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Iterative Approaches to Multiple Fixed Points in Generalized Metric Spaces

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Abstract. We combine the concept of having multiple fixed points for a mapping with the approach of obtaining these points through iterative methods. As is well known, contraction self-mappings in standard metric spaces yield unique fixed points that can be obtained iteratively. To overcome this limitation, we conduct our study within the framework of generalized MP-metric spaces, utilizing their properties and the broader concept of limits. This enables us to establish our main result: the existence of multiple fixed points that can be iteratively obtained for generalized contraction mappings satisfying specific conditions.

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Key Words and Phrases: Fixed point, Iterative methods, Generalized contractor, Multiple fixed points, Generalized metric, MP-metric

1. Introduction

Fixed-point theory in metric spaces is one of the most active research areas in mathematics due to its broad applications in both pure and applied fields. It plays a crucial role in theoretical research and practical disciplines such as physics, computing, and engineering [1], [2]. The origins of fixed-point theory trace back to the late 19th and early 20th centuries, with pioneering contributions from scholars such as Poincaré, Lefschetz–Hopf, and Leray–Schauder [3]. However, the foundation of fixed-point theory in metric spaces was largely established in 1922 when Banach published his seminal paper [4], proving the existence and uniqueness of fixed points for a special class of functions called contraction mappings. Banach's proof was more than just a demonstration of the theorem; it implicitly introduced an iterative method—specifically, Picard iteration—to obtain the fixed point. His groundbreaking result inspired numerous researchers, leading to extensive generalizations and applications of Banach's theorem. Among the major directions of research in fixed-point theory, two prominent areas stand out: the first is the study of iterative methods to obtain fixed points, and the second is the quest for the existence of more than one fixed point. In this paper, we aim to combine these two directions and integrate them.

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In this article, the notion of an iterative fixed point refers to a fixed point that can be obtained through iterative methods. An iterative fixed point is of dual importance; not only is it a fixed point, but we also have established methods for locating or approximating it, making it particularly useful in practical applications. Consequently, it is unsurprising that significant attention has been devoted to iterative methods in fixed point theory, with various approaches being implemented within this framework. In addition to the straightforward Picard iterative method, more complex iterative techniques—such as the Krasnoselskij, Mann, and Ishikawa methods—have been developed and successfully employed to obtain fixed points. (see, for example, [5], [6], [7], [8] and [9]).

On the other hand, the study of the existence of multiple fixed points began in the 1960s with the works of Browder and Kirk ([10], [11], [12], and [13]). Their research demonstrated that non-expansive mappings can possess more than one fixed point, thus establishing a foundation for examining the potential of multiple fixed points in specific contexts. Subsequently, the explicit investigation of the existence of multiple fixed points gained traction. For instance, results concerning multiple fixed points for monotone and positive mappings are available in [14] and [15]. Moreover, further applications have illustrated that the significance of multiple fixed points is partially derived from the fact that, in many practical scenarios, the solution to a system corresponds to a fixed point of a particular mapping. This underscores a side of the importance of studying the existence of more than one fixed point.

However, in the standard metric space, the existence of an iterative fixed point often implies its uniqueness. It is common to combine contraction conditions and iteration methods to construct a Cauchy sequence, ensuring its convergence through the space's completeness, with the limit serving as the desired fixed point. Therefore, we cannot get multiple fixed points in such a setting, as the uniqueness of the fixed point primarily stems from the contraction conditions and implicitly from the uniqueness of the limit, a single point in ordinary metric spaces. We address this issue by conducting our study on a generalized metric space, where we consider the MP-metric space and exploit its properties to show the existence of multiple iterative fixed points under certain conditions. Compared to other results in the field, we observe that the finding presented in this work offers a distinct advantage: it establishes the existence of multiple fixed points for a given function while leveraging iterative methods and their notable practical benefits. The concept of multiple iterative fixed points is a potentially valuable tool in both pure and applied mathematics, as it facilitates the analysis of systems where uniqueness is either undesirable or cannot be guaranteed.

2. Preliminaries

Numerous studies have highlighted the advantages of studying fixed points in generalized metric spaces, which help overcome some of the limitations of ordinary metrics and provide a more suitable framework for achieving desired results. For example, one can refer to [16],[17] ,[18] , [19], and [20] for results on fixed-point theory in generalized metric spaces, such as G-metric and b-metric spaces.

In this section, we review the generalized MP-metric space, highlight its key properties, and present essential theorems relevant to this study.

Definition 1 ([21]). *Let $P^*(X)$ denote the set of all non-empty finite subsets of X. Then,*

$$d : P^*(X) \longrightarrow [0, \infty)$$

is called an MP-metric (a multiple point-metric) if for all $A, B \in P^*(X)$, the following hold:

- *($\mathcal{A}1$)* $d(A) = 0 \iff |A| = 1$,
- *($\mathcal{A}2$)* $A \subseteq B \Longrightarrow d(A) \leq d(B)$,
- *($\mathcal{A}3$)* $A \cap B \neq \phi \Longrightarrow d(A \cup B) \leq d(A) + d(B)$,
- *($\mathcal{A}4$)* $d(A \cup \{b\}) = d(A)$ *for all* $b \in B \Longrightarrow d(A \cup B) = d(A)$,

where $|A|$ represents the cardinality of the set A. Furthermore, (X, d) is called an MP-metric space.

To simplify the presentation below, it is assumed that $d(a_1, ..., a_k)$, $d(\{a_1, ..., a_k\})$, and $d(A)$ for $A = \{a_1, ..., a_k\}$ are identical. If we let $d_{(2)}(a, b) = d(\{a, b\})$, we can check that $d_{(2)}$ is a standard metric that is referred to as the associated metric. The MP-metric concept can be considered an extension to some other types of metrics, including the standard metric (see [21]).

Example 1 ([21]). *A natural MP-metric $d : P^*(\mathbb{R}) \longrightarrow [0, \infty)$ on $\mathbb{R}$ can be defined in the following way: $d(A) = Max(A) - Min(A)$. The restriction of this metric to two points gives the usual distance metric on $\mathbb{R}$, where $d_{(2)}(a, b) = d(\{a, b\}) = Max(\{a, b\}) - Min(\{a, b\}) = |a - b|$.*

Next, we present the concept of convergence in MP-metric spaces. It allows us to directly obtain a generalized form of a sequence's limit, where the limit can be a set (a compact set) rather than a single point. Such a way of recognizing the limit will be crucial in achieving our main result.

Definition 2 ([21]). *Let (X, d) be an MP-metric space and $A \in P^*(X)$. Then*

$$D(A) = \{x \in X : d(A \cup \{x\}) = d(A)\}.$$

is the set of all d-dependent points on A.

Example 2 ([21]). *(i)- Let (X, d) be the MP-metric space introduced in Example 1 and $a \leq b$. Then, $D(\{a, b\}) = \{x \in X : d(\{a, b, x\}) = d(\{a, b\})\} = [a, b]$.*

(ii)- In any MP-metric space and for any singleton set, it follows from ($\mathcal{A}1$) that $D(\{a\}) = \{a\}$, since $D(\{a\}) = \{x \in X : d(\{a, x\}) = d(\{a\}) = 0\} = \{a\}$.

Definition 3 ([21]). *Let (X, d) be an MP-metric space and $D \subseteq X$. Then, we say that a sequence x_n is d-convergent to D and write $\overset{d}{\lim\limits_{n\to\infty}} x_n = D$ if there is $A \in P^*(X)$ such that $D = D(A)$ and:*

(i)- $\lim\limits_{n_i\to\infty} d(\{x_{n_1}, x_{n_2}..., x_{n_k}\} \cup A) = d(A)$ for all $k \in \mathbb{N}$,

(ii)- $\liminf\limits_{n\longrightarrow\infty} d(x_n, a) = 0$ for all $a \in A$.

We will say that x_n converges to $D(A)$, and we will write $\lim x_n = D(A)$ to indicate that A meets the convergence conditions in the preceding formulation. The above definition preserves the characteristics of convergence in ordinary metric spaces and, at the same time, provides a natural generalization to the concept of limit.

Proposition 1 ([21]). *Let (X, d) be an MP-metric space, and let $(X, d_{(2)})$ be the associated metric space. A sequence x_n converges to a in $(X, d_{(2)})$ if and only if it is d-convergent to $D(\{a\}) = \{a\}$ in (X, d).*

The MP-metric, like the standard metric, is considered continuous with respect to all of its components.

Proposition 2 ([21]). *Let (X, d) be an MP-metric space, and let $(X, d_{(2)})$ be the associated metric space.*

(i)- If x_n converges to a, then $\lim\limits_{n\longrightarrow\infty} d(\{x_n, a_1..., a_k\}) = d(\{a, a_1..., a_k\})$.

(ii)- If x_{n_i} converges to a_i, $\forall i = 1, ..., k$, then $\lim\limits_{n_i\longrightarrow\infty} d(\{x_{n_1}, ..., x_{n_k}\}) = d(\{a_1..., a_k\})$.

The Cauchy convergence and completeness concepts are fundamental tools in any type of metric space. The next definition provides a generalization of the notion of Cauchy sequences that aligns with the concept of convergence in the generalized MP-metric spaces.

Definition 4 ([21]). *Assume that (X, d) is an MP-metric space. Then, x_n is called r-Cauchy for some $r \geq 0$ if there exists $N \in \mathbb{N}$ such that*

$$\lim_{m\longrightarrow\infty} \sup_{n_i>m} d(x_{n_1}, x_{n_2}, ..., x_{n_k}) = r \text{ for all } k \geq N.$$

The convergence in MP-metric spaces implies some essential results, such as boundedness, r-Cauchy, and other findings that are analogous to those in standard metric theory.

Theorem 1 ([21]). *Let (X, d) be an MP-metric space. If $\lim x_n = D(A)$ for some $A \in P^*(X)$, then (x_n) is r-Cauchy, and moreover $r = d(A)$.*

Proposition 3 ([21]). *Let (X, d) be an MP-metric space. If x_n is r-Cauchy for some $r \geq 0$, then x_n is bounded.*

In an obvious way, the completeness of the MP-metric spaces is defined, where we require the equivalence between the convergence in MP-metric spaces and the satisfaction of the r-Cauchy criterion.

Definition 5. *An MP-metric space (X,d) is said to be complete MP-metric space if, for all r-Cauchy sequence (x_n) in (X,d) with $r \geq 0$, there exists a finite nonempty subset $A \in P^*(X)$ such that $d(A) = r$ and the sequence (x_n) is d-convergent to $D(A)$; that is,* $\overset{d}{\lim_{n\to\infty}} x_n = D(A)$.

The following theorem provides an example of a complete MP-metric space. It shows that the Euclidean space is complete with respect to the MP-metric introduced in Example 1. This result is valid for higher dimensions when appropriate MP-metrics are implemented in $\mathbb{R}^n$ (see [21]).

Theorem 2 ([21])**.** *Let $(\mathbb{R}, d)$ be the MP-metric space introduced in Example 1. If x_n is r-Cauchy for some $r > 0$ in $(\mathbb{R}, d)$, then there are $a, b \in \mathbb{R}$ such that x_n is d-convergent to $D(\{a, b\})$.*

Another notion that we will need in the next section is the notion of the boundedness of an MP-metric space. Therefore, we introduce the following definition.

Definition 6. *An MP-metric space (X,d) is called bounded MP-metric space if there is a positive constant C such that $d(A) < C$ for all $A \in P^*(X)$.*

3. Construction and Main Results

We start this section by introducing some constructive definitions and propositions that will help simplify the presentation of the main result. For a mapping T, we use T^n to denote applying the mapping T n times. Also, we note that for any $A \in P^*(X)$, we can always enumerate its elements and write $A = \{a_1, a_2, ..., a_k\}$.

Definition 7. *Let (X,d) be an MP-metric space and T is a self mapping on X, and $A = \{a_1, ..., a_k\} \in P^*(X)$. Then for all $j \in \mathbb{N} \cup \{0\}$, we define the set*

$$S_T^j(A) = \bigcup_{n=j}^{\infty} T^n(A), \tag{1}$$

and for $j > 0$ and $1 \leq l \leq k$, we define the set

$$S_T^{j,l}(A) = (\bigcup_{n=j}^{\infty} T^n(A)) \cup T^{j-1}(\{a_l, ..., a_k\}) \tag{2}$$

The sets introduced above have some properties that are directly derived from their definitions. The following proposition discusses some of these properties, including the effect of the mapping T.

Proposition 4. *Let (X,d) be an MP-metric space, and let T be a self mapping on X, and $A = \{a_1, ..., a_k\} \in P^*(X)$. Then*

(i) $T(S_T^j(A)) = S_T^{j+1}(A)$ *for all* $j \in \mathbb{N} \cup \{0\}$,

(ii) $T(S_T^{j,l}(A)) = S_T^{j+1,l}(A)$ *for all* $j \in \mathbb{N}$ *and* $1 \leq l \leq k$,

(iii) $S_T^j(A) = S_T^{j+1,1}(A)$ *for all* $j \in \mathbb{N}$,

(iv) $S_T^{j_1}(A) \subseteq S_T^{j_2}(A)$ *for all* $j_1, j_2 \in \mathbb{N} \cup \{0\}$ *and* $j_2 \leq j_1$

(v) $S_T^{j,l_1}(A) \subseteq S_T^{j,l_2}(A)$ *for all* $j \in \mathbb{N}$, $1 \leq l_1, l_2 \leq k$ *and* $j_2 \leq j_1$

(vi) $S_T^{j_1,l_1}(A) \subseteq S_T^{j_2,l_2}(A)$ *for all* $j_1, j_2 \in \mathbb{N}$, $1 \leq l_1, l_2 \leq k$, $j_2 \leq j_1$ *and* $j_2 \leq j_1$

Proof. The results of this proposition follow directly from straightforward calculations using Definition 7. However, for the convenience of the reader, we provide a proof for part (iii); the remaining parts can be established in a similar manner.

$$S_T^j(A) = \bigcup_{n=j}^{\infty} T^n(A) = (\bigcup_{n=j+1}^{\infty} T^n(A)) \cup T^j(A)$$

$$= (\bigcup_{n=j+1}^{\infty} T^n(A)) \cup T^{j+1-1}(\{a_1, ..., a_k\}) = S_T^{j+1,1}(A).$$

Note that the elements of the set $S_T^0(A)$ can be written as $\{a_1, ..., a_k, T(a_1), ..., T(a_k), T^2(a_1), ...\}$, suggesting that we may think of S^0 as a sequence while $S_T^{j,l}(A)$ as tail subsequences for all $j \in \mathbb{N}$. In particular, we have the following definition.

Definition 8. *Let* (X, d) *be an MP-metric space and* T *is a self mapping on* X, *and* $A = \{a_1, ..., a_k\} \in P^*(X)$. *Then for all* $j \in \mathbb{N} \cup \{0\}$, *we define the sequence* (α_n) *as*

$$\alpha_n = \alpha_n(A, T) = T^j(a_{n-jk}) \tag{3}$$

where $j = Int(\frac{n-1}{k})$ *(the function of the greatest integer less than or equal)*

The following proposition describes the relationship between the sequence $\alpha_n(A, T)$ and the sets $S_T^{s,l}(A)$.

Proposition 5. *Let* (X, d) *be an MP-metric space, and let* T *be a self mapping on* X, *and* $A = \{a_1, ..., a_k\} \in P^*(X)$. *Then*

(i) $S_T^{s,l}(A) = \{\alpha_n : n \geq m\}$ *where* $m = l + (s-1)k$, $(s = Int(\frac{m-1}{k}) + 1\ )$

(ii) $S_T^s(A) = \{\alpha_n : n > sk\}$,

(iii) $S_T^0(A) = \{\alpha_n : n \in \mathbb{N}\}$,

Proof. To show (i), let $x \in S_T^{s,l}(A) = S_T^s(A) \cup T^{s-1}(\{a_l, a_{l+1} ..., a_k\})$. Therefore, either $x = T^t(a_r)$ for some $t \geq s$ and $1 \leq r \leq k$, or $x = T^{s-1}(a_r)$ for some $r \in \{l, ..., k\}$. In the first case, we choose $n = r + tk \geq l + (s-1)k = m$, while in the second case, we set $n = r + (s-1)k \geq l + (s-1)k = m$. Hence, we get $x = \alpha_n$ and $n \geq m$.

For the other inclusion, let $\alpha_n = T^t(a_{n-tk})$ where $t = Int(\frac{n-1}{k})$ for some $n \geq m$. We set $s = 1 + Int(\frac{m-1}{k})$ and $l = m - (s-1)k$ to get $t \geq s - 1$. If $t \geq s$, then $\alpha_n = T^t(a_{n-tk}) \in S_T^s(A) \subseteq S_T^{s,l}(A)$. If $t = s - 1$, then $n - tk \geq m - (s-1)k = l$, and hence $\alpha_n = T^t(a_{n-tk}) \in T^{s-1}(\{a_l, a_{l+1}..., a_k\}) \subseteq S_T^{s,l}(A)$. (ii) and (iii) follow from (i) using Proposition 4-(iii).

Proposition 6. *Let (X, d) be an MP-metric space, and let T be a self mapping on X, and $A = \{a_1, ..., a_k\} \in P^*(X)$. If (α_{n_u}) is a subsequence of $(\alpha_n(A, T))$. Then there are $a \in A$ and $s_v \in \mathbb{N}$ such that $T^{s_v}(a)$ is a subsequence of (α_{n_u}).*

Proof. This result is a direct consequence of the finiteness of A, as assuming the opposite leads to a contradiction with (α_{n_u}) being a subsequence. The conclusion of this proposition is equivalent to the following statement:

$$\exists\, a_i \in A \quad s.t \quad \forall\, M \in \mathbb{N} \quad \exists\, s_v, n_u > M, \quad s.t. \quad T^{s_v}(a_i) = \alpha_{n_u}.$$

Thus, its negation is:

$$\forall\, a_i \in A \quad \exists\, M_i \in \mathbb{N} \quad s.t. \quad T^{s_v}(a_i) \neq \alpha_{n_u}. \quad \forall\, s_v, n_u > M_i.$$

Using the finiteness of $A = \{a_1, ..., a_k\}$, we define

$$M = \max_{i=1,...,k} M_i.$$

Therefore,

$$T^{s_v}(a_i) \neq \alpha_{n_u}. \quad \forall\, s_v, n_u > M, \quad and\, \forall\, a_i \in A,$$

which contradicts the assumption that (α_{n_u}) is a subsequence of $(\alpha_n(A, T)) = (T^j(a_{n-jk}))$ where it directly conflicts with the construction of (α_{n_u}) as a subsequence formed by applying the iteration of T over elements of A (see Definition 8, Equation (3)).

We now present our main result, which provides sufficient conditions for a mapping to attain several iterative fixed points.

Theorem 3. *Let (X, d) be a bounded complete MP-metric space, and let $T : X \to X$ be a continuous mapping. If there exist positive constants r, p, and q, and $N \in \mathbb{N}$ such that $p + q \leq 1$ and the following inequalities hold*

$$r \leq \sup_{x_i \in S_T^{s,l}(A)} d(T(\{x_1, ..., x_u\})) \leq q \sup_{x_i \in S_T^{s,l}(A)} d(\{x_1, ..., x_u\}) + pr \text{ for all } u \geq N. \tag{4}$$

for some $A \in P^*(X)$, then T has at least two distinct iterative fixed points.

Proof. Assume $A = \{a_1, ..., a_k\}$ satisfying (4) and set $\alpha_n = T^s(a_{n-ks})$ with $s = Int(\frac{n-1}{k})$ as defined in (3). Then

$$\begin{aligned}
\sup_{n_i \geq m} d(\alpha_{n_1}, ..., \alpha_{n_u}) &= \sup_{n_i \geq m} d(T^{s_1}(a_{n_1-ks_1}), ..., T^{s_u}(a_{n_u-ks_u})) \\
&= \sup_{x_i \in S_T^{s,l}(A)} d(\{x_1, ..., x_u\}) \text{ (by Proposition 5-(i).)} \\
&= \sup_{x_i \in S_T^{s-1,l}(A)} d(T(\{x_1, ..., x_u\})) \text{ (by Proposition 4-(ii).)}
\end{aligned} \tag{5}$$

Using (4) and applying the standard iteration keeping in mind the properties of the sets $S_T^{s,l}$, we get

$$\begin{aligned}\sup_{x_i\in S_T^{s-1,l}(A)} d(T(\{x_1,...,x_u\})) &\leq q \sup_{x_i\in S_T^{s-1,l}(A)} d(\{x_1,...,x_u\}) + pr \textit{ for all } u\geq N.\\ &= q \sup_{x_i\in S_T^{s-2,l}(A)} d(T(\{x_1,...,x_u\})) + pr\\ &\leq q[q \sup_{x_i\in S_T^{s-2,l}(A)} d(\{x_1,...,x_u\}) + pr] + pr\\ &= q^2 \sup_{x_i\in S_T^{s-2,l}(A)} d(\{x_1,...,x_u\}) + qpr + pr\\ &= q^2 \sup_{x_i\in S_T^{s-3,l}(A)} d(T(\{x_1,...,x_u\})) + (q^0+q^1)pr.\end{aligned} \tag{6}$$

Repeating the above process, we get

$$\begin{aligned}\sup_{x_i\in S_T^{s-1,l}(A)} d(T(\{x_1,...,x_u\})) &\leq q^s \sup_{x_i\in S_T^{0}(A)} d(\{x_1,...,x_u\}) + (q^0+q^1+...+q^{s-1})pr\\ &\leq q^s \sup_{x_i\in S_T^{0}(A)} d(\{x_1,...,x_u\}) + r \textit{ for all } u\geq N.\end{aligned} \tag{7}$$

where in the last step we used

$$(q^0+q^1+...+q^{s-1})pr \leq \frac{pr}{1-q} \leq r, \textbf{ since } p+q\leq 1 \iff \frac{p}{1-q}\leq 1\textbf{.}$$

Combining (4), (5) and (7), we get

$$r \leq \sup_{n_i\geq m} d(\alpha_{n_1},...,\alpha_{n_u}) \leq q^s \sup_{x_i\in S_T^{0}(A)} d(\{x_1,...,x_u\}) + r, \qquad \textit{for all } u\geq N. \tag{8}$$

Using the boundedness of (X,d) and the fact $m\to\infty \iff s\to\infty$, we get

$$\lim_{m\to\infty}\sup_{n_i\geq m} d(\alpha_{n_1},...,\alpha_{n_u}) = r, \qquad \textit{for all } u\geq N. \tag{9}$$

Therefore, (α_n) is r-Cauchy sequence and hence by the completeness of the MP-metric space (X,d) there is $B\in P^*(X)$ such that

$$\overset{d}{\lim_{n\to\infty}} \alpha_n = D(B) \tag{10}$$

in particular $\liminf_{n\to\infty} d(\alpha_n, b) = 0$ for all $b\in B$ (see Definition 3). Hence, there is a subsequence (α_{n_m}) such that $\lim_{n_m\to\infty} d(\alpha_{n_m}, b) = 0$. By Proposition 6, and since A is finite, there is $a\in A$ such that $(T^{s_l}(a))$ is a subsequence of (α_{n_m}). Thus, using the continuity of d and T, we obtain

$$0 = \lim_{s_l\to\infty} d(T^{s_l}(a), b) = \lim_{s_l\to\infty} d(T(T^{s_l-1}(a)), b) = d(T(\lim_{s_l\to\infty} T^{s_l-1}(a)), b) = d(T(b), b)$$

That is, b is a fixed point for all $b \in B$.

It remains to estimate the number of the fixed points. Using the properties of the MP-metric, we have

$$d(\alpha_{n_1}, ..., \alpha_{n_u}) \leq d(\{\alpha_{n_1}, ..., \alpha_{n_u}\} \cup B) \tag{11}$$

which in turn gives

$$r = \lim_{m\to\infty} \sup_{n_i \geq m} d(\alpha_{n_1}, ..., \alpha_{n_u}) \leq \lim_{m.\to\infty} \sup_{n_i \geq m} d(\{\alpha_{n_1}, ..., \alpha_{n_u}\} \cup B) \tag{12}$$

Moreover, in view of Equation (10) and Definition 3, we observe that $\lim_{n_i\to\infty} d(\{\alpha_{n_1}, ..., \alpha_{n_u}\} \cup B)$ exists and is equal to $d(B)$. Hence, $\lim_{m\to\infty} \sup_{n_i \geq m} d(\{\alpha_{n_1}, ..., \alpha_{n_u}\} \cup B)$ also exists, and we have that

$$\lim_{m\to\infty} \sup_{n_i \geq m} d(\{\alpha_{n_1}, ..., \alpha_{n_u}\} \cup B) = \lim_{n_i\to\infty} d(\{\alpha_{n_1}, ..., \alpha_{n_u}\} \cup B) = d(B) \tag{13}$$

Combining (12) and (13), we get

$$r \leq d(B)) \tag{14}$$

As a result, $d(B) > 0$ (since r is positive), which according to axiom $(\mathcal{A}1)$ implies that B contains at least two distinct elements, that are fixed points. This concludes the proof.

Example 3. *Consider the MP-Metric $d : P^*([0, 1]) \longrightarrow [0, \infty)$ defined as $d(A) = Max(A) - Min(A)$. The space $([0, 1], d)$ is a bounded complete MP-metric space. If x_n is r-Cauchy for some $r > 0$ in $([0, 1], d)$, then there are $a, b \in [0, 1]$ such that x_n is d-convergent to $D(\{a, b\})$. Let $F : [0, 1] \longrightarrow [0, 1]$ be defined as $F(x) = x^t$ for some $t > 1$. Then, choosing $A = \{\frac{1}{2}, 1\}$, we have the sequence $(\alpha_n) = \{\frac{1}{2}, 1, (\frac{1}{2})^t, 1, (\frac{1}{2})^{t^2}, 1, ...\}$. Therefore, if we let $r = 1$, $N = 2$ and $p = 1 - q$ where q can be any number in $(0, 1)$, we get*

$$1 \leq \sup_{x_i \in S_F^{s,l}(A)} d(F(\{x_1, ..., x_u\})) \leq q \sup_{x_i \in S_F^{s,l}(A)} d(\{x_1, ..., x_u\}) + (1 - q) \text{ for all } u \geq 2.$$

Thus, F and $([0, 1], d)$ satisfy the conditions of Theorem 3. We can check that

$$\overset{d}{\lim_{n\to\infty}} \alpha_n = D(\{0, 1\}).$$

It is clear that 0 and 1 are fixed points for F. Moreover, it can be observed that they can be obtained as iterative fixed points: $\lim_{n\to\infty} F^n(\frac{1}{2}) = 0$, and $\lim_{n\to\infty} F^n(1) = 1$.

4. Conclusion

In this research paper, we examine a class of functions that possess multiple fixed points, which can be obtained through iterative methods. To achieve this objective, we conducted our study within the framework of generalized MP-metric spaces, leveraging their comprehensive convergence concepts. Furthermore, by implementing an iterative process, we established the existence of multiple fixed points for this class of functions that satisfy general contraction conditions.

Our results open several avenues for further research.For instance, potential applications can be explored in function spaces where an MP-metric d on a set X is extended to an MP-metric δ on certain spaces of functions over X by employing integration and setting

$$\delta(f_1, f_2, ..., f_k) = \int_X d(f_1(x), f_2(x), ..., f_k(x)).$$

In this context, we consider two functions to be equal if they coincide almost everywhere. Furthermore, we can verify that the space of continuous functions $C[a, b]$, equipped with the MP-metric δ, is not a complete MP-metric space. Thus, a prominent area of research is the investigation of the MP-completion of the space $C[a, b]$. This exploration will lay a foundation for studying MP-metric spaces of functions, while also facilitating the examination of potential applications for our results, such as studying the existence of multiple solutions for certain systems.

Another potential direction for future work is the investigation of the geometric aspects of multiple fixed points. In this regard, the concept of fixed discs, where individual points are replaced by discs, becomes particularly relevant. This idea has been studied in both standard metric spaces and some generalized metric spaces (see, for example, [22] and [23]). In connection with our work, the theory of fixed discs appears to be compatible with the framework of MP-metric spaces and aligns naturally with the set-valued nature of limits within this setting.

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