EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6243
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Some Fixed Point Results for Hybrid Contraction in Metric Spaces and Ulam-Hyers Stability

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Abstract. In the present manuscript, we introduce a new notion of $(\beta, \phi)-$ admissible hybrid contractions in metric spaces and establish fixed point results in the setting of these spaces. The derived results extend the reported findings of the past. The derived result is supplemented with a non-trivial example. We have also analyzed the Ulam-Hyers stability and well-poisedness as an application to the derived results.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: $(\beta, \phi)-$admissible hybrid contraction, fixed point, Ulam - Hyers stability, metric space

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1. Introduction

Fixed point theory simply deals with the solution of the equation $Tx = x$ where T is a self-map on a non-empty set X. The fixed point problem first appeared in the solution of an initial value problem. Liouville [1] in 1837 and Picard [2] in 1890 solved the problem using the successive approximation method that also provided the solution of the fixed point equation. Before 1922, there was no any direct method to evaluate the fixed point of

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6243

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a map. In 1922, Stephen Banach [3] was the first to introduce the contraction principle to evaluate the fixed point in the setting of complete metric spaces. Metric fixed point theory has been investigated since then, by many researchers, as it is the natural and strong connection of the theoretical results in non-linear functional analysis with applied sciences. Later a lot of generalization of Banach contraction principle was done by extending and generalizing the topological spaces as well as the contraction conditions.

In the recent past, interpolative contractive conditions were reported by some researchers and fixed point results please see [4–12]. The notion of Ulam stability was proposed by by Ulam [13] and developed by Hyers [14], Ulam [15], Rassias [16], etc. In 2023, Manoj et al [17] reported Ulam-Hyer's stability and well-posedness of fixed point problems in the setting of c^* - Algebra valued bipolar metric spaces.

Inspired, in this paper, we introduce the notion of "(β, ϕ) - admissible hybrid contraction" that combines and unifies several existing linear and nonlinear contractions and also extends fixed point results of such contraction conditions. We also analyze the Ulam-Hyer's stability and well-posedness of fixed point problems by applying the derived results.

Accordingly, the rest of the paper is organised as follows: In section-2, we review some preliminaries and monograph which are required in the sequel. In Section-3, we present our main results and establish fixed point results using the " (β, ϕ) - admissible hybrid contraction" and supplement the results with non-trivial example. In Section-4, we present an application to analyse Ulam-Hyer's stability and welll posedness of fixed point problems.

2. Preliminaries

The following are required in the sequel.

Definition 1. *[7, 18] Let Φ be the set of functions $\phi : [0, +\infty) \to [0, +\infty)$ such that*

(i) ϕ is non-decreasing;

(ii) there exists $n_0 \in N$ and $\delta \in (0, 1)$ and a convergent series $\sum_{i=0}^{+\infty} v_i$ with $v_i \geq 0$ such that

$$\phi^{i+1}(t) \quad \leq \quad \delta\phi(t) + v_i, \tag{1}$$

for $i \geq i_o$ and $t \geq 0$. Each $\phi \in \Phi$ is called a $(c)-$comparison function.

Lemma 1. *[18] If $\phi \in \Phi$, then* *(i)* $(\phi^n(t))_{n\in N}$ *converges to* 0 *as* $n \to +\infty$ *for* $t \geq 0$*;*

(ii) $\phi(t) < t$, *for any* $t \in \mathbb{R}^+$*;*

(iii) ϕ *is continuous at* 0*;*

(iv) the series $\sum_{k=0}^{+\infty} \phi^k(t)$ *is convergent for* $t \geq 0$.

Lemma 2. *[19] Let* $\alpha : X \times X \to [0, +\infty)$ *be a function. We say that a mapping* $T : X \to X$ *is* $\alpha-$*orbital admissible if*

$$\alpha(x, Tx) \geq 1 \quad \textit{implies} \quad \alpha(Tx, T^2x) \geq 1, \quad \textit{for all} \quad x \in X. \tag{2}$$

An $\alpha-$*orbital admissible mapping* f *is called triangular* $\alpha-$*orbital admissible if* $\alpha(x, y) \geq 1$ *and*

$$\alpha(y, Ty) \geq 1 \quad \textit{implies} \quad \alpha(x, y) \geq 1, \tag{3}$$

for every $x, y \in X$.

Lemma 3. *[19] Suppose that for a triangular* $\alpha-$*orbital admissible mapping* $f : X \to X$ *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$. *Then*

$$\alpha(x_n, x_m) \quad \geq \quad 1, \tag{4}$$

for all $n, m \in N$, *where the sequence* $\{x_n\}$ *is defined by* $x_{n+1} = Tx_n, n \in N$.

Definition 2. *Let* $\alpha : X \times X \to [0, +\infty)$ *be a mapping. The set* X *is called regular with respect to* α *if for a sequence* $\{x_n\}$ *in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$, *for all* n *and* $x_n \to x \in X$ *as* $n \to +\infty$ *we have* $\alpha(x_n, x) \geq 1$ *for all* n.

3. Main Results

In this section, we shall introduce a new notion of (β, ϕ) admissible hybrid contraction and prove some fixed point results for such types of contraction in metric spaces. In addition to this, an example is also provided for the validity of our result.

Definition 3. *Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is said to be an* (β, ϕ) *admissible hybrid contraction, if there exists* $\phi \in \Phi$ *and* $\beta : X \times X \to [0, +\infty)$ *such that*

$$\beta(x, y)d(Tx, Ty) \leq \phi(J_S^T(x, y)), \tag{5}$$

for all distinct $x, y \in X$, where $s \geq 0$ and $\alpha_i \geq 0$ for $i = 1, 2$ such that $\alpha_1 + \alpha_2 = 1$ and

$$J_S^T(x,y) = \begin{cases} \alpha_1(\frac{(d(x,Tx)d(y,Ty))}{d(x,y)})^s + \alpha_2(d(x,y))^s]^{\frac{1}{s}} & \text{if } s > 0 \\ (d(x,Tx))^{\alpha_1}(d(y,Ty))^{\alpha_2} & \text{if } s = 0 \end{cases} \tag{6}$$

Here $Fix_T(X) := \{x \in X : Tx = x\}$.

Remark 1. *The concept of "admissible hybrid contraction" is inspired from the notion of "interpolative contractions", see e.g. [1-3, 7-9] The main results of this manuscript is the following theorem:*

Theorem 1. *Let (X, d) be complete metric space and let $T : X \to X$ be $(\beta, \phi)-$ admissible hybrid contraction satisfying the followings;*

(i) T is triangular β- orbital admissible;

(ii) there exists $x_0 \in X$ s.t. $\beta(x_0, Tx_0) \geq 1$;

(iii) Either T is continuous, or

(iv) T^2 is continuous and $\beta(Tx, x) \geq 1$ for any $x \in F_T(X) = \{x \in X : Tx = x\}$.

Then T has a unique fixed point.

Proof. We recursively construct up the sequence $\{x_n\}$, starting from any random point x_0 in X, such that $x_n = T^n x_0$ for every $n \in N$. Assuming that there is some $m \in N$ such that $Tx_m = x_{m+1} = x_m$, we conclude the proof by finding that x_m is a fixed point of T. Thus, for all $n \in N$, we can assume going forward that $x_n \neq x_{n-1}$. Assuming (i) that T is an admissible hybrid contraction, we obtain by replacing x by x_{n-1}. and y by x_n in Equation (5)

$$\beta(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n) \leq \phi(J_S^T(x_{n-1}, x_n)). \tag{7}$$

Considering that T is triangular $\beta-$ orbital admissible, along with (4) holding, the above inequality becomes

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n). \\ &< \phi(J_S^T(x_{n-1}, x_n)). \end{aligned} \tag{8}$$

Case 1:
For the case $s > 0$ we have

$$J_S^T(x_{n-1}, x_n) = [\alpha_1(\frac{(d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n))})^s + \alpha_2(d(x_{n-1}, x_n))^s]^{\frac{1}{s}}$$

$$
\begin{aligned}
&= [\alpha_1(\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n))}^s + \alpha_2(d(x_{n-1},x_n))^s]^{\frac{1}{s}} \\
&= [\alpha_1(d(x_n,x_{n+1}))^s + \alpha_2(d(x_{n-1},x_n))^s]^{\frac{1}{s}}.
\end{aligned}
$$

And from (8) we get

$$
\begin{aligned}
d(x_n,x_{n+1}) &\leq \beta(x_{n-1},x_n)d(Tx_{n-1},Tx_n) \\
&< \phi(J_S^T(x_{n-1},x_n)) \qquad (9)\\
&= \phi[\alpha_1(d(x_n,x_{n+1}))^s + \alpha_2(d(x_{n-1},x_n))^s]^{\frac{1}{s}}.
\end{aligned}
$$

Since β is a non-decreasing function, let us assume that

$$
\begin{aligned}
d(x_{n-1},x_n) &\leq d(x_n,x_{n+1}), \\
d(x_n,x_{n-1}) &\leq \beta(x_{n-1},x_n)d(Tx_{n-1},Tx_n) \\
&\leq \phi[\alpha_1(d(x_n,x_{n+1}))^s + \alpha_2(d(x_{n-1},x_n))^s]^{\frac{}{s}} \\
&\leq \phi[(\alpha_1+\alpha_2)(d(x_n,x_{n+1}))^s]^{\frac{1}{s}} \qquad (10)\\
&= \phi[[(\alpha_1+\alpha_2)]^{\frac{1}{s}}d(x_n,x_{n+1})] \\
&< (\alpha_1+\alpha_2)]^{\frac{1}{s}}d(x_n,x_{n+1}) \\
&\leq d(x_n,x_{n+1}),
\end{aligned}
$$

which contradicts itself. Consequently, for any $n \in N$,we have

$$
d(x_n,x_{n+1}) \leq d(x_{n-1},x_n),
$$

and the inequality (8) yields

$$
\begin{aligned}
d(x_n,x_{n+1}) &\leq \phi[\alpha_1(d(x_n,x_{n+1}))^s + \alpha_2(d(x_{n-1},x_n))^s]^{\frac{1}{s}} \\
&\leq \phi[(\alpha_1+\alpha_2)(d(x_{n-1},x_n))^s]^{\frac{1}{s}} \\
&\leq \phi(\alpha_1+\alpha_2)]^{\frac{1}{s}}d(x_{n-1},x_n) \qquad (11)\\
&\leq \phi(d(x_{n-1},x_n)). \\
&\leq \phi^2(d(x_{n-2},x_{n-1})) \\
&... \\
&< \phi^n(d(x_0,x_1)).
\end{aligned}
$$

Assume that $p > m$ for any $m, p \in N$. Given that $d(x_m,x_{m+1}) < \phi(d(x_0,x_1))$ for each x, the triangle inequality. Given $m \in N$, we have

$$
d(x_m,x_p) \leq d(x_m,x_{m+1}) + d(x_{m+1},x_{m+2}) + ... + d(x_{p-1},x_p)
$$

$$= \sum_{j=m}^{p-1} d(x_j, x_{j+1}) \leq \sum_{j=m}^{p-1} \phi^j(d(x_0, x_1)).$$

Given that ϕ functions as a $c-$comparison, the series is $\sum_{j=0}^{+}\infty\phi^j(d(x_0, x_1))$ convergent, $s_n = \sum_{j=0}^{n}\phi^j(d(x_0, x_1))$ transforms the inequality above into:

$$d(x_m, x_p) \leq \delta_{p-1} - \delta_{m-1},$$

and as $m, p \to +\infty$ we get

$$d(x_m, x_p) \to 0. \tag{12}$$

This indicates that there exists z such that $\{x_n\}$ is a Cauchy sequence on a complete metric space

$$\lim_{n\to+\infty} d(x_m, z) = 0. \tag{13}$$

We'll demonstrate that z is a fixed point of T at this point. Given assumption (3), if T is continuous, then

$$\lim_{n\to+\infty} d(x_{n+1}, Tz) = \lim_{n\to+\infty} d(x_n, Tx_n) = 0.$$

so we get that $Tz = z$, that is, z is a fixed point of T. In the alternative hypothesis, that T^2 is continuous we have $T^2z = lim_{n\to+\infty}T^2x_n = z$ and we want to show that $Tz = z$. Assuming that, on the contrary, $Tz \neq z$, we have from (5)

$$\begin{aligned}
d(z, Tz) &= d(T^2z, Tz) \leq \beta(Tz, z)d(Tz, z) \\
&\leq \phi(J_S^T(Tz, z)) < J_S^T(Tz, z) \\
&= [\alpha_1(\frac{d(Tz, T^2z)d(z, Tz)}{d(Tz, z)})^s + \alpha_2(d(Tz, z)^s]^{\frac{1}{s}}. \\
&= [\alpha_1(\frac{d(Tz, z)d(z, Tz)}{d(Tz, z)})^s + \alpha_2(d(Tz, z)^s]^{\frac{1}{s}} \\
&= [\alpha_1(d(z, Tz))^s + \alpha_2(d(Tz, z)^s]^{\frac{1}{s}} \\
&= [(\alpha_1 + \alpha_2)(d(Tz, z))^s]^{\frac{1}{s}} \\
&= (\alpha_1 + \alpha_2)^{\frac{1}{s}}(d(Tz, z)) \\
&\leq d(Tz, z).
\end{aligned}$$

This is a contradiction, so that $Tz = z$.
Case 2:
For the case $s = 0$ taking $x = x_{n-1}$ and $y = x_n$ we have

$$\begin{aligned} J_S^T(x_{n-1}, x_n) &= [(d(x_{n-1}, Tx_{n-1}))^{\alpha_1} + (d(x_n, Tx_n))^{\alpha_2}] \\ &= (d(x_{n-1}, x_n))^{\alpha_1} + (d(x_n, x_{n+1}))^{\alpha_2} \end{aligned}$$

and from (5)

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n) \\ &\leq \phi(J_S^T(x_{n-1}, x_n)). \end{aligned} \tag{14}$$

For the same reason as the previous case, $d(x_{n-1}, x_n) > d(x_n, x_{n+1})$ because the other case contradicts itself. Furthermore, if we assume absurdum that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, we obtain.

$$\begin{aligned} d(x_n, x_{n+1}) &< \phi(J_S^T(x_{n-1}, x_n)) \\ &< (d(x_n, x_{n+1}))^{\alpha_1+\alpha_2} \\ &= d(x_n, x_{n+1}). \end{aligned}$$

This is a contradiction. Then from (14) we obtain the following:

$$d(x_n, x_{n+1}) \leq \phi(J_S^T(x_{n-1}, x_n)) < \phi(x_{n-1}, x_n) \tag{15}$$

and inductively we get

$$d(x_n, x_{n+1}) \leq \phi^n(d(x_n, x_{n+1})).$$

We can readily determine that $\{x_n\}$ is a Cauchy sequence in a complete metric space by applying the same arguments as in the case $s > 0$. Consequently, there exists z such that $lim_{n\to+\infty} x_n = z$.
We claim that z is a fixed point of T under the assumption that T is continuous we have

$$\lim_{n\to+\infty} d(x_{n+1}, Tz) = \lim_{n\to+\infty} d(Tx_n, Tz) = 0,$$

And also together with the uniqueness of limit, $Tz = z$. Also, if T^2 is continuous as in case (1) we have that $Tz = z$ then

$$\begin{aligned} d(z, Tz) &= d(T^2z, Tz) \\ &\leq \beta(Tz, z)d(T^2z, Tz) \\ &\leq \phi(J_S^T(T^2z, Tz)) \end{aligned}$$

$$\begin{aligned} &\leq \phi(d(x_n, x_{n+1}))^{\alpha_1+\alpha_2} < d(z, Tz) \\ &< d(z, Tz). \end{aligned}$$

This contradiction shows that $z = Tz$.

Example 1. *Let $X = [0, 2]$, $d : X \times X \to [0, +\infty)$ be the usual metric, $d(x, y) = |x - y|$ for all $x, y \in X$ and the mapping $T : X \to X$ be define by*

$$T(x) = \begin{cases} \frac{2}{3}, & \text{if } x \in [0, 1] \\ \frac{x}{2}, & \text{if } x \in (1, 2]. \end{cases}$$

Consider also a function

$$\beta(x, y) = \begin{cases} 2, & \text{if } x, y \in [0, 1] \\ 1, & \text{if } x = 0,\ y = 2 \end{cases}$$

and the comparison function $\phi : [0, +\infty) \to [0, +\infty), \phi(t) = t/5$. The assumptions (1) *and* (2) *are readily shown to be true, and as $T^2(x) = 2/3$ is continuous, the assumption* (4) *is likewise confirmed. Since $d(Tx, Ty) = 0$ holds for every $x, y \in [0, 1]$, the inequality* (5) *is true. Assuming $y = 2$ and $x = 0$, we get*

$$\begin{aligned} \beta(0, 2)d(T0, T2) &= \beta(0, 2)d(\frac{2}{3}, 1) = \frac{1}{3} < \frac{1}{5}\sqrt{(\frac{1}{9} + 4)} \\ &= ((\frac{d(x_n, Tx_n)d(z, Tz)}{d(Tx_n, z)})^s)^{\frac{1}{2}} \\ &= (\alpha_1(\frac{d(x_n, x_{n+1})d(z, Tz)}{d(x_n, z)})^s + \alpha_2(d(x_n, z)^s)^{\frac{1}{2}}. \end{aligned}$$

In all other cases, $\beta(x, y) = 0$ and (5) *is obviously satisfied. Because T is an admissible hybrid contraction and satisfies assumptions* (1), (2), *and* (4) *of Theorem* 1, *we may determine that $x = 0$ is the fixed point of T by letting $\beta_1 = \beta_2 = 1$, and $s = 2$.*

Theorem 2. *Let (X, d) be complete metric space and let $T : X \to X$ be $(\beta, \phi)-$ admissible hybrid contraction satisfying the followings;*

(i) T is triangular $\beta-$ orbital admissible;

(ii) there exists $x_0 \in X$ s.t. $\beta(x_0, Tx_0) \leq 1$;

(iii) (X, d) *is regular with respect* β.

Then T *possesses a fixed point.*

Proof. As we can see from the lines in the proof of Theorem 1, the sequence $\{x_n\}$ is Cauchy for any $s > 0$, and there exists a point z such that $lim_{n\to+\infty}d(x_n, z) = 0$ because the metric space (X, d) is complete. Given that the space x is regular with regard to β, inequality (5) together with the triangular inequality gives

$$\begin{aligned} d(z, Tz) &\leq (d(z, x_{n+1})) + (d(x_{n+1}, Tz)) \\ &\leq \beta(x_n, z) + d(Tx_n, Tz) \\ &\leq \phi(J_S^T(x_n, z)) \leq J_S^T(x_n, z). \end{aligned} \tag{16}$$

Again, we have to consider two separate cases. For the case $s > 0$,

$$\begin{aligned} J_S^T(x_n, z) &= [\alpha_1(\frac{d(x_n, Tx_n)d(z, Tz)}{d(Tx_n, z)})^s + \alpha_2(d(z, Tz)^s]^{\frac{1}{s}} \\ &= [\alpha_1(\frac{d(x_n, x_{n+1})d(z, Tz)}{d(x_n, z)})^s + \alpha_2(d(x_n, z)^s]^{\frac{1}{s}}. \end{aligned}$$

Since $lim_{n\to+\infty}J_S^T(x_n, z) = (\alpha_1(d(z, Tz)))$, consider $n \to +\infty$ in (16) we obtain $d(z, Tz) \leq d(z, Tz)$. Which implies that $Tz = z$ similarly, for the case $s = 0$, we get $lim_{n\to+\infty}J_S^T(x_n, z) = 0$ then $d(z, Tz) = 0$.

Theorem 3. *If in Theorem* 1 *and* 2*, in the case* $s > 0$*, we assume supplementary that* $\beta(x, y) \geq 1$
For any $x, y \in F_T(x)$ *then the fixed point of* T *is unique.*

Proof. Let $v \in X$ be a different fixed point of T from z. Taking into consideration the extra hypotheses and substituting in (5), we have

$$\begin{aligned} d(z, v) &\leq \beta(z, v)(Tz, Tv) \leq \beta(J_S^T(z, v)) < J_S^T(z, v) \\ &= [\alpha_1(\frac{d(z, Tz)d(v, Tv)}{d(z, v)})^s + \alpha_2(d(z, v))^s]^{\frac{1}{s}} \\ &= [\alpha_1(\frac{d(z, z)d(v, v)}{d(z, v)})^s + \alpha_2(d(z, v)^s]^{\frac{1}{s}} \\ &= \alpha_2^{\frac{1}{s}}d(z, v) \leq d(z, v), \end{aligned}$$

which is a contradiction. This implies that T has exactly one fixed point.

Example 2. *Let* $X = \{a, b, c, e\}$ *and* $d : X \times X \to [0, +\infty)$ *such that* $d(y, x) = d(y, x), d(x, x) = 0$ *for any* $x, y \in X$ *and*

$$d(x,y) = \begin{cases} 1, & \text{if } (x,y) \in (a,b),(b,c),(c,e) \\ 2, & \text{if } (x,y) \in (a,c),(b,e) \\ 3, & \text{if } (x,y) \in (a,e) \end{cases}$$

Let's define the self-mapping T *on metric space* (X, d) *as follows:* $T(a) = T(b) = a$, $T(c) = e$, $T(e) = b$. *The function* $\beta : X \times X \to [0, +\infty)$ *is also considered, along with the comparison function* $\phi : [0, +\infty) \to [0, +\infty)$, $\phi(t) = \frac{1}{\sqrt{2}}$ *where* $\beta(x, a) = \beta(a, x) = 3$ *for every* $x \in X$, $\beta(b, e) = 1$, *and* $\beta(x, y) = 0$ *in all other cases. The application of Theorem* 1 *is not possible since neither* T *nor* T^2 *are continuous. However, the triangular* $\beta-$ *orbital admissibility of* T *is readily apparent, and the assumptions* (2) *and* (3) *from Theorem* 2 *are likewise met. Considering* $s = 0$, $\alpha(1) = \alpha(2) = 1$ *and taking into account the definition of function* β, *we remark that the only interesting case is for* $x = b$ *and* $y = e$. *We have in this case:*

$$\begin{aligned} \beta(b,e)d(b,Te) &= d(a,b) = 1 < \sqrt{2} = \frac{1}{\sqrt{2}}(2^1 \cdot 1^1) \\ &= \frac{1}{\sqrt{2}}(d(b,Tb))^{\alpha_1}(d(e,Te))^{\alpha_2} \\ &= \phi((d(b,Tb))^{\alpha_1}(d(e,Te))^{\alpha_2}. \end{aligned}$$

3.1. Application

3.1.1. Ulam type stability

In this section we investigate the general Ulam type stability in sense of a fixed point problem. Suppose that $T : X \to X$ is a self - mapping on a metric space (X, d). The fixed point problem

$$X = TX \tag{17}$$

has the general Ulam type stability if and only if there exists an increasing function $\tau : [0, +\infty) \leftrightarrow (0, +\infty)$, continuous at 0 with $\tau(0) = 0$ such that for every $\epsilon > 0$ and for each $y^* \in X$ which satisfies the inequality

$$d(y^*, fy^*) \leq \epsilon, \tag{18}$$

there exists a solution $z \in X$ of (17) such that

$$d(z, y^*) \leq \tau(\epsilon). \tag{19}$$

In case that for $C > 0$, we consider $\tau(t) = Ct$ for all $t \geq 0$ then the fixed point equation (17) is said to be Ulam type stable. On a metric space (X, d), the fixed point problem (17), where $T : X \to X$, is said to be well-posed if the following assumptions are satisfy:

(i) T has a unique fixed point z in X ;

(ii) $d(x_n, z) = 0$ for each sequence $\{x_n\} \in X$ such that $lim_{n\to+\infty}(d(x_n, Tx_n)) = 0$.

Theorem 4. *Let (X, d) be a complete metric space. If we add the condition $\alpha_2 < \frac{1}{c(s)}$, where $c(s) = max\{1, \sqrt{2^{s-1}}\}$ to the assumptions of Theorem 3, then the following affirmations hold:*

(i) the fixed point equation (17) is Ulam - Hyers stable if $\beta(u, v) \geq 1$ for any u, v satisfying the inequality (18);

(ii) the fixed point equation (17) is well- posed if $\beta(x_n, z) \geq 1$ for any sequence $\{x_n\} \in X$ such that $lim_{n\to+\infty}d(x_n, Tx_n) = 0$ and $Fix_T(x) = z$.

Proof. Case 1:
Since from Theorem 3 we know that there is unique $z \in X$ such that $Tz = z$, let $y^* \in X$ such that

$$d(y^*, Ty^*) \leq \epsilon \quad \text{for all} \quad \epsilon > 0.$$

Obvious, z verifies (18) so we have that $\beta(y^*, z) \geq 1$ and then by using the triangular inequality we get

$$\begin{aligned}
d(z, y^*) &\leq d(Tz, Ty^*) + d(Ty^*, y^*) \leq \beta(y^*, z)d(Ty^*, Tz) + d(Ty^*, y^*) \\
&\leq \phi(J_S^T(y^*, z)) + d(Ty^*, y^*) \\
&< \phi(J_S^T(y^*, z)) + d(Ty^*, y^*) \\
&\leq [\alpha_1(\frac{(d(y^*, Ty^*)d(z, Tz)}{d(z, y^*)})^s + \alpha_2(d(z, y^*))^s]^{\frac{1}{s}} + d(Ty^*, y^*) \\
&= [\alpha_2(d(z, y^*))^s]^{\frac{1}{s}} + d(Ty^*, y^*) \\
&\leq [\alpha_2(d(z, y^*))^s]^{\frac{1}{s}} + \epsilon.
\end{aligned}$$

Therefore,

$$(d(z, y^*))^s \leq c(s)[\alpha_2(d(z, y^*))^s + \epsilon^s],$$

where $c(s) = \max\{1, \sqrt{2^{s-1}}\}$ By simple calculation, from the above inequality we have

$$d(z, y^*)^s \leq \frac{c(s)}{(1 - c(s)\alpha_2)}\epsilon^s,$$

which is equivalent to

$$d(z, y^*) \quad \leq \quad C\epsilon.$$

Where $C = (\frac{c(s)}{(1-c(s)\alpha_2})^{\frac{1}{s}}$ for any $s > 0$ and $\alpha_2 \in [0,1)$ such that $\alpha_2 < \frac{1}{c(s)}$.
Case 2:
Taking into account the supplementary condition and since $Fix_T(x) = z$ we have

$$\begin{aligned}
d(x_n, z) &\leq d(x_n, Tx_n) + d(Tx_n, Tz) \leq d(x_n, Tx_n) + \beta(x_n, z)d(Tx_n, Tz) \\
&\leq d(x_n, Tx_n) + \phi(J_S^T(x_n, z)) < d(x_n, Tx_n) + J_S^T(x_n, z) \\
&\leq [\alpha_1(\frac{d(x_n, Tx_n)d(z, Tz))}{d(x_n, z)})^s + \alpha_2(d(x_n, z))^s]^{\frac{1}{s}} + d(x_n, Tx_n) \\
&= [\alpha_2(d(x_n, z))^s]^{\frac{1}{s}} + d(x_n, Tx_n) \\
(d(x_n, z))^s &\leq \alpha_2(d(x_n, z))^s + (d(x_n, Tx_n))^s \\
(d(x_n, z))^s &\leq \frac{c(s)}{(1-c(s))\alpha_2}(d(x_n, Tx_n))^s.
\end{aligned}$$

Letting $n \to +\infty$ in the above inequality and keeping in mind that $\lim_{n\to+\infty} d(x_n, Tx_n) = 0$, we obtain

$$\lim_{n\to+\infty} d(x_n, z) = 0.$$

That is, the fixed point equation (17) is well - posed.

4. Conclusions

In our work we introduced the $(\beta, \phi)-$ admissible hybrid contractions in metric spaces and establish fixed point results in the setting of these spaces and the derived results have been supplemented with suitable example and an application to Ulam-Hyers stability and well-poisedness has also been provided. It is an open problem to extend and generalize our result in the setting of other topological spaces and some other generalized contractions.

Acknowledgements

(i) This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2025/R/1446).

(ii) The authors convey their sincere appreciation to the anonymous reviewers for their comments, which helped to improve the manuscript to its present form.

Conflict of interests

The authors declare no conflicts of interest.

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