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Hemicomplemented Almost Distributive Lattices

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Abstract. This paper introduces the concept of condensed elements in an Almost Distributive Lattice(ADL) and explores their fundamental properties. It also presents the concept of hemicomplemented ADL and characterizes these ADLs using ideals, congruences, and minimal prime $\mathcal{D}$-filters. Furthermore, a collection of equivalent criteria is established to determine when a hemicomplemented ADL qualifies as a quasicomplemented ADL.

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Key Words and Phrases: Almost distributive lattice, hemicomplemented ADL, $\mathcal{D}$-filter, condensed element

1. Introduction

Swamy U.M. and Rao G.C. defined the concept of an Almost Distributive Lattice(ADL) as a common abstraction that includes various ring-theoretic generalizations of Boolean algebras and distributive lattices [1]. They defined ideals in ADLs analogous to those in distributive lattices and showed that the collection of principal ideals constitutes a distributive lattice, thereby facilitating the extension of lattice theory concepts to ADLs. In [2] presented the concept of $\mathcal{D}$-filters in ADLs, examining their important properties. In [3], introduced the concept of a hemicomplemented lattice and studied their properties. In this article, a new type of element, termed condensed elements, is defined within ADLs, and several of their essential characteristics are examined. The concept of hemicomplemented ADL is also introduced, along with the derivation of a number of equivalent statements

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determining when such ADLs can be regarded as quasicomplemented. Additionally, established a criterion that is both necessary and sufficient for a hemicomplemented ADL to satisfy the conditions of a Boolean algebra. Providing a characterization of hemicomplemented ADLs is proven using $\mathcal{D}$-filters and minimal prime $\mathcal{D}$-filters. This particular class of ADLs was further described through the ideals and congruences.

2. Preliminaries

This section presents fundamental definitions and key results from [1, 4], which will be referenced throughout the document.

Definition 1. *[1] A structure $(\mathcal{L}, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice(ADL) with zero if it fulfills the following conditions:*

(1) $(\theta \vee \vartheta) \wedge \sigma = (\theta \wedge \sigma) \vee (\vartheta \wedge \sigma)$;

(2) $\theta \wedge (\vartheta \vee \sigma) = (\theta \wedge \vartheta) \vee (\theta \wedge \sigma)$;

(3) $(\theta \vee \vartheta) \wedge \vartheta = \vartheta$;

(4) $(\theta \vee \vartheta) \wedge \theta = \theta$;

(5) $\theta \vee (\theta \wedge \vartheta) = \theta$;

(6) $0 \wedge \theta = 0$, *for any* $\theta, \vartheta, \sigma \in \mathcal{L}$.

To define a partial order $\leq$ on $\mathcal{L}$, consider the condition $\theta = \theta \wedge \vartheta$ or equivalently $\theta \vee \vartheta = \vartheta$ for every $\theta, \vartheta \in \mathcal{L}$. This condition ensures that $\theta \leq \vartheta$, establishing $\leq$ as a partial order on $\mathcal{L}$. When $m \in \mathcal{L}$ is maximal with respect to this partial order, it is referred to as *maximal*. The collection of all such maximal elements in $\mathcal{L}$ is indicated by $\mathfrak{M}(\mathcal{L})$.

ADL $\mathcal{L}$ exhibits many properties of a distributive lattice[5, 6], with the exception of non-commutativity of $\vee$ and $\wedge$ and lack of right distributivity of $\vee$ over $\wedge$, as highlighted in Swamy's work[1]. If either of these properties held, $\mathcal{L}$ would be classified as a distributive lattice. We define a non-void subset $\mathcal{I}$ of $\mathcal{L}$ as an ideal(a filter) if it satisfies that for any elements $\theta, \vartheta \in \mathcal{I}$ and $\mu \in \mathcal{L}$, the subset $\mathcal{I}$ must include $\theta \wedge \mu$ and $\theta \vee \vartheta$ ($\mu \vee \theta$ and $\theta \wedge \vartheta$). A maximal ideal (filter) contains every proper ideal (filter) of $\mathcal{L}$. The smallest ideal containing a subset $\mathcal{S}$ of $\mathcal{L}$ is defined as $(\mathcal{S}] := \{(\bigvee_{i=1}^{n} \theta_i) \wedge \mu \mid \theta_i \in \mathcal{S}, \mu \in \mathcal{L}, n \in \mathbb{N}\}$. A principal ideal generated by an element θ is denoted as $(\theta]$. Similarly, for each subset $\mathcal{S}$ of $\mathcal{L}$, the smallest filter containing $\mathcal{S}$ is defined as $[\mathcal{S}) := \{\mu \vee (\bigwedge_{i=1}^{n} \theta_i) \mid \theta_i \in \mathcal{S}, \mu \in \mathcal{L}, n \in \mathbb{N}\}$. A principal filter generated by an element θ is denoted as $[\theta)$. It is established that $(\theta] \vee (\vartheta] = (\theta \vee \vartheta]$ and $(\theta] \cap (\vartheta] = (\theta \wedge \vartheta]$ for any $\theta, \vartheta \in \mathcal{L}$. Represented all principal ideals of $\mathcal{L}$ by the set $(\mathcal{PI}(\mathcal{L}), \vee, \cap)$, this brings out a sublattice of the distributive lattice $(\mathcal{I}(\mathcal{L}), \vee, \cap)$ of all ideals of $\mathcal{L}$. Furthermore, the set $(\mathcal{F}(\mathcal{L}), \vee, \cap)$ of all filters of $\mathcal{L}$ forms a bounded distributive lattice. In an ADL[7], a prime ideal $\mathcal{Q}$ of $\mathcal{L}$ exists if and only if $\mathcal{L} \setminus \mathcal{Q}$ is a prime filter of $\mathcal{L}$. A prime ideal $\mathcal{Q}$ of an ADL is a minimal prime ideal if and only if to each $\mu \in \mathcal{Q}$ there exists $\pi \notin \mathcal{Q}$ such that $\mu \wedge \pi = 0$ (or equivalently, for any $\mu \in \mathcal{L}, \mu \notin \mathcal{Q}$ if and only if $(\mu)^* \subseteq \mathcal{Q}$). For each non-void subset $\mathcal{S}$ of $\mathcal{L}$, the set

$\mathcal{S}^* = \{\mu \in \mathcal{L} \mid \theta \wedge \mu = 0, \text{ for all } \theta \in \mathcal{S}\}$ is an ideal of $\mathcal{L}$. Generally, for every $\theta \in \mathcal{L}$, $\{\theta\}^* = (\theta)^*$, where $(\theta) = (\theta]$. The *annihilator* of an element $\theta \in \mathcal{L}$ is defined as the set $(\theta)^* = \{\mu \in \mathcal{L} \mid \mu \wedge \theta = 0\}$. If $(e)^* = \{0\}$ then an element $e \in \mathcal{L}$ is considered dense. Within $\mathcal{L}$, the set $\mathcal{D}$ is the set of dense elements. A filter of an ADL $\mathcal{L}$ can be obtained by the set $\mathcal{D}$. A ADL $\mathcal{L}$ is called normal [8] if every prime ideal contains a unique minimal prime ideal. An ADL $\mathcal{L}$ is called quasi-complemented [9] if to each $\theta \in \mathcal{L}$, $\theta \wedge \theta' = 0$ and $\theta \vee \theta' \in \mathcal{D}$, for some $\theta' \in \mathcal{L}$. An ADL with dense elements is considered quasi-complemented if and only if, for every $\mu \in \mathcal{L}$, there exists some $\mu' \in \mathcal{L}$ such that $(\mu)^{**} = (\mu')^*$. The ADL $\mathcal{L}$ is termed a generalized Stone ADL [10] when it satisfies the property $(\mu)^* \vee (\mu)^{**} = \mathcal{L}$ for all $\mu \in \mathcal{L}$. A unary operation $\theta \mapsto \theta^*$ on an ADL $\mathcal{L}$ is said to be *pseudo-complementation*[11] on $\mathcal{L}$ if, for any $\theta, \vartheta \in \mathcal{L}$, (1) $\theta \wedge \vartheta = 0$ implies $\theta^* \wedge \vartheta = \vartheta$; (2) $\theta \wedge \theta^* = 0$; (3) $(\theta \vee \vartheta)^* = \theta^* \wedge \vartheta^*$. It becomes clear that every pseudo-complemented ADL is quasi-complemented.

According to [2], a filter $\mathcal{G}$ of an ADL $\mathcal{L}$ is known as a *$\mathcal{D}$-filter* if $\mathcal{D} \subseteq \mathcal{G}$. The smallest $\mathcal{D}$-filter is $\mathcal{D}$. For any non empty subset $\mathcal{S}$ of $\mathcal{L}$, consider the set $(\mathcal{S}, \mathcal{D}) = \{\mu \in \mathcal{L} \mid \theta \vee \mu \in \mathcal{D} \text{ for all } \theta \in \mathcal{S}\}$. It is noted that $(\mathcal{L}, \mathcal{D}) = \mathcal{D}$ and $(\mathcal{D}, \mathcal{D}) = \mathcal{L}$. Furthermore, for any subset $\mathcal{S}$ of $\mathcal{L}$, $\mathcal{D} \subseteq (\mathcal{S}, \mathcal{D})$. For every $\theta \in \mathcal{L}$, $(\{\theta\}, \mathcal{D})$ is denoted as $(\theta, \mathcal{D})$. Therefore, $(m, \mathcal{D}) = \mathcal{L}$ for any $m \in \mathfrak{M}(\mathcal{L})$. $(\mathcal{S}, \mathcal{D})$ forms a $\mathcal{D}$-filter in $\mathcal{L}$ for each subset $\mathcal{S}$ in $\mathcal{L}$.

Lemma 1. [2] *Given two subsets $\mathcal{S}$, $\mathcal{T}$ of an ADL $\mathcal{L}$, the following holds*

(1) $\mathcal{S} \subseteq \mathcal{T} \Rightarrow (\mathcal{T}, \mathcal{D}) \subseteq (\mathcal{S}, \mathcal{D})$;
(2) $\mathcal{S} \subseteq ((\mathcal{S}, \mathcal{D}), \mathcal{D})$;
(3) $(\mathcal{S}, \mathcal{D}) = (((\mathcal{S}, \mathcal{D}), \mathcal{D}), \mathcal{D})$;
(4) $\mathcal{S} \subseteq \mathcal{D} \Leftrightarrow (\mathcal{S}, \mathcal{D}) = \mathcal{L}$.

Proposition 1. [2] *Given filters $\mathcal{G}$,$\mathcal{U}$ of an ADL $\mathcal{L}$, the following holds*

(1) $(\mathcal{G}, \mathcal{D}) \cap ((\mathcal{G}, \mathcal{D}), \mathcal{D}) = \mathcal{D}$;
(2) $\mathcal{G} \cap \mathcal{U} \subseteq \mathcal{D} \Rightarrow \mathcal{G} \subseteq (\mathcal{U}, \mathcal{D})$;
(3) $((\mathcal{G} \vee \mathcal{U}), \mathcal{D}) = (\mathcal{G}, \mathcal{D}) \cap (\mathcal{U}, \mathcal{D})$;
(4) $((\mathcal{G} \cap \mathcal{U}, \mathcal{D}), \mathcal{D}) = ((\mathcal{G}, \mathcal{D}), \mathcal{D}) \cap ((\mathcal{U}, \mathcal{D}), \mathcal{D})$.

The idea that $([\mu), \mathcal{D}) = (\mu, \mathcal{D})$ is obvious. It follows that $(0, \mathcal{D}) = \mathcal{D}$. The previously noted observations directly lead to the corollary that follows.

Corollary 1. *[2] For any $\mu, \pi, \psi \in \mathcal{L}$ we have the following*

(1) $([\mu), \mathcal{D}) = (\mu, \mathcal{D})$;

(2) $\mu \leq \pi \Rightarrow (\mu, \mathcal{D}) \subseteq (\pi, \mathcal{D})$;

(3) $(\mu \wedge \pi, \mathcal{D}) = (\mu, \mathcal{D}) \cap (\pi, \mathcal{D})$;

(4) $((\mu \vee \pi, \mathcal{D}), \mathcal{D}) = ((\mu, \mathcal{D}), \mathcal{D}) \cap ((\pi, \mathcal{D}), \mathcal{D})$;

(5) $(\mu, \mathcal{D}) = \mathcal{L} \Leftrightarrow \mu \in \mathcal{D}$;

(6) $(\mu, \mathcal{D}) = (\pi, \mathcal{D}) \Leftrightarrow (\mu \wedge \psi, \mathcal{D}) = (\pi \wedge \psi, \mathcal{D})$;

(7) $(\mu, \mathcal{D}) = (\pi, \mathcal{D}) \Leftrightarrow (\mu \vee \psi, \mathcal{D}) = (\pi \vee \psi, \mathcal{D})$.

3. Hemicomplemented ADLs

In this paper, the concept of hemicomplemented ADLs is presented, and these particular types of ADLs are described through the ideals, $\mathcal{D}$-filters, congruence relations, and minimal prime $\mathcal{D}$-filters. A group of equivalent conditions is also formulated to identify when a hemicomplemented ADL can be converted as a quasicomplemented ADL.

Now we begin with the following definition.

Definition 2. *An element μ in an ADL $\mathcal{L}$ is said to be condensed if the filter extension $(\mu, \mathcal{D})$ is equal to $\mathcal{D}$.*

It is evident that the zero element 0 of an ADL $\mathcal{L}$ is always a condensed element. Let us use the notation $\mathcal{D}^{\infty}$ to represent the collection of all condensed elements in $\mathcal{L}$. With this definition, we arrive at the following result.

Proposition 2. *The following statements are valid in an ADL $\mathcal{L}$*

(1) $\mathcal{D} \cap \mathcal{D}^{\infty} = \emptyset$;

(2) $\mathcal{D}^{\infty}$ *is an ideal in* $\mathcal{L}$.

Proof. (1) Assume that $\mu \in \mathcal{D} \cap \mathcal{D}^{\infty}$. By the definition of condensed elements, this implies that $(\mu, \mathcal{D}) = \mathcal{D}$. Since μ also belongs to $\mathcal{D}$, it follows that $\mathcal{L} = (\mu, \mathcal{D}) = \mathcal{D}$. Consequently, the zero element 0 must be in $\mathcal{D}$, which contradicts the assumption that $\mathcal{D}$ is a proper filter. Thus, we conclude that $\mathcal{D} \cap \mathcal{D}^{\infty} = \emptyset$.
(2) It is clear that 0 belongs to $\mathcal{D}^{\infty}$. Suppose $\mu, \pi \in \mathcal{D}^{\infty}$. By the definition of condensed elements, we have $(\mu, \mathcal{D}) = \mathcal{D}$ and $(\pi, \mathcal{D}) = \mathcal{D}$, which implies that $((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$ and $((\pi, \mathcal{D}), \mathcal{D}) = \mathcal{L}$. Therefore,$((\mu \vee \pi), \mathcal{D}), \mathcal{D}) = ((\mu, \mathcal{D}), \mathcal{D}) \cap ((\pi, \mathcal{D}), \mathcal{D}) = \mathcal{L}$. This leads to $((\mu \vee \pi), \mathcal{D}) = (\mathcal{L}, \mathcal{D}) = \mathcal{D}$, showing that $\mu \vee \pi \in \mathcal{D}^{\infty}$. Now, let $\mu \in \mathcal{D}^{\infty}$. Then $(\mu, \mathcal{D}) = \mathcal{D}$. For any $\pi \in \mathcal{L}$, we have $(\pi \wedge \mu, \mathcal{D}) \subseteq (\mu, \mathcal{D}) = \mathcal{D}$. Since $\mathcal{D}$ is always contained in $(\pi \wedge \mu, \mathcal{D})$, it follows that $(\pi \wedge \mu, \mathcal{D}) = \mathcal{D}$, so $\pi \wedge \mu \in \mathcal{D}^{\infty}$. Hence, $\mathcal{D}^{\infty}$ is an ideal in $\mathcal{L}$.

Proposition 3. 0 *is the unique condensed element in any dense ADL.*

Proof. Consider a dense ADL $\mathcal{L}$. In such ADLs, all elements except zero are dense. Suppose that μ is a condensed element in $\mathcal{L}$, meaning that $(\mu, \mathcal{D}) = \mathcal{D}$. Assume, contrary to what we aim to prove, that $\mu \neq 0$. Since $\mathcal{L}$ is dense, this implies that μ belongs to $\mathcal{D}$. Applying Corollary 1(5), we then obtain $(\mu, \mathcal{D}) = \mathcal{L}$, which contradicts the earlier conclusion that $(\mu, \mathcal{D}) = \mathcal{D}$. This contradiction forces us to reject the assumption that $\mu \neq 0$, so we conclude that $\mu = 0$. Therefore, the only condensed element in a dense ADL is the zero element, or equivalently, $\mathcal{D}^{\infty} = \{0\}$.

The following example illustrates that an ADL with a unique condensed element 0 does not necessarily become as a dense ADL.

Example 1. *Consider the set* $\mathcal{L} = \{0, 1, 2, 3, 4, 5, 6\}$, *with the operations* $\vee$ *(join) and* $\wedge$ *(meet) defined on* $\mathcal{L}$ *as follows:*

$\wedge$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	1	2	3	4	5	6
3	0	3	3	3	0	3	3
4	0	4	4	0	4	4	4
5	0	5	5	3	4	5	5
6	0	6	6	3	4	5	6

$\vee$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	1	2	3	5	5	6
4	4	1	2	5	4	5	6
5	5	1	2	5	5	5	6
6	6	1	2	6	6	6	6

Thus, $(\mathcal{L}, \vee, \wedge)$ *is an ADL. Clearly, we have the dense set* $\mathcal{D} = \{1, 2, 5, 6\}$. *It is evident that* $(0, \mathcal{D}) = \mathcal{D}$, *and* $\mathcal{L}$ *is not a dense ADL because non zero elements* $3, 4$ *are not dense.*

Proposition 4. *Consider an Almost Distributive Lattice (ADL)* $\mathcal{L}$, *and introduce a binary relation* Ψ *on* $\mathcal{L}$ *defined as follows:*

$$(\theta, \vartheta) \in \Psi \text{ if and only if } (\theta, \mathcal{D}) = (\vartheta, \mathcal{D}),$$

for every pair of elements $\theta, \vartheta \in \mathcal{L}$. *This relation* Ψ *constitutes a congruence on ADL* $\mathcal{L}$. *Under this congruence, the set* $\mathcal{D}^{\infty}$ *forms the least congruence class, while the class corresponding to the greatest element is given by* $\mathcal{D}$.

Proof. It is straightforward to verify that the relation Ψ defines an equivalence on the set $\mathcal{L}$. Moreover, using items (6) and (7) from Corollary 1, we conclude that Ψ indeed satisfies the necessary conditions to be a congruence relation on ADL $\mathcal{L}$. Now, suppose $\mu, \pi \in \mathcal{D}^{\infty}$. Since both elements generate the same extension filter, we have $(\mu, \pi) \in \Psi$, which implies that $\mathcal{D}^{\infty}$ forms a congruence class under this relation. Let θ be any element in $\mathcal{D}^{\infty}$. Given that $\mathcal{D}^{\infty}$ is an ideal, the meet $\theta \wedge \mu$ must also lie in $\mathcal{D}^{\infty}$ for every $\mu \in \mathcal{L}$. This implies that the intersection $[\theta]_{\Psi} \cap [\mu]_{\Psi} = [\theta \wedge \mu]_{\Psi} = [\theta]_{\Psi}$, since both θ and $\theta \wedge \mu$ belong to $\mathcal{D}^{\infty}$. Therefore, the equivalence class $[\theta]_{\Psi}$ is precisely $\mathcal{D}^{\infty}$, confirming that this is the minimal congruence class under Ψ. Conversely, since $\mathcal{D}$ is a filter, a dual argument reveals that $\mathcal{D}$ corresponds to the greatest congruence class with respect to the congruence Ψ.

We now define the concept of a hemicomplemented ADL.

Definition 3. *An ADL* $\mathcal{L}$ *is termed hemicomplemented if for every element* μ *in* $\mathcal{L}$, *there is an element* $\pi \in \mathcal{L}$ *such that* $\mu \wedge \pi \in \mathcal{D}^{\infty}$ *and* $\mu \vee \pi \in \mathcal{D}$.

It can be observed that every pseudocomplemented ADL is also hemicomplemented. To see this, let $\mathcal{L}$ be a pseudocomplemented ADL. Then, for any $\mu \in \mathcal{L}$, there exists an element $\mu^* \in \mathcal{L}$ such that $\mu \wedge \mu^* = 0$, where $0 \in \mathcal{D}^{\infty}$. Additionally, it is clear that $\mu \vee \mu^* \in \mathcal{D}$. Thus, $\mathcal{L}$ satisfies the condition for being hemicomplemented. In the same way, it follows that every quasicomplemented ADL also possesses the hemicomplemented property. In the upcoming theorem, we present a collection of equivalent statements that characterize when a hemicomplemented ADL qualifies as a quasicomplemented ADL.

Theorem 1. *For a hemicomplemented ADL $\mathcal{L}$, the following conditions are all equivalent.*

(1) $\mathcal{L}$ is quasicomplemented;

(2) for any $\mu, \pi \in \mathcal{L} \setminus \mathcal{D}$, $(\mu, \mathcal{D}) = (\pi, \mathcal{D})$ implies $\mu = \pi$;

(3) $\mathcal{L}$ has a unique condensed element.

Proof. (1) $\Rightarrow$ (2) : Assume (1). Let $\mu, \pi \in \mathcal{L} \setminus \mathcal{D}$ with $(\mu, \mathcal{D}) = (\pi, \mathcal{D})$. Assume, for contradiction, that $\mu \neq \pi$. Then either $(\mu] \cap ([\pi) \vee \mathcal{D}) = \emptyset$ or $(\pi] \cap ([\mu) \vee \mathcal{D}) = \emptyset$. Suppose $(\pi] \cap ([\mu) \vee \mathcal{D}) = \emptyset$. In this case, there exists a prime filter $\mathcal{Q}$ containing $([\mu) \vee \mathcal{D})$ but disjoint from the principal ideal $(\pi]$. This gives $\mu \in \mathcal{Q}$ (since $\mu \in ([\mu) \vee \mathcal{D})$) and $\pi \notin \mathcal{Q}$. Now, as $\mathcal{L}$ is quasicomplemented, there exists an element $\mu' \in \mathcal{L}$ such that $\mu \wedge \mu' = 0$ and $\mu \vee \mu' \in \mathcal{D}$. Since $(\mu, \mathcal{D}) = (\pi, \mathcal{D})$, we conclude $\mu' \in (\pi, \mathcal{D})$, and therefore $\mu' \vee \pi \in \mathcal{D} \subseteq \mathcal{Q}$. Because $\mathcal{Q}$ is a prime filter and $\mu' \vee \pi \in \mathcal{Q}$ while $\pi \notin \mathcal{Q}$, it must be that $\mu' \in \mathcal{Q}$. But this implies that $\mu \wedge \mu' = 0 \in \mathcal{Q}$, which contradicts the definition of a proper filter. Thus, our initial assumption that $\mu \neq \pi$ is incorrect. Therefore, $\mu = \pi$.
(2) $\Rightarrow$ (3) : Suppose that the second condition is valid. Assume, for contradiction, that there exist two different condensed elements in $\mathcal{L}$, denoted by μ and π. Since both are condensed, it follows that $(\mu, \mathcal{D}) = \mathcal{D} = (\pi, \mathcal{D})$. According to condition (2), this implies that μ and π must be equal, contradicting our assumption that they are distinct. Thus, it follows that $\mathcal{L}$ contains exactly one condensed element.
(3) $\Rightarrow$ (1) : Let us assume that the only condensed element in $\mathcal{L}$ is 0. Consider any element μ in $\mathcal{L}$. Since $\mathcal{L}$ is hemicomplemented , there exists an element $\mu' \in \mathcal{L}$ such that $\mu \wedge \mu'$ belongs to $\mathcal{D}^{\infty}$ and $\mu \vee \mu'$ lies in $\mathcal{D}$. Given that $\mathcal{D}^{\infty} = 0$, we conclude that $\mu \wedge \mu' = 0$. This confirms that $\mathcal{L}$ is a quasicomplemented ADL.

Corollary 2. *A hemicomplemented ADL $\mathcal{L}$ is a Boolean algebra precisely when it has exactly one condensed element and exactly one dense element.*

Based on Lemma 1, one can readily establish that the collection $\mathcal{D}^{\circ}(\mathcal{L})$—comprising all filters of the form $(\mu, \mathcal{D})$ for elements $\mu \in \mathcal{L}$ constitutes an Almost Distributive Lattice (ADL) under the binary operations:

$$(\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = (\mu \wedge \pi, \mathcal{D}) \text{ and } (\mu, \mathcal{D}) \sqcup (\pi, \mathcal{D}) = (\mu \vee \pi, \mathcal{D}),$$

for every $\mu, \pi \in \mathcal{L}$. In the subsequent theorem, a collection of equivalent criteria is provided that determine when the lattice $(\mathcal{D}^{\circ}(\mathcal{L}), \sqcup, \cap)$ becomes a Boolean algebra resulting in a characterization of hemicomplemented ADLs.

Theorem 2. *The following assertions are equivalent in an ADL $\mathcal{L}$:*

(1) $\mathcal{L}$ is hemicomplemented;

(2) $(\mathcal{D}^{\circ}(\mathcal{L}), \sqcup, \cap)$ is a Boolean algebra;

(3) $\mathcal{L}/\Psi$ is a Boolean algebra;

(4) for each $\mu \in \mathcal{L}$*, there exists* $\pi \in \mathcal{L}$ *such that* $((\mu, \mathcal{D}), \mathcal{D}) = (\pi, \mathcal{D})$*;*

(5) For any $\mathcal{D}$*-filter* $\mathcal{G}$ *of* $\mathcal{L}$ *with* $\mathcal{G} \cap \mathcal{D}^{\infty} = \emptyset$*, there exists a minimal prime* $\mathcal{D}$*-filter* $\mathcal{Q}$ *of* $\mathcal{L}$ *such that* $\mathcal{G} \subseteq \mathcal{Q}$*.*

Proof. (1) $\Rightarrow$ (2) : Assume (1). Let $(\mu, \mathcal{D}) \in \mathcal{D}^{\circ}(\mathcal{L})$. Then there exists $\pi \in \mathcal{L}$ such that $\mu \wedge \pi \in \mathcal{D}^{\infty}$ and $\mu \vee \pi \in \mathcal{D}$. Hence $(\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = (\mu \wedge \pi, \mathcal{D}) = \mathcal{D}$ and $(\mu, \mathcal{D}) \sqcup (\pi, \mathcal{D}) = (\mu \vee \pi, \mathcal{D}) = \mathcal{L}$. Hence $(\mathcal{D}^{\circ}(\mathcal{L}), \sqcup, \cap)$ is a Boolean algebra.
(2) $\Rightarrow$ (3) : Assume (2). Consider any element $[\mu]_{\Psi}$ in $\mathcal{L}/\Psi$. Given that $\mathcal{D}^{\circ}(\mathcal{L})$ forms a Boolean algebras, there exists some $\pi \in \mathcal{L}$ such that $(\mu \wedge \pi, \mathcal{D}) = (\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = \mathcal{D}$, and $(\mu \vee \pi, \mathcal{D}) = (\mu, \mathcal{D}) \sqcup (\pi, \mathcal{D}) = \mathcal{L}$. This leads to $\mu \wedge \pi \in \mathcal{D}^{\infty}$ and $\mu \vee \pi \in \mathcal{D}$. Consequently, we obtain $[\mu]_{\Psi} \cap [\pi]_{\Psi} = [\mu \wedge \pi]_{\Psi} = \mathcal{D}^{\infty}$ and $[\mu]_{\Psi} \vee [\pi]_{\Psi} = [\mu \vee \pi]_{\Psi} = \mathcal{D}$. Therefore, $\mathcal{L}/\Psi$ satisfies all the condition for being a Boolean algebra.
(3) $\Rightarrow$ (4) : Assume (3). Let $\mu \in \mathcal{L}$. Then there is $[\pi]_{\Psi} \in \mathcal{L}/\Psi$ such that $[\mu \wedge \pi]_{\Psi} = [\mu]_{\Psi} \cap [\pi]_{\Psi} = \mathcal{D}^{\infty}$ and $[\mu \vee \pi]_{\Psi} = [\mu]_{\Psi} \vee [\pi]_{\Psi} = \mathcal{D}$. Therefore $\mu \wedge \pi \in \mathcal{D}^{\infty}$ and $\mu \vee \pi \in \mathcal{D}$. Now,

$$\begin{aligned} \mu \wedge \pi \in \mathcal{D}^{\infty} &\Rightarrow (\mu \wedge \pi, \mathcal{D}) = \mathcal{D} \\ &\Rightarrow (\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = \mathcal{D} \\ &\Rightarrow (\pi, \mathcal{D}) \subseteq ((\mu, \mathcal{D}), \mathcal{D}) \end{aligned}$$

$$\begin{aligned} \mu \vee \pi \in \mathcal{D} &\Rightarrow \mu \in (\pi, \mathcal{D}) \\ &\Rightarrow [\mu) \subseteq (\pi, \mathcal{D}) \\ &\Rightarrow ((\mu, \mathcal{D}), \mathcal{D}) \subseteq (\pi, \mathcal{D}). \end{aligned}$$

Hence, to every $\mu \in \mathcal{L}$, there exists $\pi \in \mathcal{L}$ such that $((\mu, \mathcal{D}), \mathcal{D}) = (\pi, \mathcal{D})$.
(4) $\Rightarrow$ (5) : Assume (4). Let $\mathcal{F}$ be a $\mathcal{D}$-filter in the ADL $\mathcal{L}$ such that $\mathcal{F} \cap \mathcal{D}^{\infty} = \emptyset$. Then, there exists a prime filter $\mathcal{P}$ in $\mathcal{L}$ that contains $\mathcal{F}$ and also remains disjoint from $\mathcal{D}^{\infty}$, i.e., $\mathcal{F} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{D}^{\infty} = \emptyset$. It is clear that such a $\mathcal{P}$ qualifies as a $\mathcal{D}$-filter. Our next objective is to show that $\mathcal{P}$ is minimal with this property. Take any element $\mu \in \mathcal{P}$. By assumption (specifically, condition (4)), there exists some $\pi \in \mathcal{L}$ satisfying $((\mu, \mathcal{D}), \mathcal{D}) = (\pi, \mathcal{D})$. Since μ belongs to $((\mu, \mathcal{D}), \mathcal{D})$, we get $\mu \in (\pi, \mathcal{D})$, which implies $\mu \vee \pi \in \mathcal{D}$. Now consider the meet $\mu \wedge \pi$. Using the properties of filters, we have:

$$(\mu \wedge \pi, \mathcal{D}) = (\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = (\mu, \mathcal{D}) \cap ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{D},$$

which leads to $\mu \wedge \pi \in \mathcal{D}^{\infty}$. However, since $\mathcal{P}$ is disjoint from $\mathcal{D}^{\infty}$, we conclude that $\mu \wedge \pi \notin \mathcal{P}$. Suppose now that $\pi \in \mathcal{P}$. Then $x \wedge y \in \mathcal{P}$, contradicting our earlier conclusion. Hence, $\pi \notin \mathcal{P}$. Thus, $\mathcal{P}$ is a minimal prime $\mathcal{D}$-filter of $\mathcal{L}$.
(5) $\Rightarrow$ (1) : Assume (5). Let $\mu \in \mathcal{L}$. It is evident that $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D})$ forms a $\mathcal{D}$-filter within $\mathcal{L}$. Suppose there exists a minimal prime $\mathcal{D}$-filter $\mathcal{Q}$ such that $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. Then, since $\mu \in ((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$, and also $(\mu, \mathcal{D}) \subseteq \mathcal{Q}$, the minimality of $\mathcal{Q}$ implies that $\mu \notin \mathcal{Q}$, which results in a contradiction. Therefore, the filter $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D})$

cannot be contained in any minimal prime $\mathcal{D}$-filter. Applying hypothesis (5), it gives that $\{(\mu,\mathcal{D}) \vee ((\mu,\mathcal{D}),\mathcal{D})\} \cap \mathcal{D}^{\infty} \neq \emptyset$. Let us select an element σ in $\{(\mu,\mathcal{D}) \vee ((\mu,\mathcal{D}),\mathcal{D})\} \cap \mathcal{D}^{\infty}$. Then $(\sigma,\mathcal{D}) = \mathcal{D}$, and there exist elements $\theta \in (\mu,\mathcal{D})$ and $\vartheta \in ((\mu,\mathcal{D}),\mathcal{D})$ such that $\sigma = \theta \wedge \vartheta$.

Since $\theta \in (\mu,\mathcal{D})$, it immediately follows that $\theta \vee \mu \in \mathcal{D}$. Moreover, as $\vartheta \in ((\mu,\mathcal{D}),\mathcal{D})$, we deduce that $(\mu,\mathcal{D}) \subseteq (\vartheta,\mathcal{D})$. From this, we have the following:

$$\begin{aligned}
\sigma = \theta \wedge \vartheta \quad &\Rightarrow \quad (\theta \wedge \vartheta,\mathcal{D}) = (\sigma,\mathcal{D}) = \mathcal{D} \\
&\Rightarrow \quad (\theta,\mathcal{D}) \cap (\vartheta,\mathcal{D}) = \mathcal{D} \\
&\Rightarrow \quad (\theta,\mathcal{D}) \cap (\mu,\mathcal{D}) \qquad \text{since } (\mu,\mathcal{D}) \subseteq (\vartheta,\mathcal{D}) = \mathcal{D} \\
&\Rightarrow \quad (\theta \wedge \mu,\mathcal{D}) = \mathcal{D} \\
&\Rightarrow \quad \theta \wedge \mu \in \mathcal{D}^{\infty}.
\end{aligned}$$

Thus $\mathcal{L}$ is hemicomplented.

In the following, another characterization is given for the congruence Ψ in a hemicomplemented ADL. For this, we observe another congruence defined in terms of $\mathcal{D}^{\infty}$.

Definition 4. *For every $\theta \in \mathcal{L}$, defined $\langle \theta,\mathcal{D}^{\infty}\rangle = \{\mu \in \mathcal{L} \mid \mu \wedge \theta \in \mathcal{D}^{\infty}\}$.*

Lemma 2. *For any $\theta,\vartheta \in \mathcal{L}$, We obtain the following:*

(1) $\langle \theta,\mathcal{D}^{\infty}\rangle$ is an ideal in $\mathcal{L}$;

(2) $\mathcal{D}^{\infty} \subseteq \langle \theta,\mathcal{D}^{\infty}\rangle$;

(3) $\theta \leq \vartheta$ implies $\langle \vartheta,\mathcal{D}^{\infty}\rangle \subseteq \langle a,\mathcal{D}^{\infty}\rangle$;

(4) $\langle \theta \vee \vartheta,\mathcal{D}^{\infty}\rangle = \langle \theta,\mathcal{D}^{\infty}\rangle \cap \langle \vartheta,\mathcal{D}^{\infty}\rangle$;

(5) $\theta \in \mathcal{D}^{\infty}$ iff $\langle \theta,\mathcal{D}^{\infty}\rangle = \mathcal{L}$.

Proof. (1) It is evident that $0 \in \langle \theta,\mathcal{D}^{\infty}\rangle$. Let $\mu,\pi \in \langle \theta,\mathcal{D}^{\infty}\rangle$. Then $\theta \wedge \mu \in \mathcal{D}^{\infty}$ and $\theta \wedge \pi \in \mathcal{D}^{\infty}$. Hence $((\theta \wedge (\mu \vee \pi),\mathcal{D}),\mathcal{D}) = (((\theta \wedge \mu) \vee (\theta \wedge \pi),\mathcal{D}),\mathcal{D}) = (((\theta \wedge \mu),\mathcal{D}),\mathcal{D}) \cap (((\theta \wedge \pi),\mathcal{D}),\mathcal{D}) = \mathcal{L} \cap \mathcal{L} = \mathcal{L}$. Hence $(\theta \wedge (\mu \vee \pi),\mathcal{D}) = (\mathcal{L},\mathcal{D}) = \mathcal{D}$, which gives that $\theta \wedge (\mu \vee \pi) \in \mathcal{D}^{\infty}$. Thus $\mu \vee \pi \in \langle \theta,\mathcal{D}^{\infty}\rangle$. Let $\mu \in \langle \theta,\mathcal{D}^{\infty}\rangle$ and $\pi \leq \mu$. Then $\pi \wedge \theta \leq \mu \wedge \theta \in \mathcal{D}^{\infty}$. Hence $\pi \in \langle \theta,\mathcal{D}^{\infty}\rangle$, which leads that $\langle \theta,\mathcal{D}^{\infty}\rangle$ is an ideal in $\mathcal{L}$.
(2) It follows directly.
(3) It follows directly.
(4) It can be seen that $\langle \theta \vee \vartheta,\mathcal{D}^{\infty}\rangle \subseteq \langle \theta,\mathcal{D}^{\infty}\rangle \cap \langle \vartheta,\mathcal{D}^{\infty}\rangle$. Let $\mu \in \langle \theta,\mathcal{D}^{\infty}\rangle \cap \langle \vartheta,\mathcal{D}^{\infty}\rangle$. Then $\theta \wedge \mu \in \mathcal{D}^{\infty}$ and $\vartheta \wedge \mu \in \mathcal{D}^{\infty}$. As $\mathcal{D}^{\infty}$ forms an ideal, we obtain $(\theta \vee \vartheta) \wedge \mu = (\theta \wedge \mu) \vee (\vartheta \wedge \mu) \in \mathcal{D}^{\infty}$. Thus $\mu \in \langle \theta \vee \vartheta,\mathcal{D}^{\infty}\rangle$.
(5) Assume that $\theta \in \mathcal{D}^{\infty}$. As $\mathcal{D}^{\infty}$ forms an ideal, we obtain $\theta \wedge \mu \in \mathcal{D}^{\infty}$ for each $\mu \in \mathcal{L}$. Hence $\langle \theta,\mathcal{D}^{\infty}\rangle = \mathcal{L}$. On the other hand, suppose that $\langle \theta,\mathcal{D}^{\infty}\rangle = \mathcal{L}$. Then $m \in \langle \theta,\mathcal{D}^{\infty}\rangle$, for all maximal elements m in $\mathcal{L}$. Hence $\theta = m \wedge \theta \in \mathcal{D}^{\infty}$.

Proposition 5. *Define a relation $\Psi_{\mathcal{D}^\infty}$ on $\mathcal{L}$ as follows:*

$$(\theta, \vartheta) \in \Psi_{\mathcal{D}^\infty} \text{ if and only if } \langle \theta, \mathcal{D}^\infty \rangle = \langle \vartheta, \mathcal{D}^\infty \rangle$$

for all $\theta, \vartheta \in \mathcal{L}$. Then $\Psi_{\mathcal{D}^\infty}$ is congruence on $\mathcal{L}$.

Proof. It is evident that $\Psi_{\mathcal{D}^\infty}$ is an equivalence relation on $\mathcal{L}$. Let $(\mu, \pi) \in \Psi_{\mathcal{D}^\infty}$. For each $\sigma \in \mathcal{L}$, we obtain $\langle \mu \vee \sigma, \mathcal{D}^\infty \rangle = \langle \mu, \mathcal{D}^\infty \rangle \cap \langle \sigma, \mathcal{D}^\infty \rangle = \langle \pi, \mathcal{D}^\infty \rangle \cap \langle \sigma, \mathcal{D}^\infty \rangle = \langle \pi \vee \sigma, \mathcal{D}^\infty \rangle$. Thus $(\mu \vee \sigma, \pi \vee \sigma) \in \Psi_{\mathcal{D}^\infty}$. For any $\nu \in \mathcal{L}$, we get $\nu \in \langle \mu \wedge \sigma, \mathcal{D}^\infty \rangle \Leftrightarrow \nu \wedge \mu \wedge \sigma \in \mathcal{D}^\infty \Leftrightarrow \nu \wedge \sigma \in \langle \mu, \mathcal{D}^\infty \rangle = \langle \pi, \mathcal{D}^\infty \rangle \Leftrightarrow \nu \wedge \sigma \wedge \pi \in \mathcal{D}^\infty \Leftrightarrow \nu \in \langle \pi \wedge \sigma, \mathcal{D}^\infty \rangle$. Thus $\langle \mu \wedge \sigma, \mathcal{D}^\infty \rangle = \langle \pi \wedge \sigma, \mathcal{D}^\infty \rangle$. Therefore $(\mu \wedge \sigma, \pi \wedge \sigma) \in \Psi_{\mathcal{D}^\infty}$. Thus $\Psi_{\mathcal{D}^\infty}$ is a congruence.

Theorem 3. *In a hemicomplemented ADL $\mathcal{L}$, we have $\theta = \Psi_{\mathcal{D}^\infty}$.*

Proof. Let $(\mu, \pi) \in \theta$. Then we obtain $(\mu, \mathcal{D}) = (\pi, \mathcal{D})$. For any $\nu \in \mathcal{L}$, we have the following:

$$\begin{aligned}
\nu \in \langle \mu, \mathcal{D}^\infty \rangle &\Leftrightarrow \mu \wedge \nu \in \mathcal{D}^\infty \\
&\Leftrightarrow (\mu, \mathcal{D}) \cap (\nu, \mathcal{D}) = (\mu \wedge \nu, \mathcal{D}) = \mathcal{D} \\
&\Leftrightarrow (\pi \wedge \nu, \mathcal{D}) = (\pi, \mathcal{D}) \cap (\nu, \mathcal{D}) = \mathcal{D} \\
&\Leftrightarrow \pi \wedge \nu \in \mathcal{D}^\infty \\
&\Leftrightarrow \nu \in \langle \pi, \mathcal{D}^\infty \rangle.
\end{aligned}$$

Hence $\langle \mu, \mathcal{D}^\infty \rangle = \langle \pi, \mathcal{D}^\infty \rangle$. Thus $(\mu, \pi) \in \Psi_{\mathcal{D}^\infty}$. Therefore $\theta \subseteq \Psi_{\mathcal{D}^\infty}$. Conversely, let $(\mu, \pi) \in \Psi_{\mathcal{D}^\infty}$ for $\mu, \pi \in \mathcal{L}$. Then $\langle \mu, \mathcal{D}^\infty \rangle = \langle \pi, \mathcal{D}^\infty \rangle$. Since $\mathcal{L}$ is hemicomplemented, there exists $\mu' \in \mathcal{L}$ such that $\mu \wedge \mu' \in \mathcal{D}^\infty$ and $\mu \vee \mu' \in \mathcal{D}$. Hence $\mu' \in \langle \mu, \mathcal{D}^\infty \rangle = \langle \pi, \mathcal{D}^\infty \rangle$. Hence $\mu' \wedge \pi \in \mathcal{D}^\infty$. Hence $(\mu', \mathcal{D}) \cap (\pi, \mathcal{D}) = \mathcal{D}$. Let $\nu \in (\pi, \mathcal{D})$. Since $(\pi, \mathcal{D})$ is a filter, which gives $\nu \vee \mu \in (\pi, \mathcal{D})$. As $\mu \vee \mu' \in \mathcal{D}$, we obtain $\mu \in (\mu', \mathcal{D})$. Hence $\nu \vee \mu \in (\mu', \mathcal{D})$. Thus $\nu \vee \mu \in (\mu', \mathcal{D}) \cap (\pi, \mathcal{D}) = \mathcal{D}$. Hence $\nu \in (\mu, \mathcal{D})$. Therefore $(\pi, \mathcal{D}) \subseteq (\mu, \mathcal{D})$. With a similar approach, we get $(\mu, \mathcal{D}) \subseteq (\pi, \mathcal{D})$. Hence $(\mu, \pi) \in \theta$. Therefore $\Psi_{\mathcal{D}^\infty} \subseteq \theta$.

To characterize hemicomplemented ADLs, we begin with the following lemma.

Lemma 3. *For any ideal $\mathcal{K}$ of $\mathcal{L}$, the set $\mathcal{D}(\mathcal{K}) = \{\mu \in \mathcal{L} \mid \mu \vee \theta \in \mathcal{D}, \text{ for some } \theta \in \mathcal{K}\}$ is a filter of $\mathcal{L}$.*

Proof. It is evident that $\mathcal{D} \subseteq \mathcal{D}(\mathcal{K})$. Let $\mu, \pi \in \mathcal{D}(\mathcal{K})$. Then $\mu \vee \theta, \pi \vee \vartheta \in \mathcal{D}$, for some $\theta, \vartheta \in \mathcal{K}$. Hence $(\mu \wedge \pi) \vee (\theta \vee \vartheta) = (\mu \vee \theta \vee \vartheta) \wedge (\pi \vee \theta \vee \vartheta) \in \mathcal{D}$. As $\mathcal{K}$ is an ideal, we obtain $\mu \wedge \pi \in \mathcal{D}(\mathcal{K})$. Let $\mu \in \mathcal{D}(\mathcal{K})$ and $\mu \leq \pi$. Then $\mu \vee \theta \in \mathcal{D}$, for all $\theta \in \mathcal{K}$. Hence $\pi \vee \theta \in \mathcal{D}$, which gives that $\pi \in \mathcal{D}(\mathcal{K})$. Thus $\mathcal{D}(\mathcal{K})$ is a filter of $\mathcal{L}$.

Theorem 4. *For an ADL $\mathcal{L}$, the following properties hold equivalently*

(1) $\mathcal{L}$ is hemicomplemented;

(2) for every filter $\mathcal{G}$, there exists an ideal $\mathcal{K}$ such that $((\mathcal{G}, \mathcal{D}), \mathcal{D}) = \mathcal{D}(\mathcal{K})$;

(3) for every filter $\mathcal{G}$, $((\mathcal{G},\mathcal{D}),\mathcal{D}) = \mathcal{D}(\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})})$, where $\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$ is the ideal defined as $\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})} = \{\mu \in \mathcal{L} \mid (\theta,\mathcal{D}) \subseteq ((\mu,\mathcal{D}),\mathcal{D}),\ \text{for some } \theta \in ((\mathcal{G},\mathcal{D}),\mathcal{D})\}$.

Proof. (1) $\Rightarrow$ (3) : Assume (1). Define $\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})} = \{\mu \in \mathcal{L} \mid (\theta,\mathcal{D}) \subseteq ((\mu,\mathcal{D}),\mathcal{D}),$ for some $\theta \in ((\mathcal{G},\mathcal{D}),\mathcal{D})\}$. It can be seen that $0 \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$. Let $\mu,\pi \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$. Then $(\theta,\mathcal{D}) \subseteq ((\mu,\mathcal{D}),\mathcal{D})$ and $(\vartheta,\mathcal{D}) \subseteq ((\pi,\mathcal{D}),\mathcal{D})$, for some $\theta,\vartheta \in ((\mathcal{G},\mathcal{D}),\mathcal{D})$. Now $(\theta \wedge \vartheta,\mathcal{D}) = (\theta,\mathcal{D}) \cap (\vartheta,\mathcal{D}) \subseteq ((\mu,\mathcal{D}),\mathcal{D}) \cap ((\pi,\mathcal{D}),\mathcal{D}) = ((\mu \vee \pi,\mathcal{D}),\mathcal{D})$ and $\theta \wedge \vartheta \in ((\mathcal{G},\mathcal{D}),\mathcal{D})$. Hence $\mu \vee \pi \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$. Let $\mu \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$ and $\tau \in \mathcal{L}$. Then $(\theta,\mathcal{D}) \subseteq ((\mu,\mathcal{D}),\mathcal{D})$, for some $\theta \in ((\mathcal{G},\mathcal{D}),\mathcal{D})$. Now $(\theta,\mathcal{D}) \subseteq ((\mu,\mathcal{D}),\mathcal{D}) \subseteq ((\mu \wedge \tau,\mathcal{D}),\mathcal{D})$. Thus $\mu \wedge \tau \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$. Therefore $\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$ is an ideal of $\mathcal{L}$. Next, we show that $((\mathcal{G},\mathcal{D}),\mathcal{D}) = \mathcal{D}(\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})})$. Let $\mu \in ((\mathcal{G},\mathcal{D}),\mathcal{D})$. Since $\mathcal{L}$ is hemicomplemented, there exists $\pi \in \mathcal{L}$ such that $((\mu,\mathcal{D}),\mathcal{D}) = (\pi,\mathcal{D})$. As $\mu \in ((\mathcal{G},\mathcal{D}),\mathcal{D})$, we gives that $\pi \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$. Now, $\mu \in ((\mu,\mathcal{D}),\mathcal{D}) = (\pi,\mathcal{D})$ gives that $\mu \vee \pi \in \mathcal{D}$ for some $\pi \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$. Therefore $((\mathcal{G},\mathcal{D}),\mathcal{D}) \subseteq \mathcal{D}(\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})})$. Let $\mu \in \mathcal{D}(\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})})$. Then $\mu \vee \rho \in D$ for some $\rho \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$. Hence $\mu \in (\rho,\mathcal{D})$ for some $\rho \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}$. Now we obtain

$$\begin{aligned} \rho \in \mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})} &\Rightarrow (\theta,\mathcal{D}) \subseteq ((\rho,\mathcal{D}),\mathcal{D}) \text{ for some } \theta \in ((\mathcal{G},\mathcal{D}),\mathcal{D}) \\ &\Rightarrow (\rho,\mathcal{D}) \subseteq ((\theta,\mathcal{D}),\mathcal{D}) \text{ for some } \theta \in ((\mathcal{G},\mathcal{D}),\mathcal{D}) \\ &\Rightarrow \theta \in ((\theta,\mathcal{D}),\mathcal{D}) \subseteq ((\mathcal{G},\mathcal{D}),\mathcal{D}). \end{aligned}$$

Hence $\mathcal{D}(\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})}) \subseteq ((\mathcal{G},\mathcal{D}),\mathcal{D})$. Therefore $((\mathcal{G},\mathcal{D}),\mathcal{D}) = \mathcal{D}(\mathcal{K}_{((\mathcal{G},\mathcal{D}),\mathcal{D})})$.
(3) $\Rightarrow$ (2) : It is clear.
(2) $\Rightarrow$ (1) : Assume (2). Let $\mu \in \mathcal{L}$. Then there is an ideal $\mathcal{K}$ such that $((\mu,\mathcal{D}),\mathcal{D}) = \mathcal{D}(\mathcal{K})$.

$$\begin{aligned} \mu \in \mathcal{D}(\mathcal{K}) &\Rightarrow \mu \vee i \in \mathcal{D} \text{ for some } i \in \mathcal{K} \\ &\Rightarrow [\mu) \subseteq (i,\mathcal{D}) \\ &\Rightarrow ((\mu,\mathcal{D}),\mathcal{D}) \subseteq (((i,\mathcal{D}),\mathcal{D}),\mathcal{D}) = (i,\mathcal{D}). \end{aligned}$$

Let $\theta \in (i,\mathcal{D})$. Then $\theta \vee i \in \mathcal{D}$ and $i \in \mathcal{K}$. Thus $\theta \in \mathcal{D}(\mathcal{K}) = ((\mu,\mathcal{D}),\mathcal{D})$. Thus $(i,\mathcal{D}) \subseteq ((\mu,\mathcal{D}),\mathcal{D})$. Therefore $((\mu,\mathcal{D}),\mathcal{D}) = (i,\mathcal{D})$. Hence $\mathcal{L}$ is hemicomplemented.

4. Conclusions

This paper introduces the concept of condensed elements in an Almost Distributive Lattice(ADL) and explores their fundamental properties. It also presents the concept of hemicomplemented ADL and characterizes these ADLs using ideals, congruences, and minimal prime $\mathcal{D}$-filters. Furthermore, a collection of equivalent criteria is established to determine when a hemicomplemented ADL qualifies as a quasicomplemented ADL. In future work, we plan to study the properties of hemi-complemented ADLs and $\mathcal{D}$-Stone ADLs using congruences, which may help in understanding different structures of ADLs.

Conflicts of interest or competing interests

The authors declare that they have no conflicts of interest.

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