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Direct Product of Complex Neutrosophic Subrings

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Abstract. The complex neutrosophic set is a generalization of the neutrosophic set with the addition of three phase terms. The complex neutrosophic set deals with periodic data that contains uncertainty, indeterminacy, and falsity. The complex neutrosophic set has a variety of applications, such as signal processing, hospital infrastructure design, medical image denoising, segmentation distance measurement, and the game of loser, neutral, and winner. This article presents a novel concept for complex neutrosophic subrings and illustrates how these subrings can generate two other neutrosophic subrings. Additionally, we prove that the intersection of two neutrosophic subrings is a neutrosophic subring. We expand this idea to talk about the abstraction of level subsets of complex neutrosophic sets and look into the basic algebraic properties of this event. We prove that the level subset of the complex neutrosophic subring is a subring. Moreover, we demonstrate that the product of two complex neutrosophic subrings is also a complex neutrosophic subring and explore some novel consequences about the direct product of complex neutrosophic subrings. Our findings generalize and extend the existing ring theory results within a complex neutrosophic framework.

2020 Mathematics Subject Classifications: 13E15, 08A72, 03E72

Key Words and Phrases: Complex neutrosophic subring, level subsets of complex neutrosophic subrings, product of complex neutrosophic subrings

1. Introduction

In the beginning, initiatives to demonstrate Fermat's last theorem gave rise to the idea of a ring in the 1880s, commencing with Dedekind [1]. In the 1920s, Noether and Krull [2]

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generalized and firmly established the idea of a ring after contributions from other domains, especially number theory. Modern ring theory, a relatively active mathematical field, investigates rings independently. Fuzzy set theory conceptualized by Zadeh [3] as an extension of the idea of classical set theory, It is extremely important for managing uncertainty in practical applications and described as $\kappa :\to \mathbb{Q}$ such that $\{\mathfrak{y} : \ (\mathfrak{y},\ \kappa_{\mathbb{Q}}(\mathfrak{y}))\ \forall\, \mathfrak{y}, \kappa_{\mathbb{Q}}(\mathfrak{y}) \in [01]\}$. As an extension of the fuzzy set, Atanassov [4] presented an intuitionistic fuzzy ($\mathbb{IF}$) set and characterize as $\chi : \mathfrak{T} \to \{(x, \mu(\mathfrak{y}), \overline{\mu}(\mathfrak{y})) : \mathfrak{y} \in \mathfrak{T}\}$ where $, \mu(\mathfrak{y}), \overline{\mu}(\mathfrak{y}) \in [0\ 1]$ such that $0 \leq \mu(\mathfrak{y}) + \overline{\mu}(\mathfrak{y}) \leq 1$. In decision-making challenges, the positive and negative membership functions of $\mathbb{IF}$ sets in contrast to classical fuzzy sets, provide that both are able to manage situations that are unclear and uncertain in physical issues. Alolaiyan [5] et al. discussed decision making problems by employing linguistic intuitionistic fuzzy set with a fuzzy Dombi weighted geometric operator. Rosenfeld [6] proposed the fuzzy subgroup in 1971 by applying the fuzzy set theory on algebra. $\mathbb{IF}$ subgroups and the algebraic structure of intuitionistic fuzzification introduced by Biswas [7].
Smarandache [8] was the first to introduce neutrosophy as a discipline of philosophy that investigated the origin, nature, and scope of neutralities as well as how they interacted with various ideational spectra. A belonging membership function, a not belonging membership function, and an indeterminacy membership function define a neutrosophic set ($\mathbb{NS}$). Agboola et al. [9] provided the idea of neutrosophic BCI/BCK algebras and discussed some fundamental characteristics of neutrosophic BCI/BCK algebras. The group structure of single valued $\mathbb{NS}$s was investigated by Cetkin and Aygun [10]. Additionally, they scrutinized the essential features of the neutrosophic subgroup and showcased the homomorphic image and pre-image of a neutrosophic (normal) subgroup. Song et al. [11] offered the idea of a neutrosophic distributive N-ideal in BCK-algebras and explored a number of its features. Additionally, they engaged in a discussion about the connections between a neutrosophic commutative N-ideal and a neutrosophic N-ideal. Chalapathi and Kumar [12] discussed the finite groups through graphs under the framework of $\mathbb{NS}$. The idea of the neutrosophic triplet group, It includes a novel extension of the traditional group idea, was derived from the fundamental idea of the $\mathbb{NS}$ and the structural characteristics of the neutrosophic triplet group are examined in detail [13]. In a BCK-algebra, Borzooei et al. [14] developed the idea of an extended neutrosophic commutative ideal, and associated features were demonstrated. Moreover, some equivalence relations have been introduced, and several features of the family of all commutative modified neutrosophic ideals in BCK-algebras are investigated. Interval neutrosophic subalgebra was introduced by Jun et al. [15]. In BCK/BCI-algebra, their characteristics and relationships are studied. Additionally, we introduce the concept of an interval neutrosophic length and review its associated features. The idea of a neutrosophic positive implicative N-ideal in BCK-algebras was suggested by Jun et al. [16], and numerous features were studied. Bender et al. [17] investigated the complex anti-fuzzy subgroups and proved some important results. Gulzar et al. [18] discussed the Q-fuzzy subrings under the framework of complex fuzzy (CF) sets.

Alghazzawi et al. [19] purposed the optimal solution for energy crises by employing the interval-valued intuitionistic fuzzy sets. Alolaiyan et al. [20] studied the algebraic structure of bipolar fuzzy subrings. Additionally, a detailed discussion of the algebraic attributes took place. Altassan et al. [21] discussed the algebraic product of fuzzy subrings. Furthermore, the fundamental theorems of the fuzzy isomorphism subring extend to algebraic products. Dilshad et al. [22] introduced the novel idea of q-rung orthopair fuzzy (ROF) subrings. Under the influence of ring theory, they also established various algebraic operations, ideals, homomorphic images, and pre-images. Mateen et al. initiated the novel algebraic structure of complex Pythagorean fuzzy subfield. They also explored the direct product and homomorphism within the context of a complex Pythagorean fuzzy set. Bal et al. [23] defined the neutrosophic extended triplet subgroup, neutrosophic kernel, neutrosophic inverse-image, and neutrosophic image in this study using the idea of a neutrosophic extended triplet.

Smarandache et al. [24] examined the neutrosophic triplet G-module and described its characteristics. Also, the definitions of neutrosophic triplet G-modules that are reducible, irreducible, and completely reducible were given, along with an analysis of the connections between these structures. Ali et al. established a neutrosophic triplet subring and neutrosophic triplet subfield, as well as some of its fundamental characteristics. In a neutrosophic group, Abobala et al. [25] defined certain novel substructures (AH-substructures). It also covers some basic AH-subgroup characteristics, AH-normality, AH-homomorphisms, AH-quotients, and AH-direct products. Kandasamy et al. [26] discussed the neutrosophic triplets in neutrosophic rings $< \mathbb{Z} \cup \mathbb{I} >$ $< \mathbb{Q} \cup \mathbb{I} >$ or $< \mathbb{R} \cup \mathbb{I} >$. Furthermore, the study revealed the existence of three distinct types of neutrosophic triplets in these neutrosophic subrings, all of which generate torsion-free component-wise product abelian groups. The idea of a generalized neutrosophic extended triplet group is introduced by Ma et al. [27], and some of its features are addressed. Researchers have demonstrated that the generalized neutrosophic extended triplet group and the weak commutative generalized neutrosophic extended triplet group are equivalent to the quasi-completely regular semigroup and the quasi-clifford semigroup, respectively.

Numerous characteristics of implicative neutrosophic quadruple BCK-algebras were investigated by Muhiuddin et al. [28], and developed criteria for the neutrosophic quadruple set to be a neutrosophic quadruple BCI-algebra. Kandasamy et al. [26] examined the semi-idempotents in neutrosophic subrings and studied the vital characteristics in details. Bashir et al. [29] introduced the subsemirings, ideals, generalized bi-ideals, and quasi-ideals under the framework of a m-polar fuzzy set in semirings. Bashir et al. [30] purposed the ternary multiplication to extend the roughness of a fuzzy set in three dimensions. A number of vital characteristics were discussed by using the idea of set-valued and strong set-valued homomorphism.

The notion of CF sets and their fundamental algebraic operations were investigated in [31, 32], extending the range of the belonging function from real numbers to complex numbers with the unit disc. Due to the CF set only considering the degree of belonging and giving no consideration to the data entities that are not members, which are equally important in the process of making decisions on system evaluation. However, it is usually difficult to measure the true value of a truth estimate by the exact value of a fuzzy set in real life. In some circumstances, it could be simpler to express the ambiguity and vagueness that characterize real-world situations using two-dimensional data rather than one. The complex intuitionistic fuzzy ($\mathbb{CIF}$) sets and their set operations were presented by Alkouri and Salleh [33, 34]. Also, it indicated the unpredictability of complex-valued functions in a variety of physical measurements. As an example, complex intensity, impedance and wave function in the fields of quantum physics and electronics.

The following are the motivations for the new work.

(i) The concept of a $\mathbb{CIF}$ subgroup and $\mathbb{CIF}$ level subsets were introduced by Gulzar et al. [35]. Additionally, the Cartesian product of two $\mathbb{CIF}$ subgroups was introduced, along with the homomorphic image and inverse image of the $\mathbb{CIF}$ subgroup under group homomorphism.

(ii) Gulzar et al. [36] introduced the idea of the direct product of two $\mathbb{CIF}$ subrings, demonstrated that it is also a $\mathbb{CIF}$ subring, and discussed about its different algebraic characteristics. Hameed et al. [37] discussed the (α, β, γ) neutrosophic submodules and some fundamental algebraic properties were investigated.

(iii) Elrawy and Abdalla [38] defined the algebraic structure based on single-valued $\mathbb{NS}$s and proposed a novel method for constructing the neutrosophic subring and ideal by combining $\mathbb{NS}$s with the classical ring.

(iv) Ali and Smarandache established innovative complex neutrosophic sets ($\mathbb{CNS}$), which extend the range of components in the complex plane from the unit interval to the unit disc. Amplitude and phase values are assigned to each of its components. Furthermore, $\mathbb{CNS}$s have been used in the fields of science and engineering.

(v) Gulistana et al. [39] initiated the idea of complex neutrosophic subgroups and defined the term alpha-cut of $\mathbb{CNS}$. The cartesian product of complex neutrosophic subgroups is also defined. Rahoumah et al. [40] introduced the complex neutrosophic soft subgroups, and the fundamental results related to this phenomenon were discussed.

(vi) The complex neutrosophic set is the generalization of existing theorizes i.e., complex fuzzy sets and complex intuitionistic fuzzy sets. The concept of complex neutrosophic set is not yet applied to submodules. Our proposed model is given in Figure 1.

In this paper, we initiate the work on complex neutrosophic subrings ($CNSR$s). The paper is shaped as follow: The concept of CNS is defined in Section 2. Some important algebraic properties of this abstraction are mentioned in this section. In Section 3 the novel concept of $CNSR$ is present together with their fundamental results. We show that the intersection of two $CNSR$s is $CNSR$. Additionally, we depict level subset of complex neutrosophic(CN) subset and prove that the level subsets of $CNSR$ is CNR. The 4th Section reveals the idea about direct product of $CNSR$s and explore algebraic characteristics. We shown that direct product of $CNSR$s is $CNSR$.

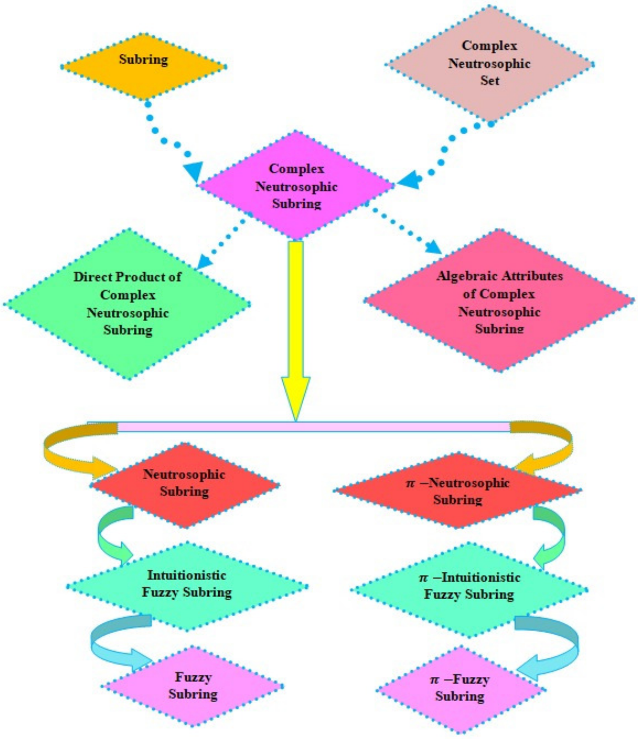

Figure 1: Flowchart of Proposed Model

2. Preliminaries

The following portion recalls some essential concepts of CNS and $CNSR$ which are necessary for our further discussion.

Definition 1. [8] $\mathbb{NS}$ W *of universal set* P *is of the form* $W = \{< \mathrm{c}, p_W(\mathrm{c}),\ q_W(\mathrm{c}), r_W(\mathrm{c}) >: \mathrm{c} \in P\}$, *where* $p_W(\mathrm{c})$, $q_W(\mathrm{c})$ *and* $r_W(\mathrm{c})$ *give the degree of accuracy, level of indeterminacy and degree of falsehood of* c *from unit interval, respectively such that* $0 \leq p_W(\mathrm{c}) + q_W(\mathrm{c}) + r_W(\mathrm{c}) \leq 3$, *for any* $\mathrm{c} \in P$.

Definition 2. [38] *A* $\mathbb{NS}$ W *of a ring* R *is known a* NSR *of a* R, *if these conditions are valid:*

(i) $p_W(m - \mathrm{d}) \geq min\{p_W(m), p_W(\mathrm{d})\}$, *for all* $m, \mathrm{d} \in R$.

(ii) $p_W(m\mathrm{d}) \geq min\{p_W(m), p_W(\mathrm{d})\}$,

(iii) $q_W(m\mathrm{d}) \leq max\{q_W(m), q_W(\mathrm{d})\}$,

(iv) $q_W(m - \mathrm{d}) \leq max\{q_W(m), q_W(\mathrm{d})\}$,

(v) $r_W(m - \mathrm{d}) \leq max\{r_W(m), r_W(\mathrm{d})\}$,

Table 1: Comparison of complex neutrosophic sets to the existing approaches

Sets	Domain	Co-domain	Truth	Falsity	Indeterminacy	Truth with periodicity	Falsity with periodicity	Indeterminacy with periodicity
Fuzzy sets	Universel set	Unit interval	✓	×	×	×	×	×
Intuitionistic fuzzy sets	Universel set	unit interval	✓	✓	×	×	×	×
Neutrosophic sets	Universel set	Unit interval	✓	✓	✓	×	×	×
Complex fuzzy sets	Universel set	Unit Disc	✓	×	×	✓	×	×
Complex intuitionistic fuzzy sets	Universel set	Unit Disc	✓	✓	×	✓	✓	×
Complex Neutrosophic fuzzy sets	Universel set	Unit Disc	✓	✓	✓	✓	✓	✓

(vi) $r_W(m\mathrm{d}) \leq max\{r_W(m), r_W(\mathrm{d})\}$.

Example 1. *Consider the ring $R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ under addition and multiplication modulo 6. Define a neutrosophic subring N of R as follows:*

$$p_W(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0.88, & \text{if } x \in \{2, 4\}, \\ 0, & \text{otherwise.} \end{cases}$$

$$q_W(x) = \begin{cases} 0, & \text{if } x = 0, \\ 0.11, & \text{if } x \in \{2, 4\}, \\ 0.33, & \text{otherwise.} \end{cases}$$

$$r_W(x) = \begin{cases} 0, & \text{if } x = 0, \\ 0.11, & \text{if } x \in \{2, 4\}, \\ 0.77, & \text{otherwise.} \end{cases}$$

Clearly, N is NSR of subring R.

Theorem 1. [38] *Intersection of two $NSRs$ of ring R is NSR.*

Definition 3. [31, 32] *A CNS $\mathbb{W}$ of universal set P is an object of the form $\mathbb{W} = \{< f, \mathbb{T}_{\mathbb{W}}(f), \mathbb{I}_{\mathbb{W}}(f), \mathbb{F}_{\mathbb{W}}(f) >: f \in P\}$, where the degree of truth $\mathbb{T}_{\mathbb{W}}(f) = p_{\mathbb{W}}(f)e^{i\theta_{\mathbb{W}}(f)}$, and is expressed as $\mathbb{T}_{\mathbb{W}} : P \to \{\tau \in C : |\tau| \leq 1\}$, degree of indeterminacy $\mathbb{I}_{\mathbb{W}}(f) = q_{\mathbb{W}}(f)e^{i\phi_{\mathbb{W}}(f)}$ is expressed as $\mathbb{I}_{\mathbb{W}} : P \to \{\tau \in C : |\tau| \leq 1\}$ and degree of falsity $\mathbb{F}_{\mathbb{W}}(f) = r_{\mathbb{W}}(f)e^{i\omega_{\mathbb{W}}(f)}$ and is defined as $\mathbb{F}_{\mathbb{W}} : P \to \{\tau \in C : |\tau| \leq 1\}$, where $|\mathbb{T}_{\mathbb{W}}(f) + \mathbb{I}_{\mathbb{W}}(f) + \mathbb{F}_{\mathbb{W}}(f)| \leq 3$ and C is the set of complex numbers. the degree of accuracy, level of indeterminacy and degree of falsehood receive all complex valued grade from within in the unit circle of complex plane, respectively. where $i = \sqrt{-1}$, $p_{\mathbb{W}}(f)$, $q_{\mathbb{W}}(f)$, $r_{\mathbb{W}}(f) \in [0, 1]$, and $\theta_{\mathbb{W}}(f)$, $\phi_{\mathbb{W}}(f), \omega_{\mathbb{W}}(f) \in [0, 2\pi]$ are real valued such that $0^- \leq p_{\mathbb{W}}(f) + q_{\mathbb{W}}(f) + r_{\mathbb{W}}(f) \leq 3^+$ and $0 \leq \theta_{\mathbb{W}}(f) + \phi_{\mathbb{W}}(f) + \omega_{\mathbb{W}}(f) \leq 6\pi$. For convenience we shall use $\mathbb{T}_{\mathbb{W}}(f) = p_{\mathbb{W}}(f)e^{i\theta_{\mathbb{W}}(f)}$, $\mathbb{T}_{\mathbb{X}}(f) = p_{\mathbb{X}}(f)e^{i\theta_{\mathbb{X}}(f)}$ as degree of truth, $\mathbb{I}_{\mathbb{W}}(f) = q_{\mathbb{W}}(f)e^{i\phi_{\mathbb{W}}(f)}$, $\mathbb{I}_{\mathbb{X}}(f) = q_{\mathbb{X}}(f)e^{i\phi_{\mathbb{X}}(f)}$ as degree of indeterminacy and $\mathbb{F}_{\mathbb{W}}(f) = r_{\mathbb{W}}(f)e^{i\omega_{\mathbb{W}}(f)}$, $\mathbb{F}_{\mathbb{X}}(f) = r_{\mathbb{X}}(f)e^{i\omega}_{\mathbb{X}}(f)$ as degree of falsity of $CNSs$ $\mathbb{W}$ and $\mathbb{X}$.*

Definition 4. [41] *Assume that* $\mathbb{W}$ *and* $\mathbb{X}$ *be two CNSs of set* $\mathbb{P}$*. Then the intersection of CNSs* $\mathbb{W}$ *and* $\mathbb{X}$ *is elaborated as:*

$$\begin{aligned} \mathbb{W}\cap\mathbb{X} &= \{<(\mathfrak{p}),\mathbb{T}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}),\mathbb{I}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}),\mathbb{F}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})>\}. \\ \text{Where}\quad \mathbb{T}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}) &= p_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})e^{i\theta_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})} = \min\{p_{\mathbb{W}}(\mathfrak{p}),p_{\mathbb{X}}(\mathfrak{p})\}e^{i\min\{\theta_{\mathbb{W}}(\mathfrak{p}),\theta_{\mathbb{X}}(\mathfrak{p})\}}, \\ \mathbb{I}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}) &= q_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})e^{i\phi_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})} = \max\{q_{\mathbb{W}}(\mathfrak{p}),q_{\mathbb{X}}(\mathfrak{p})\}e^{i\max\{\phi_{\mathbb{W}}(\mathfrak{p}),\phi_{\mathbb{X}}(\mathfrak{p})\}}, \\ \mathbb{F}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}) &= r_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})e^{i\omega_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})} = \max\{r_{\mathbb{W}}(\mathfrak{p}),r_{\mathbb{X}}(\mathfrak{p})\}e^{i\max\{\omega_{\mathbb{W}}(\mathfrak{p}),\omega_{\mathbb{X}}(\mathfrak{p})\}}. \end{aligned}$$

Definition 5. [41] *Let* $\mathbb{W}$ *and* $\mathbb{X}$ *be two CNSs of set* P*. Then the union of CNSs* $\mathbb{W}$ *and* $\mathbb{X}$ *is described as:*

$$\begin{aligned} \mathbb{W}\cap\mathbb{X} &= \{<(\mathfrak{p}),\mathbb{T}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}),\mathbb{I}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}),\mathbb{F}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})>\}. \\ \text{Where}\quad \mathbb{T}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}) &= p_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})e^{i\theta_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})} = \max\{p_{\mathbb{W}}(\mathfrak{p}),p_{\mathbb{X}}(\mathfrak{p})\}e^{i\max\{\theta_{\mathbb{W}}(\mathfrak{p}),\theta_{\mathbb{X}}(\mathfrak{p})\}}, \\ \mathbb{I}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}) &= q_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})e^{i\phi_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})} = \min\{q_{\mathbb{W}}(\mathfrak{p}),q_{\mathbb{X}}(\mathfrak{p})\}e^{i\min\{\phi_{\mathbb{W}}(\mathfrak{p}),\phi_{\mathbb{X}}(\mathfrak{p})\}}, \\ \mathbb{F}_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p}) &= r_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})e^{i\omega_{\mathbb{W}\cap\mathbb{X}}(\mathfrak{p})} = \min\{r_{W}(\mathfrak{p}),r_{\mathbb{X}}(\mathfrak{p})\}e^{i\min\{\omega_{\mathbb{W}}(\mathfrak{p}),\omega_{\mathbb{X}}(\mathfrak{p})\}}. \end{aligned}$$

3. Properties of Complex Neutrosophic Subrings

The investigation of CNSRs and level subsets of CNSRs is the focus of this section. We investigate that a $CNSR$s produces two neutrosophic subrings and intersection of two $CNSR$s is $CNSR$. We describe the level-subset of CNS and prove that level-subset of $CNSR$ form subring of ring.

Definition 6. *Let* $W=\{<f,\theta_W(f),\phi_W(f),\omega_W(f),>: f\in K\}$ *be a* $\mathbb{NS}$*, where* k *is the ring. Then the* π-$\mathbb{NS}$ W_π *is described as* $W_\pi=\{<f,\theta_{W_\pi}(f),\phi_{W_\pi}(f),\omega_{W_\pi}(f)>: f\in K\}$*, where the function* $\theta_{W_\pi}(f)=2\pi\theta_W(f)$*,* $\phi_{W_\pi}(f)=2\pi\phi_W(f)$ *and* $\omega_{W_\pi}(f)=2\pi\omega_W(f)$ *indicate the measure of association, measure of indeterminacy and measure of non- association of an element* f *of* K*, consequently. and fulfil the consequent criteria* $0\le\theta_{W_\pi}(f)+\phi_{W_\pi}(f)+\omega_{W_\pi}(f)\le 6\pi$.

Definition 7. *A* π-$\mathbb{NS}$ W_π *of ring* K *is known as* π*-neutrosophic subring of* K, $\forall\ \mathfrak{p},\ \mathfrak{u}\in K$ *if*

(i) $\theta_{W_\pi}(\mathfrak{p}-\mathfrak{u})\ge\min\{\theta_{W_\pi}(\mathfrak{p}),\theta_{W_\pi}(\mathfrak{u})\}$,

(ii) $\theta_{W_\pi}(\mathfrak{p}\mathfrak{u})\ge\min\{\theta_{W_\pi}(\mathfrak{p}),\theta_{W_\pi}(\mathfrak{u})\}$,

(iii) $\phi_{W_\pi}(\mathfrak{p}-\mathfrak{u})\le\max\{\phi_{W_\pi}(\mathfrak{p}),\phi_{W_\pi}(\mathfrak{u})\}$,

(iv) $\phi_{W_\pi}(\mathfrak{p}\mathfrak{u})\le\max\{\phi_{W_\pi}(\mathfrak{p}),\phi_{W_\pi}(\mathfrak{u})\}$,

(v) $\omega_{W_\pi}(\mathfrak{p}-\mathfrak{u})\le\max\{\omega_{W_\pi}(\mathfrak{p}),\omega_{W_\pi}(\mathfrak{u})\}$,

(vi) $\omega_{W_\pi}(\mathfrak{p}\mathfrak{u})\le\max\{\omega_{W_\pi}(\mathfrak{p}),\omega_{W_\pi}(\mathfrak{u})\}$.

Definition 8. *Let* W *and* X *be two CNSs of* K*. Then* *(i)* *A CNS W is homogeneous CNS, if for all* $\mathfrak{p}, a \in K$, *we have*

a $p_W(\mathfrak{p}) \leq p_W(a)$ *if and only if* $\theta_W(\mathfrak{p}) \leq \theta_W(a)$,

b $q_W(\mathfrak{p}) \leq q_W(a)$ *if and only if* $\phi_W(\mathfrak{p}) \leq \phi_W(a)$,

c $r_W(\mathfrak{p}) \geq r_W(a)$ *if and only if* $\omega_W(\mathfrak{p}) \geq \omega_W(a)$.

(ii) *A CNS W is homogeneous CNS with X, if for all* $\mathfrak{p} \in K$, *we have*

a $p_W(\mathfrak{p}) \leq p_X(\mathfrak{p})$ *iff* $\theta_W(\mathfrak{p}) \leq \theta_X(\mathfrak{p})$,

b $q_W(\mathfrak{p}) \leq q_X(\mathfrak{p})$ *iff* $\phi_W(\mathfrak{p}) \leq \phi_X(\mathfrak{p})$,

c $r_W(\mathfrak{p}) \geq r_X(\mathfrak{p})$ *iff* $\omega_W(\mathfrak{p}) \geq \omega_X(\mathfrak{p})$.

In this article, we shall take CNS as homogeneous CNS.

Definition 9. *A CNS* $W = \{< (x, \mathbb{T}_W(\mathfrak{p}), \mathbb{I}_W(\mathfrak{p}), \mathbb{F}_W(\mathfrak{p})) >: \mathfrak{p} \in K\}$ *of ring K is called a CNSR,* $\forall\ p, u \in K$ *if*

(i) $\mathbb{T}_W(\mathfrak{p} - \mathfrak{u}) \geq \min\{\mathbb{T}_W(\mathfrak{p}), \mathbb{T}_W(\mathfrak{u})\}$,

(ii) $\mathbb{T}_W(\mathfrak{p}\mathfrak{u}) \geq \min\{\mathbb{T}_W(\mathfrak{p}), \mathbb{T}_W(\mathfrak{u})\}$,

(iii) $\mathbb{I}_W(\mathfrak{p} - \mathfrak{u}) \leq \max\{\mathbb{I}_W(\mathfrak{p}), \mathbb{I}_W(\mathfrak{u})\}$,

(iv) $\mathbb{I}_W(\mathfrak{p}\mathfrak{u}) \leq \max\{\mathbb{I}_W(\mathfrak{p}), \mathbb{I}_W(\mathfrak{u})\}$,

(v) $\mathbb{F}_W(\mathfrak{p} - \mathfrak{u}) \leq \max\{\mathbb{F}_W(\mathfrak{p}), \mathbb{F}_W(\mathfrak{u})\}$,

(vi) $\mathbb{F}_W(\mathfrak{p}\mathfrak{u}) \leq \max\{\mathbb{F}_W(\mathfrak{p}), \mathbb{F}_W(\mathfrak{u})\}$,

Another way to describe the definition CNSR is given as;

(i) $p_W(\mathfrak{p} - \mathfrak{u})e^{i\theta_W(\mathfrak{p}-\mathfrak{u})} \geq \min\{p_W(\mathfrak{p}), p_W(\mathfrak{u})\}e^{i\min\{\theta_W(\mathfrak{p}),\theta_W(\mathfrak{u})\}}$,

(ii) $p_W(\mathfrak{p}\mathfrak{u})e^{i\theta_W(\mathfrak{p}\mathfrak{u})} \geq \min\{p_W(\mathfrak{p}), p_W(\mathfrak{u})\}e^{i\min\{\theta_W(\mathfrak{p}),\theta_W(\mathfrak{u})\}}$,

(iii) $q_W(\mathfrak{p} - \mathfrak{u})e^{i\phi_W(\mathfrak{p}-\mathfrak{u})} \leq \max\{q_W(\mathfrak{p}), q_W(\mathfrak{u})\}e^{i\max\{\phi_W(\mathfrak{p}),\phi_W(\mathfrak{u})\}}$,

(iv) $q_W(\mathfrak{p}\mathfrak{u})e^{i\phi_W(\mathfrak{p}\mathfrak{u})} \leq \max\{q_W(\mathfrak{p}), q_W(\mathfrak{u})\}e^{i\max\{\phi_W(\mathfrak{p}),\phi_W(\mathfrak{u})\}}$,

(v) $r_W(\mathfrak{p} - \mathfrak{u})e^{i\omega_W(\mathfrak{p}-\mathfrak{u})} \leq \max\{r_W(\mathfrak{p}), r_W(\mathfrak{u})\}\ e^{i\max\{\omega_W(\mathfrak{p}),\omega_W(\mathfrak{u})\}}$

(vi) $r_W(\mathfrak{p}\mathfrak{u})e^{i\omega_W(\mathfrak{p}\mathfrak{u})} \leq \max\{r_W(\mathfrak{p}), r_W(\mathfrak{u})\}\ e^{i\max\{\omega_W(\mathfrak{p}),\omega_W(\mathfrak{u})\}}$ *for all* $\mathfrak{p}, \mathfrak{u} \in K$.

In the following theorem, we prove that a $CNSR$ produce two neutrosophic subrings, namely neutrosophic subring and π-neutrosophic subring(π- NSR).

Theorem 2. *Let W be a CNS of ring K. Then W is a CNSR of K if and only if*

(i) *The fuzzy set* $\overline{W} = \{< k, p_W(k), q_W(k), r_W(k) >: k \in K, p_W(k), q_W(k),\ r_W(k) \in [0, 1]$ *and* $0 \leq p_W(k) + q_W(k) + r_W(k) + \leq 3\}$ *is a NSR .*

(ii) The π*-fuzzy set* $\underline{W} = \{< k, \theta_W(k), \phi_W(k), \omega_W(k) >: k \in K, \theta_W(k), \phi_W(k), \omega_W(k) \in [0, 2\pi]\}$ *is a* π*-* NSR*.*

Proof. Assume that W is a $CNSR$ and $k, l \in M$. Then we know that,

$$\begin{aligned} p_W(k-l)e^{i\theta_W(k-l)} &= \mathbb{T}_W(k-l) \geq \min\{\mathbb{T}_W(k), \mathbb{T}_W(l)\} \\ &= \min\{p_W(k)e^{i\theta_W(k)}, p_W(l)e^{i\theta_W(l)}\} \\ &= \min\{p_W(k), p_W(l)\}e^{i\min\{\theta_W(k), \theta_W(l)\}}. \end{aligned}$$

As W is homogeneous, so

$$p_W(k-l) \geq \min\{p_W(k), p_W(l)\} \text{ and } \theta_W(k-l) \geq \min\{\theta_W(k), \theta_W(l)\}.$$

$$\begin{aligned} p_W(kl)e^{i\omega_W(kl)} &= \mathbb{T}_W(kl) \geq \min\{\mathbb{T}_W(k), \mathbb{T}_W(l)\} = \min\{p_W(k)e^{i\theta_W(k)}, p_W(l)e^{i\theta_W(l)}\} \\ &= \min\{p_W(k), p_W(l)\}e^{i\min\{\theta_W(k), \theta_W(l)\}}. \end{aligned}$$

As W is homogeneous, thus

$$p_W(kl) \geq \min\{p_W(k), p_W(l)\} \text{ and } \theta_W(kl) \quad \geq \quad \min\{\theta_W(k), \theta_W(l)\}.$$

Suppose that W is a $CNSR$ and $k, l \in H$. Then we have,

$$\begin{aligned} q_W(k-l)e^{i\phi_W(k-l)} &= \mathbb{I}_W(k-l) \leq \max\{\mathbb{I}_W(k), \mathbb{I}_W(l)\} \\ &= \max\{q_W(k)e^{i\phi_W(k)}, q_W(l)e^{i\phi_W(l)}\} \\ &= \max\{q_W(k), q_W(l)\}e^{i\max\{\phi_W(k), \phi_W(l)\}}. \end{aligned}$$

As W is homogeneous, so

$$q_W(k-l) \quad \leq \quad \max\{q_W(k), q_W(l)\} \text{ and } \phi_W(k-l) \leq \max\{\phi_W(k), \phi_W(l)\}.$$

$$\begin{aligned} q_W(kl)e^{i\phi_W(kl)} &= \mathbb{I}_W(kl) \leq \max\{\mathbb{I}_W(k), \mathbb{I}_W(l)\} \\ &= \max\{q_W(k)e^{i\phi_W(k)}, q_W(l)e^{i\phi_W(l)}\} \\ &= \max\{q_W(k), q_W(l)\}e^{i\max\{\phi_W(k), \phi_W(l)\}}. \end{aligned}$$

As W is homogeneous, we have

$$q_W(kl) \quad \leq \quad \max\{q_W(k), q_W(l)\} \text{ and } \phi_W(kl) \leq \max\{\phi_W(k), \phi_W(l)\}.$$

$$\begin{aligned} r_W(k-l)e^{i\omega_W(k-l)} &= r_W(k-l) \leq \max\{\mathbb{F}_W(k), \mathbb{F}_W(l)\} \\ &= \max\{r_W(k)e^{i\omega_W(k)}, r_W(l)e^{i\omega_W(l)}\} \\ &= \max\{r_W(k), r_W(l)\}e^{i\max\{\omega_W(k), \omega_W(l)\}}. \end{aligned}$$

As W is homogeneous, so

$$r_W(k-l) \leq \max\{r_W(k), r_W(l)\} \text{ and } \omega_W(k-l) \quad \leq \quad \max\{\omega_W(k), \omega_W(l)\}.$$

$$\begin{aligned} r_W(kl)e^{i\omega_W(kl)} &= \mathbb{F}_W(kl) \leq \max\{\mathbb{F}_W(k), \mathbb{F}_W(l)\} \\ &= \max\{r_W(k)e^{i\omega_W(k)}, r_W(l)e^{i\omega_W(l)}\} \\ &= \max\{r_W(k), r_W(l)\}e^{i\max\{\omega_W(k),\omega_W(l)\}}. \end{aligned}$$

As W is homogeneous, so

$$r_W(kl) \leq \max\{r_W(k), r_W(l)\} \text{ and } \omega_W(kl) \quad \leq \quad \max\{\omega_W(k), \omega_W(l)\}.$$

Consequently, $\overline{W}$ is NSR and $\underline{W}$ is π-NSR.
Conversely, assume that $\overline{W}$ and $\underline{W}$ is NSR and π-NSR, respectively. Then, we know that

$$\begin{aligned} p_W(k-l) &\geq \min\{T_W(k), T_W(l)\},\ p_W(kl) \geq \min\{p_W(k), p_W(l)\},\ q_W(k-l) \\ &\leq \max\{q_W(k), q_W(l)\}, \\ q_W(kl) &\leq \max\{q_W(k), q_W(l)\},\ r_W(k-l) \leq \max\{r_W(k), r_W(l)\},\ r_W(kl) \\ &\leq \max\{r_W(k), r_W(l)\} \end{aligned}$$

$$\begin{aligned} \theta_W(k-l) &\geq \min\{\theta_W(k), \theta_W(l)\},\ \theta_W(kl) \geq \min\{\theta_W(k), \theta_W(l)\},\ \phi_W(k-l) \\ &\leq \max\{\phi_W(k), \phi_W(l)\}, \\ \phi_W(kl) &\leq \max\{\phi_W(k), \phi_W(l)\},\ \omega_W(k-l) \leq \max\{\omega_W(k), \omega_W(l)\},\ \omega_W(kl) \\ &\leq \max\{\omega_W(k), \omega_W(l)\}, \end{aligned}$$

For this, we consider

$$\begin{aligned} \mathbb{T}_W(k-l) &= p_W(k-l)e^{i\theta_W(k-l)} \geq \min\{p_W(k), p_W(l)\}e^{i\min\{\theta_W(k),\theta_W(l)\}} \\ &= \min\{p_W(k)e^{i\theta_W(k)}, p_W(l)e^{i\theta_W(l)}\} = \min\{\mathbb{T}_W(k), \mathbb{T}_W(l)\}\ . \end{aligned}$$

For this, we have

$$\begin{aligned} \mathbb{T}_W(kl) &= p_W(kl)e^{i\theta_W(kl)} \geq \min\{p_W(k), p_W(l)\}e^{i\min\{\theta_W(k),\theta_W(l)\}} \\ &= \min\{p_W(k)e^{i\theta_W(k)}, p_W(l)e^{i\theta_W(l)}\} = \min\{\mathbb{T}_W(k), \mathbb{T}_W(l)\}\ . \end{aligned}$$

For this, we get

$$\begin{aligned} \mathbb{I}_W(k-l) &= q_W(k-l)e^{i\phi_W(k-l)} \leq \max\{q_W(k), q_W(l)\}e^{i\max\{\phi_W(k),\phi_W(l)\}} \\ &= \max\{q_W(k)e^{i\phi_W(k)}, q_W(l)e^{i\phi_W(l)}\} = \max\{\mathbb{I}_W(k), \mathbb{I}_W(l)\}\ . \end{aligned}$$

$$\begin{aligned} \mathbb{I}_W(kl) &= q_W(kl)e^{i\phi_W(kl)} \leq \max\{q_W(k), q_W(l)\}e^{i\max\{\phi_W(k),\phi_W(l)\}} \\ &= \max\{q_W(k)e^{i\phi_W(k)}, q_W(l)e^{i\phi_W(l)}\} = \max\{\mathbb{I}_W(k), \mathbb{I}_W(l)\}\ . \end{aligned}$$

Consider,

$$\begin{aligned}
\mathbb{F}_W(k-l) &= r_W(k-l)e^{i\omega_W(k-l)} \leq \max\{r_W(k), r_W(l)\}e^{i\max\{\omega_W(k),\omega_W(l)\}} \\
&= \max\{r_W(k)e^{i\omega_W(k)}, r_W(l)e^{i\omega_W(l)}\} = \max\{\mathbb{F}_W(k), \mathbb{F}_W(l)\} . \\
\mathbb{F}_W(kl) &= r_W(kl)e^{i\omega_W(kl)} \leq \max\{r_W(k), r_W(l)\}e^{i\max\{\omega_W(k),\omega_W(l)\}} \\
&= \max\{r_W(k)e^{i\omega_W(k)}, r_W(l)e^{i\omega_W(l)}\} = \max\{\mathbb{F}_W(k), \mathbb{F}_W(l)\} .
\end{aligned}$$

Hence, W is $CNSR$.

In the following theorem, we illustrate that the intersection of two complex neutrosophic subrings is complex neutrosophic subring.

Theorem 3. *Intersection of two $CNSR$s of ring R is $CNSR$.*

Proof. The proof is so accomplished.

Remark 1. *The union of two $CNSR$s of ring R may not be $CNSR$ of ring R.*

In the upcoming example, we describe that the union of two $CNSR$s of ring may not be $CNSR$ of ring.

Example 2. *Take $\mathbb{R} = \mathbb{Z} = \{0, \pm1, \pm2, \pm3, ...\}$ is a ring of integers. Assume that $\mathbb{W}$ and $\mathbb{X}$ are two $CNSR$s of ring $\mathbb{R}$ and described as*

$$\mathbb{T}_{\mathbb{W}}(\mathfrak{q}) = \begin{cases} 0.3e^{\frac{i\pi}{3}} & \textit{if } \mathfrak{q} \in 3\mathbb{Z} \\ 0 & \textit{otherwise.} \end{cases}$$

$$\mathbb{I}_{\mathbb{W}}(\mathfrak{q}) = \begin{cases} 0.4e^{\frac{i\pi}{3}} & \textit{if } \mathfrak{q} \in 3\mathbb{Z} \\ 0 & \textit{otherwise.} \end{cases}$$

$$\mathbb{F}_{\mathbb{W}}(\mathfrak{q}) = \begin{cases} 0.2e^{\frac{i\pi}{9}} & \textit{if } q \in 3\mathbb{Z} \\ 0.6e^{\frac{i\pi}{4}} & \textit{else.} \end{cases}$$

$$\mathbb{T}_{\mathbb{X}}(\mathfrak{q}) = \begin{cases} 0.2e^{\frac{i\pi}{3}} & \textit{if } \mathfrak{q} \in 2\mathbb{Z} \\ 0.01e^{\frac{i\pi}{8}} & \textit{otherwise.} \end{cases}$$

$$\mathbb{I}_{\mathbb{X}}(\mathfrak{q}) = \begin{cases} 0.1e^{\frac{i\pi}{4}} & \textit{if } \mathfrak{q} \in 2\mathbb{Z} \\ 0.01e^{\frac{i\pi}{8}} & \textit{otherwise.} \end{cases}$$

$$\mathbb{F}_{\mathbb{X}}(\mathfrak{q}) = \begin{cases} 0.4e^{\frac{i\pi}{8}} & \textit{if } \mathfrak{q} \in 2\mathbb{Z} \\ 0.2e^{\frac{i\pi}{9}} & \textit{otherwise.} \end{cases}$$

It is simple to determine that $\mathbb{W}$ *and* $\mathbb{X}$ *are two CNSRs of ring* $\mathbb{R}$*. Using Definition 5* $\mathbb{W} \cup \mathbb{X} = \{(\mathfrak{q}, \mathbb{T}_{\mathbb{W}\cup\mathbb{X}}, \mathbb{I}_{\mathbb{W}\cup\mathbb{X}}, \mathbb{F}_{\mathbb{W}\cup\mathbb{X}})\}$*. Therefore,*

$$\mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(\mathfrak{q}) = \begin{cases} 0.3e^{\frac{i\pi}{3}} & \text{if } \mathfrak{q} \in 3\mathbb{Z} \\ 0.2e^{\frac{i\pi}{3}} & \text{if } \mathfrak{q} \in 2\mathbb{Z} - 3\mathbb{Z} \\ 0.01e^{\frac{i\pi}{8}} & \text{otherwise.} \end{cases}$$

$$\mathbb{I}_{\mathbb{W}\cup\mathbb{X}}(\mathfrak{q}) = \begin{cases} 0.4e^{\frac{i\pi}{3}} & \text{if } \mathfrak{q} \in 3\mathbb{Z} \\ 0.1e^{\frac{i\pi}{4}} & \text{if } \mathfrak{q} \in 2\mathbb{Z} - 3\mathbb{Z} \\ 0.01e^{\frac{i\pi}{4}} & \text{else.} \end{cases}$$

$$\mathbb{F}_{\mathbb{W}\cup\mathbb{X}}(\mathfrak{q}) = \begin{cases} 0.2e^{\frac{i\pi}{9}} & \text{if } \mathfrak{q} \in 3\mathbb{Z} \\ 0.4e^{\frac{i\pi}{9}} & \text{if } \mathfrak{q} \in 2\mathbb{Z} - 3\mathbb{Z} \\ 0.2e^{\frac{i\pi}{9}} & \text{else.} \end{cases}$$

Assume $s = 15$ *and* $t = 10$*. Then* $\mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(15) = 0.3e^{\frac{i\pi}{3}}$ *and* $\mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(10) = 0.2e^{\frac{i\pi}{3}}$*, then* $\mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(15-10) = \mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(5) = 0.01e^{\frac{i\pi}{8}}$ *and* $min\{\mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(15), \mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(10)\} = min\{0.3e^{\frac{i\pi}{3}}, 0.2e^{\frac{i\pi}{3}}\}$ $= 0.2e^{\frac{i\pi}{3}}$*. Clearly,* $\mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(15-10) < min\{\mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(15), \mathbb{T}_{\mathbb{W}\cup\mathbb{X}}(10)\}$*. This condition does not holds. Consequently,* $\mathbb{W} \cup \mathbb{X}$ *is not CNSRs of* $\mathbb{R}$*.*

Now, we define the idea of complex neutrosophic level subset of $\mathbb{CNS}$ and discuss its vital results under the framework of complex neutrosophic subring.

Definition 10. *Let* $\mathbb{W} = \{< \mathfrak{j}, \mathbb{T}_{\mathbb{W}}(\mathfrak{j}), \mathbb{I}_{\mathbb{W}}(\mathfrak{j}), \mathbb{F}_{\mathbb{W}}(\mathfrak{j}) >: \mathfrak{j} \in R\}$ *be a CNS of R, for all* $\nu,\ \eta,\ \rho \in [0,1]$*, and* $\hat{\nu},\ \hat{\eta},\ \hat{\rho} \in [0, 2\pi]$*. The level subset of CNS is described as;*

$$\mathbb{W}^{(\nu,\ \eta,\ \rho)}_{(\hat{\nu},\ \hat{\eta},\ \hat{\rho})} = \{\mathfrak{j} \in R : p_{\mathbb{W}}(\mathfrak{j}) \geq \nu, \theta_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\nu}, q_{\mathbb{W}}(\mathfrak{j}) \geq \eta, \phi_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\eta}, r_{\mathbb{W}}(\mathfrak{j}) \leq \rho, \omega_{\mathbb{W}}(\mathfrak{j}) \leq \hat{\rho}\}.$$

For $\eta = \rho = 0 = \hat{\eta} = \hat{\rho}$*, we get* $\mathbb{W}^{\nu}_{\hat{\nu}} = \{\mathfrak{j} \in R : p_{\mathbb{W}}(\mathfrak{j}) \geq \nu, \theta_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\nu}\}$*, for* $\nu = \rho = 0 = \hat{\nu} = \hat{\rho}$*, we get,* $\mathbb{W}^{\eta}_{\hat{\eta}} = \{\mathfrak{j} \in R : q_{\mathbb{W}}(\mathfrak{j}) \geq \eta, \phi_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\eta}\}$ *and for* $\nu = \eta = 0 = \hat{\nu} = \hat{\eta}$*, we get* $\mathbb{W}^{\rho}_{\hat{\rho}} = \{\mathfrak{j} \in R : r_{\mathbb{W}}(\mathfrak{j}) \leq \rho, \omega_{\mathbb{W}}(\mathfrak{j}) \leq \hat{\rho}\}$*.*

Theorem 4. *Let* $\mathbb{W}$ *be CNNSR of ring R. Then* $\mathbb{W}^{(\nu,\ \eta,\ \rho)}_{(\hat{\nu},\ \hat{\eta},\ \hat{\rho})}$ *is a subring of ring R, for all* $\nu,\ \eta,\ \rho \in [0,1]$*, and* $\hat{\nu},\ \hat{\eta},\ \hat{\rho} \in [0, 2\pi]$*, where* $p_{\mathbb{W}}(\mathfrak{j}) \geq \nu, \theta_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\nu}, q_{\mathbb{W}}(\mathfrak{j}) \geq \eta, \phi_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\eta}, r_{\mathbb{W}}(\mathfrak{j}) \leq \rho, \omega_{\mathbb{W}}(\mathfrak{j}) \leq \hat{\rho}$*.*

Proof. We know that $\mathbb{W}^{(\nu,\ \eta,\ \rho)}_{(\hat{\nu},\ \hat{\eta},\ \hat{\rho})}$ is nonempty, as $e \in A^{(\nu,\ \eta,\ \rho)}_{(\hat{\nu},\ \hat{\eta},\ \hat{\rho})}$. Let $f, y \in A^{(\nu,\ \eta,\ \rho)}_{(\hat{\nu},\ \hat{\eta},\ \hat{\rho})}$ be any two elements. Then

$$p_{\mathbb{W}}(\mathfrak{j}) \geq \nu, \theta_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\nu}, q_{\mathbb{W}}(\mathfrak{j}) \geq \eta, \phi_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\eta}, r_{\mathbb{W}}(\mathfrak{j}) \leq \rho, \omega_{\mathbb{W}}(\mathfrak{j}) \leq \hat{\rho}.$$

Now we suppose that,

$$p_{\mathbb{W}}(\mathfrak{j} - g)e^{i\theta_{\mathbb{W}}(\mathfrak{j}-g)} = \mathbb{T}_{\mathbb{W}}(\mathfrak{j} - g) \geq \min\{\mathbb{T}_{\mathbb{W}}(\mathfrak{j}), \mathbb{T}_{\mathbb{W}}(g)\} = \min\{p_{\mathbb{W}}(\mathfrak{j})e^{i\theta_{\mathbb{W}}(\mathfrak{j})}, p_{\mathbb{W}}(g)e^{i\theta_{\mathbb{W}}(g)}\}$$

$$= \min\{p_{\mathbb{W}}(\mathfrak{j}), p_{\mathbb{W}}(g)\}\ e^{i\min\{\theta_{\mathbb{W}}(\mathfrak{j}),\theta_{\mathbb{W}}(g)\}}\ .$$

As $\mathbb{W}$ is homogeneous, so

$$\begin{aligned} p_{\mathbb{W}}(\mathfrak{j}-g) &\geq \min\{p_{\mathbb{W}}(\mathfrak{j}), p_{\mathbb{W}}(g)\} \ = \min\{\nu,\nu\} \ = \nu, \\ \theta_{\mathbb{W}}(\mathfrak{j}-g) &\geq \min\{\theta_{\mathbb{W}}(\mathfrak{j}), \theta_{\mathbb{W}}(g)\} \ = \min\{\hat{\nu},\hat{\nu}\} \ = \hat{\nu}. \end{aligned}$$

$$\begin{aligned} p_{\mathbb{W}}(\mathfrak{j}g)e^{i\theta_{\mathbb{W}}(\mathfrak{j}g)} &= \mathbb{T}_{\mathbb{W}}(\mathfrak{j}g) \geq \min\{\mathbb{T}_{\mathbb{W}}(\mathfrak{j}), \mathbb{T}_{\mathbb{W}}(g)\} \ = \min\{p_{\mathbb{W}}(\mathfrak{j})e^{i\theta_{\mathbb{W}}(\mathfrak{j})}, p_{\mathbb{W}}(g)e^{i\theta_{\mathbb{W}}(g)}\} \\ &= \min\{p_{\mathbb{W}}(\mathfrak{j}), p_{\mathbb{W}}(g)\}\ e^{i\min\{\theta_{\mathbb{W}}(\mathfrak{j}),\theta_{\mathbb{W}}(g)\}}\ . \end{aligned}$$

As $\mathbb{W}$ is homogeneous, so

$$\begin{aligned} p_{\mathbb{W}}(\mathfrak{j}g) &\geq \min\{p_{\mathbb{W}}(\mathfrak{j}), p_{\mathbb{W}}(g)\} \ = \min\{\nu,\nu\} \ = \nu, \\ \theta_{\mathbb{W}}(\mathfrak{j}g) &\geq \min\{\theta_{\mathbb{W}}(\mathfrak{j}), \theta_{\mathbb{W}}(g)\} \ = \min\{\hat{\nu},\hat{\nu}\} \ = \hat{\nu}. \end{aligned}$$

Further,

$$\begin{aligned} q_{\mathbb{W}}(\mathfrak{j}-g)e^{i\phi_{\mathbb{W}}(\mathfrak{j}-g)} &= \mathbb{I}_{\mathbb{W}}(\mathfrak{j}-g) \geq \min\{\mathbb{I}_{\mathbb{W}}(\mathfrak{j}), \mathbb{I}_{\mathbb{W}}(g)\} \ = \min\{q_{\mathbb{W}}(\mathfrak{j})e^{i\phi_{\mathbb{W}}(\mathfrak{j})}, q_{\mathbb{W}}(g)e^{i\phi_{\mathbb{W}}(g)}\} \\ &= \min\{q_{\mathbb{W}}(\mathfrak{j}), q_{\mathbb{W}}(g)\}\ e^{i\min\{\phi_{\mathbb{W}}(\mathfrak{j}),\phi_{\mathbb{W}}(g)\}}\ . \end{aligned}$$

As $\mathbb{W}$ is homogeneous, so

$$\begin{aligned} q_{\mathbb{W}}(\mathfrak{j}-g) &\geq \min\{q_{\mathbb{W}}(\mathfrak{j}), q_{\mathbb{W}}(g)\} \ = \min\{\nu,\nu\} \ = \nu, \\ \phi_{\mathbb{W}}(\mathfrak{j}-g) &\geq \min\{\phi_{\mathbb{W}}(\mathfrak{j}), \phi_{\mathbb{W}}(g)\} \ = \min\{\hat{\nu},\hat{\nu}\} \ = \hat{\nu}. \end{aligned}$$

$$\begin{aligned} q_{\mathbb{W}}(\mathfrak{j}g)e^{i\phi_{\mathbb{W}}(\mathfrak{j}g)} &= \mathbb{I}_{\mathbb{W}}(\mathfrak{j}g) \geq \min\{\mathbb{I}_{\mathbb{W}}(\mathfrak{j}), \mathbb{I}_{\mathbb{W}}(g)\} \ = \min\{q_{\mathbb{W}}(\mathfrak{j})e^{i\phi_{\mathbb{W}}(\mathfrak{j})}, q_{\mathbb{W}}(g)e^{i\phi_{\mathbb{W}}(g)}\} \\ &= \min\{q_{\mathbb{W}}(\mathfrak{j}), q_{\mathbb{W}}(g)\}\ e^{i\min\{\phi_{\mathbb{W}}(\mathfrak{j}),\phi_{\mathbb{W}}(g)\}}\ . \end{aligned}$$

As $\mathbb{W}$ is homogeneous, so

$$\begin{aligned} q_{\mathbb{W}}(\mathfrak{j}g) &\geq \min\{q_{\mathbb{W}}(\mathfrak{j}), q_{\mathbb{W}}(g)\} \ = \min\{\nu,\nu\} \ = \nu, \\ \phi_{\mathbb{W}}(\mathfrak{j}g) &\geq \min\{\phi_{\mathbb{W}}(\mathfrak{j}), \phi_{\mathbb{W}}(g)\} \ = \min\{\hat{\nu},\hat{\nu}\} \ = \hat{\nu}. \end{aligned}$$

Further,

$$\begin{aligned} r_{\mathbb{W}}(\mathfrak{j}-g)e^{i\omega_{\mathbb{W}}(\mathfrak{j}-g)} &= \mathbb{F}_{\mathbb{W}}(\mathfrak{j}-g) \leq \max\{\mathbb{F}_{\mathbb{W}}(\mathfrak{j}), \mathbb{F}_{\mathbb{W}}(g)\} \ = \max\{r_{\mathbb{W}}(\mathfrak{j})e^{i\omega_{\mathbb{W}}(\mathfrak{j})}, r_{\mathbb{W}}(g)e^{i\omega_{\mathbb{W}}(g)}\} \\ &= \max\{r_{\mathbb{W}}(\mathfrak{j}), r_{\mathbb{W}}(g)\}\ e^{i\max\{\omega_{\mathbb{W}}(\mathfrak{j}),\omega_{\mathbb{W}}(g)\}}\ . \end{aligned}$$

By homogeneity, so

$$\begin{aligned} r_{\mathbb{W}}(\mathfrak{j}-g) &\leq \max\{r_{\mathbb{W}}(\mathfrak{j}), r_{\mathbb{W}}(g)\} \ = \max\{\rho,\rho\} \ = \rho, \\ \omega_{\mathbb{W}}(\mathfrak{j}-g) &\leq \max\{\omega_{\mathbb{W}}(\mathfrak{j}), \omega_{\mathbb{W}}(g)\} \ = \max\{\hat{\rho},\hat{\rho}\} \ = \hat{\rho}. \end{aligned}$$

$$\begin{aligned} r_{\mathbb{W}}(\mathrm{j}g)e^{i\omega_{\mathbb{W}}(\mathrm{j}g)} &= \mathbb{F}_{\mathbb{W}}(\mathrm{j}g) \leq \max\{\mathbb{F}_{\mathbb{W}}(\mathrm{j}), \mathbb{F}_{\mathbb{W}}(g)\} = \max\{r_{\mathbb{W}}(\mathrm{j})e^{i\omega_{\mathbb{W}}(\mathrm{j})}, r_{\mathbb{W}}(g)e^{i\omega_{\mathbb{W}}(g)}\} \\ &= \max\{r_{\mathbb{W}}(\mathrm{j}), r_{\mathbb{W}}(g)\}\ e^{i\max\{\omega_{\mathbb{W}}(\mathrm{j}),\omega_{\mathbb{W}}(g)\}}\ . \end{aligned}$$

By homogeneity, so

$$\begin{aligned} r_{\mathbb{W}}(\mathrm{j}g) &\leq \max\{r_{\mathbb{W}}(\mathrm{j}), r_{\mathbb{W}}(g)\} = \max\{\rho, \rho\} = \rho, \\ \omega_{\mathbb{W}}(\mathrm{j}g) &\leq \max\{\omega_{\mathbb{W}}(\mathrm{j}), \omega_{\mathbb{W}}(g)\} = \max\{\hat{\rho}, \hat{\rho}\} = \hat{\rho}. \end{aligned}$$

This implies that $mn \in \mathbb{W}^{(\nu,\ \eta,\ \rho)}_{(\hat{\nu},\ \hat{\eta},\ \hat{\rho})}$. Hence, $\mathbb{W}^{(\nu,\ \eta,\ \rho)}_{(\hat{\nu},\ \hat{\eta},\ \hat{\rho})}$ is subring.

Theorem 5. *Let* $\mathbb{W}^{(\alpha,\ \beta,\ \gamma)}_{(\hat{\alpha},\ \hat{\beta},\ \hat{\gamma})}$ *be a subring of ring R, then* $\mathbb{W}$ *is CNSR of R if* $p_{\mathbb{W}}(\mathrm{j}) \geq \alpha, \theta_{\mathbb{W}}(\mathrm{j}) \geq \hat{\alpha}, q_{\mathbb{W}}(\mathrm{j}) \geq \beta, \phi_{\mathbb{W}}(\mathrm{j}) \geq \hat{\beta}, r_{\mathbb{W}}(\mathrm{j}) \leq \gamma, \omega_{\mathbb{W}}(\mathrm{j}) \leq \hat{\gamma},\ \forall\ \alpha,\ \beta,\ \gamma \in [0,1]$*, and* $\hat{\alpha},\ \hat{\beta},\ \hat{\gamma} \in [0, 2\pi]$.

Proof. Suppose that $\min\{p_{\mathbb{W}}(\mathrm{j}), p_{\mathbb{W}}(\mathbf{g})\} = \alpha$, $\min\{\theta_{\mathbb{W}}(\mathrm{j}), \theta_{\mathbb{W}}(\mathbf{g})\} = \hat{\alpha}$, $\min\{q_{\mathbb{W}}(\mathrm{j}), q_{\mathbb{W}}(\mathbf{g})\} = \beta$, $\min\{\phi_{\mathbb{W}}(\mathrm{j}), \phi_{\mathbb{W}}(\mathbf{g})\} = \hat{\beta}$ and $\max\{r_{\mathbb{W}}(\mathrm{j}), r_{\mathbb{W}}(\mathbf{g})\} = \gamma$, $\max\{\omega_{\mathbb{W}}(\mathrm{j}), \omega_{\mathbb{W}}(\mathbf{g})\} = \hat{\gamma}$. Then we have $p_{\mathbb{W}}(\mathrm{j}) \geq \alpha$, $q_{\mathbb{W}}(\mathrm{j}) \geq \beta$, $r_{\mathbb{W}}(\mathrm{j}) \leq \gamma$, $\theta_{\mathbb{W}}(\mathrm{j}) \geq \hat{\alpha}$, $\phi_{\mathbb{W}}(\mathrm{j}) \geq \hat{\beta}$, $\omega_{\mathbb{W}}(\mathrm{j}) \leq \hat{\gamma}$ and $p_{\mathbb{W}}(\mathbf{g}) \geq \alpha$, $q_{\mathbb{W}}(\mathbf{g}) \geq \beta$, $r_{\mathbb{W}}(\mathbf{g}) \leq \gamma$, $\theta_{\mathbb{W}}(\mathbf{g}) \geq \alpha$, $\phi_{\mathbb{W}}(\mathbf{g}) \geq \beta$, $\omega_{\mathbb{W}}(\mathbf{g}) \leq \hat{\gamma}$. This means that $f \in \mathbb{W}^{(\alpha,\ \beta,\ \gamma)}_{(\hat{\alpha},\ \hat{\beta},\ \hat{\gamma})}$ and $n \in \mathbb{W}^{(\alpha,\ \beta,\ \gamma)}_{(\hat{\alpha},\ \hat{\beta},\ \hat{\gamma})}$. As $\mathbb{W}^{(\alpha,\ \beta,\ \gamma)}_{(\hat{\alpha},\ \hat{\beta},\ \hat{\gamma})}$ is subring, so $mn \in \mathbb{W}^{(\alpha,\ \beta,\ \gamma)}_{(\hat{\alpha},\ \hat{\beta},\ \hat{\gamma})}$. Then we have

$$p_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \geq \alpha \text{ and } \theta_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \geq \hat{\alpha},\ q_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \geq \beta \text{ and } \phi_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \geq \hat{\beta},\ r_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \leq \gamma \text{ and } \omega_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \leq \hat{\gamma}$$

Implies that

$$\begin{aligned} & p_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \geq \min\{p_{\mathbb{W}}(\mathrm{j}), p_{\mathbb{W}}(\mathbf{g})\} \text{ and } \theta_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \geq \min\{\theta_{\mathbb{W}}(\mathrm{j}), \theta_{\mathbb{W}}(\mathbf{g})\}\ , \\ & q_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \geq \min\{q_{\mathbb{W}}(\mathrm{j}), q_{\mathbb{W}}(\mathbf{g})\} \text{ and } \phi_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \geq \min\{\phi_{\mathbb{W}}(\mathrm{j}), \phi_{\mathbb{W}}(\mathbf{g})\}\ , \\ & r_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \leq \max\{r_{\mathbb{W}}(\mathrm{j}), r_{\mathbb{W}}(\mathbf{g})\}\ , \omega_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) \leq \max\{\omega_{\mathbb{W}}(\mathrm{j}), \omega_{\mathbb{W}}(\mathbf{g})\}\ . \\ & p_{\mathbb{W}}(\mathbf{jg}) \geq \alpha \text{ and } \theta_{\mathbb{W}}(\mathbf{jg}) \geq \hat{\alpha},\ q_{\mathbb{W}}(\mathbf{jg}) \geq \beta \text{ and } \phi_{\mathbb{W}}(\mathbf{jg}) \geq \hat{\beta}\ r_{\mathbb{W}}(\mathbf{jg}) \leq \gamma \\ & \text{and } \omega_{\mathbb{W}}(\mathbf{jg}) \leq \hat{\gamma} \end{aligned}$$

Implies that

$$\begin{aligned} & p_{\mathbb{W}}(\mathbf{jg}) \geq \min\{p_{\mathbb{W}}(\mathrm{j}), p_{\mathbb{W}}(\mathbf{g})\} \text{ and } \theta_{\mathbb{W}}(\mathbf{jg}) \geq \min\{\theta_{\mathbb{W}}(\mathrm{j}), \theta_{\mathbb{W}}(\mathbf{g})\} \\ & q_{\mathbb{W}}(\mathbf{jg}) \geq \min\{q_{\mathbb{W}}(\mathrm{j}), q_{\mathbb{W}}(\mathbf{g})\} \text{ and } \phi_{\mathbb{W}}(\mathbf{jg}) \geq \min\{\phi_{\mathbb{W}}(\mathrm{j}), \phi_{\mathbb{W}}(\mathbf{g})\} \\ & r_{\mathbb{W}}(\mathbf{jg}) \leq \max\{r_{\mathbb{W}}(\mathrm{j}), r_{\mathbb{W}}(\mathbf{g})\} \text{ and } \omega_{\mathbb{W}}(\mathbf{jg}) \leq \max\{\omega_{\mathbb{W}}(\mathrm{j}), \omega_{\mathbb{W}}(\mathbf{g})\}\ . \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{T}_{\mathbb{W}}(\mathrm{j}-\mathbf{g}) &= p_{\mathbb{W}}(\mathrm{j}-\mathbf{g})e^{i\theta_{\mathbb{W}}(\mathrm{j}-\mathbf{g})} \geq \min\{p_{\mathbb{W}}(\mathrm{j}), p_{\mathbb{W}}(\mathbf{g})\}\ e^{i\min\{\theta_{\mathbb{W}}(\mathrm{j}),\theta_{\mathbb{W}}(\mathbf{g})\}} \\ &= \min\{p_{\mathbb{W}}(\mathrm{j})e^{i\theta_{\mathbb{W}}(\mathrm{j})}, p_{\mathbb{W}}(\mathbf{g})e^{i\theta_{\mathbb{W}}(\mathbf{g})}\} \end{aligned}$$

$$\mathbb{T}_{\mathbb{W}}(\mathfrak{j}-\mathbf{g}) \geq \min\{\mathbb{T}_{\mathbb{W}}(\mathfrak{j}), \mathbb{T}_{\mathbb{W}}(\mathbf{g})\} .$$

$$\begin{aligned}
\mathbb{I}_{\mathbb{W}}(\mathfrak{j}-\mathbf{g}) &= q_{\mathbb{W}}(\mathfrak{j}-\mathbf{g})e^{i\phi_{\mathbb{W}}(\mathfrak{j}-\mathbf{g})} \geq \min\{q_{\mathbb{W}}(\mathfrak{j}), q_{\mathbb{W}}(\mathbf{g})\}\ e^{i\min\{\phi_{\mathbb{W}}(\mathfrak{j}),\phi_{\mathbb{W}}(\mathbf{g})\}} \\
&= \min\{q_{\mathbb{W}}(\mathfrak{j})e^{i\phi_{\mathbb{W}}(\mathfrak{j})}, q_{\mathbb{W}}(\mathbf{g})e^{i\phi_{\mathbb{W}}(\mathbf{g})}\} \\
\mathbb{I}_{\mathbb{W}}(\mathfrak{j}-\mathbf{g}) &\geq \min\{\mathbb{I}_{\mathbb{W}}(\mathfrak{j}), \mathbb{I}_{\mathbb{W}}(\mathbf{g})\} . \\
\mathbb{F}_{\mathbb{W}}(\mathfrak{j}-\mathbf{g}) &= r_{\mathbb{W}}(\mathfrak{j}-\mathbf{g})e^{i\omega_{\mathbb{W}}(\mathfrak{j}-\mathbf{g})} \leq \max\{r_{\mathbb{W}}(\mathfrak{j}), r_{\mathbb{W}}(\mathbf{g})\}\ e^{i\max\{\omega_{\mathbb{W}}(\mathfrak{j}),\omega_{\mathbb{W}}(\mathbf{g})\}} \\
&= \max\{r_{\mathbb{W}}(\mathfrak{j})e^{i\omega_{\mathbb{W}}(\mathfrak{j})}, r_{\mathbb{W}}(\mathbf{g})e^{i\omega_{\mathbb{W}}(\mathbf{g})}\} \\
\mathbb{F}_{\mathbb{W}}(\mathfrak{j}-\mathbf{g}) &\leq \max\{\mathbb{F}_{\mathbb{W}}(\mathfrak{j}), \mathbb{F}_{\mathbb{W}}(\mathbf{g})\}
\end{aligned}$$

$$\begin{aligned}
\mathbb{T}_{\mathbb{W}}(\mathfrak{j}\mathbf{g}) &= p_{\mathbb{W}}(\mathfrak{j}\mathbf{g})e^{i\theta_{\mathbb{W}}(\mathfrak{j}\mathbf{g})} \geq \min\{p_{\mathbb{W}}(\mathfrak{j}), p_{\mathbb{W}}(\mathbf{g})\}\ e^{i\min\{\theta_{\mathbb{W}}(\mathfrak{j}),\theta_{\mathbb{W}}(\mathbf{g})\}} \\
&= \min\{p_{\mathbb{W}}(\mathfrak{j})e^{i\theta_{\mathbb{W}}(\mathfrak{j})}, p_{\mathbb{W}}(\mathbf{g})e^{i\theta_{\mathbb{W}}(\mathbf{g})}\} \\
\mathbb{T}_{\mathbb{W}}(\mathfrak{j}\mathbf{g}) &\geq \min\{\mathbb{T}_{\mathbb{W}}(\mathfrak{j}), \mathbb{T}_{\mathbb{W}}(\mathbf{g})\} .
\end{aligned}$$

$$\begin{aligned}
\mathbb{I}_{\mathbb{W}}(\mathfrak{j}\mathbf{g}) &= q_{\mathbb{W}}(\mathfrak{j}\mathbf{g})e^{i\phi_{\mathbb{W}}(\mathfrak{j}\mathbf{g})} \geq \min\{q_{\mathbb{W}}(\mathfrak{j}), q_{\mathbb{W}}(\mathbf{g})\}\ e^{i\min\{\phi_{\mathbb{W}}(\mathfrak{j}),\phi_{\mathbb{W}}(\mathbf{g})\}} \\
&= \min\{q_{\mathbb{W}}(\mathfrak{j})e^{i\phi_{\mathbb{W}}(\mathfrak{j})}, q_{\mathbb{W}}(\mathbf{g})e^{i\phi_{\mathbb{W}}(\mathbf{g})}\} \\
\mathbb{I}_{\mathbb{W}}(\mathfrak{j}\mathbf{g}) &\geq \min\{\mathbb{I}_{\mathbb{W}}(\mathfrak{j}), \mathbb{I}_{\mathbb{W}}(\mathbf{g})\} .
\end{aligned}$$

$$\begin{aligned}
\mathbb{F}_{\mathbb{W}}(\mathfrak{j}\mathbf{g}) &= r_{\mathbb{W}}(\mathfrak{j}\mathbf{g})e^{ir_{\mathbb{W}}(\mathfrak{j}\mathbf{g})} \leq \max\{r_{\mathbb{W}}(\mathfrak{j}), r_{\mathbb{W}}(\mathbf{g})\}\ e^{i\max\{\omega_{\mathbb{W}}(\mathfrak{j}),\omega_{\mathbb{W}}(\mathbf{g})\}} \\
&= \max\{r_{\mathbb{W}}(\mathfrak{j})e^{i\omega_{\mathbb{W}}(\mathfrak{j})}, r_{\mathbb{W}}(\mathbf{g})e^{i\omega_{\mathbb{W}}(\mathbf{g})}\} \\
\mathbb{F}_{\mathbb{W}}(\mathfrak{j}\mathbf{g}) &\leq \max\{\mathbb{F}_{\mathbb{W}}(\mathfrak{j}), \mathbb{F}_{\mathbb{W}}(\mathbf{g})\} .
\end{aligned}$$

Further, let $f \in H$ be any element. Let $p_{\mathbb{W}}(\mathfrak{j}) = \alpha$, $\theta_{\mathbb{W}}(\mathfrak{j}) = \hat{\alpha}$, $q_{\mathbb{W}}(\mathfrak{j}) = \beta$, $\phi_{\mathbb{W}}(\mathfrak{j}) = \hat{\beta}$, $r_{\mathbb{W}}(\mathfrak{j}) = \gamma$, and $\omega_{\mathbb{W}}(\mathfrak{j}) = \hat{\gamma}$. Then, $p_{\mathbb{W}}(\mathfrak{j}) \geq \alpha$, $\theta_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\alpha}$, $q_{\mathbb{W}}(\mathfrak{j}) \geq \beta$, $\phi_{\mathbb{W}}(\mathfrak{j}) \geq \hat{\beta}$, and $r_{\mathbb{W}}(\mathfrak{j}) \leq \hat{\gamma}$, $\omega_{\mathbb{W}}(\mathfrak{j}) \leq \hat{\gamma}$ is true. Implies that $\mathfrak{j} \in W^{(\alpha,\ \beta,\ \gamma)}_{(\hat{\alpha},\ \hat{\beta},\ \hat{\gamma})}$.

4. Properties of the Direct Product of Complex Neutrosophic Subrings

In this part, we describe the direct product of $CNSR$s. We use the abstraction of $CNSs$ to explore the fundamental properties the direct product of $CNSR$.

Definition 11. *Assume that $\mathbb{W}$ and $\mathbb{X}$ be any two π-$\mathbb{NS}$ of sets K_1 and K_2, consequently. The cartesian product of π-NS $\mathbb{W}$ and $\mathbb{X}$ is expressed as* $(\mathbb{W}_\pi \times \mathbb{X}_\pi)(\mathfrak{f},\ \mathfrak{g}) = \{< (\mathfrak{f},\ \mathfrak{g}),\ \mathbb{T}_{\mathbb{W}_\pi\times\mathbb{X}_\pi}(\mathfrak{f},\ \mathfrak{g}),\ \mathbb{I}_{\mathbb{W}_\pi\times\mathbb{X}_\pi}(\mathfrak{f},\ \mathfrak{g}), \mathbb{F}_{\mathbb{W}_\pi\times\mathbb{X}_\pi}(\mathfrak{f},\mathfrak{g}) >\}, \forall\ \mathfrak{f} \in K_1,\ \mathfrak{g} \in K_2$.

Remark 2. *Let $\mathbb{W}$ and $\mathbb{X}$ be two π-NSRs of K_1 and K_2, respectively. Then $\mathbb{W}_\pi \times \mathbb{X}_\pi$ is π-$CNSR$ of $K_1 \times K_2$.*

Remark 3. *A π-$CNSR$ $\mathbb{W}_\pi \times \mathbb{X}_\pi$ of ring $K_1 \times K_2$ is a π-$CNSR$ of $K_1 \times K_2$ if and only if $\mathbb{W} \times \mathbb{X}$ is $CNSR$ of $K_1 \times K_2$*

Definition 12. *Let $\mathbb{W}$ and $\mathbb{X}$ be two CNSs of set P. The cartesian product of CNSSs $\mathbb{W}$ and $\mathbb{X}$ is expressed by a function*

$$\begin{aligned}
\mathbb{W}\times\mathbb{X} &= \{<(\vartheta,\jmath),\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath),\mathbb{I}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath),\mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)>\},\\
\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath) &= p_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)e^{i\theta_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)}=\min\{p_{\mathbb{W}}(\vartheta),p_{\mathbb{X}}(\jmath)\}e^{i\min\{\theta_{\mathbb{W}}(\vartheta),\theta_{\mathbb{X}}(\jmath)\}},\\
\mathbb{I}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath) &= q_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)e^{i\varphi_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)}=\min\{q_{\mathbb{W}}(\vartheta),q_{\mathbb{X}}(\jmath)\}e^{i\min\{\phi_{\mathbb{W}}(\vartheta),\phi_{\mathbb{X}}(\jmath)\}},\\
\mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath) &= r_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)e^{ir_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)}=\max\{r_{\mathbb{W}}(\vartheta),r_{\mathbb{X}}(\jmath)\}e^{i\max\{\omega_{\mathbb{W}}(\vartheta),\omega_{\mathbb{X}}(\jmath)\}}.
\end{aligned}$$

In this paper we shall take
$\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)=p_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)e^{i\theta_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)}$, $\mathbb{I}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)=q_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)e^{i\phi_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)}$ *and* $\mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)=r_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)e^{i\omega_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath)}$ *for the level of truth, level of neutral and level of falsehood of* $\mathbb{W}\times\mathbb{X}$.

The upcoming theorem explain that the cartesian product of two $CNSR$s is $CNSR$.

Theorem 6. *Let $\mathbb{W}$ and $\mathbb{X}$ be two CNSRs of R_1 and R_2, consequently. Then $\mathbb{W}\times\mathbb{X}$ is complex neutrosophic subrings of $R_1\times R_2$.*

Proof. Let $\vartheta,k\in R_1$ and $\jmath,l\in R_2$ be an elements. Then $(\vartheta,\jmath)$, $(k,l)\in R_1\times R_2$. Consider

$$\begin{aligned}
\mathbb{T}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)-(k,l)) &= \mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\vartheta-k,\jmath-l)=p_{\mathbb{W}\times\mathbb{X}}(\vartheta-k,\jmath-l)e^{i\theta_{\mathbb{W}\times\mathbb{X}}(x-k,y-l)}\\
&= \min\{p_{\mathbb{W}}(\vartheta-k),p_{\mathbb{X}}(\jmath-l)\}\ e^{i\min\{\theta_{\mathbb{W}}(\vartheta-k),\theta_{\mathbb{X}}(\jmath-l)\}}\\
&= \min\{p_{\mathbb{W}}(\vartheta-k)e^{i\theta_{\mathbb{W}}(\vartheta-k)},p_{\mathbb{X}}(\jmath-l)e^{i\theta_{\mathbb{W}}(\jmath-l)}\}\\
&= \min\{\mathbb{T}_{\mathbb{W}}(\vartheta-k),\mathbb{T}_{\mathbb{X}}(\jmath-l)\}\\
&\geq \min\{\min\{\mathbb{T}_{\mathbb{W}}(\vartheta),\mathbb{T}_{\mathbb{W}}(k)\}\ ,\min\{\mathbb{T}_{\mathbb{X}}(\jmath),\mathbb{T}_{\mathbb{X}}(l)\}\ \}\\
&= \min\{\min\{\mathbb{T}_{\mathbb{W}}(\vartheta),\mathbb{T}_{\mathbb{X}}(\jmath)\}\ ,\min\{\mathbb{T}_{\mathbb{W}}(k),\mathbb{T}_{\mathbb{X}}(l)\}\ \}\\
&\geq \min\{\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath),\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(k,l)\}\\
\mathbb{T}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)-(k,l)) &\geq \min\{\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath),\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(k,l)\}\ .
\end{aligned}$$

$$\begin{aligned}
\mathbb{T}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)(k,l)) &= \mathbb{T}_{\mathbb{W}\times X}((\vartheta,\jmath)(\vartheta,\jmath))\mathbb{T}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)(k,l))=\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\vartheta k,\jmath l)\\
&= p_{\mathbb{W}\times\mathbb{X}}(\vartheta k,\jmath l)e^{i\theta_{\mathbb{W}\times\mathbb{X}}(\vartheta k,\jmath l)}=\min\{p_{\mathbb{W}}(\vartheta k),p_{\mathbb{X}}(\jmath l)\}\ e^{i\min\{\theta_{\mathbb{W}}(\vartheta k),\theta_{\mathbb{X}}(\jmath l)\}}\\
&= \min\{p_{\mathbb{W}}(\vartheta k)e^{i\theta_{\mathbb{W}}(\vartheta k)},p_{\mathbb{X}}(\jmath l)e^{i\theta_{W}(\jmath l)}\}\quad=\min\{\mathbb{T}_{\mathbb{W}}(\vartheta k),\mathbb{T}_{\mathbb{X}}(\jmath l)\}\\
&\geq \min\{\min\{\mathbb{T}_{\mathbb{W}}(\vartheta),\mathbb{T}_{\mathbb{W}}(k)\}\ ,\min\{\mathbb{T}_{\mathbb{X}}(\jmath),\mathbb{T}_{\mathbb{X}}(l)\}\ \}\\
&= \min\{\min\{\mathbb{T}_{\mathbb{W}}(\vartheta),\mathbb{T}_{\mathbb{X}}(\jmath)\}\ ,\min\{\mathbb{T}_{\mathbb{W}}(k),\mathbb{T}_{\mathbb{X}}(l)\}\ \}\\
&\geq \min\{\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath),\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(k,l)\}\\
\mathbb{T}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)(k,l)) &\geq \min\{\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath),\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(k,l)\}\ .
\end{aligned}$$

Consider

$$\begin{aligned}
\mathbb{I}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)-(k,l)) &= \chi_{\mathbb{W}\times X}((\vartheta,\jmath)(\vartheta,\jmath))\mathbb{I}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)-(k,l))=\mathbb{I}_{\mathbb{W}\times\mathbb{X}}(\vartheta-k,\jmath-l)\\
&= q_{\mathbb{W}\times\mathbb{X}}(\vartheta-k,\jmath-l)e^{i\phi_{\mathbb{W}\times\mathbb{X}}(\vartheta-k,\jmath-l)}
\end{aligned}$$

$$
\begin{aligned}
&= \min\{q_{\mathbb{W}}(\vartheta-k), q_{\mathbb{X}}(\jmath-l)\}\ e^{i\min\{\phi_{\mathbb{W}}(\vartheta-k),\phi_{\mathbb{X}}(g-l)\}} \\
&= \min\{q_{\mathbb{W}}(\vartheta-k)e^{i\phi_{\mathbb{W}}(\vartheta-k)}, q_{\mathbb{X}}(\jmath-l)e^{i\phi_{W}(\jmath-l)}\} \\
&= \min\{\mathbb{I}_{\mathbb{W}}(\vartheta-k), \mathbb{I}_{\mathbb{X}}(\jmath-l)\} \\
&\geq \min\{\min\{\mathbb{I}_{\mathbb{W}}(\vartheta), \mathbb{I}_{\mathbb{W}}(k)\}\ , \min\{\mathbb{I}_{\mathbb{X}}(\jmath), \mathbb{I}_{\mathbb{X}}(l)\}\ \} \\
&= \min\{\min\{\mathbb{I}_{\mathbb{W}}(\vartheta), \mathbb{I}_{\mathbb{X}}(\jmath)\}\ , \min\{\mathbb{I}_{\mathbb{W}}(k), \mathbb{I}_{\mathbb{X}}(l)\}\ \} \\
&\geq \min\{\mathbb{I}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath), \mathbb{I}_{\mathbb{W}\times\mathbb{X}}(k,l)\} \\
\mathbb{I}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)-(k,l)) &\geq \min\{\mathbb{I}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath), \mathbb{I}_{\mathbb{W}\times\mathbb{X}}(k,l)\}\ .
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{I}_{\mathbb{W}\times\mathbb{X}}\big((\vartheta,\jmath)(k,l)\big) &= \mathbb{I}_{\mathbb{W}\times\mathbb{X}}(\vartheta k,\jmath l) = q_{\mathbb{W}\times\mathbb{X}}(\vartheta k,\jmath l)e^{i\phi_{\mathbb{W}\times\mathbb{X}}(\vartheta k,\jmath l)} \\
&= \min\big\{q_{\mathbb{W}}(\vartheta k),\ q_{\mathbb{X}}(\jmath l)\big\}e^{i\min\{\phi_{\mathbb{W}}(\vartheta k),\ \phi_{\mathbb{X}}(\jmath l)\}} \\
&= \min\big\{q_{\mathbb{W}}(\vartheta k)e^{i\phi_{\mathbb{W}}(\vartheta k)},\ q_{\mathbb{X}}(\jmath l)e^{i\phi_{\mathbb{X}}(\jmath l)}\big\} = \min\big\{\mathbb{I}_{\mathbb{W}}(\vartheta k),\ \mathbb{I}_{\mathbb{X}}(\jmath l)\big\} \\
&\geq \min\Big\{\ \min\{\mathbb{I}_{\mathbb{W}}(\vartheta),\ \mathbb{I}_{\mathbb{W}}(k)\}, \min\{\mathbb{I}_{\mathbb{X}}(\jmath),\ \mathbb{I}_{\mathbb{X}}(l)\}\Big\} \\
&\geq \min\big\{\mathbb{I}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath),\ \mathbb{I}_{\mathbb{W}\times\mathbb{X}}(k,l)\big\}.
\end{aligned}
$$

Assume that,

$$
\begin{aligned}
\mathbb{F}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)-(k,l)) &= \mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\vartheta-k,\jmath-l) = r_{\mathbb{W}\times\mathbb{X}}(\vartheta-k,\jmath-l)e^{i\omega_{\mathbb{W}\times\mathbb{X}}(\vartheta-k,\jmath-l)} \\
&= \max\{r_{\mathbb{W}}(\vartheta-k), r_{\mathbb{X}}(\jmath-l)\}\ e^{i\max\{\omega_{\mathbb{W}}(\vartheta-k),\omega_{\mathbb{X}}(\jmath-l)\}} \\
&= \max\{r_{\mathbb{W}}(\vartheta-k)e^{i\omega_{\mathbb{W}}(\vartheta-k)}, r_{\mathbb{X}}(\jmath-l)e^{i\omega_{\mathbb{W}}(\jmath-l)}\} \\
&= \max\{\mathbb{F}_{\mathbb{W}}(\vartheta-k), \mathbb{F}_{\mathbb{X}}(\jmath-l)\} \\
&\leq \max\{\max\{\mathbb{F}_{\mathbb{W}}(\vartheta), \mathbb{F}_{\mathbb{W}}(k)\}\ , \max\{\mathbb{F}_{\mathbb{X}}(\jmath), \mathbb{F}_{\mathbb{X}}(l)\}\ \} \\
&= \max\{\max\{\mathbb{F}_{\mathbb{W}}(\vartheta), \mathbb{F}_{\mathbb{X}}(\jmath)\}\ , \max\{\mathbb{F}_{\mathbb{W}}(k), \mathbb{F}_{\mathbb{X}}(l)\}\ \} \\
&\leq \max\{\mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath), \mathbb{F}_{\mathbb{W}\times\mathbb{X}}(k,l)\}
\end{aligned}
$$

$$
\mathbb{F}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)-(k,l)) \leq \max\{\mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath), \mathbb{F}_{\mathbb{W}\times\mathbb{X}}(k,l)\}\ .
$$

Now, we take,

$$
\begin{aligned}
\mathbb{F}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)(k,l)) &= \mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\vartheta k,\jmath l) = r_{\mathbb{W}\times\mathbb{X}}(\vartheta k,\jmath l)e^{i\omega_{\mathbb{W}\times\mathbb{X}}(\vartheta k,\jmath l)} = \max\{r_{\mathbb{W}}(\vartheta k), r_{\mathbb{X}}(\jmath l)\} \\
&* \quad e^{i\max\{\omega_{\mathbb{W}}(\vartheta k),\omega_{\mathbb{X}}(\jmath l)\}} \\
&= \max\{r_{\mathbb{W}}(\vartheta k)e^{i\omega_{\mathbb{W}}(\vartheta k)}, r_{\mathbb{X}}(\jmath l)e^{i\omega_{\mathbb{W}}(\jmath l)}\} \quad = \max\{\mathbb{F}_{\mathbb{W}}(\vartheta k), \mathbb{F}_{\mathbb{X}}(\jmath l)\} \\
&\leq \max\{\max\{\mathbb{F}_{\mathbb{W}}(\vartheta), \mathbb{F}_{\mathbb{W}}(k)\}\ , \max\{\mathbb{F}_{\mathbb{X}}(\jmath), \mathbb{F}_{\mathbb{X}}(l)\}\ \} \\
&= \max\{\max\{\mathbb{F}_{\mathbb{W}}(\vartheta), \mathbb{F}_{\mathbb{X}}(\jmath)\}\ , \max\{\mathbb{F}_{\mathbb{W}}(k), \mathbb{F}_{\mathbb{X}}(l)\}\ \} \\
&\leq \max\{\mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath), \mathbb{F}_{\mathbb{W}\times\mathbb{X}}(k,l)\} \\
\mathbb{F}_{\mathbb{W}\times\mathbb{X}}((\vartheta,\jmath)(k,l)) &\leq \max\{\mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\vartheta,\jmath), \mathbb{F}_{\mathbb{W}\times\mathbb{X}}(k,l)\}.
\end{aligned}
$$

Hence the desired result is obtained.

Corollary 1. *Let* $\mathbb{W}_1, \ \mathbb{W}_2, \ldots, \mathbb{W}_n$ *be CNSRs of* $K_1, \ K_2, \ldots, K_n$*, respectively. Then* $\mathbb{W}_1 \times \mathbb{W}_2 \times, \ldots, \times \mathbb{W}_n$ *is* $CNSR$ *of* $K_1 \times K_2 \times \ldots \ \times K_n$.

Remark 4. *Let* $\mathbb{W}$ *and* $\mathbb{X}$ *be two CNSRs of* K_1 *and* K_2*, consequently and* $\mathbb{W}_1 \times \mathbb{W}_2$ *be* $CNSR$ *of* $K_1 \times K_2$*. Then it is not compulsory both* $\mathbb{W}_1$ *and* $\mathbb{W}_2$ *should be CNSRs of* K_1 *and* K_2*, consequently.*

Remark 5. *Let* $\mathbb{W} \times \mathbb{X}$ *be* $CNSR$ *of ring* $K_1 \times K_2$*. Then* $p_{\mathbb{W}\times\mathbb{X}}(0, 0^{'}) \geq p_{\mathbb{W}\times\mathbb{X}}(\mathfrak{f}, \mathfrak{g})$, $\theta_{\mathbb{W}\times\mathbb{X}}(0, 0^{'}) \geq \theta_{\mathbb{W}\times\mathbb{X}}(\mathfrak{f}, \mathfrak{g})$, $q_{\mathbb{W}\times\mathbb{X}}(0, 0^{'}) \geq q_{\mathbb{W}\times\mathbb{X}}(\mathfrak{f}, \mathfrak{g})$, $\phi_{\mathbb{W}\times\mathbb{X}}(0, 0^{'}) \geq \phi_{\mathbb{W}\times\mathbb{X}}(\mathfrak{f}, \mathfrak{g})$, $r_{\mathbb{W}\times\mathbb{X}}(0, 0^{'}) \leq r_{\mathbb{W}\times\mathbb{X}}(\mathfrak{f}, \mathfrak{g})$, *and* $\omega_{\mathbb{W}\times\mathbb{X}}(0, 0^{'}) \leq \omega_{\mathbb{W}\times\mathbb{X}}(\mathfrak{f}, \mathfrak{g})$. $\forall \mathfrak{f} \in K_1, \ \mathfrak{g} \in K_2$*. Here* 0 *and* $0^{'}$ *are neutral elements of* K_1 *and* K_2*, respectively.*

Theorem 7. *Let* $\mathbb{W}$ *and* $\mathbb{X}$ *be two CNS of rings* K_1 *and* K_2*. If* $\mathbb{W} \times \mathbb{X}$ *is a complex neutrosophic subring of* $K_1 \times K_2$*, then one of the following statements must be satisfied.*

(i) $p_{\mathbb{W}}(0) \geq p_{\mathbb{X}}(\mathfrak{g})$, $\theta_{\mathbb{W}}(0) \geq \theta_{\mathbb{X}}(\mathfrak{g})$, $q_{\mathbb{W}}(0) \geq q_{\mathbb{X}}(\mathfrak{g})$, $\phi_{\mathbb{W}}(0) \geq \phi_{\mathbb{X}}(\mathfrak{g})$ *and* $r_{\mathbb{W}}(0) \leq r_{\mathbb{X}}(\mathfrak{g})$, $\omega_{\mathbb{W}}(0) \leq \omega_{\mathbb{X}}(\mathfrak{g})$, $\forall \mathfrak{g} \in K_2$.

(ii) $p_{\mathbb{X}}(0^{'}) \geq p_{\mathbb{W}}(\mathfrak{f})$, $\theta_{\mathbb{X}}(0^{'}) \geq \theta_{\mathbb{W}}(\mathfrak{f})$, $q_{\mathbb{X}}(0^{'}) \geq q_{\mathbb{W}}(\mathfrak{f})$, $\phi_{\mathbb{X}}(0^{'}) \geq \phi_{\mathbb{W}}(\mathfrak{f})$, $r_{\mathbb{X}}(0^{'}) \leq r_{\mathbb{W}}(\mathfrak{f})$, $\omega_{\mathbb{X}}(0^{'}) \leq \omega_{\mathbb{W}}(\mathfrak{f})$, $\forall \mathfrak{f} \in K_1$.

Here 0 *and* $0^{'}$ *are neutral elements of* K_1 and K_2.

Proof. Let $\mathbb{W} \times \mathbb{X}$ be a $CNSR$ of $K_1 \times K_2$. On contrary, assume that the statements [1] and [2] do not hold. Then there exist $\mathfrak{f} \in K_1$ and $\mathfrak{g} \in K_2$ such that

(i) $p_{\mathbb{W}}(0) \leq p_{\mathbb{X}}(\mathfrak{g})$, $\theta_{\mathbb{W}}(0) \leq \theta_{\mathbb{X}}(\mathfrak{g})$, $q_{\mathbb{W}}(0) \leq q_{\mathbb{X}}(\mathfrak{g})$, $\phi_{\mathbb{W}}(0) \leq \phi_{\mathbb{X}}(\mathfrak{g})$ and $r_{\mathbb{W}}(0) \geq r_{\mathbb{X}}(\mathfrak{g})$, $\omega_{\mathbb{W}}(0) \geq \omega_{\mathbb{X}}(\mathfrak{g})$, $\forall \mathfrak{g} \in K_2$.

(ii) $p_{\mathbb{X}}(0^{'}) \leq p_{\mathbb{W}}(\mathfrak{f})$, $\theta_{\mathbb{X}}(0^{'}) \leq \theta_{\mathbb{W}}(\mathfrak{f})$, $q_{\mathbb{X}}(0^{'}) \leq q_{\mathbb{W}}(\mathfrak{f})$, $\phi_{\mathbb{X}}(0^{'}) \leq \phi_{\mathbb{W}}(\mathfrak{f})$, $r_{\mathbb{X}}(0^{'}) \geq r_{\mathbb{W}}(\mathfrak{f})$, $\omega_{\mathbb{X}}(0^{'}) \geq \omega_{\mathbb{W}}(\mathfrak{f})$, $\forall \mathfrak{f} \in K_1$.

Consider, $\mathbb{T}_{\mathbb{W}\times\mathbb{X}}(\mathfrak{f}, \mathfrak{g}) = \min\{p_{\mathbb{W}}(\mathfrak{f}), p_{\mathbb{X}}(\mathfrak{g})\} \ e^{i\min\{\theta_{\mathbb{W}}(\mathfrak{f}), \theta_{\mathbb{X}}(\mathfrak{g})\}} \geq \min\{p_{\mathbb{W}}(0), p_{\mathbb{X}}(0^{'})\}$ $e^{imin\{\theta_{\mathbb{W}}(0), \theta_{\mathbb{X}}(0^{'})\}} = \mathbb{T}_{\mathbb{W}\times\mathbb{X}}(0, 0^{'})$. $\mathbb{I}_{\mathbb{W}\times\mathbb{X}}(\mathfrak{f}, \mathfrak{g}) = \min\{q_{\mathbb{W}}(\mathfrak{f}), q_{\mathbb{X}}(\mathfrak{g})\} \ e^{i\min\{\phi_{\mathbb{W}}(\mathfrak{f}), \phi_{\mathbb{X}}(\mathfrak{g})\}}$ $\geq \min\{q_{\mathbb{W}}(0), q_{\mathbb{X}}(0^{'})\} e^{imin\{\phi_{\mathbb{W}}(0), \phi_{\mathbb{X}}(0^{'})\}} = \mathbb{I}_{\mathbb{W}\times\mathbb{X}}(0, 0^{'})$ and $\mathbb{F}_{\mathbb{W}\times\mathbb{X}}(\mathfrak{f}, \mathfrak{g}) = \max\{r_W(\mathfrak{f}), r_{\mathbb{X}}(\mathfrak{g})\}$ $e^{i\max\{\omega_W(\mathfrak{f}), \omega_{\mathbb{X}}(\mathfrak{g})\}} \leq \max\{r_{\mathbb{W}}(0), r_{\mathbb{X}}(0^{'})\} e^{imax\{\omega_{\mathbb{W}}(0), \omega_{\mathbb{X}}(0^{'})\}} = \mathbb{F}_{\mathbb{W}\times\mathbb{X}}(0, 0^{'})$. But $\mathbb{W} \times \mathbb{X}$ is $CNSR$. Hence, it is proved that at least one statements must be satisfied.

(i) $p_{\mathbb{W}}(0) \geq p_{\mathbb{X}}(\mathfrak{g})$, $\theta_{\mathbb{W}}(0) \geq \theta_{\mathbb{X}}(\mathfrak{g})$, $q_{\mathbb{W}}(0) \geq q_{\mathbb{X}}(\mathfrak{g})$, $\phi_{\mathbb{W}}(0) \geq \phi_{\mathbb{X}}(\mathfrak{g})$ and $r_{\mathbb{W}}(0) \leq r_{\mathbb{X}}(\mathfrak{g})$, $\omega_{\mathbb{W}}(0) \leq \omega_{\mathbb{X}}(\mathfrak{g})$, $\forall \mathfrak{g} \in K_2$.

(ii) $p_{\mathbb{X}}(0^{'}) \geq p_{\mathbb{W}}(\mathfrak{f})$, $\theta_{\mathbb{X}}(0^{'}) \geq \theta_{\mathbb{W}}(\mathfrak{f})$, $q_{\mathbb{X}}(0^{'}) \geq q_{\mathbb{W}}(\mathfrak{f})$, $\phi_{\mathbb{X}}(0^{'}) \geq \phi_{\mathbb{W}}(\mathfrak{f})$, $r_{\mathbb{X}}(0^{'}) \leq r_{\mathbb{W}}(\mathfrak{f})$, $\omega_{\mathbb{X}}(0^{'}) \leq \omega_{\mathbb{W}}(\mathfrak{f})$, $\forall \mathfrak{f} \in K_1$.

Theorem 8. *Let* $\mathbb{W}$ *and* $\mathbb{X}$ *be CNSs of* K_1, K_2 *and* $p_{\mathbb{X}}(0^{'}) \geq p_{\mathbb{W}}(\vartheta)$, $\theta_{\mathbb{X}}(0^{'}) \geq \theta_{\mathbb{W}}(\vartheta)$, $q_{\mathbb{X}}(0^{'}) \geq q_{\mathbb{W}}(\vartheta)$, $\phi_{\mathbb{X}}(0^{'}) \geq \phi_{\mathbb{W}}(\vartheta)$, $r_{\mathbb{X}}(0^{'}) \leq r_{\mathbb{W}}(\vartheta)$, $\omega_{\mathbb{X}}(0^{'}) \leq \omega_{\mathbb{W}}(\vartheta)$, $\forall \ \vartheta \in K_1$, $0^{'}$ *is identity of* K_2*. If* $\mathbb{W} \times \mathbb{X}$ *is* $CNSR$ *of* $K_1 \times K_2$*, then* $\mathbb{W}$ *is* $CNSR$ *of* K_1.

Proof. Let $(\vartheta, 0^{'}), (a, 0^{'})$ be elements of $K_1 \times K_2$. By given condition $p_{\mathbb{X}}(0^{'}) \geq p_{\mathbb{W}}(\vartheta)$ and $\theta_{\mathbb{X}}(0^{'}) \geq \theta_{\mathbb{W}}(\vartheta)$, for all $\vartheta, a \in K_1$ and $0^{'} \in K_2$. Consider,

$$\begin{aligned}
\mathbb{T}_{\mathbb{W}}(\vartheta - a) &= p_{\mathbb{W}}(\vartheta - a)e^{i\theta_{\mathbb{W}}(\vartheta - a)} = \min\{p_{\mathbb{W}}(\vartheta - a)e^{i\theta_{\mathbb{W}}(\vartheta - a)}, p_{\mathbb{X}}(0^{'} - 0^{'})e^{i\theta_{\mathbb{X}}(0^{'} - 0^{'})}\} \\
&= p_{\mathbb{W}\times\mathbb{X}}((\vartheta, 0^{'})(\vartheta, 0^{'}))e^{i\theta_{\mathbb{W}\times\mathbb{X}}((\vartheta,0^{'})(\vartheta,0^{'}))} \\
&\geq \min\{p_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'}), p_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'})\}\ e^{i\min\{\theta_{\mathbb{W}\times\mathbb{X}}(a,0^{'}), \theta_{\mathbb{W}\times\mathbb{X}}(\vartheta,0^{'})\}} \\
&= \min\{\min\{p_{\mathbb{W}}(\vartheta), p_{\mathbb{X}}(0^{'})\}\ , \min\{p_{\mathbb{W}}(a), p_{\mathbb{X}}(0^{'})\}\ \} \\
&* \quad e^{i\min\{\min\{\theta_{\mathbb{X}}(0^{'})\},\ \min\{\theta, \min\{\theta_{\mathbb{W}}(a), \theta_{\mathbb{X}}(0^{'})\}\ \}} \\
&= \min\{\mathbb{T}_{\mathbb{W}}(\vartheta), \mathbb{T}_{\mathbb{W}}(a)\}.
\end{aligned}$$

Thus, $\mathbb{T}_{\mathbb{W}}(\vartheta - a) \geq \min\{\mathbb{T}_{\mathbb{W}}(\vartheta), \mathbb{T}_{\mathbb{W}}(a)\}$.

$$\begin{aligned}
\mathbb{T}_{\mathbb{W}}(\vartheta a) &= p_{\mathbb{W}}(\vartheta a)e^{i\theta_{\mathbb{W}}(\vartheta a)} = \min\{p_{\mathbb{W}}(\vartheta a)e^{i\theta_{\mathbb{W}}(\vartheta a)}, p_{\mathbb{X}}(0^{'}0^{'})e^{i\theta_{\mathbb{X}}(0^{'}0^{'})}\} \\
&= p_{\mathbb{W}\times\mathbb{X}}((\vartheta, 0^{'})(\vartheta, 0^{'}))e^{i\theta_{\mathbb{W}\times\mathbb{X}}((\vartheta,0^{'})(\vartheta,0^{'}))} \\
&\geq \min\{p_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'}), p_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'})\}\ e^{i\min\{\theta_{\mathbb{W}\times\mathbb{X}}(a,0^{'}), \theta_{\mathbb{W}\times\mathbb{X}}(\vartheta,0^{'})\}} \\
&= \min\{\min\{p_{\mathbb{W}}(\vartheta), p_{X}(0^{'})\}\ , \min\{p_{\mathbb{W}}(a), p_{\mathbb{X}}(0^{'})\}\ \} \\
&* \quad e^{i\min\{\min\{\theta_{\mathbb{X}}(0^{'})\}\ , \min\{\theta, \min\{\theta_{\mathbb{W}}(a), \theta_{\mathbb{X}}(0^{'})\}\ \}} \\
&= \min\{\mathbb{T}_{\mathbb{W}}(\vartheta), \mathbb{T}_{\mathbb{W}}(a)\}.
\end{aligned}$$

Thus, $\mathbb{T}_{\mathbb{W}}(\vartheta a) \geq \min\{\mathbb{T}_{\mathbb{W}}(\vartheta), \mathbb{T}_{\mathbb{W}}(a)\}$. Consider,

$$\begin{aligned}
\mathbb{I}_{\mathbb{W}}(\vartheta - a) &= q_{\mathbb{W}}(\vartheta - a)e^{i\theta_{\mathbb{W}}(\vartheta - a)} \\
&= \min\{q_{\mathbb{W}}(\vartheta - a)e^{i\theta_{\mathbb{W}}(\vartheta - a)}, q_{\mathbb{X}}(0^{'} - 0^{'})e^{i\theta_{\mathbb{X}}(0^{'} - 0^{'})}\} \qquad (-6) \\
&= q_{\mathbb{W}\times\mathbb{X}}((\vartheta, 0^{'})(\vartheta, 0^{'}))e^{i\theta_{\mathbb{W}\times\mathbb{X}}((\vartheta,0^{'})(\vartheta,0^{'}))} \\
&\geq \min\{q_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'}), q_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'})\}\ e^{i\min\{\theta_{\mathbb{W}\times\mathbb{X}}(a,0^{'}), \theta_{\mathbb{W}\times\mathbb{X}}(\vartheta,0^{'})\}} \\
&= \min\{\min\{q_{W}(\vartheta), q_{\mathbb{X}}(0^{'})\}\ , \min\{q_{\mathbb{W}}(a), p_{\mathbb{X}}(0^{'})\}\ \} \\
&* \quad e^{i\min\{\min\{\theta_{\mathbb{X}}(0^{'})\}\ , \min\{\theta, \min\{\theta_{\mathbb{W}}(a), \theta_{\mathbb{X}}(0^{'})\}\ \}} \\
&= \min\{\mathbb{I}_{\mathbb{W}}(\vartheta), \mathbb{I}_{\mathbb{W}}(a)\}.
\end{aligned}$$

Thus, $\mathbb{I}_{\mathbb{W}}(\vartheta - a) \geq \min\{\mathbb{I}_{\mathbb{W}}(\vartheta), \mathbb{I}_{\mathbb{W}}(a)\}$.

$$\begin{aligned}
\mathbb{I}_{\mathbb{W}}(\vartheta a) &= q_{\mathbb{W}}(\vartheta a)e^{i\theta_{\mathbb{W}}(\vartheta a)} = \min\{q_{\mathbb{W}}(\vartheta a)e^{i\theta_{\mathbb{W}}(\vartheta a)}, q_{\mathbb{X}}(0^{'}0^{'})e^{i\theta_{\mathbb{X}}(0^{'}0^{'})}\} \\
&= q_{\mathbb{W}\times\mathbb{X}}((\vartheta, 0^{'})(\vartheta, 0^{'}))e^{i\theta_{\mathbb{W}\times\mathbb{X}}((\vartheta,0^{'})(\vartheta,0^{'}))} \\
&\geq \min\{q_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'}), q_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'})\}\ e^{i\min\{\theta_{\mathbb{W}\times\mathbb{X}}(a,0^{'}), \theta_{\mathbb{W}\times\mathbb{X}}(\vartheta,0^{'})\}} \\
&= \min\{\min\{q_{\mathbb{W}}(\vartheta), q_{\mathbb{X}}(0^{'})\}\ , \min\{q_{\mathbb{W}}(a), p_{\mathbb{X}}(0^{'})\}\ \} \\
&* \quad e^{i\min\{\min\{\theta_{\mathbb{X}}(0^{'})\}\ , \min\{\theta, \min\{\theta_{\mathbb{W}}(a), \theta_{\mathbb{X}}(0^{'})\}\ \}}
\end{aligned}$$

$$= \min\{\mathbb{I}_{\mathbb{W}}(\vartheta), \mathbb{I}_{\mathbb{W}}(a)\}.$$

Thus, $\mathbb{I}_{\mathbb{W}}(\vartheta a) \geq \min\{\mathbb{I}_{\mathbb{W}}(\vartheta), \mathbb{I}_{\mathbb{W}}(a)\}$.
Further,

$$\begin{aligned}
\mathbb{F}_{\mathbb{W}}(\vartheta - a) &= r_{\mathbb{W}}(\vartheta - a)e^{i\omega_{\mathbb{W}}(\vartheta - a)} \\
&= \{\max\{r_{\mathbb{W}}(\vartheta - a)e^{i\omega_{\mathbb{W}}(\vartheta - a)}, r_{\mathbb{X}}(0^{'} - 0^{'})e^{i\omega_{\mathbb{X}}(0^{'} - 0^{'})}\}\} \qquad (-17) \\
&= \{r_{\mathbb{W}\times\mathbb{X}}((\vartheta, 0^{'})(a, 0^{'}))\}e^{i\{\omega_{\mathbb{W}\times\mathbb{X}}((\vartheta,0^{'})(a,0^{'}))\}} \\
&\leq \max\{r_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'}), r_{\mathbb{W}\times\mathbb{X}}(a, 0^{'})\}\ e^{i\max\{\omega_{\mathbb{W}\times\mathbb{X}}(\vartheta,0^{'}),\omega_{\mathbb{W}\times\mathbb{X}}(a,0^{'})\}} \\
&= \max\{\max\{r_{W}(\vartheta), r_{\mathbb{X}}(0^{'})\}\ , \max\{r_{\mathbb{W}}(a), r_{\mathbb{X}}(0^{'})\}\ \} \\
&* \quad e^{i\max\{\max\{\omega_{\mathbb{X}}(0^{'})\},\ \max\{\omega,\max\{\omega_{\mathbb{W}}(a),\omega_{\mathbb{X}}(0^{'})\}\ \}} \\
&= \max\{\mathbb{F}_{\mathbb{W}}(\vartheta), \mathbb{F}_{\mathbb{W}}(a)\}.
\end{aligned}$$

Thus, $\mathbb{F}_{\mathbb{W}}(\vartheta - a) \leq \max\{\mathbb{F}_{\mathbb{W}}(\vartheta), \mathbb{F}_{\mathbb{W}}(a)\}$.
Further,

$$\begin{aligned}
\mathbb{F}_{\mathbb{W}}(\vartheta a) &= r_{\mathbb{W}}(\vartheta a)e^{i\omega_{\mathbb{W}}(\vartheta a)} = \{\max\{r_{\mathbb{W}}(\vartheta a)e^{i\omega_{\mathbb{W}}(\vartheta a)}, r_{\mathbb{X}}(0^{'}0^{'})e^{i\omega_{\mathbb{X}}(0^{'}0^{'})}\}\} \\
&= \{r_{\mathbb{W}\times\mathbb{X}}((\vartheta, 0^{'})(a, 0^{'}))\}e^{i\{\omega_{\mathbb{W}\times\mathbb{X}}((\vartheta,0^{'})(a,0^{'}))\}} \\
&\leq \max\{r_{\mathbb{W}\times\mathbb{X}}(\vartheta, 0^{'}), r_{\mathbb{W}\times\mathbb{X}}(a, 0^{'})\}\ e^{i\max\{\omega_{\mathbb{W}\times\mathbb{X}}(\vartheta,0^{'}),\omega_{\mathbb{W}\times\mathbb{X}}(a,0^{'})\}} \\
&= \max\{\max\{r_{\mathbb{W}}(\vartheta), r_{\mathbb{X}}(0^{'})\}\ , \max\{r_{\mathbb{W}}(a), r_{\mathbb{X}}(0^{'})\}\ \} \\
&* \quad e^{i\max\{\max\{\omega_{\mathbb{X}}(0^{'})\},\ \max\{\omega,\max\{\omega_{\mathbb{W}}(a),\omega_{\mathbb{X}}(0^{'})\}\ \}} \\
&= \max\{\mathbb{F}_{\mathbb{W}}(\vartheta), \mathbb{F}_{\mathbb{W}}(a)\}.
\end{aligned}$$

Thus, $\mathbb{F}_{\mathbb{W}}(\vartheta a) \leq \max\{\mathbb{F}_{\mathbb{W}}(\vartheta), \mathbb{F}_{\mathbb{W}}(a)\}$. Hence, we obtained the result.

Theorem 9. *Let $\mathbb{W}$ and $\mathbb{X}$ two $CNSSs$ of K_1 and K_2 such that $p_{\mathbb{W}}(0) \geq p_{\mathbb{X}}(\mathfrak{g})$, $q_{\mathbb{W}}(0) \geq q_{\mathbb{X}}(\mathfrak{g})$ and $r_{\mathbb{W}}(0) \geq r_{\mathbb{X}}(\mathfrak{g})$, $\forall \mathfrak{g} \in K_2$ and 0 is identity of K_1. If $\mathbb{W}\times\mathbb{X}$ is $CNSR$ of $K_1 \times K_2$, then $\mathbb{X}$ is a $CNSR$ of K_2.*

Proof. The proof is on similar lines as Theorem 4.6.

Corollary 2. *Let $\mathbb{W}$ and $\mathbb{X}$ be two $CNSSs$ of K_1 and K_2, respectively. If $\mathbb{W}\times\mathbb{X}$ is $CNSR$ of $K_1 \times K_2$, then $\mathbb{W}$ is a $CNSR$ of K_1 or $\mathbb{X}$ is a $CNSR$ of K_2.*

5. Conclusion

In this manuscript, we have discussed the complex neutrosophic subring, intersection, and level subset of complex neutrosophic subring. We have demonstrated that every complex neutrosophic subring generates two neutrosophic subrings and examined important aspects of this fact. We have shown that the level subset of the complex neutrosophic subring forms the subring of the ring and we have talked about some of the algebraic characteristics of the level subset. In future we intend to extend this approch to subfield, submodules and BCK/BCI algebra. Also we intend to introduce applications of these defined algebraic structures to practical and theoretical problems.

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