$$

Hence, the theorem.

The converse of the above theorem is need not be true. This illustrated in the following example.

Example 4. *let,*

$$u_\mu = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}, \quad u_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix}$$

$$u_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \neq u_\mu$$

$$u_\mu P_1 \mu u_\mu \neq u_\mu$$

$$u_\mu P_2 \mu u_\mu \neq u_\mu$$

$$u_\mu P_3 \mu u_\mu \neq u_\mu$$

$$u_\mu P_4 \mu u_\mu \neq u_\mu$$

$$u_\mu P_5 \mu u_\mu \neq u_\mu$$

$$u_\mu P_6 \mu u_\mu \neq u_\mu$$

$$u_\mu^2 u_\mu^T u_\mu = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} = u_\mu^2$$

Thus, u_μ is 2-reg and u_μ^T is the 2-g-inv of u_μ.

$$u_\nu^2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \neq u_\nu$$

$$u_\nu P_1 \nu u_\nu \neq u_\nu$$

$$u_\nu P_2 \nu u_\nu \neq u_\nu$$

$$u_\nu P_3 \nu u_\nu \neq u_\nu$$

$$u_\nu P_4 \nu u_\nu \neq u_\nu$$

$$u_\nu P_5 \nu u_\nu \neq u_\nu$$

$$u_\nu P_6 \nu u_\nu \neq u_\nu$$

$$u_\nu^2 u_\nu^T u_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

Therefore, u_ν is 2-reg and u_ν^T is the 2-g-inv of u_ν. Let,

$$v_\mu = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \quad \textit{and} \quad B_\nu = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

$$v_\mu^2 = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = v_\mu$$

$$v\nu^2 = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = v_\nu$$

Therefore, v is regular.

$$u_\mu^2 u_\mu^T u_\mu = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = u_\mu^2.$$

$$u_\nu^2 u_\nu^T u_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

$$u_\mu u_\mu^T u_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = u_\mu^2$$

$$u_\nu u_\nu^T u_\nu^2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

$$(u_\mu^2 u_\mu^T)^T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$u_\mu^2 u_\mu^T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$(u_\nu^2 u_\nu^T)^T = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$u_\nu^2 u_\nu^T = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$\textit{Therefore,} \quad u^2 u^T u = u^2 \quad \textit{and} \quad u u^T u^2 = u^2.$$

$$(u^2 u^T)^T = u^2 u^T.$$

So, u^T *is a 2-Moore-Penrose inv of* u.

$$u_\mu u_\mu^T v_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = u_\mu^2$$

$$u_\nu u_\nu^T v_\nu^2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

$$v_\mu^2 u_\mu^T u_\mu = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = u_\mu^2$$

$$v_\nu^2 u_\nu^T u_\nu = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

$$u_\mu^2 u_\mu^T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$v_\mu^2 u_\mu^T = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$$

$$u_\nu^2 u_\nu^T = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$v_\nu^2 u_\nu^T = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$u_\mu^T u_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$u_\mu^T v_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$u_\nu^T u_\nu^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$u_\nu^T v_\nu^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

Therefore, $u \not<_k^T v$

Remark 6. *For $k = 1$, Theorem 5 and Theorem 4 reduces to the following.*

Theorem 6. *Let $A, B \in (IFM)_{mn}$ and A^+ exists. Then the following conditions are equivalent:*

(i) $A <^T B$

(ii) $A^+A = A^+B$ *and* $AA^+ = BA^+$

(iii) $AA^+B = A = BA^+A$

Theorem 7. *For $u, v \in (IFM)_n$, u_k^+ and v_k^+ both exist:*

$$u <_k^T v \Rightarrow \left(u^T\right)^k = (v^T)^k v v^T = v^T v (u^T)^k.$$

Proof.

$$u^k u^T = v^k u^T \quad \text{and} \quad u^T u^k = u^T v^k,$$

u_k^+ and v_k^+ exist.

Take $u_k^+ = u^T$ and $v_k^+ = v^T$.

$$\begin{aligned}
u^T u^k &= \left(u^T u^k\right)^T, \\
&= \left(u^T v^k\right)^T, \\
&= \left(u^T \left(v^k v^T v\right)\right)^T, \\
&= \left(v^T v\right)^T \left(u^T v^k\right)^T, \\
&= v^T v \left(u^T u^k\right)^T, \\
\left(u^T u^k\right)^T &= v^T v \left(u^T u^k\right)^T, \\
\left(u^T\right)^k u &= v^T v \left(u^T\right)^k u, \\
\left(u^T\right)^k u u^T &= v^T v \left(u^T\right)^k u u^T.
\end{aligned}$$

Thus,

$$\left(u^T\right)^k = v^T v \left(u^T\right)^k.$$

Similarly,

$$\left(u^T\right)^k = \left(u^T\right)^k v v^T.$$

Theorem 8. *In $(IFM)_n^+$, the set of all matrices $u \in (IFM)_n$ for which u_k^+ exists, $<_k^T$ is not a partial ordering.*

Proof. $u <_k^T u$ is obvious.
Hence $<_k^T$ is reflexive.
By Theorem 5,

$$\begin{aligned}
& u <_k^T v \Rightarrow u^k = v^k u^T u = u u^T v^k \\
\text{and} \quad & v <_k^T u \Rightarrow v^k = u^k v^T v = v v^T u^k
\end{aligned}$$

Now,

$$\begin{aligned}
u^k &= v^k u^T u \\
&= \left(v v^T u^k\right) u^T u \\
&= v v^T \left(u^k u^T u\right) \\
&= v v^T u^k \\
&= v^k
\end{aligned}$$

Hence, $<_k^T$ is anti-symmetric.

$$u <_k^T v \text{ and } v <_k^T w \Rightarrow u \not<_k^T w$$

Hence $<_k^T$ is not transitive.
Thus, $<_k^T$ is not a partial ordering. This is explained in the following example.

Example 5. *Let,*

$$u_\mu = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}, \quad u_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix}$$

$$u_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \neq u_\mu$$

$$u_\mu P_1 \mu u_\mu \neq u_\mu$$
$$u_\mu P_2 \mu u_\mu \neq u_\mu$$
$$u_\mu P_3 \mu u_\mu \neq u_\mu$$
$$u_\mu P_4 \mu u_\mu \neq u_\mu$$
$$u_\mu P_5 \mu u_\mu \neq u_\mu$$
$$u_\mu P_6 \mu u_\mu \neq u_\mu$$

$$u_\mu^2 u_\mu^T u_\mu = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} = u_\mu^2$$

Thus, u_μ is 2-reg and u_μ^T is the 2-g-inv of u_μ.

$$u_\nu^2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \neq u_\nu$$

$$u_\nu P_1 \nu u_\nu \neq u_\nu$$
$$u_\nu P_2 \nu u_\nu \neq u_\nu$$
$$u_\nu P_3 \nu u_\nu \neq u_\nu$$
$$u_\nu P_4 \nu u_\nu \neq u_\nu$$
$$u_\nu P_5 \nu u_\nu \neq u_\nu$$
$$u_\nu P_6 \nu u_\nu \neq u_\nu$$

$$u_\nu^2 u_\nu^T u_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

Therefore, u_ν is 2-reg and u_ν^T is the 2-g-inv of u_ν.
Let,

$$v_\mu = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix}, \quad v_\nu = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.5 \end{bmatrix}$$

$$v_\mu^2 = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \neq v_\mu$$

$$v_\mu P_1 \mu v_\mu \neq v_\mu$$
$$v_\mu P_2 \mu v_\mu \neq v_\mu$$
$$v_\mu P_3 \mu v_\mu \neq v_\mu$$
$$v_\mu P_4 \mu v_\mu \neq v_\mu$$
$$v_\mu P_5 \mu v_\mu \neq v_\mu$$
$$v_\mu P_6 \mu v_\mu \neq v_\mu$$

$$v_\mu^2 v_\mu^T v_\mu = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.5 \\ 0.6 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = v_\mu^2$$

Thus, v_μ is 2-reg and v_μ^T is the 2-g-inv of v_μ.

$$v_\nu^2 = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \neq v_\nu$$

$$v_\nu P_1 \nu v_\nu \neq v_\nu$$
$$v_\nu P_2 \nu v_\nu \neq v_\nu$$
$$v_\nu P_3 \nu v_\nu \neq v_\nu$$
$$v_\nu P_4 \nu v_\nu \neq v_\nu$$
$$v_\nu P_5 \nu v_\nu \neq v_\nu$$
$$v_\nu P_6 \nu v_\nu \neq v_\nu$$

$$v_\nu^2 v_\nu^T v_\nu = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = v_\nu^2$$

Therefore v_ν is 2-reg and v_ν^T is the 2-g inv of v_ν.
Therefore, v is 2-reg and v^T is the 2-g inv of v.

$$u_\mu^T u_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$u_\mu^T v_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$u_\nu^T u_\nu^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$u_\nu^T v_\nu^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$u_\mu^2 u_\mu^T = \begin{bmatrix} 0.5 & 0.5 & \\ 0.5 & 0.5 & \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$v_\mu^2 u_\mu^T = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$u_\nu^2 u_\nu^T = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$v_\nu^2 u_\nu^T = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$\textit{Therefore,} \quad u^2 u^T = v^2 u^T \quad \textit{and} \quad u^T u^2 = u^T v^2.$$

$$So, \quad u <_k^T v.$$

$$u_\mu^2 u_\mu^T u_\mu = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = u_\mu^2.$$

$$u_\nu^2 u_\nu^T u_\nu = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

$$u_\mu u_\mu^T u_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = u_\mu^2$$

$$u_\nu u_\nu^T u_\nu^2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = u_\nu^2$$

$$(u_\mu^2 u_\mu^T)^T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$u_\mu^2 u_\mu^T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$(u_\nu^2 u_\nu^T)^T = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$u\nu^2 u_\nu^T = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$Therefore, \quad u^2 u^T u = u^2 \quad and \quad u u^T u^2 = u^2.$$

$$(u^2 u^T)^T = u^2 u^T.$$

So, u^T is a 2-Moore-Penrose inv of u.

Let,

$$w_\mu = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \quad and \quad w_\nu = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}$$

$$w_\mu^2 = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = w_\mu$$

$$w_\nu^2 = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = w_\nu$$

Therefore, w is regular.

$$w_\mu^2 v_\mu^T = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.5 \\ 0.6 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$v_\nu^2 v_\nu^T = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$w_\nu^2 v_\nu^T = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$v_\mu^T v_\mu^2 = \begin{bmatrix} 0.7 & 0.5 \\ 0.6 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.6 & 0.6 \end{bmatrix}$$

$$v_\mu^T w_\mu^2 = \begin{bmatrix} 0.7 & 0.5 \\ 0.6 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.6 & 0.6 \end{bmatrix}$$

$$v_\nu^T v_\nu^2 = \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$v_\nu^T w_\nu^2 = \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

Therefore, $v <_k^T w$.

$$v_\mu^2 v_\mu^T v_\mu = \begin{bmatrix} 0.7 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = v_\mu^2$$

$$v_\nu^2 v_\nu^T v_\nu = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = v_\nu^2$$

$$v_\mu v_\mu^T v_\mu^2 = \begin{bmatrix} 0.7 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = v_\mu^2$$

$$v_\nu v_\nu^T v_\nu^2 = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = v_\nu^2$$

$$(v_\mu^2 v_\mu^T)^T = \begin{bmatrix} 0.7 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = v_\mu^2 v_\mu^T$$

$$(v_\nu^2 v_\nu^T)^T = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0.2 \end{bmatrix} = v_\nu^2 v_\nu^T$$

Therefore v^T is a 2-Moore Penrose inv of v.

$$u_\mu^2 u_\mu^T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$w_\mu^2 u_\mu^T = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$$

$$u_\nu^2 u_\nu^T = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$w_\nu^2 u_\nu^T = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

$$u_\mu^T u_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$u_\mu^T w_\mu^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$u_\nu^T u_\nu^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

$$u_\nu^T w_\nu^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

Therefore, $u \nless_k^T w$
So, T-ordering is not a partial ordering.

Theorem 9. *Let $u \in (IFM)_n^+$ and $v \in (IFM)_n$. Then the following are equivalent:*
(i) $u <_k^T v \iff u^T <_k^T v^T$
(ii) $u <_k^T v \iff QuQ^T <_k^T QvQ^T$ for some permutation matrix Q

Proof. (i)

$$u <_k^T v \Leftarrow u^k u^T = v^k u^T \text{ and } u^T u_k = u^T v_k$$

Since u_k^+ exists, $u_k^+ = u^T$. By Theorem 1,

$$u^T \in u\{1_r^k\} \iff u \in u^T\{1_l^k\}$$

$$u^T \in u\{1_l^k\} \iff u \in u^T\{1_r^k\}$$

$$u^T \in u\{3^k\} \iff (u^k u^T)^T = u^k u^T$$

$$\iff u(u^T)^k = u^k u^T$$

$$\iff (u(u^T)^k)^T = (u^k u^T)^T = u(u^T)^k$$

$$\iff u \in u^T\{4^k\}$$

Similarly,

$$u^T \in u\{4^k\} \iff u \in u^T\{3^k\}$$
$$u^k u^T = v^k u^T \iff (u^k u^T)^T = (v^k u^T)^T$$

$$\iff u(u^T)^k = u(v^T)^k$$

$$\text{and} \quad u^T u^k = u^T v^k \iff (u^T u^K)^T = (u^T v^k)^T$$

$$\iff (u^T)^k u = (v^T)^k u$$

$$\text{Here,} \quad u <_k^T v \iff u^T <_k^T v^T$$

(ii) Claim:
If $u_k^+ = u^T$, then $(QuQ^T)_k^+ = (QuQ^T)^T = Qu^TQ^T$

$$(QuQ^T)^k(Qu^TQ^T)(QuQ^T)$$

$$= (Qu^kQ^T)(Qu^TQ^T)(QuQ^T)$$

$$= Qu^k(Q^TQ)u^T(Q^TQ)uQ^T$$

$$= Qu^ku^TuQ^T$$

$$= Qu^kQ^T$$

$$= (QuQ^T)^k$$

Similarly,

$$(QuQ^T)(Qu^TQ^T)(QuQ^T)^k = (QuQ^T)^k$$

$$((QuQ^T)^k(Qu^TQ^T))^T = (Qu^TQ^T)^T((QuQ^T)^T)^k$$

$$= (QuQ^T)(Qu^TQ^T)^k$$

$$= (QuQ^T)\left(Q(u^T)^kQ^T\right)$$

$$= Qu(Q^TQ)(u^T)^kQ^T$$

$$= Qu(u^T)^kQ^T$$

$$= Q(u^ku^T)^TQ^T$$

$$= Qu^ku^TQ^T$$

$$(QuQ^T)^k(QuQ^T) = (Qu^kQ^T)Qu^TQ^T$$

$$= Qu^k(Q^TQ)u^TQ^T$$

$$= Qu^ku^TQ^T$$

Therefore,

$$((QuQ^T)^k(Qu^TQ^T))^T = (QuQ^T)^k(Qu^TQ^T)$$

Similarly,

$$((Qu^TQ^T)(QuQ^T)^k)^T = (Qu^TQ^T)(QuQ^T)^k$$

Now,

$$(Qu^TQ^T)(QuQ^T)^k = (Qu^TQ^T)(Qu^kQ^T)$$

$$= Qu^T(Q^TQ)u^kQ^T$$

$$= Q(u^Tu^k)Q^T$$

$$= Q(u^Tv^k)Q^T$$

$$= (Qu^TQ^T)(Qv^kQ^T)$$

$$= (Qu^TQ^T)(QvQ^T)^k$$

Similarly,

$$(QuQ^T)^k(Qu^TQ^T) = (QvQ^T)^k(Qu^TQ^T)$$

Hence,

$$u <_k^T u \;\Rightarrow\; QuQ^T <_k^T QvQ^T$$

Conversely,

$$QuQ^T <_k^T QvQ^T \;\Rightarrow\; Q^T(QuQ^T)Q <_k^T Q^T(QvQ^T)Q \;\Rightarrow\; u <_k^T v$$

Theorem 10. *Let $u, v \in (IFM)_n^+$.*

$$v \in u^T\{1^k, 3^k, 4^k\} \Leftrightarrow v^T \in u\{1^k, 3^k, 4^k\}$$

Proof. By Theorem 1,

$$v \in u^T\{1^k\} \Leftrightarrow v^T \in u\{1^k\}$$

$$v \in u^T\{3^k\} \Leftrightarrow ((u^T)^k v)^T = (u^T)^k v$$

$$\Leftrightarrow v^T u^k = (u^T)^k v$$

$$\Leftrightarrow (v^T u^k)^T = ((u^T)^k v)^T = v^T u^k$$

$$\Leftrightarrow v^T \in u\{4^k\}$$

Similarly,

$$v \in u^T\{4^k\} \Leftrightarrow v^T \in u\{3^k\}$$

Hence the proof.

Theorem 11. *Let $u \in (IFM)_n^+$ and $v \in (IFM)_n$, $u <_k^T v \Rightarrow u \in v^T\{3^k, 4^k\}$*

Proof.

$$u <_k^T v \Leftrightarrow\; u^k u^T = v^k u^T \quad \text{and} \quad u^T u^k = u^T v^k$$

$$\begin{aligned}(u^T v^k)^T &= (u^T u^k)^T \\ &= u^T u^k \\ &= u^T v^k \\ &\Rightarrow u^T \in v\{4^k\} \\ (v^k u^T)^T &= (u^k u^T)^T \\ &= u^k u^T \\ &= v^k u^T \\ &\Rightarrow u^T \in v\{3^k\}\end{aligned}$$

By Theorem 10, $u^T \in v\{3^k, 4^k\} \Leftrightarrow u \in v^T\{3^k, 4^k\}$.
Hence the proof.

Theorem 12. *If $u <_k^T v$, then we have the following:*

(i) If $\left(v^k\right)^2 = 0$, then $\left(u^k\right)^2 = 0$.

(ii) If $v^k = \left(v^k\right)^2$, then $u^k = \left(u^k\right)^2$.

Proof. By Theorem 5,

$$u <_k^T v \ \Rightarrow \ u^k = uu^T v^k = v^k u^T u$$

(i)

$$\begin{aligned}\left(u^k\right)^2 &= u^k u^k \\ &= (uu^T v^k)(v^k u^T u) \\ &= uu^T \left(v^k\right)^2 u^T u \\ &= 0\end{aligned}$$

(ii)

$$\begin{aligned}\left(u^k\right)^2 &= u^k u^k \\ &= (uu^T v^k)(v^k u^T u) \\ &= uu^T \left(v^k\right)^2 u^T u \\ &= uu^T v^k u^T u \\ &= u^k u^T u = u^k\end{aligned}$$

4. Conclusion

The concept of intuitionistic fuzzy matrix was defined as a generalization of fuzzy matrix utilizing the notion of intuitionistic fuzzy sets. We proved that T-ordering is identical for certain class of intuitionistic fuzzy matrices. And also, we learned about the $k - T$ ordering on intuitionistic fuzzy matrices as a generalization of the T-ordering on intuitionistic fuzzy matrices. In many applications, the parameters of the system should be represented by intuitionistic fuzzy rather than crisp or fuzzy numbers. Hence, it is important to develop the mathematical procedures that would appropriately treat intuitionistic fuzzy linear systems to solve them. Further, we can introduce the concept of k-regularity for fuzzy and intuitionistic fuzzy soft matrices and neutrosophic matrices.

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