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Solving Fractional Differential Equations and Integral Equations via Neutrosophic Bipolar Metric Space

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Abstract. The theory of metric spaces forms the basis of metric fixed point theory, which has varied applications in areas such as engineering, economics, medicine, and even space science (e.g., satellite launch). In many generalizations of metric and metric-like spaces, fuzzy metric spaces, intuitionistic fuzzy sets, and neutrosophic sets have evolved. While both metric and bipolar metric are distance functions, the classical metric considers a single set, whereas the bipolar metric considers the distance between two potentially different sets. It is also well known that fractional calculus has broad applications. In this work, we introduce neutrosophic bipolar metric spaces and establish fixed point theorems in these spaces. Our main results generalize several proven results in the literature. The derived results are supported with non-trivial illustrations. Three applications are presented to supplement the theoretical findings.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Fixed point, neutrosophic metric space, neutrosophic bipolar metric space, integral equation

1. Introduction

The foundation of metric fixed point theory lies on the concept of metric spaces and the Banach contraction principle [1] . An axiomatic grasp of metric space draws thousands of scholars to spaciousness. Metric spaces have seen various changes in the past. Here, we notify that the beauty, attraction, and expansion of the concept of metric spaces. Fractal or the Hausdorff derivative of mathematical analysis, which is a non neutonion derivative, deals with fractals defined in fractal geometry. It has vast applications. To know about certain fundamentals of fractal calculus and its application, one can refer to [2–4].

The notion of fuzzy set (FS) was introduced by L. Zadeh [5] in 1965, where each element had a degree of membership (t). The intuitionistic fuzzy set (IFS) on a universe X was introduced by K. Atanassov [6] in 1986 as a generalization of FS, where besides

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the degree of membership $\mu_A(x) \in [0,1]$ of each element $x \in X$ to a set A ,there was considered a degree of non-membership $\nu_A(x) \in [0,1]$,

$$\forall x \in X, \mu_A(x) + \nu_A(x) \leq 1 \tag{1}$$

The neutrosophic set (NS) was introduced by F. Smarandache [7] degree of indeterminacy as independent component. In the study on analytical and practical applicability of fuzzy sets and their generalisations, mathematicians reported many results including analysing the impact of fuzzy ideal extension, evaluating solutions of generalised fuzzy differential equations, intutionistic fuzzy linear system of equations, etc., to name a few., see [8–11].

In contemporary examinations of the set-theoretical and logical underpinnings of mathematics, the word "fuzzy" appears to be prevalent. The primary rationale for this unexpected development, our judgment, is straightforward. Because the information we collect from our surroundings, we employ the idea of resulting from our observations or measurements are all hazy and erroneous, the world around us is full of ambiguity. So, every usual illustration is an approximation or idealization of truth of the real world or a part of it. Fuzzy sets (orderings, languages, etc.) and other ideas allow us to deal with analyze the mentioned in a purely mathematical and formal manner of uncertainty.

Many mathematical structures have moved within the concept of the fuzzy set. Schweizer et al. [12] pioneered the conceit of continuous criteria. Kramosil et al. [13] initiated fuzzy metric spaces (shortly, FMS). They used continued norms to apply the idea of fuzziness to standard concepts of probabilistic, statistical extensions of metric spaces and compared the results to these obtained from other. In [14], Garbiec established the Banach contraction concept in FMS. Rehmam et al. [15] discovered numerous $\alpha - \phi$ contraction in fuzzy cone using the integral type, mostly considered membership functions in FMS. Park [16] developed an intuitionistic FMS for dealing without membership and nonmembership functions. Konwar [17] introduced an intuitionistic fuzzy b-metric space (shortly, F_bMS) and proved many fixed-point theorems.

Mutlu et al. [18], initiated the concept of bipolar metric spaces (shortly, BMS) and established fixed point theorems. Many researchers have recently produced a slew of fixed point outcomes in the constructions of BMS using various extension of these spaces using different contractions [19–30].

In 2019, Kiricsci et al. [31] introduced the concept of neutrosophic metric spaces (NMS), which deals with membership, non-membership, and naturalness. Again in 2020, Simsek et al. [32] established various fixed point results in the setting of NMS. Later in 2020, Sowndarrajan et al. [33] showed several fixed point discoveries in the context of neutrosophic metric spaces.

In the recent past, applications of fixed point thoerems to fractal calculus is a matter of interest. Many mathematicians have applied the results of fixed point theorems to fractional calculus. Baleanu et al. [34] studied the existence and uniqueness of a solution to non-linear fractional differential equation. More recently, in 2021, Zhu et al. [35] applied some fixed-point theorems to discuss the existence of solutions for fractional m-point boundary value problems. More recently, Chandok et al. [36] presented application to fractional calculus via orthogonal contractions. In the recent past, Mani et al. [37] improved fixed point results and to find analytical solution of integral equation using Neutrosophic triple controlled metric spaces.

Inspired by the reported results in the setting of Bipolar as well as the Neutrospohic metric spaces, in the present work, the notion of NBMS and some related topological concepts are introduced and fixed point results have been established in the setting of this space and the derived results are supplemented with non-trivial examples. This work also presents three types of applications of the derived fixed point results: (a) to find analytical and closed form solutions of an Integral Equation, (b) to find the voltage in an electrical circuit and finally (c) to find the analytical solution of a fractional differential equation. The results proven in this manuscript are extensions or generalizations of the result proven in the past.

The rest of the paper is organized as follows: Some definitions and theories are reviewed in Section 2.In Section 3, the proposed neutrosophic bipolar metric space and associated concepts are defined and discussed. Furthermore, the main fixed point result is presented in this section supported with non trivial examples to supplement the derived results. In Section 4, an application of the derived fixed point result to find the solution of the Fredholm integral equation is given. This is followed by the finding an analytical solution for the voltage in an electric circuit in Section-5 along with the closed form of BVP and finally, an application to find the analytical solution of the fractional differential equation is also presented.

2. Preliminaries

We commence this section, with certain abbreviations and some symbols used in the manuscript:

Table 1: List of Acronyms

Acronym	**Full Form**
FMS	Fuzzy Metric Space
NBMS	Neutrosophic Bipolar Metric Space
NMS	Neutrosophic Metric Space
NB	Neutrosophic Bipolar

Table 2: List of symbols

Symbol	**Meaning**
$c_{t-\|\|.\|\|}$	Continuous triangle norm
$c_{t-co-\|\|.\|\|}$	Continuous triangle co-norm
Π, Ψ, Ξ, Ω	Functions
$\digamma, \mathcal{S}$	Non empty sets
$\ast$	Continuous triangle norm
$\Diamond$	Continuous triangle co-norm
$\dot{u}, \dot{\mathfrak{z}}, \dot{t}, \dot{s}, \dot{w}$	Positive Reals

The following definitions are required in the sequel.

Definition 1. *([16]) A binary operation* $\ast\colon [0,1]^2 \to [0,1]$ *is said to be a continuous triangle norm (shortly,* $c_{t-||.||}$*) if:*

i. $\mathfrak{i} \ast \flat = \flat \ast \mathfrak{i}, (\forall)\mathfrak{i}, \flat \in [0,1]$*;*

ii. $\ast$ *is continuous;*

iii. $\mathfrak{i} \ast 1 = \mathfrak{i}, (\forall)\mathfrak{i} \in [0,1]$*;*

iv. $(\mathfrak{i} \ast \flat) \ast \hbar = \mathfrak{i} \ast (\flat \ast \hbar)$*,* $\forall\ \mathfrak{i}, \flat, \hbar \in [0,1]$*;*

v. If $\mathfrak{i} \leq \hbar$ *and* $\flat \leq \mathfrak{j}$*, with* $\mathfrak{i}, \flat, \hbar, \mathfrak{j} \in [0,1]$*, then* $\mathfrak{i} \ast \flat \leq \hbar \ast \mathfrak{j}$.

Definition 2. *([16]) A binary operation* $\Diamond\colon [0,1]^2 \to [0,1]$ *is said to be a continuous triangle co-norm (shortly,* $c_{t-co-||.||}$*) if:*

i. $\mathfrak{i}\Diamond\flat = \flat\Diamond\mathfrak{i}$*,* $\forall\ \mathfrak{i}, \flat \in [0,1]$*;*

ii. $\Diamond$ *is continuous;*

iii. $\mathfrak{i}\Diamond 0 = 0$*,* $\forall\ \mathfrak{i} \in [0,1]$*;*

iv. $(\mathfrak{i}\Diamond\flat)\Diamond\hbar = \mathfrak{i}\Diamond(\flat\Diamond\hbar)$*,* $\forall\ \mathfrak{i}, \flat, \hbar \in [0,1]$*;*

v. If $\mathfrak{i} \leq \hbar$ *and* $\flat \leq \mathfrak{j}$*, with* $\mathfrak{i}, \flat, \hbar, \mathfrak{j} \in [0,1]$*, then* $\mathfrak{i}\Diamond\flat \leq \hbar\Diamond\mathfrak{j}$.

Definition 3. *([17]) Take* $\digamma \neq \emptyset$*. Let* $\ast$ *be a* $c_{t-||.||}$*,* $\Diamond$ *be a* $c_{t-co-||.||}$*,* $\mathfrak{b} \geq 1$ *and* Π, Ψ *be defined on fuzzy sets on* $\digamma \times \digamma \times (0, +\infty)$*. If* $(\digamma, \Pi, \Psi, \ast, \Diamond)$ *fulfills all* $\varsigma, \varpi \in \digamma$ *and* $\dot{u}, \dot{\mathfrak{z}} > 0$*:*

i. $\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) + \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) \leq 1$*;*

ii. $\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) > 0$*;*

iii. $\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) = 1 \Leftrightarrow \varsigma = \varpi$*;*

iv. $\Pi(\varsigma, \varpi, \mathfrak{z}) = \Pi(\varpi, \varsigma, \mathfrak{z})$;

v. $\Pi(\varsigma, \curlywedge, \mathfrak{b}(\mathfrak{z} + \dot{u})) \geq \Pi(\varsigma, \varpi, \mathfrak{z}) \circledast \Pi(\varpi, \curlywedge, \dot{u})$;

vi. $\Pi(\varsigma, \varpi, \cdot)$ *is a non-decreasing function of* $\mathbb{R}^+$ *and* $\lim_{\mathfrak{z}\to+\infty} \Pi(\varsigma, \varpi, \mathfrak{z}) = 1$;

vii. $\Psi(\varsigma, \varpi, \mathfrak{z}) > 0$;

viii. $\Psi(\varsigma, \varpi, \mathfrak{z}) = 0 \Leftrightarrow \varsigma = \varpi$;

ix. $\Psi(\varsigma, \varpi, \mathfrak{z}) = \Psi(\varpi, \varsigma, \mathfrak{z})$;

x. $\Psi(\varsigma, \curlywedge, \mathfrak{b}(\mathfrak{z} + \dot{u})) \leq \Psi(\varsigma, \varpi, \mathfrak{z}) \Diamond \Psi(\varpi, \curlywedge, \dot{u})$;

xi. $\Psi(\varsigma, \varpi, \cdot)$ *is a function of non-increasing* $\mathbb{R}^+$ *and* $\lim_{\mathfrak{z}\to+\infty} \Psi(\varsigma, \varpi, \mathfrak{z}) = 0$,

Then, $(F, \Pi, \Psi, \circledast, \Diamond)$ *is an intuitionistic* F_bMS.

Definition 4. *([31]) Let* $F \neq \emptyset, \circledast$ *and* $\Diamond$ *are a* $c_{t-||\cdot||}$ *and* $c_{t-co-||\cdot||}$. *Here* Π, Ψ, Ω *defined on the neutrosophic sets* $F \times F \times (0, +\infty)$ *is said to be a neutrosophic metric on* F*, if* $\forall\ \varsigma, \varpi, \curlywedge \in F$*, the following axioms are fulfilled:*

i. $\Pi(\varsigma, \varpi, \mathfrak{z}) + \Psi(\varsigma, \varpi, \mathfrak{z}) + \Omega(\varsigma, \varpi, \mathfrak{z}) \leq 3$;

ii. $\Pi(\varsigma, \varpi, \mathfrak{z}) > 0$;

iii. $\Pi(\varsigma, \varpi, \mathfrak{z}) = 1\ \forall\ \mathfrak{z} > 0 \Leftrightarrow \varsigma = \varpi$;

iv. $\Pi(\varsigma, \varpi, \mathfrak{z}) = \Pi(\varpi, \varsigma, \mathfrak{z})$;

v. $\Pi(\varsigma, \curlywedge, \mathfrak{z} + \dot{u}) \geq \Pi(\varsigma, \varpi, \mathfrak{z}) \circledast \Pi(\varpi, \curlywedge, \dot{u})$;

vi. $\Pi(\varsigma, \varpi, \cdot) \colon (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\mathfrak{z}\to+\infty} \Pi(\varsigma, \varpi, \mathfrak{z}) = 1$;

vii. $\Psi(\varsigma, \varpi, \mathfrak{z}) < 1$;

viii. $\Psi(\varsigma, \varpi, \mathfrak{z}) = 0\ \forall\ \mathfrak{z} > 0 \Leftrightarrow \varsigma = \varpi$;

ix. $\Psi(\varsigma, \varpi, \mathfrak{z}) = \Psi(\varpi, \varsigma, \mathfrak{z})$;

x. $\Psi(\varsigma, \curlywedge, \mathfrak{z} + \dot{u}) \leq \Psi(\varsigma, \varpi, \mathfrak{z}) \Diamond \Psi(\varpi, \curlywedge, \dot{u})$;

xi. $\Psi(\varsigma, \varpi, \cdot) \colon (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\mathfrak{z}\to+\infty} \Psi(\varsigma, \varpi, \mathfrak{z}) = 0$;

xii. $\Omega(\varsigma, \varpi, \mathfrak{z}) < 1$;

xiii. $\Omega(\varsigma, \varpi, \mathfrak{z}) = 0\ \forall\ \mathfrak{z} > 0 \Leftrightarrow \varsigma = \varpi$;

xiv. $\Omega(\varsigma, \varpi, \mathfrak{z}) = \Omega(\varpi, \varsigma, \mathfrak{z})$;

xv. $\Omega(\varsigma, \curlywedge, \mathfrak{z} + \dot{u}) \leq \Omega(\varsigma, \varpi, \mathfrak{z}) \Diamond \Omega(\varpi, \curlywedge, \dot{u})$;

xvi. $\Omega(\varsigma, \varpi, \cdot)\colon (0,+\infty) \to [0,1]$ *is continuous and* $\lim_{\mathfrak{z}\to+\infty} \Omega(\varsigma, \varpi, \mathfrak{z}) = 0$;

xvii. *If* $\mathfrak{z} \leq 0$, *then* $\Pi(\varsigma, \varpi, \mathfrak{z}) = 0, \Psi(\varsigma, \varpi, \mathfrak{z}) = 0$;

Then, $(\digamma, \Pi, \Psi, \Omega, \circledast, \Diamond)$ *is said to be a neutrosophic metric space.*

Definition 5. *[18] Let* $\digamma$ *and* $\mathcal{S}$ *be non-void sets and* $\varrho : \digamma \times \mathcal{S} \to [0,+\infty)$ *be a function, such that*

i. $\varrho(\varsigma, \varpi) = 0$ *iff* $\varsigma = \varpi$, $\forall\ (\varsigma, \varpi) \in \digamma \times \mathcal{S}$

ii. $\varrho(\varsigma, \varpi) = \varrho(\varsigma, \varpi)$, $\forall\ (\varsigma, \varpi) \in \digamma \cap \mathcal{S}$

iii. $\varrho(\varsigma, \varpi) \leq \varrho(\varsigma, \gamma) + \varrho(\varsigma_1, \gamma) + \varrho(\varsigma_1, \varpi)$, $\forall\ \varsigma, \varsigma_1 \in \digamma$ *and* $\gamma, \varpi \in \mathcal{S}$.

We say that the pair $(\digamma, \mathcal{S}, \varrho)$ *is a bipolar metric space.*

3. Main Results

In this section, we define NBMS and prove few fixed-point theorems.

Definition 6. *Let* $\digamma \neq \emptyset$, $\mathcal{S} \neq \emptyset$ *be two sets and* $\circledast$ *be a* $c_{t-||.||}$, $\Diamond$ *be a* $c_{t-co-||.||}$. *Then,* Π, Ψ, Ξ *defined on neutrosophic sets* $\digamma \times \mathcal{S} \times (0,+\infty)$ *is called a neutrosophic bipolar metric on* $\digamma \times \mathcal{S}$, *if* $\forall\ \varsigma, \mathfrak{x} \in \digamma$, $\varpi, \curlywedge \in \mathcal{S}$ *and* $\mathfrak{z}, \hat{s}, \hat{w} > 0$, *the following axioms are fulfilled:*

i. $\Pi(\varsigma, \varpi, \mathfrak{z}) + \Psi(\varsigma, \varpi, \mathfrak{z}) + \Xi(\varsigma, \varpi, \mathfrak{z}) \leq 3$;

ii. $\Pi(\varsigma, \varpi, \mathfrak{z}) > 0$;

iii. $\Pi(\varsigma, \varpi, \mathfrak{z}) = 1\ \forall\ \mathfrak{z} > 0 \Leftrightarrow \varsigma = \varpi$;

iv. $\Pi(\varsigma, \varpi, \mathfrak{z}) = \Pi(\varpi, \varsigma, \mathfrak{z})$;

v. $\Pi(\varsigma, \curlywedge, \mathfrak{z} + \dot{u} + \dot{t}) \geq \Pi\big(\varsigma, \varpi, \mathfrak{z}\big) \circledast \Pi\big(\mathfrak{x}, \varpi, \dot{u}\big) \circledast \Pi\big(\mathfrak{x}, \curlywedge, \dot{t}\big)$;

vi. $\Pi(\varsigma, \varpi, \cdot)\colon (0,+\infty) \to [0,1]$ *is continuous and* $\lim\limits_{\mathfrak{z}\to+\infty} \Pi(\varsigma, \varpi, \mathfrak{z}) = 1$;

vii. $\Psi(\varsigma, \varpi, \mathfrak{z}) < 1$;

viii. $\Psi(\varsigma, \varpi, \mathfrak{z}) = 0\ \forall\ \mathfrak{z} > 0 \Leftrightarrow \varsigma = \varpi$;

ix. $\Psi(\varsigma, \varpi, \mathfrak{z}) = \Psi(\varpi, \varsigma, \mathfrak{z})$;

x. $\Psi(\varsigma, \curlywedge, \mathfrak{z} + \dot{u} + \dot{t}) \leq \Psi\big(\varsigma, \varpi, \mathfrak{z}\big) \Diamond \Psi\big(\mathfrak{x}, \varpi, \dot{u}\big) \Diamond \Psi\big(\mathfrak{x}, \curlywedge, \dot{t}\big)$;

xi. $\Psi(\varsigma, \varpi, \cdot)\colon (0,+\infty) \to [0,1]$ *is continuous and* $\lim\limits_{\mathfrak{z}\to+\infty} \Psi(\varsigma, \varpi, \mathfrak{z}) = 0$;

xii. $\Xi(\varsigma, \varpi, \mathfrak{z}) < 1$;

xiii. $\Xi(\varsigma, \varpi, \mathfrak{z}) = 0\ \forall\ \mathfrak{z} > 0 \Leftrightarrow \varsigma = \varpi$;

xiv. $\Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) = \Xi(\varpi, \varsigma, \dot{\mathfrak{z}})$;

xv. $\Xi(\varsigma, \curlywedge, \dot{\mathfrak{z}} + \dot{u} + \dot{t}) \leq \Xi\big(\varsigma, \varpi, \dot{\mathfrak{z}}\big) \Diamond \Xi\big(\mathfrak{x}, \varpi, \dot{u}\big) \Diamond \Xi\big(\mathfrak{x}, \curlywedge, \dot{t}\big)$;

xvi. $\Xi(\varsigma, \varpi, \cdot)\colon (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\dot{\mathfrak{z}} \to +\infty} \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) = 0$;

xvii. If $\dot{\mathfrak{z}} \leq 0$, *then* $\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) = 0, \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) = 1$ *and* $\Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) = 1$.

Then, $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is said to be a NBMS.*

An illustrative example of NBMS is presented below:

Example 1. *Let* $\digamma = \{1, 3, 5, 7\}$, $\mathcal{S} = \{1, 2, 6, 4\}$. *Define* $\Pi, \Psi, \Xi\colon \digamma \times \mathcal{S} \times (0, +\infty) \to [0, 1]$ *as*

$$\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) = \begin{cases} 1, & \textit{if} \quad \varsigma = \varpi \\ \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + \max\{\varsigma, \varpi\}}, & \textit{if} \quad \textit{otherwise}, \end{cases}$$

$$\Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) = \begin{cases} 0, & \textit{if} \quad \varsigma = \varpi \\ \frac{\max\{\varsigma, \varpi\}}{\dot{\mathfrak{z}} + \max\{\varsigma, \varpi\}}, & \textit{if} \quad \textit{otherwise}, \end{cases}$$

and

$$\Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) = \begin{cases} 0, & \textit{if} \quad \varsigma = \varpi \\ \frac{\max\{\varsigma, \varpi\}}{\dot{\mathfrak{z}}}, & \textit{if} \quad \textit{otherwise}. \end{cases}$$

Let $\varsigma = 1, \varpi = 2$, $\mathfrak{x} = 3$ *and* $\curlywedge = 4$. *Then from, (v), (x) and (xv) and obviously others.*

$$\Pi(1, 4, \dot{\mathfrak{z}} + \dot{u} + \dot{t}) = \frac{\dot{\mathfrak{z}} + \dot{u} + \dot{t}}{\dot{\mathfrak{z}} + \dot{u} + \dot{t} + \max\{1, 4\}} = \frac{\dot{\mathfrak{z}} + \dot{u} + \dot{t}}{\dot{\mathfrak{z}} + \dot{u} + \dot{t} + 4}.$$

Further,

$$\Pi\big(1, 2, \dot{\mathfrak{z}}\big) = \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + \max\{1, 2\}} = \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + 2} = \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + 2},$$

$$\Pi\big(2, 3, \dot{u}\big) = \frac{\dot{u}}{\dot{u} + \max\{2, 3\}} = \frac{\dot{u}}{\dot{u} + 3} = \frac{\dot{u}}{\dot{u} + 3}$$

and

$$\Pi\big(3, 4, \dot{t}\big) = \frac{\dot{t}}{\dot{t} + \max\{3, 4\}} = \frac{\dot{t}}{\dot{t} + 4} = \frac{\dot{t}}{\dot{t} + 4}.$$

That is,

$$\frac{\dot{\mathfrak{z}}+\dot{u}+\dot{t}}{\dot{\mathfrak{z}}+\dot{u}+\dot{t}+3} \geq \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}}+2}\cdot\frac{\dot{u}}{\dot{u}+3}.\frac{\dot{t}}{\dot{t}+4}.$$

Since each of $\dot{\mathfrak{z}},\dot{u},\dot{t}>0$, *the above inequality holds.*

$$\Pi(\varsigma,\curlywedge,\dot{\mathfrak{z}}+\dot{u}+\dot{t}) \geq \Pi\big(\varsigma,\varpi,\dot{\mathfrak{z}}\big) \circledast \Pi\big(\varpi,\mathfrak{x},\dot{u}\big) \circledast \Pi\big(\mathfrak{x},\curlywedge,\dot{t}\big).$$

Now,

$$\Psi(1,4,\dot{\mathfrak{z}}+\dot{u}+\dot{t}) = \frac{\max\{1,4\}}{\dot{\mathfrak{z}}+\dot{u}+\dot{t}+\max\{1,4\}} = \frac{4}{\dot{\mathfrak{z}}+\dot{u}+\dot{t}+4}.$$

On the other hand,

$$\Psi\big(1,2,\dot{\mathfrak{z}}\big) = \frac{\max\{1,2\}}{\dot{\mathfrak{z}}+\max\{1,2\}} = \frac{2}{\dot{\mathfrak{z}}+2} = \frac{2}{\dot{\mathfrak{z}}+2},$$

$$\Psi\big(2,3,\dot{u}\big) = \frac{\max\{2,3\}}{\dot{u}+\max\{2,3\}} = \frac{3}{\dot{u}+3} = \frac{3}{\dot{u}+3}$$

and

$$\Psi\big(3,4,\dot{t}\big) = \frac{\max\{3,4\}}{\dot{t}+\max\{3,4\}} = \frac{4}{\dot{t}+4} = \frac{4}{\dot{t}+4}.$$

That is,

$$\frac{4}{\dot{\mathfrak{z}}+\dot{u}+\dot{t}+4} \leq \max\left\{\frac{2}{\dot{\mathfrak{z}}+2},\frac{3}{\dot{u}+3},\frac{4}{\dot{t}+4}\right\}.$$

Here again, $\dot{\mathfrak{z}},\dot{u},\dot{t}>0$ *the above inequality is true and so,*

$$\Psi(\varsigma,\curlywedge,\dot{\mathfrak{z}}+\dot{u}+\dot{t}) \leq \Psi\big(\varsigma,\varpi,\dot{\mathfrak{z}}\big)\Diamond\Psi\big(\mathfrak{x},\curlywedge,\dot{u}\big)\Diamond\Psi\big(\mathfrak{x},\curlywedge,\dot{t}\big).$$

Finally,

$$\Xi(1,3,\dot{\mathfrak{z}}+\dot{u}+\dot{t}) = \frac{\max\{1,3\}}{\dot{\mathfrak{z}}+\dot{u}+\dot{t}} = \frac{3}{\dot{\mathfrak{z}}+\dot{u}+\dot{t}}.$$

On the other hand,

$$\Xi\big(1,2,\dot{\mathfrak{z}}\big) = \frac{\max\{1,2\}}{\dot{\mathfrak{z}}} = \frac{2}{\dot{\mathfrak{z}}},$$

$$\Xi\big(2,3,\dot{u}\big) = \frac{\max\{2,3\}}{\dot{u}} = \frac{3}{\dot{u}} = \frac{3}{\dot{u}}$$

and

$$\Xi(3,4,\dot{t}) = \frac{\max\{3,4\}}{\dot{t}} = \frac{4}{\dot{t}} = \frac{4}{\dot{t}}.$$

That is,

$$\frac{3}{\dot{\mathfrak{z}} + \dot{u} + \dot{t}} \leq \max\left\{\frac{2}{\dot{\mathfrak{z}}}, \frac{3}{\dot{u}}, \frac{4}{\dot{t}}\right\}.$$

The above inequality holds as $\dot{\mathfrak{z}}, \dot{u} > 0$. *Thus,,*

$$\Xi(\varsigma, \curlywedge, \dot{\mathfrak{z}} + \dot{u} + \dot{t}) \leq \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) \Diamond \Xi(\varpi, \mathfrak{x}, \dot{u}) \Diamond \Xi(\mathfrak{x}, \curlywedge, \dot{u}).$$

Thus, $(\digamma, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a NBMS with* $\mathfrak{i} \circledast \flat = \mathfrak{i}\flat$ *and* $\mathfrak{i} \Diamond \flat = \max\{\mathfrak{i}, \flat\}$ *by* $c_{t-||.||}$ *and* $c_{t-co-||.||}$, *respectively.*

Remark 1. *It is to be noted that every NBMS is a neutrosophic metric space NMS, but the converse is not always true because, NMS is a particular case of NBMS where* $F = S$.

Definition 7. *Let* $\mathfrak{p} : \digamma_1 \cup \mathcal{S}_1 \to \digamma_2 \cup \mathcal{S}_2$ *be a mapping, where* $(\digamma_1, \mathcal{S}_1)$ *and* $(\digamma_2, \mathcal{S}_2)$ *pairs of sets.*

i. If $\mathfrak{p}(\digamma_1) \subseteq \digamma_2$ *and* $\mathfrak{p}(\mathcal{S}_1) \subseteq \mathcal{S}_2$, *then* $\mathfrak{p}$ *is said to be a covariant map, or a map from* $(\digamma_1, \mathcal{S}_1, \Pi_1, \Psi_1, \Xi_1, \circledast, \Diamond)$ *to* $(\digamma_2, \mathcal{S}_2, \Pi_2, \Psi_2, \Xi_2, \circledast, \Diamond)$ *and this is written as,* $\mathfrak{p} : (\digamma_1, \mathcal{S}_1, \Pi_1, \Psi_1, \Xi_1, \circledast, \Diamond) \rightrightarrows (\digamma_2, \mathcal{S}_2, \Pi_2, \Psi_2, \Xi_2, \circledast, \Diamond)$.

ii. If $\mathfrak{p}(\digamma_1) \subseteq \mathcal{S}_2$ *and* $\mathfrak{p}(\mathcal{S}_1) \subseteq \digamma_2$, *then* $\mathfrak{p}$ *is said to be a contravariant map from* $(\digamma_1, \mathcal{S}_1, \Pi_1, \Psi_1, \Xi_1, \circledast, \Diamond)$ *to* $(\digamma_2, \mathcal{S}_2, \Pi_2, \Psi_2, \Xi_2, \circledast, \Diamond)$ *and this is denoted as:* $\mathfrak{p} : (\digamma_1, \mathcal{S}_1, \Pi_1, \Psi_1, \Xi_1, \circledast, \Diamond) \leftrightarrows (\digamma_2, \mathcal{S}_2, \Pi_2, \Psi_2, \Xi_2, \circledast, \Diamond)$.

3.1. Some topological properties of neutrosophic bipolar metric space

Definition 8. *Let* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a NBMS, and define a right open ball* $\mathcal{B}(\varsigma, \mathfrak{r}, \dot{\mathfrak{z}})$ *with center* $\varsigma \in \digamma$, *radius* $\mathfrak{r}, \mathfrak{r} \in (0,1)$, $\dot{\mathfrak{z}} > 0$ *as follows:*

$$\mathcal{B}(\varsigma, \mathfrak{r}, \dot{\mathfrak{z}}) = \{\varpi \in \mathcal{S} \colon \Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) > 1 - \mathfrak{r}, \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) < \mathfrak{r}, \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) < \mathfrak{r}\}.$$

Definition 9. *Let* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a NBMS, and define a left open ball* $\mathcal{B}(\varpi, \mathfrak{r}, \dot{\mathfrak{z}})$ *with center* $\varpi \in \mathcal{S}$, *radius* $\mathfrak{r}, \mathfrak{r} \in (0,1)$, $\dot{\mathfrak{z}} > 0$ *as follows:*

$$\mathcal{B}(\varpi, \mathfrak{r}, \dot{\mathfrak{z}}) = \{\varsigma \in \digamma \colon \Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) > 1 - \mathfrak{r}, \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) < \mathfrak{r}, \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) < \mathfrak{r}\}.$$

Definition 10. *Let* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a NBMS. A subset* $\mathcal{P}$ *of* $\mathcal{S}$ *is said to be right open set if for every* $\varpi \in \mathcal{P}$, *there exists* $\mathfrak{r}$ *such that*

$$\varpi \in \mathcal{B}(\varsigma, \mathfrak{r}, \dot{\mathfrak{z}}) \subseteq \mathcal{P}.$$

Definition 11. *Let $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ is a NBMS. A subset $\mathcal{K}$ of $\digamma$ is said to be left open set if for every $\varsigma \in \mathcal{K}$, there exists $\mathfrak{r}$ such that*

$$\varsigma \in \mathcal{B}(\varpi, \mathfrak{r}, \dot{\mathfrak{z}}) \subseteq \mathcal{K}.$$

Definition 12. *Let $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ is a NBMS. Let $\mathcal{G} \subseteq \digamma$ and $\mathcal{H} \subseteq \mathcal{S}$. Then $\mathcal{G} \times \mathcal{H}$ is called an open set if $\mathcal{G}$ is left open set and $\mathcal{H}$ is right open set.*

Definition 13. *Let $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ is a NBMS. Let $\mathcal{G} \subseteq \digamma$ and $\mathcal{H} \subseteq \mathcal{S}$. We say that a subset $\mathcal{G} \times \mathcal{H}$ of $\digamma \times \mathcal{S}$ is closed if $(\digamma - \mathcal{G}) \times (\mathcal{S} - \mathcal{H})$ is open.*

Theorem 1. *Every right open ball $\mathcal{B}(\varsigma, \mathfrak{r}, \dot{\mathfrak{z}})$ is a right open set.*

Proof. *Take $\mathcal{B}(\varsigma, \mathfrak{r}, \dot{\mathfrak{z}})$ be a right open ball. Choose $\varpi \in \mathcal{B}(\varsigma, \mathfrak{r}, \dot{\mathfrak{z}})$. Therefore, $\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) > 1 - \mathfrak{r}, \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) < \mathfrak{r}, \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) < \mathfrak{r}$. There exists $\dot{\mathfrak{z}}_0 \in (0, \dot{\mathfrak{z}})$ such that $\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}_0) > 1-\mathfrak{r}, \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}_0) < \mathfrak{r}, \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}_0) < \mathfrak{r}$ because of $\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) > 1-\mathfrak{r}$. If we take $\mathfrak{r}_0 = \Pi(\varsigma, \varpi, \dot{\mathfrak{z}}_0)$, then for $\mathfrak{r}_0 > 1-\mathfrak{r}$, $\theta \in (0,1)$ will exist such that $\mathfrak{r}_0 > 1-\theta > 1-\mathfrak{r}$. Give $\mathfrak{r}_0$ and θ such that $\mathfrak{r}_0 > 1-\theta$. Then, $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3 \in (0,1)$ will exist such that $\mathfrak{r}_0 \circledast \mathfrak{r}_1 > 1-\theta$, $(1-\mathfrak{r}_0) \diamond (1-\mathfrak{r}_2) \leq \theta$ and $(1 - \mathfrak{r}_0) \diamond (1 - \mathfrak{r}_3) \leq \theta$. Choose $\mathfrak{r}_4 = \max\{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3\}$. Consider the right open ball $\mathcal{B}(\varpi, 1 - \mathfrak{r}_4, \dot{\mathfrak{z}} - \dot{\mathfrak{z}}_0)$. We will show that $\mathcal{B}(\varpi, 1 - \mathfrak{r}_4, \dot{\mathfrak{z}} - \dot{\mathfrak{z}}_0) \subset \mathcal{B}(\varsigma, \mathfrak{r}, \dot{\mathfrak{z}})$. If we take $\theta_1 \in \mathcal{B}(\varpi, 1-\mathfrak{r}_4, \dot{\mathfrak{z}}-\dot{\mathfrak{z}}_0)$, then $\Pi(\varpi, \theta_1, \dot{\mathfrak{z}}-\dot{\mathfrak{z}}_0) > \mathfrak{r}_4, \Psi(\varpi, \theta_1, \dot{\mathfrak{z}}-\dot{\mathfrak{z}}_0) < \mathfrak{r}_4, \Xi(\varpi, \theta_1, \dot{\mathfrak{z}}-\dot{\mathfrak{z}}_0) < \mathfrak{r}_4$. Then,*

$$\Pi(\varsigma, \theta_1, \dot{\mathfrak{z}}) \geq \Pi(\varsigma, \varpi, \dot{\mathfrak{z}}_0) \circledast \Pi(\varpi, \theta_1, \dot{\mathfrak{z}} - \dot{\mathfrak{z}}_0) \geq \mathfrak{r}_0 \circledast \mathfrak{r}_4 \geq \mathfrak{r}_0 \circledast \mathfrak{r}_1 \geq 1 - \theta > 1 - \mathfrak{r},$$

$$\Psi(\varsigma, \theta_1, \dot{\mathfrak{z}}) \leq \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}_0) \Diamond \Psi(\varpi, \theta_1, \dot{\mathfrak{z}} - \dot{\mathfrak{z}}_0) \leq (1 - \mathfrak{r}_0) \Diamond (1 - \mathfrak{r}_4) \leq (1 - \mathfrak{r}_0) \Diamond (1 - \mathfrak{r}_2) \leq \theta < \mathfrak{r},$$

$$\Xi(\varsigma, \theta_1, \dot{\mathfrak{z}}) \leq \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}_0) \Diamond \Xi(\varpi, \theta_1, \dot{\mathfrak{z}} - \dot{\mathfrak{z}}_0) \leq (1 - \mathfrak{r}_0) \Diamond (1 - \mathfrak{r}_4) \leq (1 - \mathfrak{r}_0) \Diamond (1 - \mathfrak{r}_2) \leq \theta < \mathfrak{r}.$$

Therefore $\theta_1 \in \mathcal{B}(\varsigma, \mathfrak{r}, \dot{\mathfrak{z}})$.

Similarly, we can prove the following theorems.

Theorem 2. *Every left open ball $\mathcal{B}(\varpi, \mathfrak{r}, \dot{\mathfrak{z}})$ is a left open set.*

Theorem 3. *Every open ball is an open set.*

Remark 2. *We can say that*
$\tau_p = \{\mathcal{G} \subset \digamma :$ *there exist* $\dot{\mathfrak{z}} > 0$ *and* $\mathfrak{r} \in (0,1)$ *such that* $\mathcal{B}(\varsigma, \mathfrak{r}, \dot{\mathfrak{z}}) \subseteq \mathcal{G}$ *for each* $\varsigma \in \mathcal{G}\} \times \{\mathcal{H} \subset \mathcal{S} :$ *there exist* $\dot{\mathfrak{z}} > 0$ *and* $\mathfrak{r} \in (0,1)$ *such that* $\mathcal{B}(\varpi, \mathfrak{r}, \dot{\mathfrak{z}}) \subseteq \mathcal{G}$ *for each* $\varpi \in \mathcal{H}\}$ *is a product topology on $\digamma \times \mathcal{S}$. In that case every NBMS p on $\digamma \times \mathcal{S}$ produces a product topology τ_p on $\digamma \times \mathcal{S}$ which has a base the family of open sets.*

Theorem 4. *Every NBMS is Hausdorff.*

Proof. *Let $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ is a NBMS. Choose ς and ϖ as two distinct points in $\digamma$ and $\mathcal{S}$. Hence, $0 < \Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) < 1, 0 < \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) < 1, 0 < \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) < 1$. Take*

$\mathfrak{r}_1 = \Pi(\varsigma, \varpi, \mathfrak{z}), \mathfrak{r}_2 = \Psi(\varsigma, \varpi, \mathfrak{z}), \mathfrak{r}_3 = \Xi(\varsigma, \varpi, \mathfrak{z})$ *and* $\mathfrak{r} = \max\{\mathfrak{r}_1, 1-\mathfrak{r}_2, 1-\mathfrak{r}_3\}$. *If we take* $\mathfrak{r}_0 \in (\mathfrak{r}, 1)$, *then there exist* $\mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6$ *such that* $\mathfrak{r}_4 \circledast \mathfrak{r}_4 \geq \mathfrak{r}_0$, $(1-\mathfrak{r}_5)\Diamond(1-\mathfrak{r}_5) \leq 1-\mathfrak{r}_0$,$(1-\mathfrak{r}_6)\Diamond(1-\mathfrak{r}_6) \leq 1-\mathfrak{r}_0$. *Let* $\mathfrak{r}_7 = \max\{\mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}$. *If we consider the right open ball* $\mathcal{B}(\varsigma, \mathfrak{r}_7, \frac{\mathfrak{z}}{2})$ *and left open ball* $\mathcal{B}(\varpi, \mathfrak{r}_7, \frac{\mathfrak{z}}{2})$, *then clearly* $\mathcal{B}(\varsigma, \mathfrak{r}_7, \frac{\mathfrak{z}}{2}) \cap \mathcal{B}(\varpi, \mathfrak{r}_7, \frac{\mathfrak{z}}{2}) = \emptyset$. *Suppose that* $\theta_1 \in \mathcal{B}(\varsigma, \mathfrak{r}_7, \frac{\mathfrak{z}}{2}) \cap \mathcal{B}(\varpi, \mathfrak{r}_7, \frac{\mathfrak{z}}{2})$, *then*

$$\mathfrak{r}_1 = \Pi(\varsigma, \varpi, \mathfrak{z}) \geq \Pi(\varsigma, \theta_1, \frac{\mathfrak{z}}{2}) \circledast \Pi(\theta_1, \varpi, \frac{\mathfrak{z}}{2}) \geq \mathfrak{r}_7 \circledast \mathfrak{r}_7 \geq \mathfrak{r}_4 \circledast \mathfrak{r}_4 \geq \mathfrak{r}_0 > \mathfrak{r}_1,$$

$$\begin{aligned}\mathfrak{r}_2 = \Psi(\varsigma, \varpi, \mathfrak{z}) \leq \Psi(\varsigma, \theta_1, \frac{\mathfrak{z}}{2})\Diamond\Psi(\theta_1, \varpi, \frac{\mathfrak{z}}{2}) &\leq (1-\mathfrak{r}_7)\Diamond(1-\mathfrak{r}_7) \leq (1-\mathfrak{r}_5)\Diamond(1-\mathfrak{r}_5)\\ &\leq 1-\mathfrak{r}_0 < \mathfrak{r}_2,\end{aligned}$$

$$\begin{aligned}\mathfrak{r}_3 = \Xi(\varsigma, \varpi, \mathfrak{z}) \leq \Xi(\varsigma, \theta_1, \frac{\mathfrak{z}}{2})\Diamond\Xi(\theta_1, \varpi\frac{\mathfrak{z}}{2}) &\leq (1-\mathfrak{r}_7)\Diamond(1-\mathfrak{r}_7) \leq (1-\mathfrak{r}_6)\Diamond(1-\mathfrak{r}_6)\\ &\leq 1-\mathfrak{r}_0 < \mathfrak{r}_3,\end{aligned}$$

which is a contradiction.

Definition 14. *Let* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a NBMS.*

i. *A point* $\varsigma \in \digamma \cup \mathcal{S}$ *is said to be a left point if* $\varsigma \in \digamma$, *a right point if* $\varsigma \in \mathcal{S}$ *and a central point if both hold.*

ii. *A sequence* $\{\varsigma_\mu\} \subset \digamma$ *is said to be a left sequence and a sequence* $\{\beta_n\} \subset \mathcal{S}$ *is said to be a right sequence.*

iii. *A sequence* $\{\varsigma_\mu\} \subset \digamma \cup \mathcal{S}$ *is said to converge to a point* ς *if and only if* $\{\varsigma_\mu\}$ *is a left sequence,* ς *is a right point and*

$$\lim_{\mu\to+\infty} \Pi(\varsigma_\mu, \varsigma, \mathfrak{z}) = 1, \lim_{\mu\to+\infty} \Psi(\varsigma_\mu, \varsigma, \mathfrak{z}) = 0, \lim_{\mu\to+\infty} \Xi(\varsigma_\mu, \varsigma, \mathfrak{z}) = 0 \quad \forall \quad \mathfrak{z} > 0$$

or $\{\varsigma_\mu\}$ *is a right sequence,* ς *is a left point and*

$$\lim_{\mu\to+\infty} \Pi(\varsigma, \varsigma_\mu, \mathfrak{z}) = 1, \lim_{\mu\to+\infty} \Psi(\varsigma, \varsigma_\mu, \mathfrak{z}) = 0, \lim_{\mu\to+\infty} \Xi(\varsigma, \varsigma_\mu, \mathfrak{z}) = 0 \quad \forall \quad \mathfrak{z} > 0.$$

iv. *A sequence* $\{(\varsigma_\mu, \beta_\mu)\} \subset \digamma \times \mathcal{S}$ *is said to be a bisequence. If the sequences* $\{\varsigma_\mu\}$ *and* $\{\beta_\mu\}$ *both converge then the bisequence* $\{(\varsigma_\mu, \beta_\mu)\}$ *is said to be convergent in* $\digamma \times \mathcal{S}$.

v. *If* $\{\varsigma_\mu\}$ *and* $\{\beta_\mu\}$ *both converge to a point* $\beta \in \digamma \cap \mathcal{S}$ *then the bisequence* $\{(\varsigma_\mu, \beta_\mu)\}$ *is said to be biconvergent. A sequence* $\{(\varsigma_\mu, \beta_\mu)\}$ *is a Cauchy bisequence if*

$$\lim_{\mu,\mathfrak{m}\to+\infty} \Pi(\varsigma_\mu, \beta_\mathfrak{m}, \mathfrak{z}) = 1, \lim_{\mu,\mathfrak{m}\to+\infty} \Psi(\varsigma_\mu, \beta_\mathfrak{m}, \mathfrak{z}) = 0, \lim_{\mu,\mathfrak{m}\to+\infty} \Xi(\varsigma_\mu, \beta_\mathfrak{m}, \mathfrak{z}) = 0,$$

$\forall\ \mathfrak{z} > 0$.

vi. *A NBMS is said to be complete if every Cauchy bisequence is convergent.*

Now we establish our main results.

3.2. Main Results

Lemma 1. *Let* $\{\varsigma_\mu\}$ *be a Cauchy sequence in NBMS* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *such that* $\varsigma_\mu \neq \varsigma_{\mathfrak{m}}$ *for every* $\mathfrak{m}, \mu(\neq \mathfrak{m}) \in \mathbb{N}$. *Then, at most, the sequence* $\{\varsigma_\mu\}$ *converge to one limit point.*

Proof. Conversely, suppose that $\varsigma_\mu \to \varsigma \in \mathcal{S}$ *and* $\varsigma_\mu \to \varpi \in \digamma \cap \mathcal{S}$, *for* $\varsigma \neq \varpi$. *Then,* $\lim_{\mu\to+\infty} \Pi(\varsigma_\mu, \varsigma, \dot{\mathfrak{z}}) = 1, \lim_{\mu\to+\infty} \Psi(\varsigma_\mu, \varsigma, \dot{\mathfrak{z}}) = 0, \lim_{\mu\to+\infty} \Xi(\varsigma_\mu, \varsigma, \dot{\mathfrak{z}}) = 0,$ *and* $\lim_{\mu\to+\infty} \Pi(\varsigma_\mu, \varpi, \dot{\mathfrak{z}}) = 1, \lim_{\mu\to+\infty} \Psi(\varsigma_\mu, \varpi, \dot{\mathfrak{z}}) = 0, \lim_{\mu\to+\infty} \Xi(\varsigma_\mu, \varpi, \dot{\mathfrak{z}}) = 0, \ \forall \ \dot{\mathfrak{z}} > 0.$ *Suppose*

$$\begin{aligned}
\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) &\geq \Pi\left(\varsigma, \varsigma_\mu, \frac{\dot{\mathfrak{z}}}{3}\right) \circledast \Pi\left(\varsigma_\mu, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \circledast \Pi\left(\varsigma_{\mu+1}, \varpi, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&\to 1 \circledast 1 \circledast 1, \quad as \quad \mu \to +\infty, \\
\Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) &\leq \Psi\left(\varsigma, \varsigma_\mu, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Psi\left(\varsigma_\mu, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Psi\left(\varsigma_{\mu+1}, \varpi, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&\to 0 \Diamond 0 \Diamond 0, \quad as \quad \mu \to +\infty, \\
\Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) &\leq \Xi\left(\varsigma, \varsigma_\mu, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Xi\left(\varsigma_\mu, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Xi\left(\varsigma_{\mu+1}, \varpi, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&\to 0 \Diamond 0 \Diamond 0, \quad as \quad \mu \to +\infty.
\end{aligned}$$

That is $\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) \geq 1 \circledast 1 \circledast 1 = 1, \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) \leq 0 \Diamond 0 \Diamond 0 = 0$ *and* $\Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) \leq 0 \Diamond 0 \Diamond 0 = 0$. *Hence* $\varsigma = \varpi$, *i.e., the sequence converges to unique limit point at most.*

Lemma 2. *Let* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *be a NBMS. If* $\zeta \in (0,1)$ *and for some* $\varsigma, \varpi \in \digamma, \dot{\mathfrak{z}} > 0$,

$$\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) \geq \Pi\left(\varsigma, \varpi, \frac{\dot{\mathfrak{z}}}{\zeta}\right), \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) \leq \Psi\left(\varsigma, \varpi, \frac{\dot{\mathfrak{z}}}{\zeta}\right), \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) \leq \Xi\left(\varsigma, \varpi, \frac{\dot{\mathfrak{z}}}{\zeta}\right) \tag{2}$$

then $\varsigma = \varpi$.

Proof. (2) *implies that*

$$\begin{aligned}
\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) \geq \Pi\left(\varsigma, \varpi, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right), \Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) \leq \Psi\left(\varsigma, \varpi, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right), &\Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) \\
&\leq \Xi\left(\varsigma, \varpi, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right), \mu \in \mathbb{N}, \dot{\mathfrak{z}} > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\Pi(\varsigma, \varpi, \dot{\mathfrak{z}}) &\geq \lim_{\mu\to+\infty} \Pi\left(\varsigma, \varpi, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right) = 1, \\
\Psi(\varsigma, \varpi, \dot{\mathfrak{z}}) &\leq \lim_{\mu\to+\infty} \Psi\left(\varsigma, \varpi, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right) = 0, \\
\Xi(\varsigma, \varpi, \dot{\mathfrak{z}}) &\leq \lim_{\mu\to+\infty} \Xi\left(\varsigma, \varpi, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right) = 0, \dot{\mathfrak{z}} > 0.
\end{aligned}$$

Also, by definition of iii, viii, xiii, that is, $\varsigma = \varpi$.

Theorem 5. *Suppose* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a complete NBMS with* $0 < \zeta < 1$. *Let* $\mathfrak{p}\colon \digamma \cup \mathcal{S} \to \digamma \cup \mathcal{S}$ *be a mapping satisfying*

i. $\mathfrak{p}(\digamma) \subseteq \digamma$ *and* $\mathfrak{p}(\mathcal{S}) \subseteq \mathcal{S}$;

ii.

$$\begin{gathered}\Pi(\mathfrak{p}\varsigma, \mathfrak{p}\varpi, \zeta\mathfrak{z}) \geq \Pi(\varsigma, \varpi, \mathfrak{z}),\\ \Psi(\mathfrak{p}\varsigma, \mathfrak{p}\varpi, \zeta\mathfrak{z}) \leq \Psi(\varsigma, \varpi, \mathfrak{z}) \quad \text{and} \quad \Xi(\mathfrak{p}\varsigma, \mathfrak{p}\varpi, \zeta\mathfrak{z}) \leq \Xi(\varsigma, \varpi, \mathfrak{z})\end{gathered} \tag{3}$$

$\forall\, \varsigma \in \digamma$, $\varpi \in \mathcal{S}$ *and* $\mathfrak{z} > 0$.

Then $\mathfrak{p}$ *has a unique fixed point.*

Proof. *Let* $\varsigma_0 \in \digamma$ *and* $\varpi_0 \in \mathcal{S}$ *and assume that* $\mathfrak{p}(\varsigma_\mu) = \varsigma_{\mu+1}$ *and* $\mathfrak{p}(\varpi_\mu) = \varpi_{\mu+1}$ $\forall$ $\mu \in \mathbb{N} \cup \{0\}$. *Then we get* $(\varsigma_\mu, \varpi_\mu)$ *as a bisequence on NBMS* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$. *Now, we have*

$$\Pi(\varsigma_1, \varpi_1, \mathfrak{z}) = \Pi(\mathfrak{p}\varsigma_0, \mathfrak{p}\varpi_0, \mathfrak{z}) \geq \Pi(\varsigma_0, \varpi_0, \frac{\mathfrak{z}}{\zeta}),$$

$$\Psi(\varsigma_1, \varpi_1, \mathfrak{z}) = \Psi(\mathfrak{p}\varsigma_0, \mathfrak{p}\varpi_0, \mathfrak{z}) \leq \Psi(\varsigma_0, \varpi_0, \frac{\mathfrak{z}}{\zeta})$$

and

$$\Xi(\varsigma_1, \varpi_1, \mathfrak{z}) = \Xi(\mathfrak{p}\varsigma_0, \mathfrak{p}\varpi_0, \mathfrak{z}) \leq \Xi(\varsigma_0, \varpi_0, \frac{\mathfrak{z}}{\zeta}),$$

$\forall$ $\mathfrak{z} > 0$ *and* $\mu \in \mathbb{N}$. *By simple induction, we get*

$$\begin{aligned}\Pi(\varsigma_\mu, \varpi_\mu, \mathfrak{z}) &= \Pi(\mathfrak{p}\varsigma_{\mu-1}, \mathfrak{p}\varpi_{\mu-1}, \mathfrak{z}) \geq \Pi(\varsigma_{\mu-1}, \varpi_{\mu-1}, \frac{\mathfrak{z}}{\zeta}) \geq \Pi\left(\varsigma_{\mu-2}, \varpi_{\mu-2}, \frac{\mathfrak{z}}{\zeta^2}\right)\\ &\geq \Pi\left(\varsigma_{\mu-3}, \varpi_{\mu-3}, \frac{\mathfrak{z}}{\zeta^3}\right) \geq \cdots \geq \Pi\left(\varsigma_0, \varpi_0, \frac{\mathfrak{z}}{\zeta^\mu}\right),\\ \Psi(\varsigma_\mu, \varpi_\mu, \mathfrak{z}) &= \Psi(\mathfrak{p}\varsigma_{\mu-1}, \mathfrak{p}\varpi_{\mu-1}, \mathfrak{z}) \leq \Psi(\varsigma_{\mu-1}, \varpi_{\mu-1}, \frac{\mathfrak{z}}{\zeta}) \leq \Psi\left(\varsigma_{\mu-2}, \varpi_{\mu-2}, \frac{\mathfrak{z}}{\zeta^2}\right)\\ &\leq \Psi\left(\varsigma_{\mu-3}, \varpi_{\mu-3}, \frac{\mathfrak{z}}{\zeta^3}\right) \leq \cdots \leq \Psi\left(\varsigma_0, \varpi_0, \frac{\mathfrak{z}}{\zeta^\mu}\right).\end{aligned}$$

and

$$\begin{aligned}\Xi(\varsigma_\mu, \varpi_\mu, \mathfrak{z}) &= \Xi(\mathfrak{p}\varsigma_{\mu-1}, \mathfrak{p}\varpi_{\mu-1}, \mathfrak{z}) \leq \Xi(\varsigma_{\mu-1}, \varpi_{\mu-1}, \frac{\mathfrak{z}}{\zeta}) \leq \Xi\left(\varsigma_{\mu-2}, \varpi_{\mu-2}, \frac{\mathfrak{z}}{\zeta^2}\right)\\ &\leq \Xi\left(\varsigma_{\mu-3}, \varpi_{\mu-3}, \frac{\mathfrak{z}}{\zeta^3}\right) \leq \cdots \leq \Xi\left(\varsigma_0, \varpi_0, \frac{\mathfrak{z}}{\zeta^\mu}\right).\end{aligned}$$

We obtain

$$\Pi(\varsigma_\mu, \varpi_\mu, \dot{\mathfrak{z}}) \geq \Pi\left(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right),$$
$$\Psi(\varsigma_\mu, \varpi_\mu, \dot{\mathfrak{z}}) \leq \Psi\left(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right), \Xi(\varsigma_\mu, \varpi_\mu, \dot{\mathfrak{z}}) \leq \Xi\left(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right) \tag{4}$$

and

$$\Pi(\varsigma_{\mu+1}, \varpi_\mu, \dot{\mathfrak{z}}) \geq \Pi\left(\varsigma_1, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right),$$
$$\Psi(\varsigma_{\mu+1}, \varpi_\mu, \dot{\mathfrak{z}}) \leq \Psi\left(\varsigma_1, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right), \Xi(\varsigma_{\mu+1}, \varpi_\mu, \dot{\mathfrak{z}}) \leq \Xi\left(\varsigma_1, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right). \tag{5}$$

Letting $\mu < \mathfrak{m}$, *for* $\mu, \mathfrak{m} \in \mathbb{N}$. *Then,*

$$\begin{aligned}\Pi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \geq{}& \Pi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \Pi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \Pi(\varsigma_{\mu+1}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3})\\ &\vdots\\ &\geq \Pi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \Pi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \cdots \circledast \Pi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\ &\circledast \Pi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \circledast \Pi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}),\end{aligned}$$

$$\begin{aligned}\Psi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \leq{}& \Psi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \Psi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \Psi(\varsigma_{\mu+1}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3})\\ &\vdots\\ &\leq \Psi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \Psi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \cdots \lozenge \Psi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\ &\lozenge \Psi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \lozenge \Psi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}),\end{aligned}$$

and

$$\begin{aligned}\Xi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \leq{}& \Xi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \Xi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \Xi(\varsigma_{\mu+1}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3})\\ &\vdots\\ &\leq \Xi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \Xi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \cdots \lozenge \Xi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\ &\lozenge \Xi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \lozenge \Xi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}).\end{aligned}$$

Therefore,

$$\Pi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \geq \Pi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \Pi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \cdots \circledast \Pi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$

$$
\begin{aligned}
&\circledast \Pi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \circledast \Pi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\
&\geq \Pi(\varsigma_{o}, \varpi_{o}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \circledast \Pi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \circledast \cdots \circledast \Pi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}})\\
&\circledast \Pi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}}) \circledast \Pi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}}}),
\end{aligned}
$$

$$
\begin{aligned}
\Psi(\varsigma_{\mu}, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \leq{}& \Psi(\varsigma_{\mu}, \varpi_{\mu}, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Psi(\varsigma_{\mu+1}, \varpi_{\mu}, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Psi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\
&\Diamond \Psi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \Psi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\
&\leq \Psi(\varsigma_{o}, \varpi_{o}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \Diamond \Psi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \Diamond \cdots \Diamond \Psi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}})\\
&\Diamond \Psi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}}) \Diamond \Psi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}}}),
\end{aligned}
$$

and

$$
\begin{aligned}
\Xi(\varsigma_{\mu}, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \leq{}& \Xi(\varsigma_{\mu}, \varpi_{\mu}, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Xi(\varsigma_{\mu+1}, \varpi_{\mu}, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Xi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\
&\Diamond \Xi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \Xi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\
&\leq \Xi(\varsigma_{o}, \varpi_{o}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \Diamond \Xi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \Diamond \cdots \Diamond \Xi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}})\\
&\Diamond \Xi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}}) \Diamond \Xi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}}}).
\end{aligned}
$$

Which implies that,

$$
\begin{aligned}
\Pi(\varsigma_{\mu}, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \geq{}& \Pi(\varsigma_{o}, \varpi_{o}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \circledast \Pi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \circledast \cdots \circledast \Pi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}})\\
&\circledast \Pi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}}) \circledast \Pi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}}}),
\end{aligned}
$$

$$
\begin{aligned}
\Psi(\varsigma_{\mu}, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \leq{}& \Psi(\varsigma_{o}, \varpi_{o}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \Diamond \Psi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \Diamond \cdots \Diamond \Psi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}})\\
&\Diamond \Psi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}}) \Diamond \Psi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}}}),
\end{aligned}
$$

and

$$
\begin{aligned}
\Xi(\varsigma_{\mu}, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \leq{}& \Xi(\varsigma_{o}, \varpi_{o}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \Diamond \Xi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3\zeta^{\mu}}) \Diamond \cdots \Diamond \Xi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}})\\
&\Diamond \Xi(\varsigma_{1}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}-1}}) \Diamond \Xi(\varsigma_{0}, \varpi_{0}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{\mathfrak{m}}}).
\end{aligned}
$$

As $\mu, \mathfrak{m} \to +\infty$*, we deduce*

$$\lim_{\mu,\mathfrak{m}\to+\infty} \Pi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) = 1 \circledast 1 \circledast \cdots \circledast 1 = 1,$$
$$\lim_{\mu,\mathfrak{m}\to+\infty} \Psi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) = 0\Diamond 0\Diamond \cdots \Diamond 0 = 0$$

and

$$\lim_{\mu,\mathfrak{m}\to+\infty} \Xi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) = 0\Diamond 0\Diamond \cdots \Diamond 0 = 0.$$

Which implies that bisequence $(\varsigma_\mu, \varpi_\mu)$ *is a Cauchy bisequence.*
Since $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a complete NBMS. Then,* $\{\varsigma_\mu\} \to \varsigma$ *and* $\{\varpi_\mu\} \to \varsigma$*, where* $\varsigma \in \digamma \cap \mathcal{S}$*. Using v, x and xv, we get*

$$\begin{aligned}
\Pi(\varsigma, \mathfrak{p}\varsigma, \dot{\mathfrak{z}}) &\geq \Pi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \circledast \Pi\left(\varsigma_{\mu+1}, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \circledast \Pi\left(\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right)\\
&= \Pi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \circledast \Pi\left(\mathfrak{p}\varsigma_\mu, \mathfrak{p}\varsigma_\mu, \frac{\dot{\mathfrak{z}}}{3}\right) \circledast \Pi\left(\mathfrak{p}\varsigma_\mu, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right)\\
&\to 1 \circledast 1 \circledast 1 = 1 \quad as \quad \mu \to +\infty,
\end{aligned}$$

$$\begin{aligned}
\Psi(\varsigma, \mathfrak{p}\varsigma, \dot{\mathfrak{z}}) &\leq \Psi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Psi\left(\varsigma_{\mu+1}, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Psi\left(\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right)\\
&= \Psi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Psi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Psi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right)\\
&\to 0\Diamond 0\Diamond 0 = 0 \quad as \quad \mu \to +\infty
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\varsigma, \mathfrak{p}\varsigma, \dot{\mathfrak{z}}) &\leq \Xi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Xi\left(\varsigma_{\mu+1}, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Xi\left(\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right)\\
&= \Xi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Xi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \Diamond \Xi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right)\\
&\to 0\Diamond 0\Diamond 0 = 0 \quad as \quad \mu \to +\infty.
\end{aligned}$$

Hence, $\mathfrak{p}\varsigma = \varsigma$*. Let* $\mathfrak{p}\eta = \eta$ *for any* $\eta \in \digamma \cap \mathcal{S}$*, then*

$$\begin{aligned}
1 &\geq \Pi(\eta, \varsigma, \dot{\mathfrak{z}}) = \Pi(\mathfrak{p}\eta, \mathfrak{p}\varsigma, \dot{\mathfrak{z}}) \geq \Pi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta}\right) = \Pi\left(\mathfrak{p}\eta, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{\zeta}\right)\\
&\geq \Pi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^2}\right) \geq \cdots \geq \Pi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right) \to 1 \quad as \quad \mu \to +\infty,\\
0 &\leq \Psi(\eta, \varsigma, \dot{\mathfrak{z}}) = \Psi(\mathfrak{p}\eta, \mathfrak{p}\varsigma, \dot{\mathfrak{z}}) \leq \Psi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta}\right) = \Psi\left(\mathfrak{p}\eta, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{\zeta}\right)
\end{aligned}$$

$$\leq \Psi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^2}\right) \leq \cdots \leq \Psi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right) \to 0 \quad as \quad \mu \to +\infty,$$

and

$$0 \leq \Xi(\eta, \varsigma, \dot{\mathfrak{z}}) = \Xi(\mathfrak{p}\eta, \mathfrak{p}\varsigma, \dot{\mathfrak{z}}) \leq \Xi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta}\right) = \Xi\left(\mathfrak{p}\eta, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{\zeta}\right)$$
$$\leq \Xi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^2}\right) \leq \cdots \leq \Xi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right) \to 0 \quad as \quad \mu \to +\infty,$$

since iii, viii and xiii, we get $\varsigma = \eta$.

Theorem 6. *Suppose* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a complete NBMS with* $0 < \zeta < 1$. *Let* $\mathfrak{p}\colon \digamma \cup \mathcal{S} \to \digamma \cup \mathcal{S}$ *be a mapping satisfying*

i. $\mathfrak{p}(\digamma) \subseteq \mathcal{S}$ *and* $\mathfrak{p}(\mathcal{S}) \subseteq \digamma$;

ii.

$$\Pi(\mathfrak{p}\varsigma, \mathfrak{p}\varpi, \zeta\dot{\mathfrak{z}}) \geq \Pi(\varpi, \varsigma, \dot{\mathfrak{z}}),$$
$$\Psi(\mathfrak{p}\varsigma, \mathfrak{p}\varpi, \zeta\dot{\mathfrak{z}}) \leq \Psi(\varpi, \varsigma, \dot{\mathfrak{z}}) \quad and \quad \Xi(\mathfrak{p}\varsigma, \mathfrak{p}\varpi, \zeta\dot{\mathfrak{z}}) \leq \Xi(\varpi, \varsigma, \dot{\mathfrak{z}}) \tag{6}$$

$\forall\ \varsigma \in \digamma$, $\varpi \in \mathcal{S}$ *and* $\dot{\mathfrak{z}} > 0$.

Then $\mathfrak{p}$ *has a unique fixed point.*

Proof. *Let* $\varsigma_0 \in \digamma$ *and* $\varpi_0 \in \mathcal{S}$ *and assume that* $\mathfrak{p}(\varsigma_\mu) = \varpi_\mu$ *and* $\mathfrak{p}(\varpi_\mu) = \varsigma_{\mu+1}$ $\forall$ $\mu \in \mathbb{N} \cup \{0\}$. *Then we get* $(\varsigma_\mu, \varpi_\mu)$ *as a bisequence on NBMS* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$. *Now, we have*

$$\Pi(\varsigma_1, \varpi_0, \dot{\mathfrak{z}}) = \Pi(\mathfrak{p}\varpi_0, \mathfrak{p}\varsigma_0, \dot{\mathfrak{z}}) \geq \Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta}),$$

$$\Psi(\varsigma_1, \varpi_0, \dot{\mathfrak{z}}) = \Psi(\mathfrak{p}\varpi_0, \mathfrak{p}\varsigma_0, \dot{\mathfrak{z}}) \leq \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta})$$

and

$$\Xi(\varsigma_1, \varpi_0, \dot{\mathfrak{z}}) = \Xi(\mathfrak{p}\varpi_0, \mathfrak{p}\varsigma_0, \dot{\mathfrak{z}}) \leq \Xi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta}),$$

$\forall\ \dot{\mathfrak{z}} > 0$ *and* $\mu \in \mathbb{N}$. *By simple induction, we get*

$$\Pi(\varsigma_\mu, \varpi_\mu, \dot{\mathfrak{z}}) = \Pi(\mathfrak{p}\varpi_{\mu-1}, \mathfrak{p}\varsigma_\mu, \dot{\mathfrak{z}}) \geq \Pi\left(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^{2\mu}}\right),$$
$$\Psi(\varsigma_\mu, \varpi_\mu, \dot{\mathfrak{z}}) = \Psi(\mathfrak{p}\varpi_{\mu-1}, \mathfrak{p}\varsigma_\mu, \dot{\mathfrak{z}}) \leq \Psi\left(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^{2\mu}}\right),$$

$$\Xi(\varsigma_\mu, \varpi_\mu, \dot{\mathfrak{z}}) = \Xi(\mathfrak{p}\varpi_{\mu-1}, \mathfrak{p}\varsigma_\mu, \dot{\mathfrak{z}}) \leq \Xi\left(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^{2\mu}}\right)$$

and

$$\Pi(\varsigma_{\mu+1}, \varpi_\mu, \dot{\mathfrak{z}}) = \Pi(\mathfrak{p}\varpi_\mu, \mathfrak{p}\varsigma_\mu, \dot{\mathfrak{z}}) \geq \Pi\left(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^{2\mu+1}}\right),$$
$$\Psi(\varsigma_{\mu+1}, \varpi_\mu, \dot{\mathfrak{z}}) = \Psi(\mathfrak{p}\varpi_\mu, \mathfrak{p}\varsigma_\mu, \dot{\mathfrak{z}}) \leq \Psi\left(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^{2\mu+1}}\right),$$
$$\Xi(\varsigma_{\mu+1}, \varpi_\mu, \dot{\mathfrak{z}}) = \Xi(\mathfrak{p}\varpi_\mu, \mathfrak{p}\varsigma_\mu, \dot{\mathfrak{z}}) \leq \Xi\left(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{\zeta^{2\mu+1}}\right).$$

Letting $\mu < \mathfrak{m}$*, for* $\mu, \mathfrak{m} \in \mathbb{N}$*. Then,*

$$\begin{aligned}\Pi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) &\geq \Pi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \Pi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \Pi(\varsigma_{\mu+1}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3})\\ &\vdots\\ &\geq \Pi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \Pi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \cdots \circledast \Pi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\ &\circledast \Pi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \circledast \Pi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}),\end{aligned}$$

$$\begin{aligned}\Psi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) &\leq \Psi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Psi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Psi(\varsigma_{\mu+1}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3})\\ &\vdots\\ &\leq \Psi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Psi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Psi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\ &\Diamond \Psi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \Psi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\end{aligned}$$

and

$$\begin{aligned}\Xi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) &\leq \Xi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Xi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Xi(\varsigma_{\mu+1}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3})\\ &\vdots\\ &\leq \Xi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Xi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Xi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\ &\Diamond \Xi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \Xi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}).\end{aligned}$$

Therefore,

$$\Pi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \geq \Pi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \Pi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \cdots \circledast \Pi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$

$$\circledast \Pi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \circledast \Pi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\geq \Pi(\varsigma_{\mathfrak{o}}, \varpi_{\mathfrak{o}}, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu}}) \circledast \Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu+1}}) \circledast \cdots \circledast \Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-2}})$$
$$\circledast \Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-1}}) \circledast \Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}}}),$$

$$\Psi(\varsigma_\mu, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \leq \Psi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Psi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Psi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\Diamond \Psi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \Psi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\leq \Psi(\varsigma_{\mathfrak{o}}, \varpi_{\mathfrak{o}}, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu}}) \Diamond \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu+1}}) \Diamond \cdots \Diamond \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-2}})$$
$$\Diamond \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-1}}) \Diamond \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}}}),$$

and

$$\Xi(\varsigma_\mu, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \leq \Xi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Xi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Xi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\Diamond \Xi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \Xi(\varsigma_{\mathfrak{m}}, \varpi_{\mathfrak{m}}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\leq \Xi(\varsigma_{\mathfrak{o}}, \varpi_{\mathfrak{o}}, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu}}) \Diamond \Xi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu+1}}) \Diamond \cdots \Diamond \Xi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-2}})$$
$$\Diamond \Xi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-1}}) \Diamond \Xi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}}}).$$

Which implies that,

$$\Pi(\varsigma_\mu, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \geq \Pi(\varsigma_{\mathfrak{o}}, \varpi_{\mathfrak{o}}, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu}}) \circledast \Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu+1}}) \circledast \cdots \circledast \Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-2}})$$
$$\circledast \Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-1}}) \circledast \Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}}}),$$

$$\Psi(\varsigma_\mu, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \leq \Psi(\varsigma_{\mathfrak{o}}, \varpi_{\mathfrak{o}}, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu}}) \Diamond \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu+1}}) \Diamond \cdots \Diamond \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-2}})$$
$$\Diamond \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-1}}) \Diamond \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}}}),$$

and

$$\Xi(\varsigma_\mu, \varpi_{\mathfrak{m}}, \dot{\mathfrak{z}}) \leq \Xi(\varsigma_{\mathfrak{o}}, \varpi_{\mathfrak{o}}, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu}}) \Diamond \Xi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3\zeta^{2\mu+1}}) \Diamond \cdots \Diamond \Xi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-2}})$$
$$\Diamond \Xi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}-1}}) \Diamond \Xi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}\zeta^{2\mathfrak{m}}}).$$

As $\mu, \mathfrak{m} \to +\infty$, we deduce

$$\lim_{\mu,\mathfrak{m}\to+\infty} \Pi(\varsigma_\mu, \varpi_\mathfrak{m}, \mathfrak{z}) = 1 \circledast 1 \circledast \cdots \circledast 1 = 1,$$
$$\lim_{\mu,\mathfrak{m}\to+\infty} \Psi(\varsigma_\mu, \varpi_\mathfrak{m}, \mathfrak{z}) = 0\Diamond 0\Diamond \cdots \Diamond 0 = 0$$

and

$$\lim_{\mu,\mathfrak{m}\to+\infty} \Xi(\varsigma_\mu, \varpi_\mathfrak{m}, \mathfrak{z}) = 0\Diamond 0\Diamond \cdots \Diamond 0 = 0.$$

Which implies that bisequence $(\varsigma_\mu, \varpi_\mu)$ is a Cauchy bisequence.
Since $(\mathcal{F}, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ is a complete NBMS. Then, $\{\varsigma_\mu\} \to \varsigma$ and $\{\varpi_\mu\} \to \varsigma$, where $\varsigma \in \mathcal{F} \cap \mathcal{S}$. Using v, x and xv, we get

$$\begin{aligned}\Pi(\varsigma, \mathfrak{p}\varsigma, \mathfrak{z}) &\geq \Pi\left(\varsigma, \varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \circledast \Pi\left(\varsigma_{\mu+1}, \varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \circledast \Pi\left(\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\mathfrak{z}}{3}\right)\\ &= \Pi\left(\varsigma, \varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \circledast \Pi\left(\mathfrak{p}\varsigma_\mu, \mathfrak{p}\varsigma_\mu, \frac{\mathfrak{z}}{3}\right) \circledast \Pi\left(\mathfrak{p}\varsigma_\mu, \mathfrak{p}\varsigma, \frac{\mathfrak{z}}{3}\right)\\ &\to 1 \circledast 1 \circledast 1 = 1 \quad as \quad \mu \to +\infty,\end{aligned}$$

$$\begin{aligned}\Psi(\varsigma, \mathfrak{p}\varsigma, \mathfrak{z}) &\leq \Psi\left(\varsigma, \varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \Diamond \Psi\left(\varsigma_{\mu+1}, \varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \Diamond \Psi\left(\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\mathfrak{z}}{3}\right)\\ &= \Psi\left(\varsigma, \varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \Diamond \Psi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \Diamond \Psi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\mathfrak{z}}{3}\right)\\ &\to 0\Diamond 0\Diamond 0 = 0 \quad as \quad \mu \to +\infty\end{aligned}$$

and

$$\begin{aligned}\Xi(\varsigma, \mathfrak{p}\varsigma, \mathfrak{z}) &\leq \Xi\left(\varsigma, \varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \Diamond \Xi\left(\varsigma_{\mu+1}, \varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \Diamond \Xi\left(\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\mathfrak{z}}{3}\right)\\ &= \Xi\left(\varsigma, \varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \Diamond \Xi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma_{\mu+1}, \frac{\mathfrak{z}}{3}\right) \Diamond \Xi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\mathfrak{z}}{3}\right)\\ &\to 0\Diamond 0\Diamond 0 = 0 \quad as \quad \mu \to +\infty.\end{aligned}$$

Hence, $\mathfrak{p}\varsigma = \varsigma$. Let $\mathfrak{p}\eta = \eta$ for some $\eta \in \mathcal{F} \cap \mathcal{S}$, then

$$\begin{aligned}1 \geq \Pi(\eta, \varsigma, \mathfrak{z}) &= \Pi(\mathfrak{p}\varsigma, \mathfrak{p}\eta, \mathfrak{z}) \geq \Pi\left(\eta, \varsigma, \frac{\mathfrak{z}}{\zeta}\right) = \Pi\left(\mathfrak{p}\varsigma, \mathfrak{p}\eta, \frac{\mathfrak{z}}{\zeta}\right)\\ &\geq \Pi\left(\eta, \varsigma, \frac{\mathfrak{z}}{\zeta^2}\right) \geq \cdots \geq \Pi\left(\eta, \varsigma, \frac{\mathfrak{z}}{\zeta^\mu}\right) \to 1 \quad as \quad \mu \to +\infty,\\ 0 \leq \Psi(\eta, \varsigma, \mathfrak{z}) &= \Psi(\mathfrak{p}\varsigma, \mathfrak{p}\eta, \mathfrak{z}) \leq \Psi\left(\eta, \varsigma, \frac{\mathfrak{z}}{\zeta}\right) = \Psi\left(\mathfrak{p}\varsigma, \mathfrak{p}\eta, \frac{\mathfrak{z}}{\zeta}\right)\end{aligned}$$

$$\leq \Psi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^2}\right) \leq \cdots \leq \Psi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right) \to 0 \quad as \quad \mu \to +\infty,$$

and

$$0 \leq \Xi(\eta, \varsigma, \dot{\mathfrak{z}}) = \Xi(\mathfrak{p}\varsigma, \mathfrak{p}\eta, \dot{\mathfrak{z}}) \leq \Xi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta}\right) = \Xi\left(\mathfrak{p}\varsigma, \mathfrak{p}\eta, \frac{\dot{\mathfrak{z}}}{\zeta}\right)$$
$$\leq \Xi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^2}\right) \leq \cdots \leq \Xi\left(\eta, \varsigma, \frac{\dot{\mathfrak{z}}}{\zeta^\mu}\right) \to 0 \quad as \quad \mu \to +\infty,$$

since iii, viii and xiii, we get $\varsigma = \eta$.

Definition 15. *Let* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, *, \Diamond)$ *be a NBMS. A map* $\mathfrak{p}\colon \digamma \cup \mathcal{S} \to \digamma \cup \mathcal{S}$ *is an NB(neutrosophic bipolar)-contraction if we can find* $0 < \zeta < 1$, *satisfying*

$$\frac{1}{\Pi(\mathfrak{p}\varsigma, \mathfrak{p}\varpi, \dot{\mathfrak{z}})} - 1 \leq \zeta\left[\frac{1}{\Pi(\varsigma, \varpi, \dot{\mathfrak{z}})} - 1\right] \tag{7}$$

$$\Psi(\mathfrak{p}\varsigma, \mathfrak{p}\varpi, \dot{\mathfrak{z}}) \leq \zeta\Psi(\varsigma, \varpi, \dot{\mathfrak{z}}), \tag{8}$$

and

$$\Xi(\mathfrak{p}\varsigma, \mathfrak{p}\varpi, \dot{\mathfrak{z}}) \leq \zeta\Xi(\varsigma, \varpi, \dot{\mathfrak{z}}), \tag{9}$$

$\forall\ \varsigma \in \digamma$, $\varpi \in \mathcal{S}$ *and* $\dot{\mathfrak{z}} > 0$.

Now, we present the following theorem for NB(neutrosophic bipolar)-contraction.

Theorem 7. *Let* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, *, \Diamond)$ *be a complete NBMS. Let* $\mathfrak{p}\colon \digamma \cup \mathcal{S} \to \digamma \cup \mathcal{S}$ *be a mapping satisfying*

i. $\mathfrak{p}(\digamma) \subseteq \digamma$ *and* $\mathfrak{p}(\mathcal{S}) \subseteq \mathcal{S}$;

ii. $\mathfrak{p}$ *is NB-contraction,* $\forall\ \varsigma \in \digamma$, $\varpi \in \mathcal{S}$ *and* $\dot{\mathfrak{z}} > 0$.

Then, $\mathfrak{p}$ *has a unique fixed point.*

Proof. *Let* $\varsigma_0 \in \digamma$ *and* $\varpi_0 \in \mathcal{S}$ *and assume that* $\mathfrak{p}(\varsigma_\mu) = \varsigma_{\mu+1}$ *and* $\mathfrak{p}(\varpi_\mu) = \varpi_{\mu+1}$ $\forall\ \mu \in \mathbb{N} \cup \{0\}$. *Then we get* $(\varsigma_\mu, \varpi_\mu)$ *as a bisequence on NBMS* $(\digamma, \mathcal{S}, \Pi, \Psi, \Xi, *, \Diamond)$. *By using* (7), (8) *and* (9) $\forall\ \dot{\mathfrak{z}} > 0$, *we deduce*

$$\frac{1}{\Pi(\varsigma_\mu, \varpi_\mu, \dot{\mathfrak{z}})} - 1 = \frac{1}{\Pi(\mathfrak{p}\varsigma_{\mu-1}, \mathfrak{p}\varpi_{\mu-1}, \dot{\mathfrak{z}})} - 1 \leq \zeta\left[\frac{1}{\Pi(\varsigma_{\mu-1}, \varpi_{\mu-1}, \dot{\mathfrak{z}})}\right]$$
$$= \frac{\zeta}{\Pi(\varsigma_{\mu-1}, \varpi_{\mu-1}, \dot{\mathfrak{z}})} - \zeta$$
$$\Rightarrow \frac{1}{\Pi(\varsigma_\mu, \varpi_\mu, \dot{\mathfrak{z}})} \leq \frac{\zeta}{\Pi(\varsigma_{\mu-1}, \varpi_{\mu-1}, \dot{\mathfrak{z}})} + (1 - \zeta)$$
$$\leq \frac{\zeta^2}{\Pi(\varsigma_{\mu-2}, \varpi_{\mu-2}, \dot{\mathfrak{z}})} + \zeta(1 - \zeta) + (1 - \zeta).$$

In this manner, we obtain

$$\begin{aligned}\frac{1}{\Pi(\varsigma_\mu,\varpi_\mu,\dot{\mathfrak{z}})} &\leq \frac{\zeta^\mu}{\Pi(\varsigma_0,\varpi_0,\dot{\mathfrak{z}})}+\zeta^{\mu-1}(1-\zeta)\\ &\quad+\zeta^{\mu-2}(1-\zeta)+\cdots+\zeta(1-\zeta)+(1-\zeta)\\ &\leq \frac{\zeta^\mu}{\Pi(\varsigma_0,\varpi_0,\dot{\mathfrak{z}})}+(\zeta^{\mu-1}+\zeta^{\mu-2}+\cdots+1)(1-\zeta)\\ &\leq \frac{\zeta^\mu}{\Pi(\varsigma_0,\varpi_0,\dot{\mathfrak{z}})}+(1-\zeta^\mu)\end{aligned}$$

We obtain

$$\frac{1}{\frac{\zeta^\mu}{\Pi(\varsigma_0,\varpi_0,\dot{\mathfrak{z}})}+(1-\zeta^\mu)}\leq \Pi(\varsigma_\mu,\varpi_\mu,\dot{\mathfrak{z}}) \tag{10}$$

$$\begin{aligned}\Psi(\varsigma_\mu,\varpi_\mu,\dot{\mathfrak{z}}) &= \Psi(\mathfrak{p}\varsigma_{\mu-1},\mathfrak{p}\varpi_{\mu-1},\dot{\mathfrak{z}})\leq\zeta\Psi(\varsigma_{\mu-1},\varpi_{\mu-1},\dot{\mathfrak{z}})=\Psi(\mathfrak{p}\varsigma_{\mu-2},\mathfrak{p}\varpi_{\mu-2},\dot{\mathfrak{z}})\\ &\leq\zeta^2\Psi(\varsigma_{\mu-2},\varsigma_{\varpi-2},\dot{\mathfrak{z}})\leq\cdots\leq\zeta^\mu\Psi(\varsigma_0,\varpi_0,\dot{\mathfrak{z}}),\end{aligned} \tag{11}$$

$$\begin{aligned}\Xi(\varsigma_\mu,\varpi_\mu,\dot{\mathfrak{z}}) &= \Xi(\mathfrak{p}\varsigma_{\mu-1},\mathfrak{p}\varpi_{\mu-1},\dot{\mathfrak{z}})\leq\zeta\Xi(\varsigma_{\mu-1},\varpi_{\mu-1},\dot{\mathfrak{z}})=\Xi(\mathfrak{p}\varsigma_{\mu-2},\mathfrak{p}\varpi_{\mu-2},\dot{\mathfrak{z}})\\ &\leq\zeta^2\Xi(\varsigma_{\mu-2},\varpi_{\mu-2},\dot{\mathfrak{z}})\leq\cdots\leq\zeta^\mu\Xi(\varsigma_0,\varpi_0,\dot{\mathfrak{z}})\end{aligned} \tag{12}$$

and

$$\frac{1}{\frac{\zeta^\mu}{\Pi(\varsigma_1,\varpi_0,\dot{\mathfrak{z}})}+(1-\zeta^\mu)}\leq \Pi(\varsigma_{\mu+1},\varpi_\mu,\dot{\mathfrak{z}}) \tag{13}$$

$$\begin{aligned}\Psi(\varsigma_{\mu+1},\varpi_\mu,\dot{\mathfrak{z}}) &= \Psi(\mathfrak{p}\varsigma_{\mu},\mathfrak{p}\varpi_{\mu-1},\dot{\mathfrak{z}})\leq\zeta\Psi(\varsigma_{\mu},\varpi_{\mu-1},\dot{\mathfrak{z}})=\Psi(\mathfrak{p}\varsigma_{\mu-1},\mathfrak{p}\varpi_{\mu-2},\dot{\mathfrak{z}})\\ &\leq\zeta^2\Psi(\varsigma_{\mu-1},\varsigma_{\varpi-2},\dot{\mathfrak{z}})\leq\cdots\leq\zeta^\mu\Psi(\varsigma_1,\varpi_0,\dot{\mathfrak{z}}),\end{aligned} \tag{14}$$

$$\begin{aligned}\Xi(\varsigma_{\mu+1},\varpi_\mu,\dot{\mathfrak{z}}) &= \Xi(\mathfrak{p}\varsigma_{\mu},\mathfrak{p}\varpi_{\mu-1},\dot{\mathfrak{z}})\leq\zeta\Xi(\varsigma_{\mu},\varpi_{\mu-1},\dot{\mathfrak{z}})=\Xi(\mathfrak{p}\varsigma_{\mu-1},\mathfrak{p}\varpi_{\mu-2},\dot{\mathfrak{z}})\\ &\leq\zeta^2\Xi(\varsigma_{\mu-1},\varpi_{\mu-2},\dot{\mathfrak{z}})\leq\cdots\leq\zeta^\mu\Xi(\varsigma_1,\varpi_0,\dot{\mathfrak{z}}).\end{aligned} \tag{15}$$

Letting $\mu<\mathfrak{m}$*, for* $\mu,\mathfrak{m}\in\mathbb{N}$*. Then,*

$$\begin{aligned}\Pi(\varsigma_\mu,\varpi_\mathfrak{m},\dot{\mathfrak{z}}) &\geq \Pi(\varsigma_\mu,\varpi_\mu,\frac{\dot{\mathfrak{z}}}{3})\circledast\Pi(\varsigma_{\mu+1},\varpi_\mu,\frac{\dot{\mathfrak{z}}}{3})\circledast\Pi(\varsigma_{\mu+1},\varpi_\mathfrak{m},\frac{\dot{\mathfrak{z}}}{3})\\ &\vdots\\ &\geq \Pi(\varsigma_\mu,\varpi_\mu,\frac{\dot{\mathfrak{z}}}{3})\circledast\Pi(\varsigma_{\mu+1},\varpi_\mu,\frac{\dot{\mathfrak{z}}}{3})\circledast\cdots\circledast\Pi(\varsigma_{\mathfrak{m}-1},\varpi_{\mathfrak{m}-1},\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\\ &\circledast\Pi(\varsigma_\mathfrak{m},\varpi_{\mathfrak{m}-1},\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\circledast\Pi(\varsigma_\mathfrak{m},\varpi_\mathfrak{m},\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}),\end{aligned}$$

$$\Psi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \leq \Psi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Psi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Psi(\varsigma_{\mu+1}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3})$$
$$\vdots$$
$$\leq \Psi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Psi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Psi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\Diamond \Psi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \Psi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}),$$

and

$$\Xi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \leq \Xi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Xi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Xi(\varsigma_{\mu+1}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3})$$
$$\vdots$$
$$\leq \Xi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Xi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Xi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\Diamond \Xi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \Xi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}).$$

Therefore,

$$\Pi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \geq \Pi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \Pi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \circledast \cdots \circledast \Pi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\circledast \Pi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \circledast \Pi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\geq \frac{1}{\frac{\zeta^\mu}{\Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3})} + (1-\zeta^\mu)} \circledast \frac{1}{\frac{\zeta^\mu}{\Pi(\varsigma_1, \varpi_0, \frac{\dot{\mathfrak{z}}}{3})} + (1-\zeta^\mu)} \circledast \cdots$$
$$\circledast \frac{1}{\frac{\zeta^{\mathfrak{m}-1}}{\Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})} + (1-\zeta^{\mathfrak{m}-1})} \circledast \frac{1}{\frac{\zeta^{\mathfrak{m}-1}}{\Pi(\varsigma_1, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})} + (1-\zeta^{\mathfrak{m}-1})}$$
$$\circledast \frac{1}{\frac{\zeta^{\mathfrak{m}}}{\Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})} + (1-\zeta^{\mathfrak{m}})},$$

$$\Psi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \leq \Psi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Psi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Psi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\Diamond \Psi(\varsigma_\mathfrak{m}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \Psi(\varsigma_\mathfrak{m}, \varpi_\mathfrak{m}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\leq \zeta^\mu \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \zeta^\mu \Psi(\varsigma_1, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \cdots \Diamond \zeta^{\mathfrak{m}-1} \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\Diamond \zeta^{\mathfrak{m}-1} \Psi(\varsigma_1, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}) \Diamond \zeta^{\mathfrak{m}} \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}),$$

and

$$\Xi(\varsigma_\mu, \varpi_\mathfrak{m}, \dot{\mathfrak{z}}) \leq \Xi(\varsigma_\mu, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \Xi(\varsigma_{\mu+1}, \varpi_\mu, \frac{\dot{\mathfrak{z}}}{3}) \Diamond \cdots \Diamond \Xi(\varsigma_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}-1}, \frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$

$$\Diamond\Xi(\varsigma_{\mathfrak{m}},\varpi_{\mathfrak{m}-1},\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\Diamond\Xi(\varsigma_{\mathfrak{m}},\varpi_{\mathfrak{m}},\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\leq \zeta^{\mu}\Xi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3})\Diamond\zeta^{\mu}\Xi(\varsigma_1,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\Diamond\cdots\Diamond\zeta^{\mathfrak{m}-1}\Xi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\Diamond\zeta^{\mathfrak{m}-1}\Xi(\varsigma_1,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\Diamond\zeta^{\mathfrak{m}}\Xi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}).$$

Which implies that,

$$\Pi(\varsigma_\mu,\varpi_{\mathfrak{m}},\dot{\mathfrak{z}}) \geq \frac{1}{\frac{\zeta^\mu}{\Pi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3})}+(1-\zeta^\mu)} \circledast \frac{1}{\frac{\zeta^\mu}{\Pi(\varsigma_1,\varpi_0,\frac{\dot{\mathfrak{z}}}{3})}+(1-\zeta^\mu)} \circledast\cdots$$
$$\circledast \frac{1}{\frac{\zeta^{\mathfrak{m}-1}}{\Pi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})}+(1-\zeta^{\mathfrak{m}-1})} \circledast \frac{1}{\frac{\zeta^{\mathfrak{m}-1}}{\Pi(\varsigma_1,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})}+(1-\zeta^{\mathfrak{m}-1})}$$
$$\circledast \frac{1}{\frac{\zeta^{\mathfrak{m}}}{\Pi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})}+(1-\zeta^{\mathfrak{m}})},$$

$$\Psi(\varsigma_\mu,\varpi_{\mathfrak{m}},\dot{\mathfrak{z}}) \leq \zeta^{\mu}\Psi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3})\Diamond\zeta^{\mu}\Psi(\varsigma_1,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\Diamond\cdots\Diamond\zeta^{\mathfrak{m}-1}\Psi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\Diamond\zeta^{\mathfrak{m}-1}\Psi(\varsigma_1,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\Diamond\zeta^{\mathfrak{m}}\Psi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}),$$

and

$$\Xi(\varsigma_\mu,\varpi_{\mathfrak{m}},\dot{\mathfrak{z}}) \leq \zeta^{\mu}\Xi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3})\Diamond\zeta^{\mu}\Xi(\varsigma_1,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\Diamond\cdots\Diamond\zeta^{\mathfrak{m}-1}\Xi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})$$
$$\Diamond\zeta^{\mathfrak{m}-1}\Xi(\varsigma_1,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}})\Diamond\zeta^{\mathfrak{m}}\Xi(\varsigma_0,\varpi_0,\frac{\dot{\mathfrak{z}}}{3^{\mathfrak{m}-1}}).$$

As $\mu,\mathfrak{m}\to+\infty$, we deduce

$$\lim_{\mu,\mathfrak{m}\to+\infty}\Pi(\varsigma_\mu,\varpi_{\mathfrak{m}},\dot{\mathfrak{z}}) = 1\circledast 1\circledast\cdots\circledast 1 = 1,$$
$$\lim_{\mu,\mathfrak{m}\to+\infty}\Psi(\varsigma_\mu,\varpi_{\mathfrak{m}},\dot{\mathfrak{z}}) = 0\Diamond 0\Diamond\cdots\Diamond 0 = 0$$

and

$$\lim_{\mu,\mathfrak{m}\to+\infty}\Xi(\varsigma_\mu,\varpi_{\mathfrak{m}},\dot{\mathfrak{z}}) = 0\Diamond 0\Diamond\cdots\Diamond 0 = 0.$$

Which implies that bisequence $(\varsigma_\mu,\varpi_\mu)$ is a Cauchy bisequence.
Since $(\digamma,\mathcal{S},\Pi,\Psi,\Xi,\circledast,\Diamond)$ is a complete NBMS. Then, $\{\varsigma_\mu\}\to\varsigma$ and $\{\varpi_\mu\}\to\varsigma$, where $\varsigma\in\digamma\cap\mathcal{S}$. Using v, x and xv, we get

$$\Pi(\varsigma,\mathfrak{p}\varsigma,\dot{\mathfrak{z}}) \geq \Pi\left(\varsigma,\varsigma_{\mu+1},\frac{\dot{\mathfrak{z}}}{3}\right)\circledast\Pi\left(\varsigma_{\mu+1},\varsigma_{\mu+1},\frac{\dot{\mathfrak{z}}}{3}\right)\circledast\Pi\left(\varsigma_{\mu+1},\mathfrak{p}\varsigma,\frac{\dot{\mathfrak{z}}}{3}\right)$$

$$
\begin{aligned}
&= \Pi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \circledast \Pi\left(\mathfrak{p}\varsigma_{\mu}, \mathfrak{p}\varsigma_{\mu}, \frac{\dot{\mathfrak{z}}}{3}\right) \circledast \Pi\left(\mathfrak{p}\varsigma_{\mu}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&\geq \Pi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \circledast \frac{1}{\frac{\zeta^{\mu+1}}{\Pi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3})} + (1-\zeta^{\mu+1})} \circledast \Pi\left(\mathfrak{p}\varsigma_{\mu}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&\to 1 \circledast 1 \circledast 1 = 1 \quad as \quad \mu \to +\infty,
\end{aligned}
$$

$$
\begin{aligned}
\Psi(\varsigma, \mathfrak{p}\varsigma, \dot{\mathfrak{z}}) &\leq \Psi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \Psi\left(\varsigma_{\mu+1}, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \Psi\left(\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&= \Psi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \Psi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \Psi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&\leq \Psi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \zeta^{\mu+1} \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \Psi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&\to 0 \lozenge 0 \lozenge 0 = 0 \quad as \quad \mu \to +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
\Xi(\varsigma, \mathfrak{p}\varsigma, \dot{\mathfrak{z}}) &\leq \Xi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \Xi\left(\varsigma_{\mu+1}, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \Xi\left(\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&= \Xi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \Xi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \Xi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&\leq \Xi\left(\varsigma, \varsigma_{\mu+1}, \frac{\dot{\mathfrak{z}}}{3}\right) \lozenge \zeta^{\mu+1} \Psi(\varsigma_0, \varpi_0, \frac{\dot{\mathfrak{z}}}{3}) \lozenge \Xi\left(\mathfrak{p}\varsigma_{\mu+1}, \mathfrak{p}\varsigma, \frac{\dot{\mathfrak{z}}}{3}\right) \\
&\to 0 \lozenge 0 \lozenge 0 = 0 \quad as \quad \mu \to +\infty.
\end{aligned}
$$

Hence, $\mathfrak{p}\varsigma = \varsigma$*. Let* $\mathfrak{p}\eta = \eta$ *for some* $\eta \in \digamma$*, then*

$$
\begin{aligned}
\frac{1}{\Pi(\varsigma, \eta, \dot{\mathfrak{z}})} - 1 &= \frac{1}{\Pi(\mathfrak{p}\varsigma, \mathfrak{p}\eta, \dot{\mathfrak{z}})} - 1 \\
&\leq \zeta\left[\frac{1}{\Pi(\varsigma, \eta, \dot{\mathfrak{z}})} - 1\right] < \frac{1}{\Pi(\varsigma, \eta, \dot{\mathfrak{z}})} - 1,
\end{aligned}
$$

which is a contradiction.

$$
\Psi(\varsigma, \eta, \dot{\mathfrak{z}}) = \Psi(\mathfrak{p}\varsigma, \mathfrak{p}\eta, \dot{\mathfrak{z}}) \leq \zeta\Psi(\varsigma, \eta, \dot{\mathfrak{z}}) < \Psi(\varsigma, \eta, \dot{\mathfrak{z}}),
$$

which is a contradiction and

$$
\Xi(\varsigma, \eta, \dot{\mathfrak{z}}) = \Xi(\mathfrak{p}\varsigma, \mathfrak{p}\eta, \dot{\mathfrak{z}}) \leq \zeta\Xi(\varsigma, \eta, \dot{\mathfrak{z}}) < \Xi(\varsigma, \eta, \dot{\mathfrak{z}}),
$$

which is a contradiction. Therefore, we get $\Pi(\varsigma, \eta, \dot{\mathfrak{z}}) = 1, \Psi(\varsigma, \eta, \dot{\mathfrak{z}}) = 0$ *and* $\Xi(\varsigma, \eta, \dot{\mathfrak{z}}) = 0$*, hence,* $\varsigma = \eta$*.*

Example 2. *Let* $\digamma = [0,1]$ *and* $\mathcal{S} = \{0\}\cup\mathbb{N}-\{1\}$. *Define* $\Pi, \Psi, \Xi \colon \digamma\times\mathcal{S}\times(0,+\infty) \to [0,1]$ *as*

$$\Pi(\varsigma,\varpi,\dot{\mathfrak{z}}) = \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}}+|\varsigma-\varpi|},$$
$$\Psi(\varsigma,\varpi,\dot{\mathfrak{z}}) = \frac{|\varsigma-\varpi|}{\dot{\mathfrak{z}}+|\varsigma-\varpi|},$$
$$\Psi(\varsigma,\varpi,\dot{\mathfrak{z}}) = \frac{|\varsigma-\varpi|}{\dot{\mathfrak{z}}}.$$

Then, $(\digamma,\mathcal{S},\Pi,\Psi,\Xi,\circledast,\Diamond)$ *is a complete NBMS with* $c_{t-||.||}$ $\dot{e}\circledast\dot{a} = \dot{e}\dot{a}$ *and* $c_{t-co-||.||}$ $\dot{e}\Diamond\dot{a} = \max\{\dot{e},\dot{a}\}$.
Define $\mathfrak{p} : (\digamma,\mathcal{S},\Pi,\Psi,\Xi,\circledast,\Diamond) \rightrightarrows (\digamma,\mathcal{S},\Pi,\Psi,\Xi,\circledast,\Diamond)$ *by*

$$\mathfrak{p}(\varsigma) = \begin{cases} \frac{1-3^{-\varsigma}}{5}, & \text{if } \varsigma\in[0,1], \\ 0, & \text{if } \varsigma\in\mathbb{N}-\{1\}, \end{cases}$$

$\forall\ \varsigma\in\digamma\cup\mathcal{S}$ *and take* $\zeta\in[\frac{1}{2},1)$, *then*

$$\begin{aligned}\Pi(\mathfrak{p}\varsigma,\mathfrak{p}\varpi,\zeta\dot{\mathfrak{z}}) &= \Pi\left(\frac{1-3^{-\varsigma}}{5},\frac{1-3^{-\varpi}}{5},\zeta\dot{\mathfrak{z}}\right)\\ &= \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+\left|\frac{1-3^{-\varsigma}}{5}-\frac{1-3^{-\varpi}}{5}\right|} = \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+\frac{|3^{-\varsigma}-3^{-\varpi}|}{5}}\\ &\geq \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+\frac{|\varsigma-\varpi|}{5}} = \frac{5\zeta\dot{\mathfrak{z}}}{5\zeta\dot{\mathfrak{z}}+|\varsigma-\varpi|} \geq \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}}+|\varsigma-\varpi|} = \Pi(\varsigma,\varpi,\dot{\mathfrak{z}}),\end{aligned}$$

$$\begin{aligned}\Psi(\mathfrak{p}\varsigma,\mathfrak{p}\varpi,\zeta\dot{\mathfrak{z}}) &= \Psi\left(\frac{1-3^{-\varsigma}}{5},\frac{1-3^{-\varpi}}{5},\zeta\dot{\mathfrak{z}}\right)\\ &= \frac{\left|\frac{1-3^{-\varsigma}}{5}-\frac{1-3^{-\varpi}}{5}\right|}{\zeta\dot{\mathfrak{z}}+\left|\frac{1-3^{-\varsigma}}{5}-\frac{1-3^{-\varpi}}{5}\right|} = \frac{\frac{|3^{-\varsigma}-3^{-\varpi}|}{5}}{\zeta\dot{\mathfrak{z}}+\frac{|3^{-\varsigma}-3^{-\varpi}|}{5}}\\ &= \frac{|3^{-\varsigma}-3^{-\varpi}|}{5\zeta\dot{\mathfrak{z}}+|3^{-\varsigma}-3^{-\varpi}|} \leq \frac{|\varsigma-\varpi|}{5\zeta\dot{\mathfrak{z}}+|\varsigma-\varpi|} \leq \frac{|\varsigma-\varpi|}{\dot{\mathfrak{z}}+|\varsigma-\varpi|} = \Psi(\varsigma,\varpi,\dot{\mathfrak{z}})\end{aligned}$$

and

$$\begin{aligned}\Xi(\mathfrak{p}\varsigma,\mathfrak{p}\varpi,\zeta\dot{\mathfrak{z}}) &= \Xi\left(\frac{1-3^{-\varsigma}}{5},\frac{1-3^{-\varpi}}{5},\zeta\dot{\mathfrak{z}}\right)\\ &= \frac{\left|\frac{1-3^{-\varsigma}}{5}-\frac{1-3^{-\varpi}}{5}\right|}{\zeta\dot{\mathfrak{z}}} = \frac{\frac{|3^{-\varsigma}-3^{-\varpi}|}{5}}{\zeta\dot{\mathfrak{z}}}\end{aligned}$$

$$= \frac{|3^{-\varsigma} - 3^{-\varpi}|}{5\zeta\dot{\mathfrak{z}}} \leq \frac{|\varsigma - \varpi|}{5\zeta\dot{\mathfrak{z}}} \leq \frac{|\varsigma - \varpi|}{\dot{\mathfrak{z}}} = \Xi(\varsigma, \varpi, \dot{\mathfrak{z}}).$$

Therefore, all the hypothesis of Theorem 5 are satisfied, and 0 *is the only fixed point for* $\mathfrak{p}$.

Example 3. *Let* $\mathcal{F} = \{\mathcal{U}_\mu(\mathbb{R}) : \mathcal{U}_\mu(\mathbb{R})$ *is an upper triangular matrices over* $\mathbb{R}\}$ *and* $\mathcal{S} = \{\mathcal{L}_\mu(\mathbb{R}) : \mathcal{L}_\mu(\mathbb{R})$ *is an upper triangular matrices over* $\mathbb{R}\}$.
Define $\Pi, \Psi, \Xi \colon \mathcal{F} \times \mathcal{S} \times (0, +\infty) \to [0, 1]$ *as*

$$\Pi(\mathcal{R}, \mathcal{Q}, \dot{\mathfrak{z}}) = \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + \sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|},$$
$$\Psi(\mathcal{R}, \mathcal{Q}, \dot{\mathfrak{z}}) = \frac{\sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|}{\dot{\mathfrak{z}} + \sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|},$$
$$\Psi(\mathcal{R}, \mathcal{Q}, \dot{\mathfrak{z}}) = \frac{\sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|}{\dot{\mathfrak{z}}},$$

for all $\mathcal{R} = (\mathfrak{r}_{\mathrm{ij}})_{\mu\times\mu} \in \mathcal{F}$ *and* $\mathcal{Q} = (\mathfrak{q}_{\mathrm{ij}})_{\mu\times\mu} \in \mathcal{S}$*, Then,* $(\mathcal{F}, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a complete NBMS with* $c_{t-||.||}$ $\dot{e} \circledast \dot{a} = \dot{e}\dot{a}$ *and* $c_{t-co-||.||}$ $\dot{e}\Diamond\dot{a} = \max\{\dot{e}, \dot{a}\}$.
Define $\mathfrak{p} : (\mathcal{F}, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond) \rightrightarrows (\mathcal{F}, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *by*

$$\mathfrak{p}((\mathfrak{r}_{\mathrm{ij}})_{\mu\times\mu}) = \left(\frac{\mathfrak{r}_{\mathrm{ij}}}{5}\right)_{\mu\times\mu},$$

$\forall$ $(\mathfrak{r}_{\mathrm{ij}})_{\mu\times\mu} \in \mathcal{F} \cup \mathcal{S}$ *and take* $\zeta \in [\frac{1}{2}, 1)$*, then*

$$\begin{aligned}\Pi(\mathfrak{p}\mathcal{R}, \mathfrak{p}\mathcal{Q}, \zeta\dot{\mathfrak{z}}) &= \Pi\left(\left(\frac{\mathfrak{r}_{\mathrm{ij}}}{5}\right)_{\mu\times\mu}, \left(\frac{\mathfrak{q}_{\mathrm{ij}}}{5}\right)_{\mu\times\mu}, \zeta\dot{\mathfrak{z}}\right)\\ &= \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}} + \frac{1}{5}\sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|}\\ &\geq \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}} + \sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|}\\ &\geq \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + \sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|} = \Pi(\mathcal{R}, \mathcal{Q}, \dot{\mathfrak{z}}),\end{aligned}$$

$$\begin{aligned}\Psi(\mathfrak{p}\mathcal{R}, \mathfrak{p}\mathcal{Q}, \zeta\dot{\mathfrak{z}}) &= \Pi\left(\left(\frac{\mathfrak{r}_{\mathrm{ij}}}{5}\right)_{\mu\times\mu}, \left(\frac{\mathfrak{q}_{\mathrm{ij}}}{5}\right)_{\mu\times\mu}, \zeta\dot{\mathfrak{z}}\right)\\ &= \frac{\frac{1}{5}\sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|}{\zeta\dot{\mathfrak{z}} + \frac{1}{5}\sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|}\\ &= \frac{\sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|}{5\zeta\dot{\mathfrak{z}} + \sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|} \leq \frac{\sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|}{\dot{\mathfrak{z}} + \sum_{\mathrm{i,j}=1}^{\mu} |\mathfrak{r}_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|} = \Psi(\mathcal{R}, \mathcal{Q}, \dot{\mathfrak{z}})\end{aligned}$$

and

$$\begin{aligned}\Xi(\mathfrak{p}\mathcal{R}, \mathfrak{p}\mathcal{Q}, \zeta\mathfrak{z}) &= \Xi\left(\left(\frac{\mathfrak{r}_{\mathfrak{ij}}}{5}\right)_{\mu\times\mu}, \left(\frac{\mathfrak{q}_{\mathfrak{ij}}}{5}\right)_{\mu\times\mu}, \zeta\mathfrak{z}\right)\\ &= \frac{\frac{1}{5}\sum_{\mathfrak{i},\mathfrak{j}=1}^{\mu}|\mathfrak{r}_{\mathfrak{ij}} - \mathfrak{q}_{\mathfrak{ij}}|}{\zeta\mathfrak{z}}\\ &\leq \frac{\sum_{\mathfrak{i},\mathfrak{j}=1}^{\mu}|\mathfrak{r}_{\mathfrak{ij}} - \mathfrak{q}_{\mathfrak{ij}}|}{\zeta\mathfrak{z}} \leq \frac{\sum_{\mathfrak{i},\mathfrak{j}=1}^{\mu}|\mathfrak{r}_{\mathfrak{ij}} - \mathfrak{q}_{\mathfrak{ij}}|}{\mathfrak{z}} = \Xi(\mathcal{R}, \mathcal{Q}, \mathfrak{z}).\end{aligned}$$

Therefore, all the hypothesis of Theorem 5 are satisfied, and $\mathcal{O}_{\mu\times\mu}$ is the unique fixed point for $\mathfrak{p}$, where $\mathcal{O}_{\mu\times\mu}$ is the null matrix of order μ.

4. Application 1

Consider the set of all continuous functions $\digamma = \mathcal{C}([\mathfrak{c}, \mathfrak{a}], [0, +\infty))$ defined on $[\mathfrak{c}, \mathfrak{a}]$ with values in the interval $[0, +\infty)$ and $\mathcal{S} = \mathcal{C}([\mathfrak{c}, \mathfrak{a}], (-\infty, 0])$ defined on $[\mathfrak{c}, \mathfrak{a}]$ with values in the interval $(-\infty, 0]$.
Suppose the integral equation:

$$\varsigma(\mathfrak{l}) = \wedge(\mathfrak{l}) + \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\mathfrak{l}, \varepsilon)\varsigma(\mathfrak{l})\mathfrak{d}\varepsilon \quad \text{for} \quad \mathfrak{l}, \varepsilon \in [\mathfrak{c}, \mathfrak{a}] \tag{16}$$

where $\delta > 0, \mho \colon \mathcal{C}([\mathfrak{c}, \mathfrak{a}] \times \mathbb{R}) \to \mathbb{R}^+$, $\wedge(\varepsilon)$ is a fuzzy function of $\varepsilon \colon \varepsilon \in [\mathfrak{c}, \mathfrak{a}]$. Define Π, Ψ and Ξ by

$$\Pi(\varsigma(\mathfrak{l}), \varpi(\mathfrak{l}), \mathfrak{z}) = \sup_{\mathfrak{l}\in[\mathfrak{c},\mathfrak{a}]} \frac{\mathfrak{z}}{\mathfrak{z} + |\varsigma(\mathfrak{l}) - \varpi(\mathfrak{l})|} \quad \forall \quad \varsigma, \varpi \in \digamma \text{ and } \mathfrak{z} > 0,$$

$$\Psi(\varsigma(\mathfrak{o}), \varpi(\mathfrak{o}), \mathfrak{z}) = 1 - \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\mathfrak{z}}{\mathfrak{z} + |\varsigma(\mathfrak{o}) - \varpi(\mathfrak{o})|} \quad \forall \quad \varsigma, \varpi \in \digamma \text{ and } \mathfrak{z} > 0,$$

and

$$\Xi(\varsigma(\mathfrak{o}), \varpi(\mathfrak{o}), \mathfrak{z}) = \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{|\varsigma(\mathfrak{o}) - \varpi(\mathfrak{o})|}{\mathfrak{z}} \quad \forall \quad \varsigma, \varpi \in \digamma \text{ and } \mathfrak{z} > 0,$$

with $c_{t-||.||}$ and $c_{t-co-||.||}$ define by $\mathfrak{i} \circledast \flat = \mathfrak{i}\flat$ and $\mathfrak{i}\diamondsuit\flat = \max\{\mathfrak{i}, \flat\}$. Then $(\digamma, \Pi, \Psi, \Xi, \circledast, \diamondsuit)$ is a complete NBMS. Consider $|\mho(\mathfrak{o}, \varepsilon)\varsigma(\mathfrak{o}) - \mho(\mathfrak{o}, \varepsilon)\varpi(\mathfrak{o})| \leq |\varsigma(\mathfrak{o}) - \varpi(\mathfrak{o})|$ for $\varsigma \in \digamma, \varpi \in \mathcal{S}, \zeta \in (0, 1)$ and $\forall \mathfrak{o}, \varepsilon \in [\mathfrak{c}, \mathfrak{a}]$.

Also, let $\mho(\mathfrak{o}, \varepsilon)(\delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mathfrak{d}\varepsilon) \leq \zeta < 1$. Then, the integral Equation (16) has a unique solution.

Proof. Define $\mathfrak{p} : (\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \diamondsuit) \rightrightarrows (\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \diamondsuit)$ by

$$\mathfrak{p}\varsigma(\mathfrak{o}) = \wedge(\mathfrak{o}) + \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\mathfrak{o}, \varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon \quad \forall \quad \mathfrak{o}, \varepsilon \in [\mathfrak{c}, \mathfrak{a}].$$

Now, $\forall\ \varsigma, \varpi \in \digamma \cup \mathcal{S}$, we deduce

$$\begin{aligned}
\Pi(\mathfrak{p}\varsigma(\mathfrak{o}),\mathfrak{p}\varpi(\mathfrak{o}),\zeta\dot{\mathfrak{z}}) &= \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\mathfrak{p}\varsigma(\mathfrak{o})-\mathfrak{p}\varpi(\mathfrak{o})|}\\
&= \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\wedge(\mathfrak{o})+\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon-\wedge(\mathfrak{o})-\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon|}\\
&= \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon-\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon|}\\
&= \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})-\mho(\mathfrak{o},\varepsilon)\varpi(\mathfrak{o})|(\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mathfrak{d}\varepsilon)}\\
&\geq \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}}+|\varsigma(\mathfrak{o})-\varpi(\mathfrak{o})|}\\
&\geq \Pi(\varsigma(\mathfrak{o}),\varpi(\mathfrak{o}),\dot{\mathfrak{z}}),
\end{aligned}$$

$$\begin{aligned}
\Psi(\mathfrak{p}\varsigma(\mathfrak{o}),\mathfrak{p}\varpi(\mathfrak{o}),\zeta\dot{\mathfrak{z}}) &= 1-\sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\mathfrak{p}\varsigma(\mathfrak{o})-\mathfrak{p}\varpi(\mathfrak{o})|}\\
&= 1-\sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\wedge(\mathfrak{o})+\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon-\wedge(\mathfrak{o})-\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon|}\\
&= 1-\sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon-\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon|}\\
&= 1-\sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})-\mho(\mathfrak{o},\varepsilon)\varpi(\mathfrak{o})|(\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mathfrak{d}\varepsilon)}\\
&\leq 1-\sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}}+|\varsigma(\mathfrak{o})-\varpi(\mathfrak{o})|}\\
&\leq \Psi(\varsigma(\mathfrak{o}),\varpi(\mathfrak{o}),\dot{\mathfrak{z}}),
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\mathfrak{p}\varsigma(\mathfrak{o}),\mathfrak{p}\varpi(\mathfrak{o}),\zeta\dot{\mathfrak{z}}) &= \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{|\mathfrak{p}\varsigma(\mathfrak{o})-\mathfrak{p}\varpi(\mathfrak{o})|}{\zeta\dot{\mathfrak{z}}}\\
&= \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{|\wedge(\mathfrak{o})+\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon-\wedge(\mathfrak{o})-\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon|}{\zeta\dot{\mathfrak{z}}}\\
&= \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{|\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon-\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})\mathfrak{d}\varepsilon|}{\zeta\dot{\mathfrak{z}}}\\
&= \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{|\mho(\mathfrak{o},\varepsilon)\varsigma(\mathfrak{o})-\mho(\mathfrak{o},\varepsilon)\varpi(\mathfrak{o})|(\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\mathfrak{d}\varepsilon)}{\zeta\dot{\mathfrak{z}}}\\
&\leq \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{|\varsigma(\mathfrak{o})-\varpi(\mathfrak{o})|}{\dot{\mathfrak{z}}}
\end{aligned}$$

$$\leq \Xi(\varsigma(\mathfrak{o}), \varpi(\mathfrak{o}), \mathfrak{z}).$$

Therefore, all the hypothesis of Theorem 5 are satisfied and $\mathfrak{p}$ has a unique fixed point and the integral equation (16) has a unique solution.

Example 4. *Consider the the non-linear integral equation.*

$$\varsigma(\mathfrak{o}) = |\cos \mathfrak{o}| + \frac{1}{9}\int_0^1 \varepsilon\varsigma(\varepsilon)\mathfrak{d}\varepsilon, \quad \forall \quad \varepsilon \in [0,1]$$

Then it has a solution in F.

Proof. *Let $\mathfrak{p} : (F, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond) \rightrightarrows (F, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ be defined by*

$$\mathfrak{p}\varsigma(\mathfrak{o}) = |\cos \mathfrak{o}| + \frac{1}{9}\int_0^1 \varepsilon\varsigma(\varepsilon)\mathfrak{d}\varepsilon,$$

and set $\mho(\mathfrak{o}, \varepsilon)\varsigma(\mathfrak{o}) = \frac{1}{9}\varepsilon\varsigma(\varepsilon)$ and $\mho(\mathfrak{o}, \varepsilon)\varpi(\mathfrak{o}) = \frac{1}{9}\varepsilon\varpi(\varepsilon)$, where $\varsigma, \varpi \in F \cup \mathcal{S}$, and $\forall\ \mathfrak{o}, \varepsilon \in [0,1]$. Then we have

$$\begin{aligned} |\mho(\mathfrak{o}, \varepsilon)\varsigma(\mathfrak{o}) - \mho(\mathfrak{o}, \varepsilon)\varpi(\mathfrak{o})| &= |\frac{1}{9}\varepsilon\varsigma(\varepsilon) - \frac{1}{9}\varepsilon\varpi(\varepsilon)| \\ &= \frac{\varepsilon}{9}|\varsigma(\varepsilon) - \varpi(\varepsilon)| \leq |\varsigma(\varepsilon) - \varpi(\varepsilon)|. \end{aligned}$$

Furthermore, we have $\frac{1}{9}\int_0^1 \varepsilon\mathfrak{d}\varepsilon = \frac{1}{9}\big(\frac{(1)^2}{2} - \frac{(0)^2}{2}\big) = \frac{1}{18} = \zeta < 1$, with $\delta = \frac{1}{9}$. Hence, all the conditions of the application are easy to verify and the integral equation (16) has a unique solution $F \cup \mathcal{S}$.

Using Mathematica Software, near to the unique solution for the integral equation of Example 4 is found to be

$$\mathfrak{a}(\tau) = |\cos \tau| + 0.0444392,$$

and the graph of the solution is shown in Figure 1.

5. Application 2

Consider the definition of the intensity of a series electric circuit $\mathcal{I} = \frac{\mathfrak{d}\varpi}{\mathfrak{d}\mathfrak{l}}$, where ϖ denote the electric charge and $\mathfrak{l}$-the time, let us recall the following usually formulas

- $\mathcal{V} = \mathcal{I}\mathcal{R}$;
- $\mathcal{V} = \frac{\varpi}{\mathcal{C}}$

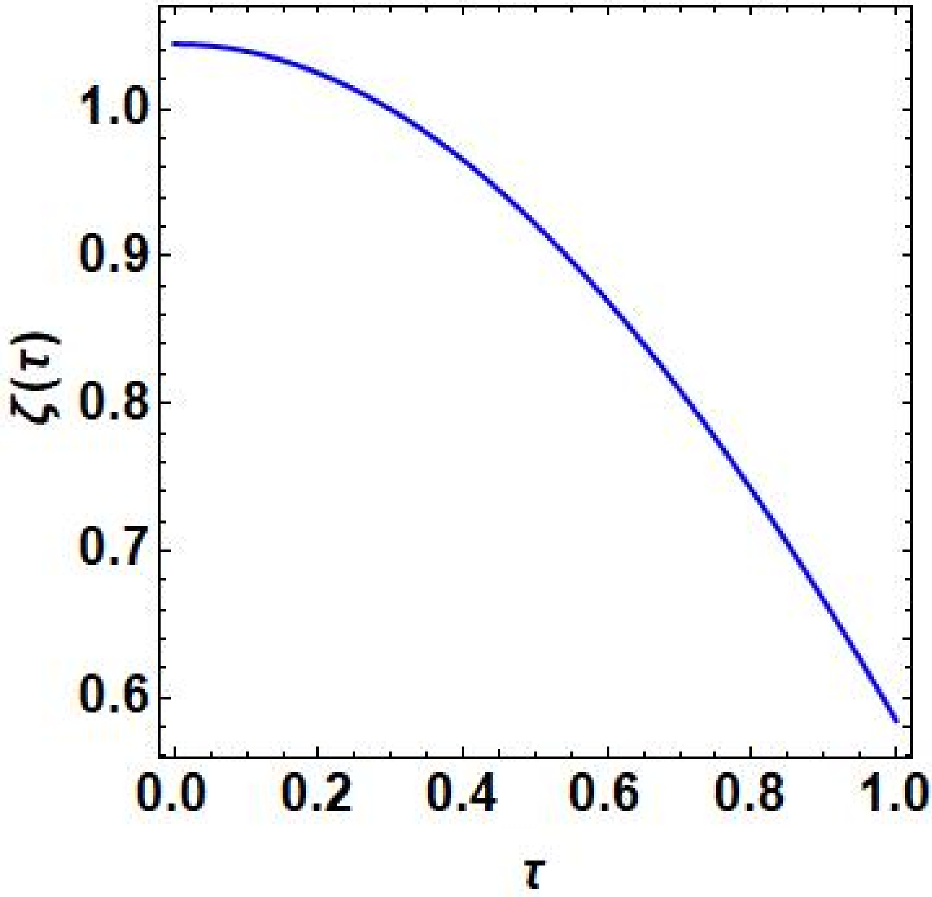

Figure 1: Solution of Example 4.1.

- $\mathcal{V} = \mathcal{L}\frac{\partial \mathcal{I}}{\partial \mathfrak{l}}$,

here

i. $\mathcal{R}$ *(Ohms) is a resistor,*

ii. $\mathcal{C}$ *(Faradays) is a capacitor,*

iii. $\mathcal{L}$ *(Henries) is an inductor,*

iv. $\mathcal{V}$ *(Volts) is an voltage and*

v. $\mathcal{E}$ *(Volts) is an electromotive force.*

Because there is only one current flowing in a series circuit, mathcalI has the same value throughout the circuit. Kirchhoff's Voltage Law is the second of his fundamental laws that can be used to analyse circuits. His voltage law states that the algebraic sum of all voltages around any closed loop in a circuit is equal to zero for a closed loop series path. The *algebraic sum of all the voltages around any closed loop in a circuit equals zero, according to Kirchhoff's Voltage Law.*

The main idea behind Kirchhoff's Voltage Law is that as you move around a closed loop/circuit, you will end up back where you started. As a result, you return to the same initial potential without any voltage losses around the loop. As a result, any voltage drop around the loop must be equal to any voltage source encountered along the way. The mathematical expression for this consequence of Kirchhoff's Voltage Law is: the sum of voltage rises across any loop is equal to the sum of voltage drops across that loop. Then we have the following relation:

$$\mathcal{IR} + \frac{\varpi}{\mathcal{C}} + \mathcal{L}\frac{\mathfrak{d}\mathcal{I}}{\mathfrak{d}\mathfrak{l}} = \mathcal{V}(\mathfrak{l}).$$

The voltage equation can be expressed as in the second-order differential equations with parameters as follows.

$$\mathcal{L}\frac{\mathfrak{d}^2\varpi}{\mathfrak{d}\mathfrak{l}^2} + \mathcal{R}\frac{\mathfrak{d}\varpi}{\mathfrak{d}\mathfrak{l}} + \frac{\varpi}{\mathcal{C}} = \mathcal{V}(\mathfrak{l}) = \mathfrak{h}(\mathfrak{l}, \varsigma(\mathfrak{l})),$$
$$\text{with the initial conditions}, \varpi(0) = 0, \varpi^{'}(0) = 0, \tag{17}$$

where $\mathcal{C} = \frac{4\mathcal{L}}{\mathcal{R}^2}$ *and* $\tau = \frac{\mathcal{R}}{2\mathcal{L}}$ *- the non dimensional time for Physics. The following Green function associated with equation 17 is*

$$\mathcal{Q}(\mathfrak{l}, \mathfrak{s}) = \begin{cases} -\mathfrak{s}\mathfrak{i}^{-\tau(\mathfrak{s}-\mathfrak{l})}, if 0 \leq \mathfrak{s} \leq \mathfrak{l} \leq 1; \\ -\mathfrak{l}\mathfrak{i}^{-\tau(\mathfrak{s}-\mathfrak{l})}, if 0 \leq \mathfrak{l} \leq \mathfrak{s} \leq 1. \end{cases}$$

In equation 17 can be expressed as in the integral equation with the above condition is

$$\varsigma(\mathfrak{l}) = \int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l}, \mathfrak{s})\mathfrak{h}(\mathfrak{s}, \varsigma(\mathfrak{s}))\mathfrak{d}\mathfrak{s}, \text{ for all } \mathfrak{l} \in [0, 1] \tag{18}$$

and $\mathfrak{h}(\mathfrak{s}, \cdot) : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *is a monotone non-decreasing mapping* $\forall\ \mathfrak{s} \in [0, 1]$.
Consider the set of all continuous functions $\digamma = (C[0, 1], [0, +\infty))$ *defined on* $[0, 1]$ *with values in* $[0, +\infty)$ *and* $\mathcal{S} = (C[0, 1], (-\infty, 0])$ *defined on* $[0, 1]$ *with values in* $(-\infty, 0]$.
Define Π, Ψ *and* Ξ *by*

$$\Pi(\varsigma(\mathfrak{o}), \varpi(\mathfrak{o}), \dot{\mathfrak{z}}) = \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + |\varsigma(\mathfrak{o}) - \varpi(\mathfrak{o})|} \quad \forall \quad \varsigma, \varpi \in \digamma \text{ and } \dot{\mathfrak{z}} > 0,$$

$$\Psi(\varsigma(\mathfrak{o}), \varpi(\mathfrak{o}), \dot{\mathfrak{z}}) = 1 - \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + |\varsigma(\mathfrak{o}) - \varpi(\mathfrak{o})|} \quad \forall \quad \varsigma, \varpi \in \digamma \text{ and } \dot{\mathfrak{z}} > 0,$$

and

$$\Xi(\varsigma(\mathfrak{o}), \varpi(\mathfrak{o}), \dot{\mathfrak{z}}) = \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{|\varsigma(\mathfrak{o}) - \varpi(\mathfrak{o})|}{\dot{\mathfrak{z}}} \quad \forall \quad \varsigma, \varpi \in \digamma \text{ and } \dot{\mathfrak{z}} > 0,$$

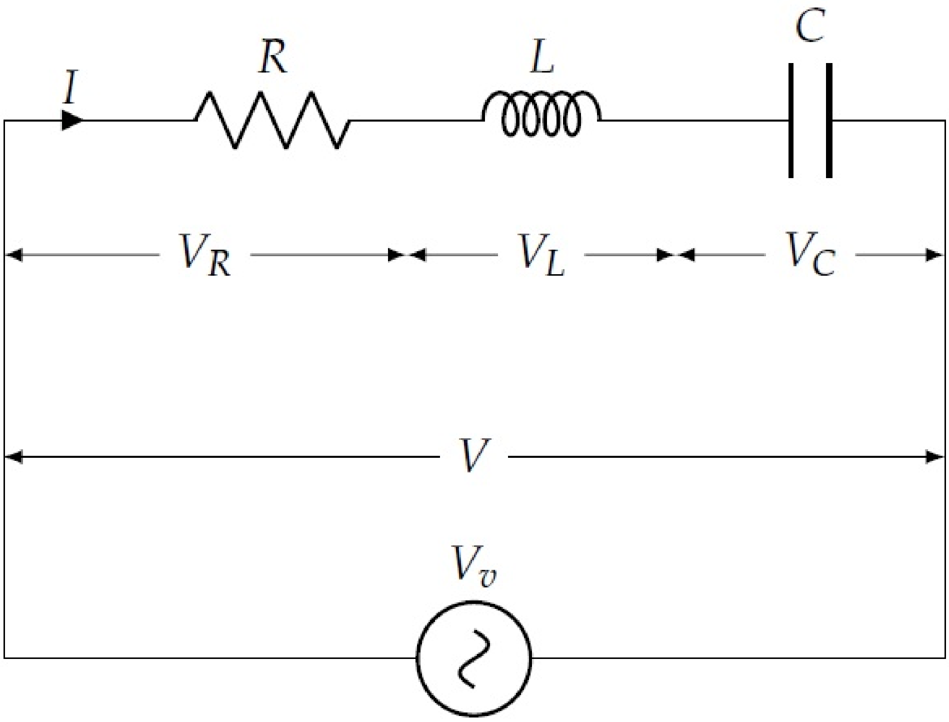

Figure 2: Series RLC circuit.

with $c_{t-||.||}$ *and* $c_{t-co-||.||}$ *define by* $\mathfrak{i} \circledast \flat = \mathfrak{i}\flat$ *and* $\mathfrak{i} \Diamond \flat = \max\{\mathfrak{i}, \flat\}$. *Then* $(\digamma, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *is a complete NBMS.*

Theorem 8. *Let* $\mathfrak{p} : (\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond) \rightrightarrows (\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *be a map such that the following axioms hold:*

i. $\mathcal{Q} \colon [0,1]^2 \to [0, \infty)$ *is a continuous function;*

ii. $\mathfrak{h}(\mathfrak{s}, \cdot) \colon [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a monotone nondecreasing mapping* $\forall\ \mathfrak{s} \in [0,1]$ *satisfying* $(\varsigma, \varpi) \in (\digamma, \mathcal{S})$,

$$|\mathfrak{h}(\mathfrak{l}, \varsigma) - \mathfrak{h}(\mathfrak{l}, \varpi) \leq |\varsigma(\mathfrak{l}) - \varpi(\mathfrak{l})|.$$

iii. $\int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l}, \mathfrak{s}) \mathfrak{d}\mathfrak{s} \leq \zeta < 1$

Then the voltage differential equation (17) *has a unique solution.*

Proof. *Define* $\mathfrak{p} : (\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond) \rightrightarrows (\digamma, \mathcal{S}, \Pi, \Psi, \Xi, \circledast, \Diamond)$ *by*

$$\mathfrak{p}\varsigma(\mathfrak{l}) = \int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l}, \mathfrak{s}) \mathfrak{h}(\mathfrak{s}, \varsigma(\mathfrak{s})) \mathfrak{d}\mathfrak{s}, \ \textit{where}\ \mathfrak{l} \in [0,1]$$

Now, $\forall\ \varsigma, \varpi \in \digamma \cup \mathcal{S}$*, we deduce*

$$\begin{aligned}
\Pi(\mathfrak{p}\varsigma(\mathfrak{l}), \mathfrak{p}\varpi(\mathfrak{l}), \zeta\dot{\mathfrak{z}}) &= \sup_{\mathfrak{l}\in[0,1]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}} + |\mathfrak{p}\varsigma(\mathfrak{l}) - \mathfrak{p}\varpi(\mathfrak{l})|} \\
&= \sup_{\mathfrak{l}\in[0,1]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}} + |\int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))\mathfrak{d}\mathfrak{s} - \int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))\mathfrak{d}\mathfrak{s}|} \\
&= \sup_{\mathfrak{l}\in[0,1]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}} + \int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})|\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s})) - \mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))|\mathfrak{d}\mathfrak{s}} \\
&= \sup_{\mathfrak{l}\in[0,1]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}} + |\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s})) - \mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))|} \\
&\geq \sup_{\mathfrak{l}\in[0,1]} \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + |\varsigma(\mathfrak{l}) - \varpi(\mathfrak{l})|} \\
&\geq \Pi(\varsigma(\mathfrak{l}), \varpi(\mathfrak{l}), \dot{\mathfrak{z}}),
\end{aligned}$$

$$\begin{aligned}
\Psi(\mathfrak{p}\varsigma(\mathfrak{l}), \mathfrak{p}\varpi(\mathfrak{l}), \zeta\dot{\mathfrak{z}}) &= 1 - \sup_{\mathfrak{l}\in[0,1]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}} + |\mathfrak{p}\varsigma(\mathfrak{l}) - \mathfrak{p}\varpi(\mathfrak{l})|} \\
&= 1 - \sup_{\mathfrak{l}\in[0,1]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}} + |\int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))\mathfrak{d}\mathfrak{s} - \int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))\mathfrak{d}\mathfrak{s}|} \\
&= 1 - \sup_{\mathfrak{l}\in[0,1]} \frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}} + \int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})|\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s})) - \mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))|\mathfrak{d}\mathfrak{s}} \\
&\leq 1 - \sup_{\mathfrak{l}\in[0,1]} \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + |\varsigma(\mathfrak{l}) - \varpi(\mathfrak{l})|} \\
&\leq \Psi(\varsigma(\mathfrak{l}), \varpi(\mathfrak{l}), \dot{\mathfrak{z}}),
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\mathfrak{p}\varsigma(\mathfrak{l}), \mathfrak{p}\varpi(\mathfrak{l}), \zeta\dot{\mathfrak{z}}) &= \sup_{\mathfrak{l}\in[0,1]} \frac{|\mathfrak{p}\varsigma(\mathfrak{l}) - \mathfrak{p}\varpi(\mathfrak{l})|}{\zeta\dot{\mathfrak{z}}} \\
&= \sup_{\mathfrak{l}\in[0,1]} \frac{|\int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))\mathfrak{d}\mathfrak{s} - \int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))\mathfrak{d}\mathfrak{s}|}{\zeta\dot{\mathfrak{z}}} \\
&= \sup_{\mathfrak{l}\in[0,1]} \frac{\int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})|\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s})) - \mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))|\mathfrak{d}\mathfrak{s}}{\zeta\dot{\mathfrak{z}}} \\
&\leq \sup_{\mathfrak{l}\in[0,1]} \frac{|\varsigma(\mathfrak{l}) - \varpi(\mathfrak{l})|}{\dot{\mathfrak{z}}} \\
&\leq \Xi(\varsigma(\mathfrak{l}), \varpi(\mathfrak{l}), \dot{\mathfrak{z}}).
\end{aligned}$$

It can be seen that all conditions of Theorem (5) *are satisfied and* $\mathfrak{p}$ *has a unique fixed point and the differential voltage equation* (17) *has a unique solution.*

Now assume that $\mathcal{R} = 10$*,* $\mathcal{C} = 1$ *and* $\mathcal{L} = 1$ *whereas the voltage source is given by* $\mathcal{V}(t) = 5sin(t)$*. Thus, the nearer form of the unique solution for circuit IVP is found using Mathematica Software and expressed as*

$$q(t) = \frac{1}{24}e^{-5t}\left(5\sqrt{6}\sinh\left(2\sqrt{6}t\right) + 12\cosh\left(2\sqrt{6}t\right)\right) - \frac{\cos(t)}{2},$$

and the graph of the solution is shown in Figure 3.

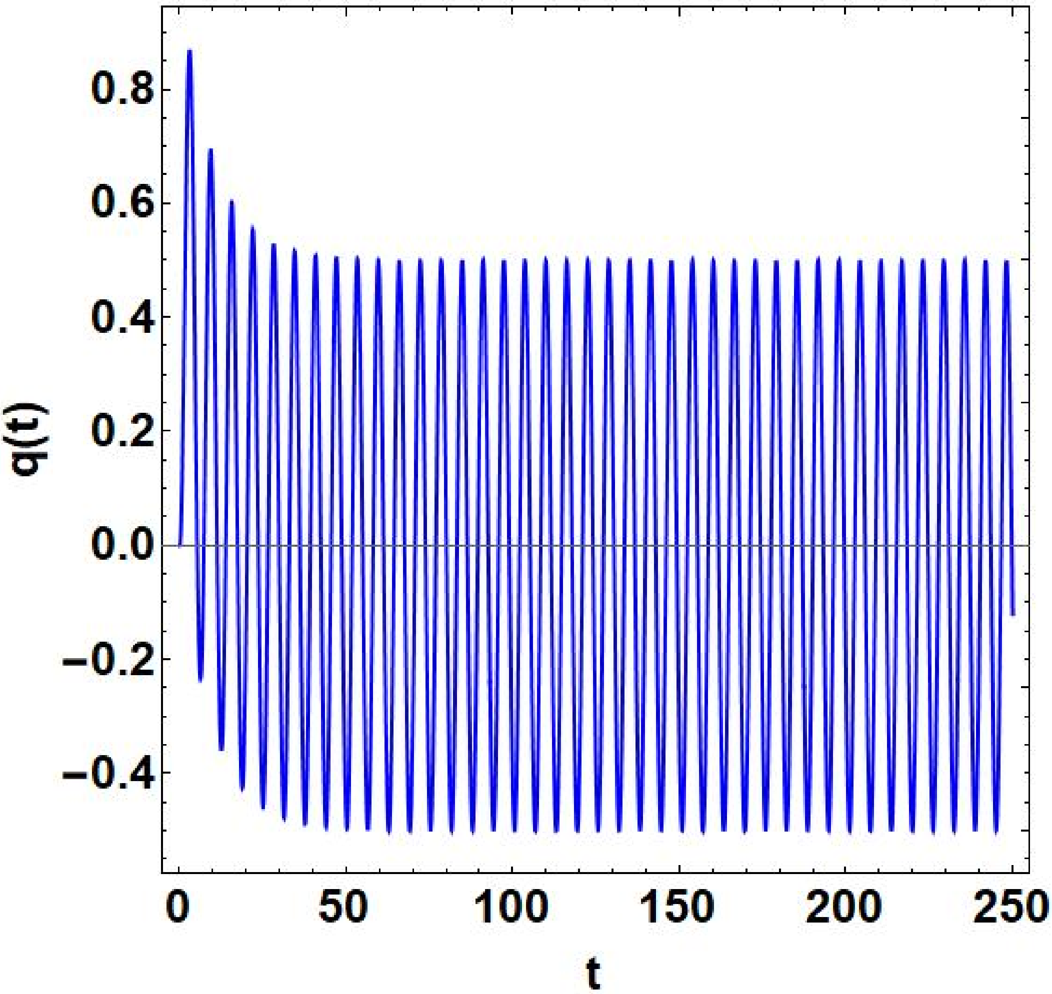

Figure 3: Solution of (5.1).

6. Application to Fractional Differential Equation

Consider the definition of Caputo derivative of a continuous function $\bar{\mathfrak{p}}\colon [0,+\infty) \to \mathbb{R}$ order $\bar{\beta} > 0$ (See[2, 3]):

$$^{C}\mathcal{D}\bar{\beta}(\bar{\mathfrak{p}}(\mathfrak{l}))\colon = \frac{1}{\Gamma(\mu-\bar{\beta})}\int_0^{\mathfrak{l}}(\mathfrak{l}-\mathfrak{s})^{\mu-\bar{\beta}-1}\bar{\mathfrak{p}}^{(\mu)}(\mathfrak{s})\mathfrak{d}\mathfrak{s}(\mu-1<\bar{\beta}<\mu, \mu=[\bar{\beta}]+1),$$

where Γ is a gamma function and $[\bar{\beta}]$ denotes the integer part of the real number $\bar{\beta} > 0$. Additionally, we provide an application of the Theorem 5 for proving the existence solution of the nonlinear fractional differential equation

$$^{C}\mathcal{D}^{\bar{\beta}}(\varsigma(\mathfrak{l})) + \mathfrak{h}(\mathfrak{l},\varsigma(\mathfrak{l})) = 0 \ (0 \le \mathfrak{l} \le 1, \bar{\beta} < 1) \tag{19}$$

with $\varsigma(0) = 0 = \varsigma(1)$ and $\mathfrak{h}\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function (See[4, 34–36]). The Green function related with (19) is

$$\mathcal{Q}(\mathfrak{l},\mathfrak{s}) = \begin{cases} (\mathfrak{l}(1-\mathfrak{s}))^{\alpha-1} - (\mathfrak{l}-\mathfrak{s})^{\alpha-1}, \text{if } 0 \le \mathfrak{s} \le \mathfrak{l} \le 1 \\ \frac{(\mathfrak{l}(1-\mathfrak{s}))^{\alpha-1}}{\Gamma(\alpha)}, \text{if } 0 \le \mathfrak{l} \le \mathfrak{s} \le 1. \end{cases}$$

Obviously $\varsigma^* \in \digamma$ is a solution of (19) if and only if $\varsigma^* \in \digamma$ is a solution of the equation

$$\varsigma(\mathfrak{l}) = \int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))\mathfrak{d}\mathfrak{s} \ \forall\ \mathfrak{l} \in [0,1].$$

Let the set of all continuous functions $\digamma = (C[0,1],[0,+\infty))$ defined on $[0,1]$ with values in $[0,+\infty)$ and $\mathcal{S} = (C[0,1],(-\infty,0])$ defined on $[0,1]$ with values in $(-\infty,0]$.

Define Π, Ψ and Ξ by

$$\Pi(\varsigma(\mathfrak{o}),\varpi(\mathfrak{o}),\dot{\mathfrak{z}}) = \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + |\varsigma(\mathfrak{o}) - \varpi(\mathfrak{o})|} \quad \forall \quad \varsigma,\varpi \in \digamma \text{ and } \dot{\mathfrak{z}} > 0,$$

$$\Psi(\varsigma(\mathfrak{o}),\varpi(\mathfrak{o}),\dot{\mathfrak{z}}) = 1 - \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}} + |\varsigma(\mathfrak{o}) - \varpi(\mathfrak{o})|} \quad \forall \quad \varsigma,\varpi \in \digamma \text{ and } \dot{\mathfrak{z}} > 0,$$

and

$$\Xi(\varsigma(\mathfrak{o}),\varpi(\mathfrak{o}),\dot{\mathfrak{z}}) = \sup_{\mathfrak{o}\in[\mathfrak{c},\mathfrak{a}]} \frac{|\varsigma(\mathfrak{o}) - \varpi(\mathfrak{o})|}{\dot{\mathfrak{z}}} \quad \forall \quad \varsigma,\varpi \in \digamma \text{ and } \dot{\mathfrak{z}} > 0,$$

with $c_{t-||.||}$ and $c_{t-co-||.||}$ define by $\mathfrak{i} \circledast \flat = \mathfrak{i}\flat$ and $\mathfrak{i}\Diamond\flat = \max\{\mathfrak{i},\flat\}$. Then $(\digamma,\Pi,\Psi,\Xi,\circledast,\Diamond)$ is a complete NBMS.

Theorem 9. *Let* $\mathfrak{p} : (\digamma,\mathcal{S},\Pi,\Psi,\Xi,\circledast,\Diamond) \rightrightarrows (\digamma,\mathcal{S},\Pi,\Psi,\Xi,\circledast,\Diamond)$ *be a map such that the following axioms hold:*

i. $\mathcal{Q}\colon [0,1]^2 \to [0,\infty)$ *is a continuous function;*

ii. $\mathfrak{h}(\mathfrak{s},\cdot)\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ *is a monotone non decreasing function* $\forall\ \mathfrak{s}\in[0,1]$ *such that* $(\varsigma,\varpi)\in(\digamma,\mathcal{S})$, *we have*

$$|\mathfrak{h}(\mathfrak{l},\varsigma)-\mathfrak{h}(\mathfrak{l},\varpi)\le|\varsigma(\mathfrak{l})-\varpi(\mathfrak{l})|;$$

iii. $\sup_{\mathfrak{l}\in[0,1]}\int_0^{\mathfrak{l}}\mathcal{Q}(\mathfrak{l},\mathfrak{s})\le\zeta<1.$

Then the fractional differential equation (19) *has a unique solution.*

Proof. Define $\mathfrak{p}:(\digamma,\mathcal{S},\Pi,\Psi,\Xi,\circledast,\Diamond)\rightrightarrows(\digamma,\mathcal{S},\Pi,\Psi,\Xi,\circledast,\Diamond)$ by

$$\mathfrak{p}\varsigma(\mathfrak{l})=\int_0^{\mathfrak{l}}\mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))\mathfrak{d}\mathfrak{s},\ \text{ where } \mathfrak{l}\in[0,1]$$

Now, $\forall\ \varsigma,\varpi\in\digamma\cup\mathcal{S}$, we deduce

$$\begin{aligned}\Pi(\mathfrak{p}\varsigma(\mathfrak{l}),\mathfrak{p}\varpi(\mathfrak{l}),\zeta\dot{\mathfrak{z}})&=\sup_{\mathfrak{l}\in[0,1]}\frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\mathfrak{p}\varsigma(\mathfrak{l})-\mathfrak{p}\varpi(\mathfrak{l})|}\\&=\sup_{\mathfrak{l}\in[0,1]}\frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\int_0^{\mathfrak{l}}\mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))\mathfrak{d}\mathfrak{s}-\int_0^{\mathfrak{l}}\mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))\mathfrak{d}\mathfrak{s}|}\\&=\sup_{\mathfrak{l}\in[0,1]}\frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+\int_0^{\mathfrak{l}}\mathcal{Q}(\mathfrak{l},\mathfrak{s})|\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))-\mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))|\mathfrak{d}\mathfrak{s}}\\&=\sup_{\mathfrak{l}\in[0,1]}\frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))-\mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))|}\\&\ge\sup_{\mathfrak{l}\in[0,1]}\frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}}+|\varsigma(\mathfrak{l})-\varpi(\mathfrak{l})|}\\&\ge\Pi(\varsigma(\mathfrak{l}),\varpi(\mathfrak{l}),\dot{\mathfrak{z}}),\end{aligned}$$

$$\begin{aligned}\Psi(\mathfrak{p}\varsigma(\mathfrak{l}),\mathfrak{p}\varpi(\mathfrak{l}),\zeta\dot{\mathfrak{z}})&=1-\sup_{\mathfrak{l}\in[0,1]}\frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\mathfrak{p}\varsigma(\mathfrak{l})-\mathfrak{p}\varpi(\mathfrak{l})|}\\&=1-\sup_{\mathfrak{l}\in[0,1]}\frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+|\int_0^{\mathfrak{l}}\mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))\mathfrak{d}\mathfrak{s}-\int_0^{\mathfrak{l}}\mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))\mathfrak{d}\mathfrak{s}|}\\&=1-\sup_{\mathfrak{l}\in[0,1]}\frac{\zeta\dot{\mathfrak{z}}}{\zeta\dot{\mathfrak{z}}+\int_0^{\mathfrak{l}}\mathcal{Q}(\mathfrak{l},\mathfrak{s})|\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))-\mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))|\mathfrak{d}\mathfrak{s}}\\&\le1-\sup_{\mathfrak{l}\in[0,1]}\frac{\dot{\mathfrak{z}}}{\dot{\mathfrak{z}}+|\varsigma(\mathfrak{l})-\varpi(\mathfrak{l})|}\\&\le\Psi(\varsigma(\mathfrak{l}),\varpi(\mathfrak{l}),\dot{\mathfrak{z}}),\end{aligned}$$

and

$$\Xi(\mathfrak{p}\varsigma(\mathfrak{l}),\mathfrak{p}\varpi(\mathfrak{l}),\zeta\dot{\mathfrak{z}})=\sup_{\mathfrak{l}\in[0,1]}\frac{|\mathfrak{p}\varsigma(\mathfrak{l})-\mathfrak{p}\varpi(\mathfrak{l})|}{\zeta\dot{\mathfrak{z}}}$$

$$
\begin{aligned}
&= \sup_{\mathfrak{l}\in[0,1]} \frac{|\int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s}))\mathfrak{d}\mathfrak{s} - \int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})\mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))\mathfrak{d}\mathfrak{s}|}{\zeta\dot{\mathfrak{z}}} \\
&= \sup_{\mathfrak{l}\in[0,1]} \frac{\int_0^{\mathfrak{l}} \mathcal{Q}(\mathfrak{l},\mathfrak{s})|\mathfrak{h}(\mathfrak{s},\varsigma(\mathfrak{s})) - \mathfrak{h}(\mathfrak{s},\varpi(\mathfrak{s}))|\mathfrak{d}\mathfrak{s}}{\zeta\dot{\mathfrak{z}}} \\
&\leq \sup_{\mathfrak{l}\in[0,1]} \frac{|\varsigma(\mathfrak{l}) - \varpi(\mathfrak{l})|}{\dot{\mathfrak{z}}} \\
&\leq \Xi(\varsigma(\mathfrak{l}),\varpi(\mathfrak{l}),\dot{\mathfrak{z}}).
\end{aligned}
$$

It can be seen that all conditions of Theorem 5 are satisfied and $\mathfrak{p}$ has a unique fixed point and the fractional differential equation (19) has a unique solution.

7. Conclusion

In this paper, the notion of neutrosophic bipolar metric space has been introduced and fixed point results in NBMS have been established. Some of the topological properties of the **NBMS** have also been presented in the manuscript. It can be seen that an analogue of the Banach Fixed Point theorem has been established supplemented with suitable non trivial examples. The results have been applied to find solution to integral equation, voltage differential equation and fractional differential equation. Simulation has also been presented for the analytical results using Mathematica Software. Since the space NBMS generalises neutrosophic metric space NMS and its seeds, the results established vide the contractions considered in this manuscript will not be satisfied in the setting of NMS or general metric spaces. It will also be an open question to establish fixed point results using different types of contractions, such as Kannan Type, Ciric Type, Reich Type, Meir-keeler type, to name a few in the setting of neutrosophic bipolar metric spaces and also finding applications in other fields such as neural networking, stochastic process etc.

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