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Total Modern Roman Dominating Functions in Graphs

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Abstract. Let $G = (V(G), E(G))$ be any connected graph. A function $f : V(G) \to \{0, 1, 2, 3\}$ is a modern Roman dominating function of G if for each $v \in V(G)$ with $f(v) = 0$, there exist $u, w \in N_G(v)$ such that $f(u) = 2$ and $f(w) = 3$; and for each $v \in V(G)$ with $f(v) = 1$, there exists $u \in N_G(v)$ such that $f(u) = 2$ or $f(u) = 3$. In addition, if every subgraph induced by the set $\{v \in V(G) : f(v) > 0\}$ is isolated-free, then we say that f is a total modern Roman dominating function of G. The minimum weight $\omega_G^{tmR}(f) = \sum_{v \in V(G)} f(v)$ of a total modern Roman dominating function f of G is called the total modern Roman domination number, $\gamma_{tmR}(G)$, of G. In this paper, we initiate the study of total modern Roman domination. We characterize graphs with smaller total modern Roman domination number and obtain the $\gamma_{tmR}(G)$ of some special graphs. Moreover, we investigate and characterize the total modern Roman domination in the join and corona of graphs.

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1. Introduction

The concept of Roman domination was introduced by Cockayne et al. [1] in 2004, inspired by the strategies for defending the Roman Empire presented in the work of ReVelle and Rosing[2] and Stewart [3]. Since then, it has become an active research field in graph theory, with numerous studies exploring this concept (see [4],[5],[6],[7],[8],[9],[10],

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[11],[12],[13],[14]). A new model of graph domination is introduced in [5] based on the Roman domination and is called modern Roman domination.

In this paper, we introduce the concept of total modern Roman domination domination in graphs. It focuses on providing the total modern Roman domination number of some special graphs and some characterizations for the total modern Roman domination of the join and corona of graphs.

2. Terminology and Notation

The symbols $V(G)$ and $E(G)$ denote the *vertex set* and *edge set*, respectively, of a graph G. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ and $|E(G)|$ are the *order* and *size*, respectively, of G. All graph terminologies that are not introduced but are being used here are adapted from [15].

The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $S \subseteq V(G)$, the *open neighborhood* of S in G is the set $N_G(S) = \bigcup_{u \in S} N_G(u)$. The *closed neighborhood* of S in G is the set $N_G[S] = N_G(S) \cup S$. For $S \subseteq V(G)$ of a connected graph G, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = S \cup N_G(S)$. A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. Thus, S is a *dominating set* in G if and only if for each $v \in V(G) \setminus S$, there exists $u \in S$, such that $uv \in E(G)$. The minimum cardinality of a dominating set in G, denoted by $\gamma(G)$, is the *domination number* of G. A dominating set S of G with $|S| = \gamma(G)$ is called a γ - set of G. The authors always refer to [16] for the introduction and more comprehensive discussion of the development of the concept of domination in graphs.

For a positive integer k, a set $D \subseteq V(G)$ is called a *k-dominating set* if each $x \in V(G) \setminus D$ is adjacent to at least k vertices in D. The *k-domination number* $\gamma_k(G)$ is then defined to be the smallest cardinality of a k-dominating set of G.

For $k = 2$, we have D as 2-*dominating set* with 2-*domination number* denoted by $\gamma_2(G)$ [8].

A *Roman dominating function* (RDF) on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex $u \in V(G)$ with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The weight of an RDF is the value $\omega_G(f) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. An RDF with $\omega_G(f) = \gamma_R(G)$ is referred to as a γ_R-*function* [8].

A *modern Roman dominating function* (MRDF) on G is a function $f : V(G) \to \{0, 1, 2, 3\}$ if

(P1) for each $v \in V(G)$ with $f(v) = 0$, there exist $u, w \in N_G(v)$ such that $f(u) = 2$ and $f(w) = 3$; and

(P2) for each $v \in V(G)$ with $f(v) = 1$, there exists $u \in N_G(v)$ such that $f(u) = 2$ or $f(u) = 3$.

The weight of a modern Roman dominating function f of G is the sum $\omega_G^{mR}(f) = \sum_{v \in V(G)} f(v)$ and its minimum weight is called the *modern Roman domination number* $\gamma_{mR}(G)$ of G [5].

For a function $f : V(G) \to \{0, 1, 2, 3\}$ on a graph G, let (V_0, V_1, V_2, V_3) be the ordered partition induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. Then we can write $f = (V_0, V_1, V_2, V_3)$. The weight of f is defined by $\omega_G(f) = |V_1| + 2|V_2| + 3|V_3|$.

3. Known Results

We make use of the following known results from [7].

Proposition 1. *Let G be any graph with no isolated vertex. If $f = (V_0, V_1, V_2, V_3)$ a γ_{mR}-function of G, then the following holds:*

(i) *$V_0 = \varnothing$ if and only if $V_3 = \varnothing$ and V_2 is a γ-set of G. Moreover, $\gamma_{mR}(G) = |V(G)| + \gamma(G)$.*

(ii) *$V_1 = \varnothing$ if and only if $V_2 \cup V_3$ is a 2-dominating set of G. Moreover, if $V_1 = \varnothing$, $\langle V_2 \cup V_3 \rangle$ is connected and V_3 is a γ-set of G, then $\gamma_{mR}(G) \geq \gamma(G) + 2\gamma_2(G)$.*

Proposition 2. *Let G and H be any graphs and let $f \in (V_0, V_1, V_2, V_3)$ be a function on $V(G + H)$ with $V_2 \neq \varnothing$ and $V_3 \neq \varnothing$. Then $f \in MRDF(G + H)$ if and only if one of the following holds:*

(i) *$f|_G \in MRDF(G)$ and one of the following holds:*

- (a) *$|V_2 \cap V(G)| \geq 1$ and $|V_3 \cap V(G)| \geq 1$*
- (b) *$V_2 \cap V(G) = \varnothing$ and each of the following holds:*
  - (b1) *V_3 is a dominating set of G.*
  - (b2) *$V_2 \cap V(H)$ is a dominating set of $V_0 \cap V(H)$.*
- (c) *$V_3 \cap V(G) = \varnothing$ and each of the following holds:*
  - (c1) *V_2 is a dominating set of G.*
  - (c2) *$V_3 \cap V(H)$ is a dominating set of $V_0 \cap V(H)$.*

(ii) *$f|_H \in MRDF(H)$ and one of the following holds:*

- (a) *$|V_2 \cap V(H)| \geq 1$ and $|V_3 \cap V(H)| \geq 1$*
- (b) *$V_2 \cap V(H) = \varnothing$ and each of the following holds:*
  - (b1) *V_3 is a dominating set of H.*

(b2) $V_2 \cap V(G)$ *is a dominating set of* $V_0 \cap V(G)$.

(c) $V_3 \cap V(H) = \varnothing$ *and each of the following holds:*

(c1) V_2 *is a dominating set of* H.

(c2) $V_3 \cap V(G)$ *is a dominating set of* $V_0 \cap V(G)$.

(iii) $f|_G \notin MRDF(G)$, $f|_H \notin MRDF(H)$ *and each of the following holds:*

(a) $V_2 \cap V(H) \neq \varnothing$ *whenever* $N_G(x) \cap V_2 = \varnothing$ *for some* $x \in V_0 \cap V(G)$.

(b) $V_3 \cap V(H) \neq \varnothing$ *whenever* $N_G(x) \cap V_3 = \varnothing$ *for some* $x \in V_0 \cap V(G)$.

(c) $V_2 \cap V(H) \neq \varnothing$ *or* $V_3 \cap V(H) \neq \varnothing$ *whenever* $\exists x \in V_1$ *with* $N_G(x) \cap V_2 = \varnothing$ *and* $N_G(x) \cap V_3 = \varnothing$

(d) $V_2 \cap V(G) \neq \varnothing$ *whenever* $N_H(x) \cap V_2 = \varnothing$ *for some* $x \in V_0 \cap V(H)$.

(e) $V_3 \cap V(G) \neq \varnothing$ *whenever* $N_H(x) \cap V_3 = \varnothing$ *for some* $x \in V_0 \cap V(H)$.

(f) $V_2 \cap V(G) \neq \varnothing$ *or* $V_3 \cap V(G) \neq \varnothing$ *whenever* $\exists x \in V_1$ *with* $N_H(x) \cap V_2 = \varnothing$ *and* $N_H(x) \cap V_3 = \varnothing$

4. Results

Definition 1. *If* $f = (V_0, V_1, V_2, V_3)$ *is a modern Roman dominating function of a non-isolated graph* G*, then it is said to be a total modern Roman dominating function* $(TMRDF(G))$ *of* G *provided it satisfies the following additional property:*

(P3) the set $\{v \in V(G) : f(v) > 0\}$ *induces an isolated-free subgraph.*

The minimum weight $\omega_G^{tmR}(f) = \sum_{v \in V(G)} f(v)$ *of a total modern Roman dominating function* f *of* G *is called the total modern Roman domination number* $\gamma_{tmR}(G)$ *of* G*. A total modern Roman dominating function of* G *with weight* $\omega_G(f) = \gamma_{tmR}(G)$ *is called a* γ_{tmR}*-function of* G.

Example 1. *Consider the graph* G *in Figure 1. The function* $f : V(G) \to \{0, 1, 2, 3\}$ *given by* $f(a) = 3, f(g) = 2, f(b) = 1$ *and* $f(c) = f(d) = f(e) = f(h) = 0$ *is a total modern Roman dominating function of* G*. It can be verified that,* $\gamma_{tmR}(G) = 6$.

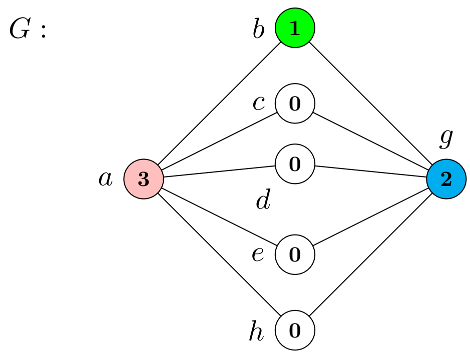

Figure 1: Graph G of order 7 with $\gamma_{tmR}(G) = 6$

Proposition 3. *Let G be any isolated-free graph. Then*

$$\gamma(G) + \gamma_t(G) \leq \gamma_{tmR}(G).$$

Moreover, this bound is sharp whenever, $G = \dot{\bigcup}_{i=1}^{n} P_{2_i}$, where $i = 1, 2, \cdots n$.

Proof. Let G be a graph with no isolated vertex and $f = (V_0, V_1, V_2, V_3)$ be a γ_{tmR}-function on G. Then $V_2 \cup V_3$ is a dominating set of G. Moreover, $V_1 \cup V_2 \cup V_3$ is a total dominating set of G. Thus,

$$\begin{aligned}\gamma_{tmR}(G) =& \omega_G^{tmR}(f)\\ =& |V_1| + 2|V_2| + 3|V_3|\\ =& (|V_1| + |V_2| + |V_3|) + (V_2| + |V_3|) + |V_3|\\ \geq& \gamma_t(G) + \gamma(G).\end{aligned}$$

Hence, we obtained the desired lower bound. Moreover, the bound is sharp by considering the disjoint union of copies of K_2. □

Proposition 4. *Let G be any graph with no isolated vertex and $f = (V_0, V_1, V_2, V_3)$ a γ_{tmR}-function of G. Then $V_0 = \varnothing$ if and only if $V_3 = \varnothing$ and V_2 is a γ-set of G.*

Proof. The result follows immediately from Proposition 1. □

Proposition 5. *Let G be any graph with no isolated vertex and $f = (V_0, V_1, V_2, V_3)$ a γ_{tmR}-function of G. Then one of the following holds:*

(i) If $|V_0| = 0$, then $|V_1| \neq 0$;

(ii) If $|V_1| = 0$, then $|V_0| \neq 0$.

Proof. (i) Let $|V_0| = 0$ and suppose that $|V_1| = 0$. Then $\gamma_{tmR}(G) = 2|V_2| + 3|V_3|$ and $\langle V_2 \cup V_3 \rangle$ is a total dominating set of G. Let $D \subseteq V_2 \cup V_3$ be a dominating set of the induced subgraph $\langle V_2 \cup V_3 \rangle$. Define a function $h = (V_0', V_1', V_2', V_3')$, where $V_0' = \varnothing, V_1' = (V_2 \cup V_3) \setminus D, V_2' = D$ and $V_3' = \varnothing$. Then $h \in TMRDF(G)$ and thus, $\omega_G^{tmR}(h) = |V_1'| + 2|V_2'| \leq 2|V_2| + 3|V_3| = \omega_G^{tmR}(f)$, a contradiction. Hence, $|V_1| \neq 0$.

(ii) Let $|V_1| = 0$ with $|V_3| \neq 0$ and suppose that $|V_0| = 0$. Then $\gamma_{tmR}(G) = \omega_G^{tmR}(f) = 2|V_2| + 3|V_3|$ and since f is a γ_{tmR}-function, $\langle V_2 \cup V_3 \rangle$ is a total dominating set in G. Consider the vertices $x \in V(G)$ such that $x \in N_G(V_2) \cap N_G(V_3)$. Let $V_1^{''} = \varnothing, V_0^{''} = N_G(V_2) \cap N_G(V_3), V_2^{''} = V_2 \setminus (N_G(V_2) \cap N_G(V_3))$. Then the function defined by $h = (V_0^{''}, V_1^{''}, V_2^{''}, V_3^{''}) \in TMRDF(G)$. Thus, $\omega_G^{tmR}(h) = 2|V_2^{''}| + 3|V_3^{''}| \leq 2|V_2| + 3|V_3| = \omega_G^{tmR}(f) = \gamma_{tmR}(G)$, a contradiction. Hence, $|V_0| \neq 0$. □

Remark 1. *The converse of* (ii) *and* (iii) *need not be true.*

Proposition 6. *Let G be a nontrivial connected graph and $f = (V_0, V_1, V_2, V_3)$ a γ_{tmR}-function of G. Then*

(i) $|V_0| = 0$ *if and only if* V_2 *is a* γ*-set of* G *and* $V(G)$ *is a total dominating set of* G*.*

(ii) $|V_1| = 0$ *if and only if* $V_2 \cup V_3$ *is both a* γ_2*-set and* γ_t*-set of* G*.*

Proof. (i) Suppose $|V_0| = 0$, then $|V_3| = 0$ by Proposition 4, V_2 is a γ-set of G. It remains to show that $V(G)$ is a γ_t-set of G. Note that $V(G) = V_1 \cup V_2$ and $\langle V_1 \cup V_2 \rangle$ is isolated-free. It follows that $V(G)$ is a total dominating set of G. Conversely, since $V(G)$ is a total dominating set of G, $f(v) \neq 0$ for all $v \in V(G)$ by definition. Hence, $|V_0| = 0$.

(ii) Suppose $|V_1| = 0$. Then $|V_0| \neq 0$ by Proposition 5 (ii). By (P1), for every $v \in V_0, |V_2 \cap N_G(v)| \geq 1$ and $|V_3 \cap N_G(v)| \geq 1$. This implies that $V_2 \cup V_3$ is a 2-dominating set of G. Suppose $V_2 \cup V_3$ is not a γ_2-set of G. Then there exists $D \subseteq V_2 \cup V_3$ such that D is a γ_2-set of G. Define a function $g = (V_0^*, V_1^*, V_2^*, V_3^*)$, such that $V_0^* = \varnothing, V_1^* = (V_2 \cup V_3) \setminus D, V_2^* = D \cap V_2$ and $V_3^* = D \cap V_3$. Then $g \in TMRDF(G)$ and so, $\omega_G^{tmR}(g) = |V_1^*| + 2|V_2^*| + 3|V_3^*| = |(V_2 \cup V_3) \setminus D| + 2|D \cap V_2| + 3|D \cap V_3| \leq 2|V_2| + 3|V_3| - |(V_2 \cup V_3) \setminus D| = \omega_G^{tmR}(f) - |(V_2 \cup V_3) \cap D| \leq \omega_G^{tmR}(f)$. This is a contradiction. Thus, $V_2 \cup V_3$ is a γ_2-set of G. Now, by (P3) $V_2 \cup V_3$ is a total dominating set of G. Suppose $V_2 \cup V_3$ is not a γ_t-set of G. Then there exists $D \subseteq V_2 \cup V_3$ such that D is a $\gamma_t - set$ of G. Let $V_0^{"} = (V_2 \cup V_3) \setminus D, V_1^{"} = \varnothing, V_2^{"} = D \cap V_2$ and $V_3^{"} = D \cap V_3$. Then $g = (V_0^{"}, \varnothing, V_2^{"}, V_3^{"}) \in TMRDF(G)$ and $\omega_G^{tmR}(g) = 2|V_2^{"}| + 3|V_3^{"}| \leq 2|V_2| + 3|V_3| = \omega_G^{tmR}(f)$, a contradiction. Hence, $V_2 \cup V_3$ is a γ_t-set of G. Conversely, let $V_2 \cup V_3$ be a γ_2-set and γ_t-set of G. Suppose $|V_1| \neq 0$. Then there exists $y \in V_1$ such that $y \in N_G(x) \cap N_G(z)$ where $x \in V_2$ and $z \in V_3$. Define a function $g = (V_0^*, V_1^*, V_2^*, V_3^*)$ for which $V_0 = V_0^*, V_1 = V_1^* \neq \varnothing, V_2 = V_2^*$ and $V_3 = V_3^*$. Then $g \in TMRDF(G)$. Thus, $\omega_G^{tmR}(g) = 3|V_3^*| + 2|V_2^*| + |V_1^*| \leq 3|V_3| + 2|V_2| = \omega_G^{tmR}(f)$. This contradicts the fact that f is a γ_{tmR}-function of G. Hence, $|V_1| = 0$. □

Proposition 7. *Let G be a connected graph of order $n \geq 2$. Then*

(i) $\gamma_{tmR}(G) = 3$ *if and only if* $G \in \{P_2, K_2\}$*.*

(ii) $\gamma_{tmR}(G) = 4$ *if and only if* $G \in \{P_3, K_3\}$*.*

Proof. (i) WLOG, Suppose $G = K_2$. Let $V(G) = \{a, b\}$, then $f = (\varnothing, \{a\}, \{b\}, \varnothing) \in TMRDF(G + H)$. Thus, $\gamma_{tmR}(G) \leq \omega_G^{tmR} = 3$. Since G cannot be trivial, $\gamma_{tmR}(G) \geq 3$. Hence, $\gamma_{tmR}(G + H) = 3$. Conversely, suppose $\gamma_{tmR}(G) = 3$ and $f = (V_0, V_1, V_2, V_3)$ a γ_{tmR}-function of G. Then $V(G) = V_1 \cup V_2$ with $|V_1| = |V_2| = 1$ and so, $|V(G + H)| = 2$. Moreover, $\langle V_1 \cup V_2 \rangle$ is connected. Thus, $G = K_2$.

(ii) WLOG, we assume that $G = K_3$. Let $V(G) = \{x, y, z\}$. Define a function $f = (V_0, V_1, V_2, V_3)$ where $V_0 = \varnothing = V_3$, $V_1 = \{x, z\}$ and $V_2 = \{y\}$. Then $f \in TMRDF(G)$ and thus, $\gamma_{tmR}(G) \leq \omega_{G+H}^{tmR}(f) = |V_1| + 2|V_2| = 4$. Since, $G \neq K_2$, $\gamma_{tmR}(G) \geq 4$ by (i). Hence, $\gamma_{tmR}(G) = 4$. Conversely, suppose that $\gamma_{tmR}(G) = 4$ and $f = (V_0, V_1, V_2, V_3)$ is a

γ_{tmR}-function of G. Then $\gamma_{tmR}(G) = |V_1| + |V_2| + |V_3| = 4$. If $|V_3| = 1$, then $|V_1| = 1$. This implies that $|V(G)| = 2$ and so, $G = K_2$ since $\langle V_1 \cup V_3 \rangle$ must be connected. But $\gamma_{tmR}(K_2) = 3$. Thus, $|V_3| = 0$. It follows that $|V_0| = 0$. Similarly, $|V_2| \neq 2$ and so, $|V_2| \leq 1$. Moreover, if $V_2 = \varnothing$, that is $V(G) = V_1$, then (P2) is not satisfied. Thus, $|V_2| = 1$ and hence, $|V_1| = 2$. Moreover, the induced subgraph $\langle V_1 \cup V_2 \rangle$ is isolated-free. This means that since G is connected, $|V(G)| = |V_1 \cup V_2| = 3$. Therefore, $G = K_3$. □

Proposition 8. *For a connected graph G, $\gamma_{tmR}(G) = 5$ if and only if $|V(G)| = 4$ and $\gamma(G) = 1$ or $\gamma_{t2}(G) = 2$ and $|V(G)| \geq 4$.*

Proof. If $\gamma_{tmR}(G) = 5$, then $|V_3| \leq 1$ and $1 \leq |V_2| \leq 2$. Also, by Proposition 7 (ii), $|V(G)| \geq 4$. Now, if $|V_3| = 0$, then $|V_0| = 0$. Hence, there are only two cases to consider, namely, $|V_2| = 1$ and $|V_2| \leq 2$. If $|V_2| = 2$, then $|V_1| = 1$. Therefore, $|V(G)| = 3$ which is not possible by Proposition 7 (ii). If $|V_2| = 1$, then $|V_1| = 3$. By (P2), $\langle V_1 \cup V_2 \rangle$ must be connected and V_2 is a dominating set in G, it follows that V_2 is a γ-set in G. Therefore, $|V(G)| = 4$ and $\gamma(G) = 1$. Now, suppose that $|V_3| = 1$. If $|V_2| = 0$, then $|V_1| = 2$. Consequently, $|V(G)| = 3$. Thus, $G \in \{K_3, P_3\}$, a contradiction by Proposition 7 (ii). If $|V_2| = 1$, then $|V_1| = 0$. It follows that $|V_2 \cup V_3| = 2$. By Proposition 6, $V_2 \cup V_3$ is a γ_{t2}-set in G. Hence, $|V(G)| \geq 4$ and $\gamma_{t2}(G) = 2$.

Conversely, suppose $|V(G)| = 4$ and $\gamma(G) = 1$. By Proposition 7 (ii), $\gamma(G) \geq 5$. Let v be a dominating vertex of G and define a function $f = (V_0, V_1, V_2, V_3)$ on $V(G)$ such that $V_0 = \varnothing = V_3, V_2 = \{v\},\ V_1 = V(G) \setminus \{v\}$. Then $f \in TMRDF(G)$ and $\omega_G^{tmR}(f) = 5$. This implies that $\gamma_{tmR}(G) = 5$. Next, suppose that $\gamma_{t2}(G) = 2$ and $|V(G)| \geq 4$. Let $D = \{u, v\}$ be the γ_{t2}-set of G. Define a function $g = (V_0, V_1, V_2, V_3)$ such that $V_1 = \varnothing$ and

$$g(x) = \begin{cases} 3, & \text{if } x = u. \\ 2, & \text{if } x = v \\ 0, & \text{if } x \in V(G) \setminus D. \end{cases}$$

Then $g \in TMRDF(G)$ and $\omega_G^{tmR}(g) = 5$. Since $G \notin \{K_3, P_3\}$, we must have $\omega_G^{tmR}(g) = 5$. Hence, $\gamma_{tmR}(G) = 5$. □

Corollary 1. *For a connected graph G of order 4, $\gamma_{tmR}(G) = 5$ if and only if $G \in \{K_1 + (K_1 \cup K_2), K_1 + K_3, K_1 + \overline{K_3}, K_1 + P_3\}$.*

Proof. The proof follows directly from Proposition 8. □

Proposition 9. *Let G be a disconnected graph with nontrivial components $G_1, G_2, \cdots, G_n$. Then*

$$\gamma_{tmR}(G) = \textstyle\sum_{i=1}^{n} \gamma_{tmR}(G_i).$$

Proof. Let $G_1, G_2, \cdots, G_n$ be the components of G. Let $f_1, f_2, \cdots, f_n$ be γ_{tmR}-functions of $G_1, G_2, \cdots, G_n$ respectively. Define a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ given by

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in V(G_1). \\ f_2(x), & \text{if } x \in V(G_2). \\ \vdots \\ f_n(x), & \text{if } x \in V(G_n). \end{cases}$$

Then f is a γ_{tmR}-function of G. Thus $\gamma_{tmR}(G) \leq \sum_{i=1}^{n} \gamma_{tmR}(G_i)$. Conversely, let f be a γ_{tmR}-function of G. Then the restriction $f|_{G_i}$ of f to G_i, where $i = 1, 2, \cdots, n$ is a γ_{tmR}-function of G_i. Thus, $\gamma_{tmR}(G_i) \leq \omega_G^{tmR}(f|_{G_i})$ for all $i = 1, 2, \cdots, n$. Hence, $\sum_{i=1}^{n} \gamma_{tmR}(G_i) \leq \gamma_{tmR}(G)$. Hence, combining the results, $\sum_{i=1}^{n} \gamma_{tmR}(G_i) \leq \gamma_{tmR}(G) \leq \sum_{i=1}^{n} \gamma_{tmR}(G_i)$. Therefore, $\gamma_{tmR}(G) = \sum_{i=1}^{n} \gamma_{tmR}(G_i)$. □

Proposition 10. *If $G \in \{P_n, C_n\}$, then $\gamma_{tmR}(G) = n + \left\lceil \frac{n}{3} \right\rceil$.*

Proof. Let $G \in \{P_n, C_n\}$. Suppose $G = [v_1, v_2, \cdots, v_n]$. If $n \equiv 0 (\text{mod } 3)$. Put $V_2 = \{v_2, v_5, \cdots, v_{n-7}, v_{n-4}, v_{n-1}\}, V_1 = V(G) \setminus V_2, |V_0| = 0 = |V_3|$. Then $f = (V_0, V_1, V_2, V_3)$ is a $TMRDF$ on G. If $n \equiv 1 (\text{mod } 3)$. Put $V_2 = \{v_2, \cdots, v_{n-8}, v_{n-5}, v_{n-2}, v_n\}, V_1 = V(G) \setminus V_2, |V_0| = 0 = |V_3|$. Then $f = (V_0, V_1, V_2, V_3)$ is a $TMRDF$ on G. If $n \equiv 2 (\text{mod } 3)$. Put $V_2 = \{v_2, v_5, \cdots, v_{n-9}, v_{n-6}, v_{n-3}, v_n\}, V_1 = V(G) \setminus V_2, |V_0| = 0 = |V_3|$. Then $f = (V_0, V_1, V_2, V_3)$ is a $TMRDF$ on G. In any case, $\gamma_{tmR}(G) \leq \omega_G^{tmR}(f) = |V_1| + |V_2| = n + \left\lceil \frac{n}{3} \right\rceil$. Since $|V_0| = 0$, by Proposition 6, V_2 is a γ-set of G and $V(G)$ is a γ_t-set of G. By Proposition 3, $\gamma(G) + \gamma_t(G) = |V_2| + |V(G)| = n + \left\lceil \frac{n}{3} \right\rceil \leq \gamma_{tmR}(G)$. Thus, $\gamma_{tmR}(G) = n + \left\lceil \frac{n}{3} \right\rceil$.

□

The *n-barbell* graph is a simple graph obtained by joining two copies of complete graph $K_{n \geq 3}$ by a bridge and is denoted by B_n. Figure 2 shows the n-barbell graphs B_3 and B_5, respectively.

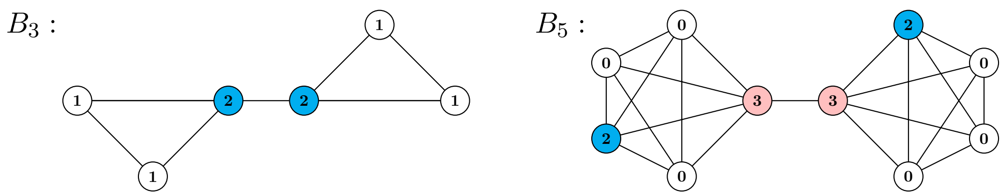

Figure 2: The graphs B_3 and B_5 with $\gamma_{tmR}(B_3) = 8$ and $\gamma_{tmR}(B_5) = 10$, respectively.

Proposition 11. *For any n-barbell graph B_n where $n \geq 3$,*

$$\gamma_{tmR}(B_n) = \begin{cases} 8, & \text{if } n = 3. \\ 10, & \text{if } n \geq 4. \end{cases}$$

Proof. Let B_n be any n-barbell graph and $uv \in E(B_n)$ be the bridge that joins the two copies of K_n. If $n = 3$, define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(x) = \begin{cases} 2, & x \in \{u, v\}. \\ 1, & \text{otherwise.} \end{cases}$$

Then f is a $TMRDF$ of B_3. Thus, $\gamma_{tmR}(B_3) \leq \omega^{tmR}_{B_3}(f) = 8$. Clearly $\gamma(B_3) = 2 = \gamma_t(B_3)$. Thus, by Proposition 3, $\gamma_{tmR}(B_3) \geq 8$. Therefore, $\gamma_{tmR}(B_3) = 8$. If $n \geq 4$. Pick any $v', u' \in V(B_n)$ such that $v' \neq u, u' \neq v$, and $v'v, u'u \in E(B_n)$. Now, define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(x) = \begin{cases} 0, & x \in V(B_n) \setminus \{u, v, u', v'\}. \\ 3, & x \in \{u, v\}. \\ 2, & x \in \{u', v'\}. \end{cases}$$

Then f is a $TMRDF$ of B_n. Thus, $\gamma_{tmR}(B_n) \leq \omega^{tmR}_{B_n}(f) = 10$. Clearly $\gamma(B_n) = 2 = \gamma_t(B_n)$. Thus, by Proposition 3, $\gamma_{tmR}(B_n) \geq 10$. Therefore, for all $n \geq 4$, $\gamma_{tmR}(B_n) = 10$. □

The *windmill graph* $Wd(k, n) = G = K_1 + nK_{k-1}$ is constructed for $k \geq 2$ and $n \geq 2$ by joining n copies of the complete graph K_k at a shared vertex. It has $n(k-1)+1$ vertices and $\frac{1}{2}nk(k-1)$ edges. The case $k = 3$ corresponds to the *dutch windmill graph* (also called Friendship graph) $G^n_3 = K_1 + nK_2$ and the case $n = 2$ corresponds to the butterfly graph $G^2_3 = K_1 + 2K_2$. The graphs in Figures 3, 4, and 5 are the windmill graph $Wd(4, 2)$, friendship graph G^4_3 and butterfly graphs G^2_3, respectively.

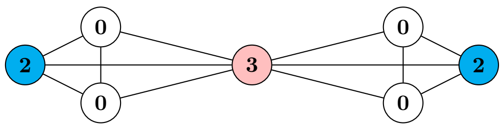

Figure 3: A windmill graph $Wd(4, 2)$ with $\gamma_{tmR}(Wd(4, 2)) = 7$

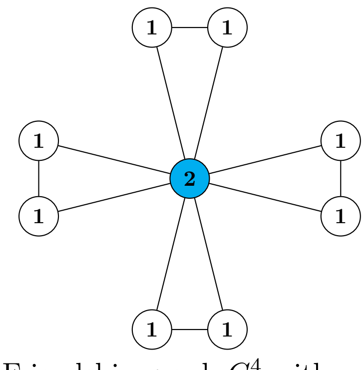

Figure 4: A Friendship graph G^4_3 with $\gamma_{tmR}(G^4_3) = 10$

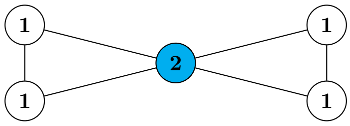

Figure 5: A butterfly graph G_3^2 with $\gamma_{tmR}(G_3^2) = 6$

Proposition 12. *For any windmill graph $G = K_1 + nK_{k-1}$, where $k \geq 4$ and $n \geq 2$, $\gamma_{tmR}(G) = 2n + 3$.*

Proof. Let $G = K_1 + nK_{k-1}$, where $k \geq 4$ and $n \geq 2$. Suppose $V(K_1) = \{u\}$ be the central vertex in G, then pick a vertex v in each copies of the complete graph K_{k-1} and define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(x) = \begin{cases} 3, & x = u. \\ 2, & x = v. \\ 0, & \textit{otherwise}. \end{cases}$$

Then $f \in TMRDF(G)$. Thus, $\gamma_{tmR}(G) \leq \omega_G^{tmR}(f) = 2n + 3$. Now, since $\gamma(G) = 1$ and $\gamma_t(G) = 2$, by Proposition 3, $\gamma_{tmR}(G) \geq 2n + 3$. Hence, $\gamma_{tmR}(G) = 2n + 3$. □

Proposition 13. *For any friendship graph G, $\gamma_{tmR}(G) = 2n + 2$.*

Proof. Let $G = K_1 + nK_2$, $n \geq 2$. Let $V(K_1) = \{u\}$ be the central vertex in G. Define a function $f = (\varnothing, V(G) \setminus \{u\}, \{u\}, \varnothing)$. Then for all $v_i \in V_1, 1 \leq i \leq n, f(N_G[v_i]) = 2n+2$. Thus, $f \in TMRDF(G)$ and $\gamma_{tmR}(G) \leq \omega_G^{tmR}(f) = 2n + 2$. Clearly, $\{u\}$ is a γ-set of G, and so $\gamma(G) = 1$. Moreover, $\gamma_t(G) = 2$ and so, by Proposition 3, $\gamma_{tmR}(G) \geq 2n + 2$. Hence, $\gamma_{tmR}(G) = 2n + 2$. □

Corollary 2. *For a butterfly graph G, $\gamma_{tmR}(G) = 6$.*

Proof. The result follows from Proposition 13. □

A graph G is called *bipartite* if the vertex set $V(G)$ of G can be partitioned into two subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2. If G is bipartite such that G contains every edge incident with any pair of vertices in V_1 and V_2, then G is a *complete bipartite graph*; in this case, $G = K_{m,n}$ if $|V_1| = m$ and $|V_2| = n$. Figure 6 shows the complete bipartite graph $K_{7,5}$.

Proposition 14. *For a complete bipartite graph $K_{m,n}$, let $p = min\{m, n\}, m, n \geq 2$. Then*

$$\gamma_{tmR}(K_{m,n}) = \begin{cases} 6, & \text{if } p = 2. \\ 8, & \text{if } p = 3 \\ 9, & \text{if } p = 4 \\ 10, & \text{if } p \geq 5. \end{cases}$$

$K_{7,5}$:

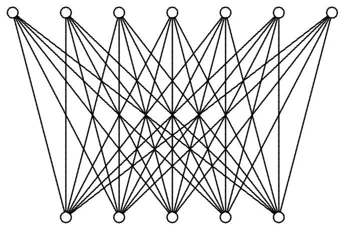

Figure 6: A complete bipartite $G = K_{7,5}$

Proof. Let G be a complete bipartite graph $K_{m,n}$ and X and Y be partite sets of $K_{m,n}$, where $|X| = m$ and $|Y| = n$, $m, n \geq 2$. Let $p = min\{m, n\}$. If $p = 2$ and WLOG, we let $X = \{x_1, x_2\}$ and pick a vertex $y_1 \in Y$ and define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(z) = \begin{cases} 3, & z = \{x_1\}. \\ 2, & z = \{x_2\}. \\ 1, & z = \{y_1\}. \\ 0, & \text{Otherwise.} \end{cases}$$

Then f is a $TMRDF$ of G. Thus, $\gamma_{tmR}(G) \leq \omega_G^{tmR}(f) = 6$. By Proposition 3, $\gamma_{tmR}(G) \geq 6$. Thus, $\gamma_{tmR}(G) = 6$. If $p = 3$ and WLOG, we let $X = \{x_1, x_2, x_3\}$ and pick a vertex $y_1 \in Y$ and define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(z) = \begin{cases} 3, & z = \{x_1\}. \\ 2, & z \in \{x_2, x_3\}. \\ 1, & z = \{y_1\}. \\ 0, & \text{Otherwise.} \end{cases}$$

Then f is a $TMRDF$ of G. Thus, $\gamma_{tmR}(G) \leq \omega_G^{tmR}(f) = 8$. By Proposition 3, $\gamma_{tmR}(G) \geq 8$. Thus, $\gamma_{tmR}(G) = 8$. If $p = 4$ and WLOG, we let $X = \{x_1, x_2, x_3, x_4\}$ and pick a vertex $y_1 \in Y$ and define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(z) = \begin{cases} 3, & z = \{x_1\}. \\ 2, & z \in \{x_2, y_1, \}. \\ 1, & z \in \{x_3, x_4\}. \\ 0, & \text{Otherwise.} \end{cases}$$

Then f is a $TMRDF$ of G. Thus, $\gamma_{tmR}(G) \leq \omega_G^{tmR}(f) = 9$. By Proposition 3, $\gamma_{tmR}(G) \geq 9$. Thus, $\gamma_{tmR}(G) = 9$. If $p \geq 5$, let $X = \overline{K_m}$ and $Y = \overline{K_n}$. WLOG, let

$\{x_1, x_2\} \in V(\overline{K_m}), \{y_1, y_2\} \in V(\overline{K_n})$ and define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(z) = \begin{cases} 3, & z \in \{x_1, y_1\}. \\ 2, & z \in \{x_2, y_2\}. \\ 0, & \text{Otherwise.} \end{cases}$$

Then f is a $TMRDF$ of G. Thus, $\gamma_{tmR}(G) \leq \omega_G^{tmR}(f) = 10$. By Proposition 3, $\gamma_{tmR}(G) \geq 10$. Thus, $\gamma_{tmR}(G) = 10$. □

The *star* S_n of order $n+1$ is the graph $\overline{K_n} + K_1$. The graph in Figures 7 is the star graph S_6.

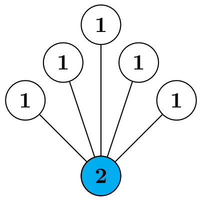

Figure 7: A star graph S_6 with $\gamma_{tmR}(S_6) = 7$

Proposition 15. *If $G = S_n$, $n \geq 1$, then $\gamma_{tmR}(G) = n + 2$.*

Proof. Let $G = S_n$ where $V(G) = V(K_1 + \overline{K_n})$ and $V(K_1) = \{u\}$ is a central vertex of G. Now, define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(x) = \begin{cases} 2, & x = \{u\}. \\ 1, & \text{otherwise.} \end{cases}$$

Then $f \in TMRDF(G)$. It follows that $\gamma_{tmR}(G) \leq n + 2$. Now, suppose that $g = (W_0, W_1, W_2, W_3)$ is a γ_{tmR}-function of G. If $W_0 = \varnothing$, then $W_3 = \varnothing$. Since g is a γ_{tmR}-function of G, $|W_2| = |V(K_1)| = 1$ and $|W_1| = |V(\overline{K_n})| = n$. Hence, $\gamma_{tmR}(G) = \omega_G^{tmR}(g) \geq n + 2$. If $|W_0| \neq 0$, then $|W_2| \geq 1$ and $|W_3| \geq 1$. It follows that $\gamma_{tmR}(G) = \omega_G^{tmR}(g) = 2|W_2| + 3|W_3| \geq n + 2$. Therefore, $\gamma_{tmR}(G) = n + 2$. □

Proposition 16. *For a complete graph K_n, $\gamma_{tmR}(K_n) = 5$ for all $n \geq 4$.*

Proof. Pick any $x, y \in V(K_n)$ with $x \neq y$. Clearly, $g = (V(K_n) \setminus \{x, y\}, \varnothing, \{x\}, \{y\}) \in TMRDF(K_n)$. It follows that $\gamma_{tmR}(K_n) \leq 5$. On the other hand, suppose that $f = (V_0, V_1, V_2, V_3)$ is a γ_{tmR}-function of K_n. If $V_0 = \varnothing$, then $V_3 = \varnothing$. Since f is a γ_{tmR}-function of $K_n, |V_2| = 1$ and $|V_1| = n - 1$. Hence, $\gamma_{tmR}(K_n) = \omega_{K_n}^{tmR}(f) = n + 1 \geq 5$. If $V_0 \neq \varnothing$, then $|V_2| \geq 1$ and $|V_3| \geq 1$. It follows that $\gamma_{tmR}(K_n) = \omega_{K_n}^{tmR}(f) = 2|V_2| + 3|V_3| \geq 5$. Therefore, $\gamma_{tmR}(K_n) = 5$. □

5. On the join of graphs

Given two graphs G and H with disjoint vertex sets, the *join* $G+H$ of graphs G and H, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$ [8].

In this section, the following proposition characterizes all TMRDF on the join of graphs.

Proposition 17. *Let G and H be any graphs and let $f \in (V_0, V_1, V_2, V_3)$ be a function on $V(G+H)$ with $V_2 \neq \varnothing$ and $V_3 \neq \varnothing$. Then $f \in TMRDF(G+H)$ if and only if one of the following holds:*

(i) $f|_G \in TMRDF(G)$ *and one of the following holds:*

- (a) $|V_2 \cap V(G)| \geq 1$ *and* $|V_3 \cap V(G)| \geq 1$
- (b) $V_2 \cap V(G) = \varnothing$ *and each of the following holds:*
  - (b1) V_3 *is a dominating set of* G.
  - (b2) $V_2 \cap V(H)$ *is a dominating set of* $V_0 \cap V(H)$.
- (c) $V_3 \cap V(G) = \varnothing$ *and each of the following holds:*
  - (c1) V_2 *is a dominating set of* G.
  - (c2) $V_3 \cap V(H)$ *is a dominating set of* $V_0 \cap V(H)$.

(ii) $f|_H \in TMRDF(H)$ *and one of the following holds:*

- (a) $|V_2 \cap V(H)| \geq 1$ *and* $|V_3 \cap V(H)| \geq 1$
- (b) $V_2 \cap V(H) = \varnothing$ *and each of the following holds:*
  - (b1) V_3 *is a dominating set of* H.
  - (b2) $V_2 \cap V(G)$ *is a dominating set of* $V_0 \cap V(G)$.
- (c) $V_3 \cap V(H) = \varnothing$ *and each of the following holds:*
  - (c1) V_2 *is a dominating set of* H.
  - (c2) $V_3 \cap V(G)$ *is a dominating set of* $V_0 \cap V(G)$.

(iii) $f|_G \notin TMRDF(G)$, $f|_H \notin TMRDF(H)$ *and each of the following holds:*

- (a) $V_2 \cap V(H) \neq \varnothing$ *whenever* $N_G(x) \cap V_2 = \varnothing$ *for some* $x \in V_0 \cap V(G)$.
- (b) $V_3 \cap V(H) \neq \varnothing$ *whenever* $N_G(x) \cap V_3 = \varnothing$ *for some* $x \in V_0 \cap V(G)$.
- (c) $V_2 \cap V(H) \neq \varnothing$ *or* $V_3 \cap V(H) \neq \varnothing$ *whenever* $\exists x \in V_1$ *with* $N_G(x) \cap V_2 = \varnothing$ *and* $N_G(x) \cap V_3 = \varnothing$
- (d) $V_2 \cap V(G) \neq \varnothing$ *whenever* $N_H(x) \cap V_2 = \varnothing$ *for some* $x \in V_0 \cap V(H)$.
- (e) $V_3 \cap V(G) \neq \varnothing$ *whenever* $N_H(x) \cap V_3 = \varnothing$ *for some* $x \in V_0 \cap V(H)$.

(f) $V_2 \cap V(G) \neq \varnothing$ *or* $V_3 \cap V(G) \neq \varnothing$ *whenever* $\exists x \in V_1$ *with* $N_H(x) \cap V_2 = \varnothing$ *and* $N_H(x) \cap V_3 = \varnothing$

(g) *For every* $x \in V_1 \cup V_2 \cup V_3$, $(V_1 \cup V_2 \cup V_3) \cap V(H) \neq \varnothing$ *whenever* $x \in V(G)$ *and* $N_G(x) \subseteq V_0$.

(h) *For every* $x \in V_1 \cup V_2 \cup V_3$, $(V_1 \cup V_2 \cup V_3) \cap V(G) \neq \varnothing$ *whenever* $x \in V(H)$ *and* $N_H(x) \subseteq V_0$.

Proof. Suppose $f|_G \in TMRDF(G)$. Assume that (i)(a) holds. Let $v \in V_0 \cap V(G)$. By assumption, there exist $u \in V_2 \cap V(G)$ and $w \in V_3 \cap V(G)$ such that $uv, wv \in E(G) \subseteq E(G+H)$. Let $v \in V_0 \cap V(H)$. Note that $|V_2 \cap V(G)| \geq 1$ and $|V_3 \cap V(G)| \geq 1$. Take $u \in V_2 \cap V(G)$ and $w \in V_3 \cap V(G)$. Then $vu, vw \in E(G+H)$. Also, since $f|_G \in TMRDF(G)$, $uw \in E(G) \subseteq E(G+H)$. Let $v \in V_1 \cap V(G)$. Since $f|_G \in TMRDF(G)$, there exists $u \in (V_2 \cup V_3) \cap V(G)$ such that $uv \in E(G) \subseteq E(G+H)$. Now, assume $v \in V_1 \cap V(H)$. Since $|V_2 \cap V(G)| \geq 1$ and $|V_3 \cap V(G)| \geq 1$, there exists $z \in V_2 \cap V(G)$ or $z \in V_3 \cap V(G)$ such that $z \in N_{G+H}(v)$. Thus, $f \in TMRDF(G+H)$. Similarly, if $f|_H \in TMRDF(H)$ with $|V_2 \cap V(H)| \geq 1$ and $|V_3 \cap V(H)| \geq 1$, then $f \in TMRDF(G+H)$. Assume (i)(b) holds. Since $f|_G \in TMRDF(G), V_0 \cap V(G) = \varnothing$. Thus, $V_0 \subseteq V(H)$. Let $v \in V_0$. By (b2), there exists $u \in V_2 \cap V(H)$ such that $uv \in E(H) \subseteq E(G+H)$. Also, since V_3 is a dominating set of G, $V_3 \cap V(G) \neq \varnothing$. Pick $w \in V_3 \cap V(G)$. Then $vw \in E(G+H)$. Moreover, $uw \in E(G+H)$. Let $v \in V_1 \cap V(G)$. By assumption, there exists $u \in V_3 \cap V(G)$ such that $uv \in E(G) \subseteq E(G+H)$. Now, let $v \in V_1 \cap V(H)$. By (b1), there exists $u \in V_3 \cap V(G)$. Then $uv \in E(G+H)$. Therefore, $f \in TMDRF(G+H)$. Assume (i)(c) holds. Since $f|_G \in TMRDF(G), V_0 \cap V(G) = \varnothing$. Thus, $V_0 \subseteq V(H)$. Let $v \in V_0$. By (c2), there exists $u \in V_3 \cap V(H)$ such that $uv \in E(H) \subseteq E(G+H)$. Also, since V_2 is a dominating set of G, $V_2 \cap V(G) \neq \varnothing$. Pick $w \in V_2 \cap V(G)$. Then $wv \in E(G+H)$. Moreover, $uw \in E(G+H)$. Let $v \in V_1 \cap V(G)$. By (c1), there exists $u \in V_2 \cap V(G)$ such that $uv \in E(G) \subseteq E(G+H)$. Now, let $v \in V_1 \cap V(H)$. By (c1), there exists $u \in V_2 \cap V(G)$. Then $uv \in E(G+H)$. Therefore, $f \in TMDRF(G+H)$. Similarly, if (ii) holds, then $f \in TMRDF(G+H)$. Suppose (iii) holds, that is $f|_G \notin TMRDF(G)$ and $f|_H \notin TMRDF(G)$. Let $v \in V_0 \cap V(G)$. If $N_G(v) \cap V_2 = \varnothing$ and $N_G(v) \cap V_3 \neq \varnothing$. Take $u \in V_3 \cap V(G)$ such that $uv \in E(G) \subseteq E(G+H)$. Since $N_G(v) \cap V_2 = \varnothing$, by assumption, there exists $w \in V_2 \cap V(H)$ such that $vw \in E(G+H)$. Moreover, $uw \in E(G+H)$. If $N_G(v) \cap V_2 \neq \varnothing$ and $N_G(v) \cap V_3 = \varnothing$. Pick $u \in V_2 \cap V(G)$ such that $uv \in E(G) \subseteq E(G+H)$. Since $N_G(v) \cap V_3 = \varnothing$, by assumption, there exists $w \in V_3 \cap V(H)$ such that $vw \in E(G+H)$. Moreover, $uw \in E(G+H)$. If $N_G(v) \cap V_2 = \varnothing$ and $N_G(v) \cap V_3 = \varnothing$. Then by assumption, $V_2 \cap V(H) \neq \varnothing$ and $V_3 \cap V(H) \neq \varnothing$ and so, there exist $u \in V_2 \cap V(H)$ and $w \in V_3 \cap V(H)$ such that $vu, vw \in E(G+H)$. Also, by assumption, there exists $x \in V(G) \setminus V_0$ such that $ux, wx \in E(G+H)$. Now, suppose $f(v) = 1$. If $N_G(v) \cap V_2 = \varnothing$ and $N_G(v) \cap V_3 = \varnothing$. Then by assumption, there exist $z \in V_2 \cap V(H)$ or $z \in V_3 \cap V(H)$ such that $vz \in E(G+H)$ satisfying (P2). Therefore, $f \in TMRDF(G+H)$. Similarly, for $v \in V(H)$ such that $f(v) \in \{0,1\}$, $f \in TMRDF(G+H)$. Now, let $v \in V_1 \cup V_2 \cup V_3$. If $v \in V(G)$ with $N_G(v) \subseteq V_0$, then by (iii)(g), $(V_1 \cup V_2 \cup V_3) \cap V(H) \neq \varnothing$. This means that $N_{G+H}(v) \cap (V_1 \cup V_2 \cup V_3) \neq \varnothing$. Similarly, $N_{G+H}(v) \cap (V_1 \cup V_2 \cup V_3) \neq \varnothing$, for each $v \in V(H)$ with $N_H(v) \subseteq V_0$. Thus,

$\langle V_1 \cup V_2 \cup V_3 \rangle$ has no isolated vertex. Therefore, $f \in TMRDF(G+H)$.

Conversely, suppose $f \in TMRDF(G+H)$. Consider the following cases:

Case 1: Suppose $f|_G \in TMRDF(G)$. If (i)(a) holds, we are done. Suppose (i)(a) does not hold. Thus, either $V_2 \cap V(G) = \varnothing$ or $V_3 \cap V(G) = \varnothing$. Suppose $V_2 \cap V(G) = \varnothing$. Necessarily, $V_0 \cap V(G) = \varnothing$. Let $v \in V_1 \cap V(G)$. Since $f|_G \in TMRDF(G)$, there exists $u \in V_3$ such that $uv \in E(G)$. Thus, V_3 is a dominating set of G, and so, (b1) holds. Also, since $V_2 \cap V(G) = \varnothing$, then $V_2 \subseteq V(H)$. Hence, $V_2 \cap V(H) \neq \varnothing$. Pick $w \in V_2$ and let $v \in V_0 \cap V(H)$. Since $f \in TMRDF(G+H)$, $vw \in E(H) \subseteq E(G+H)$. Thus, (b2) holds. Also, since V_3 is a dominating set of G, there exists $u \in V_3 \cap V(G)$ where $uv \in E(G+H)$. Suppose $V_3 \cap V(G) = \varnothing$, then similarly, (i)(c1) and (i)(c2) hold. Furthermore, the case where $f|_H \in TMRDF(H)$ is similar.

Case 2: Suppose $f|_H \in TMRDF(H)$. This can be proven similarly with Case 1.

Case 3: Assume $f|_G \notin TMRDF(G)$ and $f|_H \notin TMRDF(H)$. Suppose $N_G(x) \cap V_2 = \varnothing$ for some $x \in V_0$. Since $f \in TMRDF(G+H)$, there exists $y \in V_2 \cap N_{G+H}(x)$. Since $y \notin V(G), y \in V(H)$. Hence, $y \in V_2 \cap V(H)$. Thus, $V_2 \cap V(H) \neq \varnothing$ satisfying (iii)(a). Following similar argument, if $N_G(x) \cap V_3 = \varnothing$ for some $x \in V_0, V_3 \cap V(H) \neq \varnothing$. Thus, (iii)(b) holds. Suppose $N_G(x) \cap V_2 = \varnothing$ and $N_G(x) \cap V_3 = \varnothing$ for some $x \in V_1$. Since $f \in TMRDF(G+H)$, there exists $y \in V_2$ or $y \in V_3$ such that $xy \in E(G+H)$. Since $y \notin V(G), y \in V(H)$. Necessarily $y \in V_2 \cap V(H)$ or $y \in V_3 \cap V(H)$. Thus, $V_2 \cap V(H) \neq \varnothing$ or $V_3 \cap V(H) \neq \varnothing$ satisfying (iii)(c). Moreover, assume that $\langle (V_1 \cup V_2 \cup V_3) \cap V(G) \rangle$ has an isolated vertex. Let $v \in [(V_1 \cup V_2 \cup V_3) \cap V(G)]$ such that $N_G(v) \subseteq V_0$. Since $f \in TMRDF(G+H)$, $\langle V_1 \cup V_2 \cup V_3 \rangle$ is isolated-free vertex. Thus, $(V_1 \cup V_2 \cup V_3) \cap V(H) \neq \varnothing$. Hence, (iii)(g) holds. Similarly, (iii)(d), (iii)(e), (iii)(f), and (iii)(h) follows. □

Corollary 3. *Let G and H be any nontrivial graphs. Then*

$$3 \leq \gamma_{tmR}(G+H) \leq 10.$$

Proof. Since $G+H$ is not a trivial graphs, $\gamma_{tmR}(G+H) > 2$. That is, $\gamma_{tmR}(G+H) \geq 3$. On the other hand, let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $V(H) = \{u_1, u_2, \cdots, u_n\}$. Now, define a function $f = (V_0, V_1, V_2, V_3)$ on $V(G+H)$ given by

$$f(x) = \begin{cases} 2, & \text{if } x \in \{v_1, u_1\}. \\ 3, & \text{if } x \in \{v_2, u_2\}. \\ 0, & \text{if } x \in V(G+H) \setminus \{v_1, v_2, u_1, u_2\}. \end{cases}$$

for every $x \in V(G+H)$. Then $f \in TMRDF(G+H)$. Thus, $\gamma_{tmR}(G+H) \leq \omega_{G+H}^{tmR}(f) = 10$. Hence, $3 \leq \gamma_{tmR}(G+H) \leq 10$. □

Corollary 4. *Let G be any graph of order n, then $\gamma_{tmR}(K_1+G) \leq min\{2+n, 3+2\gamma(G)\}$, and this upper bound is sharp.*

Proof. Define a function $g = (V_0, V_1, V_2, V_3)$ for which $V_1 = V(G), V_2 = V(K_1)$ and

$V_0 = V_3 = \varnothing$. Then, $g \in TMRDF(K_1 + G)$. Thus,

$$\begin{aligned}\gamma_{tmR}(K_1+G) &\leq \omega_G^{tmR}(g)\\ &= 2|V_2| + |V_1|\\ &= 2(1) + n\\ &= 2 + n.\end{aligned}$$

Now, let $V(K_1) = \{x\}$ and D a γ-set of G. Define a function $h = (V_0', V_1', V_2', V_3')$ for which $V_0' = V(G) \setminus D, V_1' = \varnothing, V_2' = D$ and $V_3' = \{x\}$. Then, $h \in TMRDF(K_1 + G)$. Thus,

$$\begin{aligned}\gamma_{tmR}(K_1+G) &\leq \omega_G^{tmR}(h)\\ &= 3|V_3'| + 2|V_2'|\\ &= 3(1) + 2|D|\\ &= 3 + 2\gamma(G).\end{aligned}$$

Moreover, for the sharpness, consider the graphs $K_1 + G, G \in \{K_n, \overline{K_n}\}$. Strict inequality may also be attained such as the graph $K_1 + P_7$. □

Example 2. *Consider the graphs $K_1 + P_7$ and $K_1 + K_5$ in Figure 8 and Figure 9, respectively.*

$K_1 + P_7$:

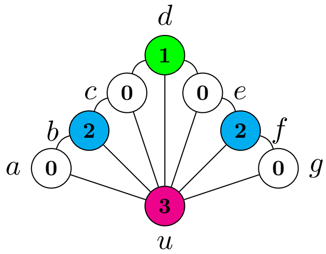

Figure 8: The graph $K_1 + P_7$ with $\gamma_{tmR}(K_1 + P_7) = 8$

In figure 8, the function $f : V(K_1 + P_7) \to \{0, 1, 2, 3\}$ given by $f(u) = 3, f(b) = f(f) = 2$, $f(a) = f(c) = f(g) = f(e) = 0$, and $f(d) = 1$. Observe that $\{b, d, f\}$ is a dominating set of P_7. Thus, the $\gamma(P_7) = 3$ and observe further that f is a total modern Roman dominating function of G with weight 8. It can be verified that $8 = \gamma_{tmR}(K_1 + P_7) < \min\{2 + n, 3 + 2(\gamma(P_7))\} = 9$.

$K_1 + K_5$:

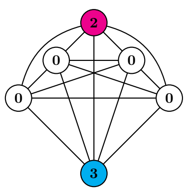

Figure 9: The graph $K_1 + K_5$ with $\gamma_{tmR}(K_1 + K_5) = 5$.

In figure 9, Observe that f is a total modern Roman dominating function of $K_1 + K_5$ with weight 5. It can be verified that $\gamma_{tmR}(K_1 + K_5) = \min\{2 + n, 3 + 2(\gamma(K_1 + K_5))\} = 5$.

Corollary 5. *Let G and H be any graphs. Then*

(i) $\gamma_{tmR}(G + H) = 3$ *if and only if* $G = K_1$ *and* $H = K_1$.

(ii) $\gamma_{tmR}(G + H) = 4$ *if and only if* $G = K_1$ *and* $H \in \{K_2, \overline{K}_2\}$ *or vise versa.*

(iii) $\gamma_{tmR}(G + H) = 5$ *if and only if one of the following holds:*

(a) $G = K_1$ *and* $H \in \{P_3, K_3, \overline{K}_3, K_1 \cup K_2\}$ *and vise versa.*

(b) $\gamma(G) = 1$ *and* $\gamma(H) = 1$.

(c) $\gamma_2(G) = 2$ *or* $\gamma_2(H) = 2$.

Proof. The proof follows immediately from Proposition 17. □

6. On the corona of graphs

Let G and H be graphs with disjoint vertex sets. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex of the i^{th} copy of H. For convenience, we adapt the notation $H^v + v$ used in [8] to denote the subgraph of $G \circ H$ corresponding to the join $H^v + \langle\{v\}\rangle, v \in V(G)$.

Proposition 18. *Let G and H be nontrivial graphs. Let $f = (V_0, V_1, V_2, V_3)$ be any function on $V(G \circ H)$. Then $f \in TMRDF(G \circ H)$ if and only if each of the following holds:*

(i) *For every $v \in (V_0 \cup V_1) \cap V(G), f|_{H^v} \in TMRDF(H^v)$. Moreover, if $v \in V_0 \cap V(G)$, then the following holds:*

(a) *If $|V_2^v| \neq 0$ and $|V_3^v| = 0$, then $|N_G(v) \cap V_3| \geq 1$; and*

(b) *If $|V_2^v| = 0$ and $|V_3^v| \neq 0$, then $|N_G(v) \cap V_2| \geq 1$.*

(ii) *For every $v \in V_2 \cap V(G), V_3 \cap V(H^v)$ is a dominating set of $\langle V_0 \cap V(H^v)\rangle$.*

(iii) *For every $v \in V_3 \cap V(G), V_2 \cap V(H^v)$ is a dominating set of $\langle V_0 \cap V(H^v)\rangle$.*

Proof. (i) Suppose $f \in TMRDF(G \circ H)$ and let $v \in (V_0 \cup V_1) \cap V(G)$. Let $u \in V_0^v$. Then $u \in V_0$ and there exist $w, z \in N_{G\circ H}(u)$ such that $w \in V_2$ and $z \in V_3$. But $N_{G\circ H}(u) = \{v\} \cup N_{H^v}(u)$ and $v \in V_0 \cup V_1$. Thus, $w, z \in N_{H^v}(u)$. Moreover, let $u \in V_1^v$. Then $u \in V_1$ and there exists $x \in V_2 \cup V_3$ such that $x \in N_{G\circ H}(u)$, $x \in (V_2^v \cup V_3^v)$ and so $x \in N_{H^v}(u)$. Now, let $u \in V_2^v \cup V_3^v$. By (P3), there exists $z' \in V(H^v) \setminus V_0^v$ such that $z \in N_{G\circ H}(u)$. Thus, $f|_{H^v} \in TMRDF(H^v)$. Now, let $v \in V_0 \cap V(G)$. Suppose that

$|V_2^v| \neq 0$ and $|V_3^v| = 0$. Then $u \in N_{G\circ H}(v)$ and $u \in V_3$. Hence, $u \in N_G(v) \cap V_3$. Thus, $|N_G(v) \cap V_3| \geq 1$. Following similar argument, if $|V_3^v| \neq 0$ and $|V_2^v| = 0$, $|N_G(v) \cap V_2| \geq 1$. This proves (i). Suppose that $v \in V_2 \cap V(G)$ and let $u \in V_0^v$. By definition, there exists $\{x, y\} \subseteq N_{G\circ H}(u) = \{v\} + N_{H^v}(u)$ such that $x \in V_2$ and $y \in V_3$. If $v \in V_2$ and take $x = v$, then $y \in V_3^v$ and $y \in N_{H^v}(u)$. Thus, V_3^v dominates V_0^v. Hence, $V_3 \cap V(H^v)$ is a dominating set of $\langle V_0 \cap V(H^v)\rangle$. This proves (ii). Similarly, (iii) follows.

Conversely, suppose (i), (ii) and (iii) hold for f. Let $u \in V_0$ and $v \in V(G)$ for which $u \in V(H^v + v)$. Consider the following cases:

Case 1: Suppose $u = v$. By (i), $f|_{H^v} \in TMRDF(H^v)$. Thus, there exist $w, z \in N_{H^v}(u)$ such that $w \in V_2^v$ and $z \in V_3^v$. Hence, $w \in V_2$ and $z \in V_3$ and $w, z \in N_{G\circ H}(u)$. Now, if $|V_3^v| \neq 0$ and $|V_2^v| = 0$, then by (i)(a), $|N_G(u) \cap V_2| \geq 1$. Hence, there exist $z \in V_3^v$ and $w \in N_G(u) \cap V_2$ and $w, z \in N_{G\circ H}(u)$. Similarly, if $|V_2^v| \neq 0$ and $|V_3^v| = 0$, then by (i)(b), $|N_G(u) \cap V_3| \geq 1$. So, there exist $w \in V_2^v$ and $z \in N_G(v) \cap V_3$ such that $w, z \in N_{G\circ H}(u)$.

Case 2: Suppose $u \neq v$. Then $u \in V_0^v$. Let $v \in V_1 \cap V(G)$. By (i), $f|_{H^v} \in TMRDF(G\circ H)$. Thus, there exist $w, z \in N_{H^v}(u)$ such that $w \in V_2^v$ and $z \in V_3^v$. If $v \in V_2 \cap V(G)$. By (ii), V_3^v dominates V_0^v. Hence, there exist $w, v \in N_{H^v}(u)$ such that $w \in V_3^v$. Also, if $v \in V_3 \cap V(G)$. By (iii), V_2^v dominates V_0^v. Thus, there exist $w, v \in N_{H^v}(u)$ such that $w \in V_2^v$.

Case 1 and Case 2 shows that there exist $w \in V_2$ and $z \in V_3$ such that $w, z \in N_{G\circ H}(u)$.

Now, let $u \in V_1$. If $u \in V(G)$, then there exist $x \in V_2^u \cup V_3^u$ such that $x \in N_{H^u}(u)$ since $f|_{H^u} \in TMRDF(H^u)$. This implies that $x \in V_2 \cup V_3$ and $x \in N_{G\circ H}(u)$. Suppose $u \in V(H^v)$ for some $v \in V(G)$. If $v \in (V_2 \cup V_3) \cap V(G)$, then $v \in (V_2 \cup V_3) \cap N_{G\circ H}(u)$. If $v \in V_0 \cup V_1$, then there exist $w \in V_2^v \cup V_3^v$ such that $w \in N_{H^v}(u)$ since $f|_{H^v} \in TMRDF(H^v)$ by (i). It implies that $w \in V_2 \cup V_3$ and $w \in N_{G\circ H}(u)$. Therefore, $f \in TMRDF(G \circ H)$. □

Corollary 6. *Let G and H be any graph with $|V(G)| = n$ and $|V(H)| = m$ and let $f = (V_0, V_1, V_2, V_3)$ be a γ_{tmR}-function of $G \circ H$. Then $3n \leq \sum_{a\in V(v+H^v)} f(a) \leq 2n + nm$, for each $v \in V(G)$.*

Proof. Let $v \in V(G)$. If $v \in V_2 \cup V_3$, then $3n \leq \sum_{p\in V(H^v)} f(p) \leq \sum_{a\in V(v+H^v)} f(a) \leq 2n + nm$. Suppose that $v \in V_0$. By Proposition 18, $f|_{H^v} \in TMRDF(H^v)$. Thus, $3n \leq \sum_{a\in V(v+H^v)} f(a) \leq 2n + nm$. If $v \in V_1$, then by Proposition 18, $f|_{H^v} \in TMRDF(H^v)$. Thus, $3n \leq \sum_{p\in V(H^v)} f(p) \leq \sum_{a\in V(v+H^v)} f(a) \leq 2n + nm$. Moreover, the bounds are sharp if $H = K_1$ and $G \in \{P_n, C_n, \overline{K_n}\}$. □

Corollary 7. *Let G be a connected graph of order $n \geq 1$ and K_m be the complete graph, then*

$$\gamma_{tmR}(G \circ K_m) = \begin{cases} 4n, & \text{if } m = 2. \\ 5n, & \text{if } m \geq 3. \end{cases}$$

Proof. If $n = 1$, then $G \circ K_m = K_{m+1}$. Hence, if $m = 2$, $\gamma_{tmR}(G \circ K_2) = \gamma_{tmR}(K_3) = 4$ by Proposition 7 (ii). If $m \geq 4$, then $\gamma_{tmR}(K_{m+1}) = 5$ by Proposition 8. Now, If $n > 1$, then for $m = 2$, let $V(K_2) = \{x, y\}$ and $V(G) = \{v_1, v_2, \cdots, v_n\}$. Define a function $f = (V_0, V_1, V_2, V_3)$ on $V(G \circ K_2)$ where $V_0 = \varnothing = V_3, V_1 = \bigcup_{v \in V(G)} V(H^v), V_2 = V(G)$. Then $f \in TMRDF(G \circ K_2)$. It follows that $\gamma_{tmR}(G \circ K_2) \leq 4n$. Now, suppose that $g = (W_0, W_1, W_2, W_3)$ is a γ_{tmR}-function of $G \circ K_2$. If $W_0 = \varnothing$, then $W_3 = \varnothing$. Since g is a γ_{tmR}-function of $G \circ K_2$, $\gamma_{tmR}(G \circ K_2) = \omega^{tmR}_{G \circ K_2}(g) \geq 4n$. If $|W_0| \neq 0$, then $|W_2| \geq 1$ and $|W_3| \geq 1$. It follows that $\gamma_{tmR}(G \circ K_2) = \omega^{tmR}_{G \circ K_2}(g) = 2|W_2| + 3|W_3| \geq 4n$. Therefore, $\gamma_{tmR}(G \circ K_2) = 4n$. For $m \geq 3$, let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and WLOG, pick a vertex $u \in V(K_m)$. Define a function $f = (V_0, V_1, V_2, V_3)$ on $V(G \circ K_m)$ by

$$f(x) = \begin{cases} 3, & \text{if } x \in V(G). \\ 2, & \text{if } x \in \bigcup_{v \in V(G)} V(u^v). \\ 0, & \text{if } x \in \bigcup_{v \in V(G)} V((H \setminus u)^v). \end{cases}$$

Then $f \in TMRDF(G \circ K_m)$. It follows that $\gamma_{tmR}(G \circ K_m) \leq 5n$. Now, suppose that $g = (W_0, W_1, W_2, W_3)$ is a γ_{tmR}-function of $G \circ K_m$. If $W_0 = \varnothing$, then $W_3 = \varnothing$. Since g is a γ_{tmR}-function of $G \circ K_m$, $\gamma_{tmR}(G \circ K_m) = \omega^{tmR}_{G \circ K_m}(g) \geq 5n$. If $|W_0| \neq 0$, then $|W_2| \geq 1$ and $|W_3| \geq 1$. It follows that $\gamma_{tmR}(G \circ K_m) = \omega^{tmR}_{G \circ K_m}(g) = 2|W_2| + 3|W_3| \geq 5n$. Therefore, $\gamma_{tmR}(G \circ K_m) = 5n$. □

Corollary 8. *If K_n is a complete graph of order $n \geq 1$, then*

(i) $\gamma_{tmR}(K_1 \circ \overline{K_n}) = n + 2$.

(ii) $\gamma_{tmR}(\overline{K_n} \circ K_1) = 3n$.

Proof. Statement (i) follows from the fact that $K_1 \circ \overline{K_n} = S_n$ and by Proposition 15, $\gamma_{tmR}(K_1 \circ \overline{K_n}) = \gamma_{tmR}(S_n) = n + 2$. For (ii), Note that $\overline{K_n} \circ K_1$ is the disjoint union n copies of K_2. Using proposition 7 (i) and Proposition 9, we have $\gamma_{tmR}(\overline{K_n} \circ K_1) = 3n$. □

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References

[1] E. J. Cockayne, P. M. Dreyer Sr., S. M. Hedetniemi, and S. T. Hedetniemi. Roman domination in graphs. *Discrete Mathematics*, 278(1-3), 2004.

[2] C. S. ReVelle and K. E. Rosing. Defendens imperium romanum: a classical problem in military strategy. *American Mathematical Monthly*, 107(7):585–594, 2000.

[3] I. Stewart. Defend the roman empire! *Scientific American*, 281(6):136–139, 1999.

[4] A. A. Hossein, A. H. Michael, S. Vladimir, and G. Y. Ismael. Total roman domination in graphs. *Applicable Analysis and Discrete Mathematics*, 10(2):501–517, 2016.

[5] A. H. Hassan and A. O. Ahmed. Modern roman domination in graphs. *Basrah Journal of Agricultural Sciences*, August 2018.

[6] A. O. Ahmed and N. A. Manal. Calculating modern roman domination of fan graph and double fan graph. *Journal of Applied Sciences and Nanotechnology*, June 2022.

[7] E. W. Chambers et al. Extremal problems for roman domination. *SIAM Journal on Discrete Mathematics*, 2004.

[8] J. B. Cariaga and F. P. Jamil. On double roman dominating functions in graphs. *European Journal of Pure and Applied Mathematics*, 16(2):847–863, 2023.

[9] L. M. Paleta and F. P. Jamil. More on perfect roman domination in graphs. *European Journal of Pure and Applied Mathematics*, 13(3):529–548, 2020.

[10] M. A. Henning and S. T. Hedetniemi. Defending the roman empire—a new strategy. *Discrete Mathematics*, 266(1-3), 2003.

[11] R. J. Fortosa, F. P. Jamil, and S. R. Canoy. Convex roman dominating functions on graphs under some binary operations. *European Journal of Pure and Applied Mathematics*, 17(2):1335–1351, 2024.

[12] S. R. Canoy Jr, F. P. Jamil, and S. M. Menchavez. Hop italian domination in graphs. *European Journal of Pure and Applied Mathematics*, 16(4):2431–2449, 2023.

[13] S. Salah, A. A. Omran, and M. N. Al-Harere. Modern roman domination on two operations in certain graphs. In *AIP Conference Proceedings*, volume 2386, page 060014, 2022.

[14] S. S. Majeed, A. A. Omran, and M. N. Yaqoob. Modern roman domination of corona of cycle graph with some certain graphs. *International Journal of Mathematics and Computer Science*, January 2022.

[15] F. Buckley and F. Harary. *Distance in Graphs.* Addison-Wesley, Redwood City, CA, 1990.

[16] E. J. Cockayne and S. T. Hedetniemi. Towards a theory of domination in graphs. *Networks: An International Journal*, 1977.