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Topological Characterization of Hemicomplemented Almost Distributive Lattices

Noorbhasha Rafi[1], Ravikumar Bandaru[2], Ravi Kumar Davala[2], Aiyared Iampan[3,*]

[1] *Department of Mathematics, Bapatla Engineering College, Bapatla-522102, Andhra Pradesh, India*
[2] *Department of Mathematics, School of Advanced Sciences, VIT-AP University, Amaravati-522237, Andhra Pradesh, India*
[3] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

Abstract. The notion of $\mathcal{D}$-stone ADLs is introduced, and their core properties are explored. It is shown that every $\mathcal{D}$-stone ADL is hemicomplemented but not vice versa. Several equivalent conditions are given for when a hemicomplemented ADL becomes $\mathcal{D}$-stone. Topological characterizations are provided via minimal prime $\mathcal{D}$-filters and their prime spectra.

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1. Introduction

Lattice theory has long served as a fundamental framework for algebraic structures, with comprehensive treatments available in Birkhoff's work [1] and Grätzer's monograph [2]. Several foundational results regarding prime spectra and congruence structures of lattices have also been developed in works by Crawley and Dilworth [3] and Grätzer and Schmidt [4]. Building upon this foundational lattice theory, Swamy and Rao [5] introduced the concept of Almost Distributive Lattices (ADLs) as an abstraction of distributive lattices and Boolean algebras. They defined ideals in ADLs analogous to those in distributive lattices. They showed that the collection of principal ideals constitutes a distributive lattice, thereby facilitating the extension of lattice theory concepts to ADLs. Rafi et al. [6] introduced the concept of $\mathcal{D}$-filters in ADLs, examining their key properties. Rafi et al. [7] studied prime E-ideals in almost distributive lattices, establishing key properties that

*Corresponding author.
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Email addresses: rafimaths@gmail.com (N. Rafi), ravimaths83@gmail.com (R. Bandaru), davalaravikumar@gmail.com (R. K. Davala), aiyared.ia@up.ac.th (A. Iampan)

parallel prime structures in classical lattice theory and contribute to the broader understanding of ideal-based frameworks in ADLs. Ramesh et al. [8] examined hierarchical classifications within almost distributive lattices, offering a refined structural perspective that complements the development of filter-based approaches in ADLs. Rafi et al. [9] introduced the concept of w-filters in almost distributive lattices, thereby expanding the theoretical landscape of generalized filter structures closely related to the notion of $\mathcal{D}$-filters considered in the present study. Rafi et al. [10] introduced the concept of hemicomplemented ADL and studied their properties. The concept of $\mathcal{D}$-stone ADLs is proposed, and their defining attributes are investigated. It is found that while every $\mathcal{D}$-stone ADL is necessarily hemicomplemented, the converse does not hold in general. Several equivalent conditions are presented that ensure a hemicomplemented ADL is, in fact, a $\mathcal{D}$-stone ADL. A condition is also identified to determine when the collection of all minimal prime $\mathcal{D}$-filters forms a compact topological space. Finally, topological characterizations of hemicomplemented and $\mathcal{D}$-stone ADLs are established, involving both the set of minimal prime $\mathcal{D}$-filters and the prime spectrum of $\mathcal{D}$-filters in ADLs.

2. Preliminaries

This section presents fundamental definitions and key results from [5, 11], which will be referenced throughout the paper.

Definition 1. *[5] A structure* $(\mathcal{L}, \vee, \wedge, 0)$ *of type* $(2, 2, 0)$ *is called an Almost Distributive Lattice (ADL) with zero if it fulfills the following conditions:*

(1) $(\theta \vee \vartheta) \wedge \sigma = (\theta \wedge \sigma) \vee (\vartheta \wedge \sigma)$,

(2) $\theta \wedge (\vartheta \vee \sigma) = (\theta \wedge \vartheta) \vee (\theta \wedge \sigma)$,

(3) $\theta \vee (\vartheta \wedge \sigma) = (\theta \vee \vartheta) \wedge (\theta \vee \sigma)$,

(4) $(\theta \vee \vartheta) \wedge \vartheta = \vartheta$,

(5) $\theta \vee 0 = \theta$,

(6) $0 \wedge \theta = 0$, *for any* $\theta, \vartheta, \sigma \in \mathcal{L}$.

To define a partial order $\leq$ on $\mathcal{L}$, consider the condition $\theta = \theta \wedge \vartheta$ or equivalently $\theta \vee \vartheta = \vartheta$ for every $\theta, \vartheta \in \mathcal{L}$. This condition ensures that $\theta \leq \vartheta$, establishing $\leq$ as a partial order on $\mathcal{L}$. When $m \in \mathcal{L}$ is maximal with respect to this partial order, it is referred to as *maximal*. The collection of all such maximal elements in $\mathcal{L}$ is indicated by $\mathfrak{M}(\mathcal{L})$.

ADL $\mathcal{L}$ exhibits many properties of a distributive lattice, with the exception of non-commutativity of $\vee$ and $\wedge$ and lack of right distributivity of $\vee$ over $\wedge$, as highlighted in Swamy's work [5]. If either of these properties held, $\mathcal{L}$ would be classified as a distributive lattice. We define a non-void subset $\mathcal{I}$ of $\mathcal{L}$ as an ideal (filter) if it satisfies that for any elements $\theta, \vartheta \in \mathcal{I}$ and $\mu \in \mathcal{L}$, the subset $\mathcal{I}$ must include $\theta \wedge \mu$ and $\theta \vee \vartheta$ ($\mu \vee \theta$ and $\theta \wedge \vartheta$). A maximal ideal (filter) contains every proper ideal (filter) of $\mathcal{L}$. The smallest ideal containing a subset $\mathcal{S}$ of $\mathcal{L}$ is defined as $(\mathcal{S}] := \{(\bigvee_{i=1}^{n} \theta_i) \wedge \mu \mid \theta_i \in \mathcal{S}, \mu \in \mathcal{L}, n \in \mathbb{N}\}$. A principal ideal generated by an element θ is denoted as $(\theta]$. Similarly, for each subset $\mathcal{S}$ of $\mathcal{L}$, the smallest filter containing $\mathcal{S}$ is defined as $[\mathcal{S}) := \{\mu \vee (\bigwedge_{i=1}^{n} \theta_i) \mid \theta_i \in \mathcal{S}, \mu \in \mathcal{L}, n \in \mathbb{N}\}$. A principal filter generated by an element θ is denoted as $[\theta)$. It is established that $(\theta] \vee (\vartheta] = (\theta \vee \vartheta]$ and $(\theta] \cap (\vartheta] = (\theta \wedge \vartheta]$ for any $\theta, \vartheta \in \mathcal{L}$. Represented all principal ideals of $\mathcal{L}$ by the set $(\mathcal{PI}(\mathcal{L}), \vee, \cap)$, this brings out a sublattice of the distributive lattice $(\mathcal{I}(\mathcal{L}), \vee, \cap)$ of all ideals of $\mathcal{L}$. Furthermore, the set $(\mathcal{F}(\mathcal{L}), \vee, \cap)$ of all filters of $\mathcal{L}$ forms a bounded distributive lattice. In an ADL [12], a prime ideal $\mathcal{Q}$ of $\mathcal{L}$ exists if and only if $\mathcal{L} \setminus \mathcal{Q}$ is a prime filter of $\mathcal{L}$. A prime ideal $\mathcal{Q}$ of an ADL is a minimal prime ideal if and only if to each $\mu \in \mathcal{Q}$ there exists $\pi \notin \mathcal{Q}$ such that $\mu \wedge \pi = 0$ (or equivalently, for any $\mu \in \mathcal{L}, \mu \notin \mathcal{Q}$ if and only if $(\mu)^* \subseteq \mathcal{Q}$). For each non-void subset $\mathcal{S}$ of $\mathcal{L}$, the set $\mathcal{S}^* = \{\mu \in \mathcal{L} \mid \theta \wedge \mu = 0, \text{ for all } \theta \in \mathcal{S}\}$ is an ideal of $\mathcal{L}$. Generally, for every $\theta \in \mathcal{L}$, $\{\theta\}^* = (\theta)^*$, where $(\theta) = (\theta]$. The *annihilator* of an element $\theta \in \mathcal{L}$ is defined as the set $(\theta)^* = \{\mu \in \mathcal{L} \mid \mu \wedge \theta = 0\}$. If $(e)^* = \{0\}$ then an element $e \in \mathcal{L}$ is considered dense. Within $\mathcal{L}$, the set $\mathcal{D}$ is the set of dense elements. A filter of an ADL $\mathcal{L}$ can be obtained by the set $\mathcal{D}$. The ADL $\mathcal{L}$ is termed a generalized stone ADL [13] when it satisfies the property $(\mu)^* \vee (\mu)^{**} = \mathcal{L}$ for all $\mu \in \mathcal{L}$.

According to [6], a filter $\mathcal{G}$ of an ADL $\mathcal{L}$ is known as a *$\mathcal{D}$-filter* if $\mathcal{D} \subseteq \mathcal{G}$. The smallest $\mathcal{D}$-filter is $\mathcal{D}$. For any nonempty subset $\mathcal{S}$ of $\mathcal{L}$, consider the set $(\mathcal{S}, \mathcal{D}) = \{\mu \in \mathcal{L} \mid \theta \vee \mu \in \mathcal{D} \text{ for all } \theta \in \mathcal{S}\}$. It is noted that $(\mathcal{L}, \mathcal{D}) = \mathcal{D}$ and $(\mathcal{D}, \mathcal{D}) = \mathcal{L}$. Furthermore, for any subset $\mathcal{S}$ of $\mathcal{L}$, $\mathcal{D} \subseteq (\mathcal{S}, \mathcal{D})$. For every $\theta \in \mathcal{L}$, $(\{\theta\}, \mathcal{D})$ is denoted as $(\theta, \mathcal{D})$. Therefore, $(m, \mathcal{D}) = \mathcal{L}$ for any $m \in \mathfrak{M}(\mathcal{L})$. $(\mathcal{S}, \mathcal{D})$ forms a $\mathcal{D}$-filter in $\mathcal{L}$ for each subset $\mathcal{S}$ in $\mathcal{L}$.

Lemma 1. [6] *Given two subsets $\mathcal{S}$, $\mathcal{T}$ of an ADL $\mathcal{L}$, the following holds:*

(1) $\mathcal{S} \subseteq \mathcal{T} \Rightarrow (\mathcal{T}, \mathcal{D}) \subseteq (\mathcal{S}, \mathcal{D})$,

(2) $\mathcal{S} \subseteq ((\mathcal{S}, \mathcal{D}), \mathcal{D})$,

(3) $(\mathcal{S}, \mathcal{D}) = (((\mathcal{S}, \mathcal{D}), \mathcal{D}), \mathcal{D})$,

(4) $\mathcal{S} \subseteq \mathcal{D} \Leftrightarrow (\mathcal{S}, \mathcal{D}) = \mathcal{L}$.

Proposition 1. *[6] Given filters $\mathcal{G}$,$\mathcal{U}$ of an ADL $\mathcal{L}$, the following holds:*

(1) $(\mathcal{G}, \mathcal{D}) \cap ((\mathcal{G}, \mathcal{D}), \mathcal{D}) = \mathcal{D}$,

(2) $\mathcal{G} \cap \mathcal{U} \subseteq \mathcal{D} \Rightarrow \mathcal{G} \subseteq (\mathcal{U}, \mathcal{D})$,

(3) $((\mathcal{G} \vee \mathcal{U}), \mathcal{D}) = (\mathcal{G}, \mathcal{D}) \cap (\mathcal{U}, \mathcal{D})$,

(4) $((\mathcal{G} \cap \mathcal{U}, \mathcal{D}), \mathcal{D}) = ((\mathcal{G}, \mathcal{D}), \mathcal{D}) \cap ((\mathcal{U}, \mathcal{D}), \mathcal{D})$.

The idea that $([\mu), \mathcal{D}) = (\mu, \mathcal{D})$ is obvious. It follows that $(0, \mathcal{D}) = \mathcal{D}$. The previously noted observations directly lead to the corollary that follows.

Corollary 1. *[6] For any $\mu, \pi, \psi \in \mathcal{L}$, we have the following:*

(1) $([\mu), \mathcal{D}) = (\mu, \mathcal{D})$,

(2) $\mu \leq \pi \Rightarrow (\mu, \mathcal{D}) \subseteq (\pi, \mathcal{D})$,

(3) $(\mu \wedge \pi, \mathcal{D}) = (\mu, \mathcal{D}) \cap (\pi, \mathcal{D})$,

(4) $((\mu \vee \pi, \mathcal{D}), \mathcal{D}) = ((\mu, \mathcal{D}), \mathcal{D}) \cap ((\pi, \mathcal{D}), \mathcal{D})$,

(5) $(\mu, \mathcal{D}) = \mathcal{L} \Leftrightarrow \mu \in \mathcal{D}$,

(6) $(\mu, \mathcal{D}) = (\pi, \mathcal{D}) \Leftrightarrow (\mu \wedge \psi, \mathcal{D}) = (\pi \wedge \psi, \mathcal{D})$,

(7) $(\mu, \mathcal{D}) = (\pi, \mathcal{D}) \Leftrightarrow (\mu \vee \psi, \mathcal{D}) = (\pi \vee \psi, \mathcal{D})$.

Definition 2. *[10] An element μ in an ADL $\mathcal{L}$ is said to be condensed if the filter extension $(\mu, \mathcal{D})$ is equal to $\mathcal{D}$.*

The zero element 0 of an ADL $\mathcal{L}$ is always condensed. Let us use the notation $\mathcal{D}^{\infty}$ to represent the collection of all condensed elements in $\mathcal{L}$. With this definition, we arrive at the following result.

Proposition 2. *[10] The following statements are valid in an ADL $\mathcal{L}$:*

(1) $\mathcal{D} \cap \mathcal{D}^{\infty} = \emptyset$,

(2) $\mathcal{D}^{\infty}$ *is an ideal in* $\mathcal{L}$.

Definition 3. *[10] An ADL $\mathcal{L}$ is termed hemicomplemented if for every element μ in $\mathcal{L}$, there is an element $\pi \in \mathcal{L}$ such that $\mu \wedge \pi \in \mathcal{D}^{\infty}$ and $\mu \vee \pi \in \mathcal{D}$.*

3. $\mathcal{D}$-Stone Almost Distributive Lattices

This section introduces the concept of $\mathcal{D}$-stone ADLs. A collection of equivalent criteria is established, characterizing when a hemicomplemented ADL qualifies as a $\mathcal{D}$-stone ADL. It is shown that a hemicomplemented ADL is $\mathcal{D}$-stone precisely when $\mathcal{D}^{\circ}(\mathcal{L})$ (see Definition 5) forms a Boolean algebra.

Lemma 2. *For all elements $a, b \in \mathcal{L}$, the conditions below are equivalent:*

(1) $\theta \vee \vartheta \in \mathcal{D}$,

(2) $((\theta, \mathcal{D}), \mathcal{D}) \cap [\vartheta) \subseteq \mathcal{D}$,

(3) $((\theta, \mathcal{D}), \mathcal{D}) \cap ((\vartheta, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{D}$.

Proof. (1) $\Rightarrow$ (2) : Assume (1). Let $\theta, \vartheta \in \mathcal{L}$. Suppose $\theta \vee \vartheta \in \mathcal{D}$. Then $\vartheta \in (\theta, \mathcal{D})$, This gives $[\vartheta) \subseteq (\theta, \mathcal{D})$. Therefore, $((\theta, \mathcal{D}), \mathcal{D}) \cap [\vartheta) \subseteq ((\theta, \mathcal{D}), \mathcal{D}) \cap (\theta, \mathcal{D}) = \mathcal{D}$.

(2) $\Rightarrow$ (3) : Assume $((\theta, \mathcal{D}), \mathcal{D}) \cap [\vartheta) \subseteq \mathcal{D}$ for any $\theta, \vartheta \in \mathcal{L}$. By corollary 1(2), we obtain $((\theta, \mathcal{D}), \mathcal{D}) \subseteq (\vartheta, \mathcal{D})$. Thus, $((\theta, \mathcal{D}), \mathcal{D}) \cap ((\vartheta, \mathcal{D}), \mathcal{D}) \subseteq (\vartheta, \mathcal{D}) \cap ((\vartheta, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{D}$.

(3) $\Rightarrow$ (1) : Assume $((\theta, \mathcal{D}), \mathcal{D}) \cap ((\vartheta, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{D}$ for every $\theta, \vartheta \in \mathcal{L}$. By corollary 1(2), we obtain $((\theta, \mathcal{D}), \mathcal{D}) \subseteq (((\vartheta, \mathcal{D}), \mathcal{D}), \mathcal{D}) = (\vartheta, \mathcal{D})$. Therefore, $\theta \in ((\theta, \mathcal{D}), \mathcal{D}) \cap (\vartheta, \mathcal{D})$, which gives $\theta \vee \vartheta \in \mathcal{D}$.

Proposition 3. *The intersection of all minimal prime $\mathcal{D}$-filters is $\mathcal{D}$.*

Proof. It is easy to verify that $\mathcal{D} \subseteq \bigcap\{\mathcal{Q} \mid \mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter$\}$. Let $\mu \notin \mathcal{D}$. Then there exists an ideal $\mathcal{I}$ such that $\mu \in \mathcal{I}$ and $\mathcal{I}$ is maximal with $\mathcal{I} \cap \mathcal{D} = \emptyset$. Clearly, $\mathcal{L} \setminus \mathcal{I}$ is a minimal prime $\mathcal{D}$-filter and $\mu \notin \mathcal{L} \setminus \mathcal{I}$. Thus, $\mu \notin \bigcap\{\mathcal{Q} \mid \mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter$\}$. Therefore, $\bigcap\{\mathcal{Q} \mid \mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter$\} \subseteq \mathcal{D}$. Hence, $\mathcal{D} = \bigcap\{\mathcal{Q} \mid \mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter$\}$.

Corollary 2. *For any $\mu \in \mathcal{L}$, we have*

$$(\mu, \mathcal{D}) = \bigcap\{\mathcal{Q} \mid \mathcal{Q} \textit{ is a minimal prime } \mathcal{D}\textit{-filter such that } \mu \notin \mathcal{Q}\}.$$

Proof. Let $\theta \in (\mu, \mathcal{D})$ and $\mathcal{Q}$ a minimal prime $\mathcal{D}$-filter with $\mu \notin \mathcal{Q}$. Then $\mu \vee \theta \in \mathcal{D} \subseteq \mathcal{Q}$. As $\mu \notin \mathcal{Q}$, we obtain $\theta \in \mathcal{Q}$ for every minimal prime $\mathcal{D}$-filters with $\mu \notin \mathcal{Q}$. Thus, $(\mu, \mathcal{D}) \subseteq \bigcap\{\mathcal{Q} \mid \mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter such that $\mu \notin \mathcal{Q}\}$. Conversely, assume that $\nu \notin (\mu, \mathcal{D})$. Then $\nu \vee \mu \notin \mathcal{D}$. By the above result, there is a minimal prime $\mathcal{D}$-filter $\mathcal{Q}$ such that $\nu \vee \mu \notin \mathcal{Q}$. Hence, $\nu \notin \bigcap\{\mathcal{Q} \mid \mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter such that $\mu \notin \mathcal{Q}\}$. Hence, $\bigcap\{\mathcal{Q} \mid \mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter such that $\mu \notin \mathcal{Q}\} \subseteq (\mu, \mathcal{D})$.

Definition 4. *An ADL $\mathcal{L}$ is said to be a $\mathcal{D}$-stone ADL if $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$ for all $\mu \in \mathcal{L}$.*

Example 1. *Consider the set $\mathcal{L} = \{0, \theta, \vartheta, \sigma, e, \pi, \rho\}$, with the operations $\vee$ (join) and $\wedge$ (meet) defined on $\mathcal{L}$ as follows:*

$\wedge$	0	θ	ϑ	σ	e	π	ρ
0	0	0	0	0	0	0	0
θ	0	θ	ϑ	σ	e	π	ρ
ϑ	0	θ	ϑ	σ	e	π	ρ
σ	0	σ	σ	σ	0	σ	σ
e	0	e	e	0	e	e	e
π	0	π	π	σ	e	π	π
ρ	0	ρ	ρ	σ	e	π	ρ

$\vee$	0	θ	ϑ	σ	e	π	ρ
0	0	θ	ϑ	σ	e	π	ρ
θ	θ	θ	θ	θ	θ	θ	θ
ϑ	ϑ	ϑ	ϑ	ϑ	ϑ	ϑ	ϑ
σ	σ	θ	ϑ	σ	π	π	ρ
e	e	θ	ϑ	π	e	π	ρ
π	π	θ	ϑ	π	π	π	ρ
ρ	ρ	θ	ϑ	ρ	ρ	ρ	ρ

Thus, $(\mathcal{L}, \vee, \wedge)$ is an ADL. Clearly, we have the dense set $\mathcal{D} = \{\theta, \vartheta, \pi, \rho\}$. It is evident that $\mathcal{L}$ is a $\mathcal{D}$-stone ADL, because $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D})\mathcal{D}) = \mathcal{L}$, for all $\mu \in \mathcal{L}$.

Proposition 4. *For any prime $\mathcal{D}$-filter of $\mathcal{L}$ is minimal, we have $\mathcal{L}$ is a $\mathcal{D}$-stone ADL.*

Proof. Suppose that every prime $\mathcal{D}$-filter of $\mathcal{L}$ is minimal. Let $\mu \in \mathcal{L}$. If $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$, then there is a prime $\mathcal{D}$-filter $\mathcal{Q}$ such that $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. Therefore $(\mu, \mathcal{D}) \subseteq \mathcal{Q}$ and $((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. By Corollary 2, we obtain $\mu \notin \mathcal{Q}$. It is evident that $\mu \in ((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. Thus, $\mu \in \mathcal{Q}$, which leads to a contradiction. Hence, $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$. Therefore, $\mathcal{L}$ is a $\mathcal{D}$-stone ADL.

Proposition 5. *Every $\mathcal{D}$-stone ADL is hemicomplemented.*

Proof. Assume that $\mathcal{L}$ is a $\mathcal{D}$-stone ADL and let $\mu \in \mathcal{L}$. By definition, we have $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$. This implies that $0 \in (\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D})$, so there exist elements $\theta \in (\mu, \mathcal{D})$ and $\vartheta \in ((\mu, \mathcal{D}), \mathcal{D})$ such that $\theta \wedge \vartheta = 0$. Consequently, $(\theta, \mathcal{D}) \cap (\vartheta, \mathcal{D}) = (\theta \wedge \vartheta, \mathcal{D}) = (0, \mathcal{D}) = \mathcal{D}$. This inclusion implies that $(\theta, \mathcal{D}) \subseteq ((\vartheta, \mathcal{D}), \mathcal{D})$, and since $\theta \in (\mu, \mathcal{D})$, we also have $((\mu, \mathcal{D}), \mathcal{D}) \subseteq (\theta, \mathcal{D})$. Hence, $((\mu, \mathcal{D}), \mathcal{D}) = (\theta, \mathcal{D})$. Therefore, $\mathcal{L}$ satisfies the condition for being hemicomplemented.

The converse of the preceding proposition does not hold universally. However, we now develop a set of equivalent conditions under which a hemicomplemented ADL becomes a $\mathcal{D}$-stone ADL. To aid this characterization, we introduce the notion of a $\mathcal{D}$-factor in an ADL $\mathcal{L}$. A $\mathcal{D}$-filter $\mathcal{G}$ of $\mathcal{L}$ is said to be a $\mathcal{D}$-factor if there exists a proper $\mathcal{D}$-filter $\mathcal{U}$ such that $\mathcal{G} \cap \mathcal{U} = \mathcal{D}$ and $\mathcal{G} \vee \mathcal{U} = \mathcal{L}$.

Before proceeding, we introduce the following notation.

Definition 5. *Let $\mathcal{L}$ be an ADL. We define*

$$\mathcal{D}^{\circ}(\mathcal{L}) := \{(\mu, D) \mid \mu \in \mathcal{L}\},$$

the collection of all principal $\mathcal{D}$-filters in $\mathcal{L}$. When equipped with the operations $\vee$ and $\cap$ inherited from the lattice of filters $\mathcal{F}(\mathcal{L})$, $\mathcal{D}^{\circ}(\mathcal{L})$ forms a substructure of $F(L)$ whose properties will be explored in the subsequent results.

Theorem 1. *The conditions listed below are equivalent in a hemicomplemented ADL:*

(1) *$\mathcal{L}$ is a $\mathcal{D}$-stone ADL,*

(2) *each $(\mu, \mathcal{D})$ is a $\mathcal{D}$-factor of $\mathcal{L}$,*

(3) *for each $\mu \in \mathcal{L}$, there exists $\mu' \in \mathcal{L}$ such that $(\mu, \mathcal{D}) \vee (\mu', \mathcal{D}) = \mathcal{L}$,*

(4) *for $\mu, \pi \in \mathcal{L}$, $(\mu, \mathcal{D}) \vee (\pi, \mathcal{D}) = (\mu \vee \pi, \mathcal{D})$,*

(5) *$\mathcal{D}^{\circ\circ}(\mathcal{L}) = \{((\mu, \mathcal{D}), \mathcal{D}) \mid \mu \in \mathcal{L}\}$ is a sublattice of $\mathcal{F}(\mathcal{L})$, where $\mathcal{F}(\mathcal{L})$ is the set of all filters of $\mathcal{L}$.*

Proof. $(1) \Rightarrow (2)$: Assume (1). Let $\mu \in \mathcal{L}$. It can be seen that $(\mu, \mathcal{D}) \cap ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{D}$. By our assumption (1), we obtain $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$. Thus, $(\mu, \mathcal{D})$ is a $\mathcal{D}$-factor of $\mathcal{L}$.

(2) $\Rightarrow$ (3) : Assume (2). Let $\mu \in \mathcal{L}$. As $\mathcal{L}$ is hemicomplemented, there is $\mu' \in \mathcal{L}$ such that $((\mu, \mathcal{D}), \mathcal{D}) = (\mu', \mathcal{D})$. By (2), there exists a $\mathcal{D}$-filter $\mathcal{U}$ such that $(\mu, \mathcal{D}) \cap \mathcal{U} = \mathcal{D}$ and $(\mu, \mathcal{D}) \vee \mathcal{U} = \mathcal{L}$. Since $(\mu, \mathcal{D}) \cap \mathcal{U} = \mathcal{D}$, we have $\mathcal{U} \subseteq ((\mu, \mathcal{D}), \mathcal{D}) = (\mu', \mathcal{D})$ Hence, $\mathcal{L} = (\mu, \mathcal{D}) \vee \mathcal{U} \subseteq (\mu, \mathcal{D}) \vee (\mu', \mathcal{D})$. Thus, $(\mu, \mathcal{D}) \vee (\mu', \mathcal{D}) = \mathcal{L}$.

(3) $\Rightarrow$ (4) : Assume (3). Let $\mu, \pi \in \mathcal{L}$. By (3), there is $\mu' \in \mathcal{L}$ such that $(\mu, \mathcal{D}) \vee (\mu', \mathcal{D}) = \mathcal{L}$. Clearly we have $(\mu, \mathcal{D}) \vee (\pi, \mathcal{D}) \subseteq (\mu \vee \pi, \mathcal{D})$. Conversely, let $\theta \in (\mu \vee \pi, \mathcal{D})$. Then $\theta \vee \mu \vee \pi \in \mathcal{D}$, which leads $\theta \vee \pi \in (\mu, \mathcal{D})$. By Corollary 1(2) and Lemma 2, we obtain

$$\begin{aligned}
\theta \vee \pi \in (\mu, \mathcal{D}) \quad &\Rightarrow \quad ((\mu, \mathcal{D}), \mathcal{D}) \subseteq (\theta \vee \pi, \mathcal{D}) \\
&\Rightarrow \quad ((\mu, \mathcal{D}), \mathcal{D}) \cap [\theta \vee \pi) \subseteq \mathcal{D} \\
&\Rightarrow \quad ((\mu, \mathcal{D}), \mathcal{D}) \cap \{[\theta) \cap [\pi)\} \subseteq \mathcal{D} \\
&\Rightarrow \quad \{((\mu, \mathcal{D}), \mathcal{D}) \cap [\theta)\} \cap [\pi) \subseteq \mathcal{D} \\
&\Rightarrow \quad \{((\mu, \mathcal{D}), \mathcal{D}) \cap [\theta)\} \subseteq (\pi, \mathcal{D}) \\
&\Rightarrow \quad \{(\mu', \mathcal{D}) \cap [\theta)\} \subseteq (\pi, \mathcal{D}).
\end{aligned}$$

Clearly, $(\mu, \mathcal{D}) \cap [\theta) \subseteq (\mu, \mathcal{D})$. Hence, $\theta \in [\theta) = \mathcal{L} \cap [\theta) = \{(\mu, \mathcal{D}) \vee (\mu', \mathcal{D})\} \cap [\theta) = \{(\mu, \mathcal{D}) \cap [\theta)\} \cap \{(\mu', \mathcal{D}) \cap [\theta)\} \subseteq (\mu, \mathcal{D}) \vee (\pi, \mathcal{D})$. Hence, $(\mu \vee \pi, \mathcal{D}) \subseteq (\mu, \mathcal{D}) \vee (\pi, \mathcal{D})$.

(4) $\Rightarrow$ (5) : For any $\mu, \pi \in \mathcal{L}$, it is clear that $((\mu, \mathcal{D}), \mathcal{D}) \cap ((\pi, \mathcal{D}), \mathcal{D}) = ((\mu \vee \pi, \mathcal{D}), \mathcal{D})$. Since $\mathcal{L}$ is hemicomplemented, there is $\mu', \pi' \in \mathcal{L}$ such that $((\mu, \mathcal{D}), \mathcal{D}) = (\mu', \mathcal{D})$ and $((\pi, \mathcal{D}), \mathcal{D}) = (\pi', \mathcal{D})$. Hence $((\mu, \mathcal{D}), \mathcal{D}) \vee ((\pi, \mathcal{D}), \mathcal{D}) = (\mu', \mathcal{D}) \vee (\pi', \mathcal{D}) = (\mu' \vee \pi', \mathcal{D}) = ((\sigma, \mathcal{D}), \mathcal{D})$ for some $\sigma \in \mathcal{L}$, as $\mathcal{L}$ is hemicomplemented. Therefore, $\mathcal{D}^{\circ\circ}(\mathcal{L})$ is a sublattice of $\mathcal{F}(\mathcal{L})$.

(5) $\Rightarrow$ (1) : Assume (5). Let $\mu \in \mathcal{L}$. Since $\mathcal{L}$ is hemicomplemented, there is $\pi \in \mathcal{L}$ such that $((\mu, \mathcal{D}), \mathcal{D}) = (\pi, \mathcal{D})$. As $\mathcal{D}^{\circ\circ}(\mathcal{L})$ is a sublattice of $\mathcal{F}(\mathcal{L})$, we obtain $((\mu, \mathcal{D}), \mathcal{D}) \vee ((\pi, \mathcal{D}), \mathcal{D}) = ((\nu, \mathcal{D}), \mathcal{D})$ for some $\nu \in \mathcal{L}$. Hence, $\mu \wedge \pi \in ((\mu, \mathcal{D}), \mathcal{D}) \vee ((\pi, \mathcal{D}), \mathcal{D}) = ((\nu, \mathcal{D}), \mathcal{D})$. Thus, $\mathcal{L}$ is a $\mathcal{D}$-stone ADL.

Corollary 3. *In any $\mathcal{D}$-stone ADL $\mathcal{L}$, we have $\mathcal{D}^{\circ}(\mathcal{L})$ is a sublattice of $\mathcal{F}(\mathcal{L})$.*

A property of $\mathcal{D}$-stone ADLs, expressed via minimal prime $\mathcal{D}$-filters, is established in the next corollary. Comaximality of two $\mathcal{D}$-filters $\mathcal{G}$ and $\mathcal{U}$ in an ADL $\mathcal{L}$ is defined by the condition $\mathcal{G} \vee \mathcal{U} = \mathcal{L}$.

Corollary 4. *Any two distinct minimal prime $\mathcal{D}$-filters of a $\mathcal{D}$-stone ADL $\mathcal{L}$ are comaximal.*

Proof. Suppose that $\mathcal{L}$ is a $\mathcal{D}$-stone ADL. According to condition (2) of the main theorem, every $(\mu, \mathcal{D})$ serves as a $\mathcal{D}$-factor of $\mathcal{L}$. Let $\mathcal{Q}$ and $\mathcal{P}$ be two distinct minimal prime $\mathcal{D}$-filters in $\mathcal{L}$. Select an element $\theta \in \mathcal{Q} \setminus \mathcal{P}$. This implies that $(\theta, \mathcal{D}) \subseteq \mathcal{P}$. Since $\mathcal{Q}$ is minimal, it follows that $((\theta, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$.

Given that $(\theta, \mathcal{D})$ is a $\mathcal{D}$-factor of $\mathcal{L}$, there exists a $\mathcal{D}$-filter $\mathcal{U}$ such that $(\theta, \mathcal{D}) \cap \mathcal{U} = \mathcal{D}$ and $(\theta, \mathcal{D}) \vee \mathcal{U} = \mathcal{L}$. This inclusion implies that $\mathcal{U} \subseteq ((\theta, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. Hence, we obtain $\mathcal{L} = (\theta, \mathcal{D}) \vee \mathcal{U} \subseteq \mathcal{P} \vee \mathcal{Q}$. Therefore, $\mathcal{Q}$ and $\mathcal{P}$ are comaximal.

Theorem 2. *In a hemicomplemented ADL $\mathcal{L}$, we have $\mathcal{L}$ is $\mathcal{D}$-stone if and only if $\mathcal{D}^{\circ}(\mathcal{L})$ is a Boolean algebra.*

Proof. Assume that $\mathcal{L}$ is a $\mathcal{D}$-stone ADL. Let $(\mu, \mathcal{D})$ and $(\pi, \mathcal{D})$ be elements in $\mathcal{D}^{\circ}(\mathcal{L})$. It is straightforward to see that $(\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = (\mu \wedge \pi, \mathcal{D})$. Since $\mathcal{L}$ is a $\mathcal{D}$-stone ADL, it satisfies that $(\mu, \mathcal{D}) \vee (\pi, \mathcal{D}) = (\mu \vee \pi, \mathcal{D})$. Therefore, $(\mathcal{D}^{\circ}(\mathcal{L}), \vee, \cap)$ forms a lattice. Moreover, observe that for every $\mu \in \mathcal{D}^{\infty}$, we have $(\mu, \mathcal{D}) = \mathcal{D}$, indicating that $\mathcal{D}$ is the least element in $\mathcal{D}^{\circ}(\mathcal{L})$. Likewise, since $(e, \mathcal{D}) = \mathcal{L}$ for all $e \in \mathcal{D}$, it follows that $\mathcal{L}$ serves as the greatest element of $\mathcal{D}^{\infty}$. Thus, the set $\mathcal{D}^{\circ}(\mathcal{L})$, equipped with the operations $\vee$ and $\cap$, forms a bounded distributive lattice. Next, consider $(\mu, \mathcal{D}) \in \mathcal{D}^{\circ}(\mathcal{L})$ with $\mu \in \mathcal{L}$. Since $\mathcal{L}$ is hemicomplemented, there exists an element $\mu' \in \mathcal{L}$ such that $((\mu, \mathcal{D}), \mathcal{D}) = (\mu', \mathcal{D})$. This implies that $(\mu, \mathcal{D}) \cap (\mu', \mathcal{D}) = \mathcal{D}$, and As $\mathcal{L}$ is a $\mathcal{D}$-stone ADL, we also have $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$, which leads to $(\mu, \mathcal{D}) \vee (\mu', \mathcal{D}) = \mathcal{L}$. Hence, $(\mu', \mathcal{D})$ serves as the complement of $(\mu, \mathcal{D})$ in $\mathcal{D}^{\circ}(\mathcal{L})$, showing that $\mathcal{D}^{\circ}(\mathcal{L})$ forms a Boolean algebra. Conversely, suppose that $\mathcal{D}^{\circ}(\mathcal{L})$ is a Boolean algebra. For any $\mu \in \mathcal{L}$, the element $(\mu, \mathcal{D})$ belongs to $\mathcal{D}^{\circ}(\mathcal{L})$. Then there exists an element $(\mu', \mathcal{D}) \in \mathcal{D}^{\circ}(\mathcal{L})$ such that $(\mu, \mathcal{D}) \cap (\mu', \mathcal{D}) = \mathcal{D}$ and $(\mu, \mathcal{D}) \vee (\mu', \mathcal{D}) = \mathcal{L}$. The former implies $(\mu', \mathcal{D}) \subseteq ((\mu, \mathcal{D}), \mathcal{D})$, and thus we obtain $\mathcal{L} = (\mu, \mathcal{D}) \vee (\mu', \mathcal{D}) \subseteq (\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D})$, which shows that $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$. Therefore, $\mathcal{L}$ satisfies the condition to be a $\mathcal{D}$-stone ADL.

4. Topological Characterizations

In this section, we provide a topological characterization of the classes of hemicomplemented ADLs and $\mathcal{D}$-stone ADLs. Let us denote by $Spec^{\mathcal{D}}_{MF}(\mathcal{L})$ the collection of all minimal prime $\mathcal{D}$-filters of $\mathcal{L}$. For any subset $\mathcal{H} \subseteq \mathcal{L}$, define the set $\mathcal{J}_m(\mathcal{H}) = \left\{\mathcal{Q} \in Spec^{\mathcal{D}}_{MF}(\mathcal{L}) \mid \mathcal{H} \nsubseteq \mathcal{Q}\right\}$. In particular, for a set $\mathcal{H} = \{\mu\}$, we use the notation $\mathcal{J}_m(\mu)$ in place of $\mathcal{J}_m(\{\mu\})$, which explicitly becomes $\mathcal{J}_m(\mu) = \left\{\mathcal{Q} \in Spec^{\mathcal{D}}_{MF}(\mathcal{L}) \mid \mu \notin \mathcal{Q}\right\}$. It follows immediately that for any subset $\mathcal{H}$ of $\mathcal{L}$, the corresponding $\mathcal{J}_m(\mathcal{H})$ can be expressed as the union $\mathcal{J}_m(\mathcal{H}) = \bigcup_{\mu \in \mathcal{H}} \mathcal{J}_m(\mu)$.

Lemma 3. *For any elements μ, π in an ADL $\mathcal{L}$, the following properties are satisfied:*

(1) $\mathcal{J}_m(\mu) \cap \mathcal{J}_m(\pi) = \mathcal{J}_m(\mu \vee \pi)$,

(2) $\mathcal{J}_m(\mu \wedge \pi) \subseteq \mathcal{J}_m(\mu) \cup \mathcal{J}_m(\pi)$,

(3) $\mathcal{J}_m(\mu) = K_m(((\mu, \mathcal{D}), \mathcal{D}))$,

(4) $(\mu, \mathcal{D}) \subseteq (\pi, \mathcal{D}) \Leftrightarrow \mathcal{J}_m(\pi) \subseteq \mathcal{J}_m(\mu)$,

(5) $\mathcal{J}_m(\mu) = \emptyset \Leftrightarrow \mu \in \mathcal{D}$,

(6) $\mathcal{J}_m(\mu) = Spec^{\mathcal{D}}_{MF}(\mathcal{L}) \Leftrightarrow \mu$ *is condensed.*

Proof. The verifications of statements (1) and (2) follow directly from the definitions.

(3) Let $\mathcal{Q} \in \mathcal{J}_m(\mu)$. This implies that $\mu \notin \mathcal{Q}$. Given that $\mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter, it follows that $((\mu, \mathcal{D}), \mathcal{D})$ is not entirely contained in $\mathcal{Q}$. Thus, $\mathcal{Q} \in \mathcal{J}_m(((\mu, \mathcal{D}), \mathcal{D}))$, and we obtain the inclusion $\mathcal{J}_m(\mu) \subseteq \mathcal{J}_m(((\mu, \mathcal{D}), \mathcal{D}))$. For the reverse inclusion, assume $\mathcal{Q} \in \mathcal{J}_m(((\mu, \mathcal{D}), \mathcal{D}))$. That is, $((\mu, \mathcal{D}), \mathcal{D}) \nsubseteq \mathcal{Q}$. Suppose, for contradiction, that $\mu \in \mathcal{Q}$. Then, due to the minimality of $\mathcal{Q}$, we must have $((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$, which contradicts our assumption. Hence, $\mu \notin \mathcal{Q}$, meaning $\mathcal{Q} \in \mathcal{J}_m(\mu)$. Therefore, we conclude that $\mathcal{J}_m(\mu) = \mathcal{J}_m(((\mu, \mathcal{D}), \mathcal{D}))$.

(4) Suppose that $(\mu, \mathcal{D}) \subseteq (\pi, \mathcal{D})$. Let $\mathcal{Q} \in \mathcal{J}_m(\pi)$. Then $\pi \notin \mathcal{Q}$, which, together with the inclusion $(\mu, \mathcal{D}) \subseteq (\pi, \mathcal{D})$, implies that $(\mu, \mathcal{D}) \subseteq \mathcal{Q}$. Since $\mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter, this leads to $\mu \notin \mathcal{Q}$, and so $\mathcal{Q} \in \mathcal{J}_m(\mu)$. Hence, we have $\mathcal{J}_m(\pi) \subseteq \mathcal{J}_m(\mu)$.

Conversely, assume $\mathcal{J}_m(\pi) \subseteq \mathcal{J}_m(\mu)$. Take any $\theta \in (\pi, \mathcal{D})$, so by definition, $\theta \vee \pi \notin \mathcal{D}$. By Proposition 3, this implies the existence of a minimal prime $\mathcal{D}$-filter $\mathcal{Q}_0$ such that $\theta \vee \pi \notin \mathcal{Q}_0$. Consequently, both $\theta \notin \mathcal{Q}_0$ and $\pi \notin \mathcal{Q}_0$, meaning $\mathcal{Q}_0 \in \mathcal{J}_m(\pi) \subseteq \mathcal{J}_m(\mu)$, so $\mu \notin \mathcal{Q}_0$ as well. It follows that $\theta \vee \mu \notin \mathcal{Q}_0$, and thus $\theta \vee \mu \notin \mathcal{D}$, which means $\theta \notin (\mu, \mathcal{D})$. Therefore, $(\mu, \mathcal{D}) \subseteq (\pi, \mathcal{D})$.

(5) This follows directly from Proposition 3.

(6) Suppose that $\mathcal{J}_m(\mu) = \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$. Then, we have

$$(\mu, \mathcal{D}) = \bigcap_{\mathcal{Q} \in \mathcal{J}_m(\mathcal{L})} \mathcal{Q} = \bigcap_{\mathcal{Q} \in \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})} \mathcal{Q} = \mathcal{D}.$$

Thus, μ is condensed. Conversely, assume that μ is condensed, i.e., $(\mu, \mathcal{D}) = \mathcal{D}$. This implies that $(\mu, \mathcal{D}) \subseteq \mathcal{Q}$ for every $\mathcal{Q} \in \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$, and hence $\mu \notin \mathcal{Q}$ for all such $\mathcal{Q}$. Therefore, $\mathcal{Q} \in \mathcal{J}_m(\mu)$ for all $\mathcal{Q} \in \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$, which yields $\mathcal{J}_m(\mu) = \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$.

From the previous result, it follows that the collection $\{\mathcal{J}_m(\mu) \mid \mu \in \mathcal{L}\}$ consists of subsets of $\mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$ that is closed under finite intersections. Moreover, since every minimal prime $\mathcal{D}$-filter is proper, we have

$$\bigcup_{\mu \in \mathcal{L}} \mathcal{J}_m(\mu) = \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L}).$$

Thus, this collection forms a basis for a topology on $\mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$.

Now, for any subset $\mathcal{H} \subseteq \mathcal{L}$, define

$$\mathcal{V}_m(\mathcal{H}) = \{\mathcal{Q} \in \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L}) \mid \mathcal{H} \subseteq \mathcal{Q}\}.$$

In particular, for a singleton $\mathcal{H} = \{\mu\}$, we write $\mathcal{V}_m(\mu)$ instead of $\mathcal{V}_m(\{\mu\})$, where

$$\mathcal{V}_m(\mu) = \{\mathcal{Q} \in \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L}) \mid \mu \in \mathcal{Q}\}.$$

With these definitions and observations in place, we arrive at the following result:

Lemma 4. *Let $\mathcal{G}$ and $\mathcal{U}$ be two $\mathcal{D}$-filters of an ADL $\mathcal{L}$. Then the following properties hold:*

(1) $\mathcal{V}_m(\mathcal{G}) \cap \mathcal{V}_m(\mathcal{U}) = \mathcal{V}_m(\mathcal{G} \vee \mathcal{U})$,

(2) $\mathcal{V}_m(\mathcal{G}) = Spec^{\mathcal{D}}_{MF}(\mathcal{L}) \Leftrightarrow \mathcal{G} = \mathcal{D}$,

(3) $\mathcal{J}_m(\mu) = \mathcal{V}_m((\mu, \mathcal{D}))$, *for all* $\mu \in \mathcal{L}$,

(4) $\mathcal{V}_m(\mu) = \mathcal{J}_m((\mu, \mathcal{D}))$.

Proof. (1) Clear.

(2) Suppose $\mathcal{V}_m(\mathcal{G}) = \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$. This means that every minimal prime $\mathcal{D}$-filter $\mathcal{Q}$ contains $\mathcal{G}$. Take any element $\mu \in \mathcal{G}$. Assume, contrary to what we want to prove, that $\mu \notin \mathcal{D}$. Then, according to Proposition 3, there exists some $\mathcal{Q} \in \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$ such that $\mu \notin \mathcal{Q}$. As a result, $\mathcal{G} \nsubseteq \mathcal{Q}$, which contradicts our initial assumption. Therefore, $\mu \in \mathcal{D}$ for all $\mu \in \mathcal{G}$, and thus $\mathcal{G} \subseteq \mathcal{D}$. Since $\mathcal{G}$ is a $\mathcal{D}$-filter, we conclude that $\mathcal{G} = \mathcal{D}$.

Conversely, if $\mathcal{G} = \mathcal{D}$, then it is immediate that $\mathcal{G} \subseteq \mathcal{Q}$ for all $\mathcal{Q} \in \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$, and hence $\mathcal{V}_m(\mathcal{G}) = \mathrm{Spec}^{\mathcal{D}}_{MF}(\mathcal{L})$.

(3) Let $\mathcal{Q} \in Spec^{\mathcal{D}}_{MF}(\mathcal{L})$. Then $\mathcal{Q} \in \mathcal{J}_m(\mu) \Leftrightarrow \mu \notin \mathcal{Q} \Leftrightarrow (\mu, \mathcal{D}) \subseteq \mathcal{Q} \Leftrightarrow \mathcal{Q} \in \mathcal{V}_m((\mu, \mathcal{D}))$.

(4) The proof proceeds similarly.

The following two theorems demonstrate some topological properties of the space $Spec^{\mathcal{D}}_{MF}(\mathcal{L})$, which represents the collection of all minimal prime $\mathcal{D}$-filters in an ADL $\mathcal{L}$.

Theorem 3. *For any ADL* $\mathcal{L}$, *the space* $Spec^{\mathcal{D}}_{MF}(\mathcal{L})$ *is compact if and only if for every filter* $\mathcal{G}$ *of* $\mathcal{L}$, *the condition* $\mathcal{V}_m(\mathcal{G}) = \emptyset$ *implies that* $\mathcal{G} \cap \mathcal{D}^{\infty} \neq \emptyset$.

Proof. Assume that $Spec^{\mathcal{D}}_{MF}(\mathcal{L})$ is compact. Let $\mathcal{G}$ be a filter of $\mathcal{L}$ such that $\mathcal{V}_m(\mathcal{G}) = \emptyset$. Then for every $\mathcal{Q} \in Spec^{\mathcal{D}}_{MF}(\mathcal{L})$, $\mathcal{G}$ is not a subset of $\mathcal{Q}$. Thus, $Spec^{\mathcal{D}}_{MF}(\mathcal{L}) = \mathcal{J}_m(\mathcal{G}) = \bigcup_{\mu \in \mathcal{G}} \mathcal{J}_m(\mu)$. Since $Spec^{\mathcal{D}}_{MF}(\mathcal{L})$ is compact, there exists $\theta \in \mathcal{G}$ such that $Spec^{\mathcal{D}}_{MF}(\mathcal{L}) = \mathcal{J}_m(\theta)$. By Lemma 3 (6), we deduce that $\theta \in \mathcal{D}^{\infty}$. Therefore, $\mathcal{G} \cap \mathcal{D}^{\infty} \neq \emptyset$. Conversely, assume that for every $\mathcal{D}$-filter $\mathcal{G}$ of $\mathcal{L}$, if $\mathcal{V}_m(\mathcal{G}) = \emptyset$, then $\mathcal{G} \cap \mathcal{D}^{\infty} \neq \emptyset$. Let $\mathcal{H} \subseteq \mathcal{L}$ be such that $Spec^{\mathcal{D}}_{MF}(\mathcal{L}) = \bigcup_{\theta \in \mathcal{H}} \mathcal{J}_m(\theta) = \mathcal{J}_m(\mathcal{H}) = \mathcal{J}_m(\mathcal{G})$, where $\mathcal{G}$ is the $\mathcal{D}$-filter generated by $\mathcal{H}$. Now, choose $\sigma \in \mathcal{G} \cap \mathcal{D}^{\infty}$. Then, we can express $\sigma = \bigwedge_{i=1}^{n} \theta_i$ for some $\theta_1, \theta_2, \ldots, \theta_n \in \mathcal{H}$ and $n \in \mathbb{N}$. Hence, by Lemma 3 (6), we obtain $Spec^{\mathcal{D}}_{MF}(\mathcal{L}) = \mathcal{J}_m(\sigma) = \mathcal{J}_m(\bigwedge_{i=1}^{n} \theta_i) \subseteq \bigcup_{i=1}^{n} \mathcal{J}_m(\theta_i)$. This proves that $Spec^{\mathcal{D}}_{MF}(\mathcal{L})$ is compact.

Theorem 4. *For any ADL* $\mathcal{L}$, $Spec^{\mathcal{D}}_{MF}(\mathcal{L})$ *is a Hausdorff space.*

Proof. Let $\mathcal{Q}$ and $\mathcal{P}$ be two distinct elements of $Spec^{\mathcal{D}}_{MF}(\mathcal{L})$. Choose $\mu \in \mathcal{L}$ such that $\mu \in \mathcal{Q}$ and $\mu \notin \mathcal{P}$. This implies that $\mathcal{P} \in \mathcal{J}_m(\mu)$. Since $\mu \in \mathcal{Q}$ and $\mathcal{Q}$ is minimal, there exists $\pi \notin \mathcal{Q}$ such that $\mu \vee \pi \in \mathcal{D}$. Thus, $\mathcal{Q} \in \mathcal{J}_m(\pi)$ and also $\mathcal{J}_m(\mu) \cap \mathcal{J}_m(\pi) = \mathcal{J}_m(\mu \vee \pi) = \emptyset$ due to Lemma 3 (5). Hence, $Spec^{\mathcal{D}}_{MF}(\mathcal{L})$ is a Hausdorff space.

In the following theorem, a set of equivalent conditions is derived for an ADL to be classified as hemicomplemented, which subsequently leads to a topological characterization.

Theorem 5. *The equivalence of the following conditions holds in an ADL* $\mathcal{L}$:

(1) $\mathcal{L}$ *is hemicomplemented,*

(2) *for each* $\mu \in \mathcal{L}$*, there exists* $\pi \in \mathcal{L}$ *such that* $\mathcal{V}_m(\mu) = \mathcal{J}_m(\pi)$*,*

(3) *for each* $\mu \in \mathcal{L}$*, there exists* $\pi \in \mathcal{L}$ *such that* $\mathcal{J}_m(\mu) = \mathcal{J}_m((\pi, \mathcal{D}))$*.*

Proof. (1) $\Rightarrow$ (2) : Assume (1). Let $\mu \in \mathcal{L}$. Then there is $\pi \in \mathcal{L}$ such that $(\mu, \mathcal{D}) = ((\pi, \mathcal{D}), \mathcal{D})$. By Lemma 3 (6) and Lemma 4 (4), we obtain $\mathcal{V}_m(\mu) = \mathcal{J}_m((\mu, \mathcal{D})) = \mathcal{J}_m((\pi, \mathcal{D}), \mathcal{D}) = \mathcal{J}_m(\pi)$.

(2) $\Rightarrow$ (3) : Assume (2). Let $\mu \in \mathcal{L}$. Then there is $\pi \in \mathcal{L}$ such that $\mathcal{J}_m(\mu) = \mathcal{V}_m(\pi)$. By Lemma 3 (6), we obtain $\mathcal{V}_m(\pi) = \mathcal{J}_m((\pi, \mathcal{D}))$. Thus, $\mathcal{J}_m(\mu) = \mathcal{J}_m((\pi, \mathcal{D}))$.

(3) $\Rightarrow$ (1) : Assume (3). Let $\mu \in \mathcal{L}$. From condition (3), there exists a $\pi \in \mathcal{L}$ such that $\mathcal{J}_m(\mu) = \mathcal{J}_m((\pi, \mathcal{D}))$. Now, let $\theta \notin (\pi, \mathcal{D})$. Since $\theta \vee \pi \notin \mathcal{D}$, by the properties of minimal prime $\mathcal{D}$-filters, there exists a minimal prime $\mathcal{D}$-filter $\mathcal{Q}$ such that $\theta \vee \pi \notin \mathcal{Q}$. This implies $\theta \notin \mathcal{Q}$ and $\pi \notin \mathcal{Q}$. Because $\pi \notin \mathcal{Q}$, we deduce that $(\pi, \mathcal{D}) \subseteq \mathcal{Q}$, which means $\mathcal{Q} \notin \mathcal{J}_m((\pi, \mathcal{D})) = \mathcal{J}_m(\mu)$. Therefore, we conclude that $\mu \in \mathcal{Q}$. Since $\mathcal{Q}$ is minimal, we get $((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. As $\theta \notin \mathcal{Q}$, it follows that $\theta \notin ((\mu, \mathcal{D}), \mathcal{D})$, so we conclude $((\mu, \mathcal{D}), \mathcal{D}) \subseteq (\pi, \mathcal{D})$. By similar reasoning, we can also obtain $(\pi, \mathcal{D}) \subseteq ((\mu, \mathcal{D}), \mathcal{D})$. Hence, we establish that $\mathcal{L}$ is hemicomplemented.

It can be easily seen that the collection $\mathcal{J}(\mathcal{L}) = \{\mathcal{J}_m(\mu) \mid \mu \in \mathcal{L}\}$ forms a distributive lattice with respect to the set operations $\cap$ and $\cup$. However, in general, $\langle\{\mathcal{J}_m(\mu) \mid \mu \in \mathcal{L}\}, \cap, \cup\rangle$ does not form a Boolean algebra for an ADL $\mathcal{L}$. The next theorem establishes a necessary and sufficient condition for this collection to become a Boolean algebra.

Theorem 6. *An ADL* $\mathcal{L}$ *is hemicomplemented if and only if* $\mathcal{J}(\mathcal{L}) = \langle\{\mathcal{J}_m(\mu) \mid \mu \in \mathcal{L}\}, \cap, \cup\rangle$ *is a Boolean algebra.*

Proof. Assume $\mathcal{L}$ is a hemicomplemented ADL. Let $\mathcal{J}_m(\mu) \in \mathcal{J}(\mathcal{L})$. Then there is $\pi \in \mathcal{L}$ such that $\mu \vee \pi \in \mathcal{D}$ and $(\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = (\mu \wedge \pi, \mathcal{D}) = \mathcal{D}$. Hence, $\mathcal{J}_m(\mu) \cap \mathcal{J}_m(\pi) = \mathcal{J}_m(\mu \vee \pi) = \emptyset$. Also $\mathcal{J}_m(\mu) \cup \mathcal{J}_m(\pi) = \mathcal{V}_m((\mu, \mathcal{D})) \cup \mathcal{V}_m((\pi, \mathcal{D})) = \mathcal{V}_m((\mu, \mathcal{D}) \cap (\pi, \mathcal{D})) = \mathcal{V}_m(\mathcal{D}) = Spec^{\mathcal{D}}_{MF}(\mathcal{L})$. Hence, $\mathcal{J}_m(\pi)$ is the complement of $\mathcal{J}_m(\mu)$ in $\mathcal{J}(\mathcal{L})$. Thus $\mathcal{J}(\mathcal{L})$ is a Boolean algebra. Conversely, assume that $\mathcal{J}(\mathcal{L})$ is a Boolean algebra. Let $\mu \in \mathcal{L}$. Then $\mathcal{J}_m(\mu) \in \mathcal{J}(\mathcal{L})$. Then there exists $\mathcal{J}_m(\pi) \in \mathcal{J}(\mathcal{L})$ such that $\mathcal{J}_m(\mu \wedge \pi) = \mathcal{J}_m(\mu) \cap \mathcal{J}_m(\pi) = \emptyset$ and $\mathcal{J}_m(\mu) \cup \mathcal{J}_m(\pi) = Spec^{\mathcal{D}}_{MF}(\mathcal{L})$. Therefore, $\mu \vee \pi \in \mathcal{D}$. Also,

$$\begin{aligned}
\mathcal{J}_m(\mu) \cup \mathcal{J}_m(\pi) = Spec^{\mathcal{D}}_{MF}(\mathcal{L}) \;&\Rightarrow\; \mathcal{V}_m((\mu, \mathcal{D})) \cup \mathcal{V}_m((\pi, \mathcal{D})) = Spec^{\mathcal{D}}_{MF}(\mathcal{L}) \\
&\Rightarrow\; \mathcal{V}_m((\mu, \mathcal{D}) \cap (\pi, \mathcal{D})) = Spec^{\mathcal{D}}_{MF}(\mathcal{L}) \\
&\Rightarrow\; (\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = \mathcal{D}. \quad \text{(by Lemma 4(2))}
\end{aligned}$$

Therefore, $\mathcal{L}$ is hemicomplemented.

We now present a topological description of $\mathcal{D}$-stone ADLs. Let $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$ represent the collection of all prime $\mathcal{D}$-filters of an ADL $\mathcal{L}$. For any subset $\mathcal{S} \subseteq \mathcal{L}$, we define

$$\mathcal{J}(\mathcal{S}) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mathcal{S} \nsubseteq \mathcal{Q}\},$$
$$\mathcal{V}(\mathcal{S}) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mathcal{S} \subseteq \mathcal{Q}\}.$$

In the case where $\mathcal{S} = \{\mu\}$, we use the following simplified notations:

$$\mathcal{J}(\mu) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mu \notin \mathcal{Q}\},$$
$$\mathcal{V}(\mu) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mu \in \mathcal{Q}\}.$$

Lemma 5. *For any pair of elements μ and π in $\mathcal{L}$, the following properties are true:*

(1) $\bigcup_{\mu \in \mathcal{L}} \mathcal{J}(\mu) = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$,

(2) $\mathcal{J}(\mu) \cap \mathcal{J}(\pi) = \mathcal{J}(\mu \vee \pi)$,

(3) $\mathcal{J}(\mu) \cup \mathcal{J}(\pi) = \mathcal{J}(\mu \wedge \pi)$,

(4) $\mathcal{J}(\mu) = \emptyset \Leftrightarrow \mu \in \mathcal{D}$,

(5) $\mathcal{J}(0) = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$.

Proof. (1) Let $\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$. Since $\mathcal{Q}$ is a proper prime $\mathcal{D}$-filter, it omits at least one element $\mu \in \mathcal{L}$, so $\mu \notin \mathcal{Q}$. Thus, $\mathcal{Q} \in \mathcal{J}(\mu)$ and therefore $\mathcal{Q} \in \bigcup_{\mu \in \mathcal{L}} \mathcal{J}(\mu)$. Conversely, any $\mathcal{Q}$ in $\bigcup_{\mu \in \mathcal{L}} \mathcal{J}(\mu)$ belongs to $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$ by definition. Hence,

$$\bigcup_{\mu \in \mathcal{L}} \mathcal{J}(\mu) = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}).$$

(2) By definition,

$$\mathcal{J}(\mu) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mu \notin \mathcal{Q}\} \text{ and } \mathcal{J}(\pi) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \pi \notin \mathcal{Q}\}.$$

Thus,

$$\mathcal{J}(\mu) \cap \mathcal{J}(\pi) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mu \notin \mathcal{Q} \text{ and } \pi \notin \mathcal{Q}\}.$$

Since $\mathcal{Q}$ is a prime $\mathcal{D}$-filter, $\mu \vee \pi \in \mathcal{Q}$ implies $\mu \in \mathcal{Q}$ or $\pi \in \mathcal{Q}$. Therefore, $\mu \notin \mathcal{Q}$ and $\pi \notin \mathcal{Q}$ together imply $\mu \vee \pi \notin \mathcal{Q}$. Conversely, if $\mu \vee \pi \notin \mathcal{Q}$, then $\mu \notin \mathcal{Q}$ and $\pi \notin \mathcal{Q}$. Hence,

$$\mathcal{J}(\mu) \cap \mathcal{J}(\pi) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mu \vee \pi \notin \mathcal{Q}\} = \mathcal{J}(\mu \vee \pi).$$

(3) By definition,

$$\mathcal{J}(\mu) \cup \mathcal{J}(\pi) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mu \notin \mathcal{Q} \text{ or } \pi \notin \mathcal{Q}\}.$$

Since filters are upwards closed, if $\mu \wedge \pi \in \mathcal{Q}$, then both $\mu, \pi \in \mathcal{Q}$. Conversely, if $\mu \wedge \pi \notin \mathcal{Q}$, at least one of μ or π is not in $\mathcal{Q}$. Thus,

$$\mathcal{J}(\mu) \cup \mathcal{J}(\pi) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mu \wedge \pi \notin \mathcal{Q}\} = \mathcal{J}(\mu \wedge \pi).$$

(4) If $\mu \in \mathcal{D}$, then $\mu \in \mathcal{Q}$ for all prime $\mathcal{D}$-filters $\mathcal{Q}$ because $\mathcal{D} \subseteq \mathcal{Q}$. Therefore, there does not exist any $\mathcal{Q}$ with $\mu \notin \mathcal{Q}$, so $\mathcal{J}(\mu) = \emptyset$.

Conversely, if $\mathcal{J}(\mu) = \emptyset$, there is no prime $\mathcal{D}$-filter omitting μ, which implies $\mu \in \mathcal{Q}$ for all prime $\mathcal{D}$-filters $\mathcal{Q}$. Thus, $\mu \in \bigcap_{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})} \mathcal{Q} = \mathcal{D}$, confirming that $\mu \in \mathcal{D}$.

(5) Since 0 is not contained in any proper filter, in particular, no prime $\mathcal{D}$-filter contains 0. Thus, for every $\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$, $0 \notin \mathcal{Q}$ and hence

$$\mathcal{J}(0) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid 0 \notin \mathcal{Q}\} = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}).$$

Based on the previous result, it is evident that the collection $\{\mathcal{J}(\mu) \mid \mu \in \mathcal{L}\}$ forms a basis for a topology on $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$. In the case of the topology on $Spec_{MF}^{\mathcal{D}}(\mathcal{L})$, the open set corresponding to any element $\mu \in \mathcal{L}$ is given by the intersection $Spec_{MF}^{\mathcal{D}}(\mathcal{L}) \cap \mathcal{J}(\mu)$, which is denoted as $\mathcal{J}_m(\mu)$. This topology on $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$ is referred to as the hull-kernel topology, where $\{\mathcal{V}(\mu) \mid \mu \in \mathcal{L}\}$ represents the hull, which serves as the basis, and $\{\mathcal{J}(\mu) \mid \mu \in \mathcal{L}\}$ corresponds to the kernel.

Lemma 6. *For arbitrary $\mathcal{D}$-filters $\mathcal{G}$ and $\mathcal{U}$ in $\mathcal{L}$, we have the following conditions:*

(1) $\mathcal{V}(\mathcal{G}) = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \Leftrightarrow \mathcal{G} = \mathcal{D}$,

(2) $\mathcal{V}(\mathcal{G}) = \emptyset \Leftrightarrow \mathcal{G} = \mathcal{L}$,

(3) $\mathcal{G} \subseteq \mathcal{U} \Rightarrow \mathcal{V}(\mathcal{U}) \subseteq \mathcal{V}(\mathcal{G})$,

(4) $\mathcal{V}(\mathcal{G}) \cap \mathcal{V}(\mathcal{U}) = \mathcal{V}(\mathcal{G} \vee \mathcal{U})$,

(5) $\mathcal{V}(\mathcal{G}) \cup \mathcal{V}(\mathcal{U}) = \mathcal{V}(\mathcal{G} \cap \mathcal{U})$.

Proof. (1) Since $\mathcal{G}$ is a $\mathcal{D}$-filter, we have $\mathcal{D} \subseteq \mathcal{G}$. Now, assume that $\mathcal{V}(\mathcal{G}) = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$. This implies that $\mathcal{G} \subseteq \mathcal{Q}$ for all $\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$. Therefore, we can conclude that $\mathcal{G} \subseteq \bigcap_{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})} \mathcal{Q} = \mathcal{D}$. Thus, we deduce that $\mathcal{G} = \mathcal{D}$. Conversely, suppose that $\mathcal{G} = \mathcal{D}$. Then, $\mathcal{G} = \mathcal{D}$ is contained in all $\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$, which implies that $\mathcal{V}(\mathcal{G}) = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$.

(2) Assume that $\mathcal{V}(\mathcal{G}) = \emptyset$. Suppose, for the sake of contradiction, that $\mathcal{G} \neq \mathcal{L}$. In this case, there exists a prime $\mathcal{D}$-filter $\mathcal{Q}$ such that $\mathcal{G} \subseteq \mathcal{Q}$. Consequently, $\mathcal{Q} \in \mathcal{V}(\mathcal{G})$, but since $\mathcal{V}(\mathcal{G}) = \emptyset$, this leads to a contradiction. Therefore, we conclude that $\mathcal{G} = \mathcal{L}$.

Conversely, let $\mathcal{G} = \mathcal{L}$. Since there is no prime $\mathcal{D}$-filter containing $\mathcal{G}$, we get $\mathcal{V}(\mathcal{G}) = \emptyset$.

(3) It is evident that, since $\mathcal{Q}$ is a prime $\mathcal{D}$-filter, properties (4) and (5) follow directly.

From the results above, it is evident that the collection $\{\mathcal{J}(\mathcal{G}) \mid \mathcal{G} \in \mathcal{G}^{\mathcal{D}}(\mathcal{L})\}$ forms a basis for a topology on $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$. In this hull-kernel topology, the open sets are of the form $\mathcal{J}(\mathcal{G})$, where $\mathcal{J}(\mathcal{G}) = \{\mathcal{Q} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \mathcal{G} \nsubseteq \mathcal{Q}\}$, and the closed sets are of the form $\mathcal{V}(\mathcal{G}) = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \setminus \mathcal{J}(\mathcal{G})$. For any subset $\mathcal{S}$ of $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$, the closure $\overline{\mathcal{S}}$ of $\mathcal{S}$ in the hull-kernel topology is given by $\overline{\mathcal{S}} = \{\mathcal{P} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \bigcap_{\mathcal{Q} \in \mathcal{S}} \mathcal{Q} \subseteq \mathcal{P}\}$.

Lemma 7. *For any element $\mu \in \mathcal{L}$, we have $\overline{\mathcal{J}(\mu)} = \mathcal{V}((\mu, \mathcal{D}))$.*

Proof. $\overline{\mathcal{J}(\mu)} = \{\mathcal{P} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \bigcap_{\mathcal{Q} \in \mathcal{J}(\mu)} \mathcal{Q} \subseteq \mathcal{P}\} = \{\mathcal{P} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid (\mu, \mathcal{D}) \subseteq \mathcal{P}\} = \mathcal{V}((\mu, \mathcal{D}))$.

Theorem 7. *An ADL $\mathcal{L}$ is $\mathcal{D}$-stone if and only if for any $\mu \in \mathcal{L}$, the closure $\overline{\mathcal{J}(\mu)}$ is open in the hull-kernel topology on $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$.*

Proof. Assume that $\mathcal{L}$ is $\mathcal{D}$-stone. Let $\mu \in \mathcal{L}$. Then

$$\begin{aligned}
\overline{\mathcal{J}(\mu)} &= \{\mathcal{P} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid \bigcap_{\mathcal{Q} \in \mathcal{J}(\mu)} \mathcal{Q} \subseteq \mathcal{P}\} \\
&= \{\mathcal{P} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid (\mu, \mathcal{D}) \subseteq \mathcal{P}\} \\
&= \{\mathcal{P} \in Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \mid ((\mu, \mathcal{D}), \mathcal{D}) \nsubseteq \mathcal{P}\} \quad \text{(since } \mathcal{L} \text{ is } \mathcal{D}\text{-stone)} \\
&= \mathcal{J}(((\mu, \mathcal{D}), \mathcal{D})).
\end{aligned}$$

Since $\mathcal{J}(((\mu, \mathcal{D}), \mathcal{D}))$ is an open set in $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$, it follows that $\overline{\mathcal{J}(\mu)}$ must be open in $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$. On the other hand, suppose $\overline{\mathcal{J}(\mu)}$ is open in the hull-kernel topology on $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$. This means that $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \setminus \overline{\mathcal{J}(\mu)}$ is a closed set in $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$. Therefore, there must exist a prime $\mathcal{D}$-filter $\mathcal{G}$ of $\mathcal{L}$ such that $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L}) \setminus \overline{\mathcal{J}(\mu)} = \mathcal{V}(\mathcal{G})$. From earlier results, we know that $\overline{\mathcal{J}(\mu)} = \mathcal{V}((\mu, \mathcal{D}))$. As a result, $\mathcal{V}(\mathcal{G})$ and $\mathcal{V}((\mu, \mathcal{D}))$ are complementary sets in $Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$, so $\mathcal{V}((\mu, \mathcal{D})) \cup \mathcal{V}(\mathcal{G}) = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$ and $\mathcal{V}((\mu, \mathcal{D})) \cap \mathcal{V}(\mathcal{G}) = \emptyset$. By Lemma 5 (4) and (6), it follows that $\mathcal{V}((\mu, \mathcal{D}) \cap \mathcal{G}) = Spec_{\mathcal{G}}^{\mathcal{D}}(\mathcal{L})$ and $\mathcal{V}((\mu, \mathcal{D}) \vee \mathcal{G}) = \emptyset$. Applying Lemma 5 (1) and (2), we get $(\mu, \mathcal{D}) \cap \mathcal{G} = \mathcal{D}$ and $(\mu, \mathcal{D}) \vee \mathcal{G} = \mathcal{L}$. Since $(\mu, \mathcal{D}) \cap \mathcal{G} = \mathcal{D}$, by Proposition 1 (2), we conclude $\mathcal{G} \subseteq ((\mu, \mathcal{D}), \mathcal{D})$. Hence, we have $\mathcal{L} = (\mu, \mathcal{D}) \vee \mathcal{G} \subseteq (\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D})$. Thus, $\mathcal{L}$ is a $\mathcal{D}$-stone ADL.

5. Conclusion

This article investigates the structural characteristics of hemicomplemented ADLs and $\mathcal{D}$-stone ADLs, elucidating their interrelationship and providing a set of equivalent conditions that characterize when a hemicomplemented ADL is, in fact, a $\mathcal{D}$-stone ADL. Additionally, it examines topological characterizations of these classes of lattices, particularly through the prime spectrum of minimal prime $\mathcal{D}$-filters. These results not only extend the foundational understanding of ADLs but also reveal intricate connections between algebraic and topological properties.

As a direction for future research, the study proposes examining how congruence relations interact with hemicomplemented and $\mathcal{D}$-stone structures, potentially revealing new topological characterizations. Extending these results to modular and non-modular generalizations of ADLs is also anticipated to broaden the understanding of algebraic-topological connections in these lattices.

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