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Global Well Posedness for Damped Wave Models with Damping in the Memory

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Abstract. This paper aims to study the Cauchy problem for damped wave models with a dissipative memory term. The main objective is to establish global (in time) well-posedness results for both energy solutions and higher-regularity solutions, determine the critical exponent in the Fujita sense, and investigate the influence of nonlinear memory on the Fujita exponent. Using modern tools from harmonic analysis and the Banach fixed point method, we show several results by taking into consideration different regularity properties of the initial data.

2020 Mathematics Subject Classifications: 35L15, 35L71, 35B44, 35L05, 33C15

Key Words and Phrases: Damped wave equation, nonlinear memory, Cauchy problem, energy solutions, Matsumura type estimates, fixed point theorem

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1. Introduction

In this paper, we are interesting in the study of a Cauchy problem for a damped wave model with dissipative term as fractional semilinear power-type non-linearity on the right-hand side. The most important gaol is to study carefully the influence of the dissipative memory term on the range of global (in time) existence of energy solutions for different regularities. In order to satisfy our aim we prove several results of global existence and find the critical exponent in Fujita sense for our model by using matsumura type estimates and Banach fixed point theorem. For physical application we refer the reader to check Eg. [1–5] and the references therein. First, let us introduce some crucial notations used throughout this paper

Notations

- $(x)_+ = \max\{x, 0\}$: The positive part of x.
- u_t: The first partial differential of u with respect to t.
- $*_x$: The convolution product with respect to the $x-$variable.
- p^* or p_{Fuj}: The critical exponent in Fujita sense.
- $|D|^\sigma$: Pseudo-differential operator of order σ.
- $\dot{H}$: Homogeneous Sobolev space.

2. Background and preliminaries

2.1. Background and Motivation

Recently the damped wave equation

$$u_{tt} - \Delta u + u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \tag{1}$$

is considered by A. Fino in [6] where he proved the global (in time) existence of energy solution by using the weighted energy method and blow-up results by the test function method. In particular he showed that the critical exponent in Fujita sense in the $L^1 \cap L^2-$theory for the Cauchy problem of equation (1) is

$$p^*(n, \gamma) = \max\Big\{p_\gamma(n); \frac{1}{\gamma}\Big\}, \text{ where } p_\gamma(n) = 1 + \frac{2(2-\gamma)}{(n - 2(1-\gamma))_+}, \tag{2}$$

where $(x)_+$ stands for the positive part of x,this means that $(n - 2(1-\gamma))_+ = \max\{n - 2(1-\gamma), 0\}$. In [7], where the same model is considered, M. D'Abbicco has improved some results of global existence by using Matsumura type estimates but he didn't investigate blow-up. In particular he proposed the same critical exponent proposed in [6]. In fact the authors proved in [6] and [7] that the term u_{tt} does not influence the critical exponent since

Cazenave *et al* proved in [8] that the critical exponent is the same (2) for the corresponding heat equation with non linear memory

$$u_t - \Delta u = \int_0^t (t-\tau)^{-\gamma} u(\tau,\cdot)|u(\tau,\cdot)|^{p-1} d\tau, \;\; (t,x) \in (0,\infty)\times\mathbb{R}^n. \tag{3}$$

In the other hand, the semilinear Cauchy problem corresponding to (1)

$$u_{tt} - \Delta u + u_t = |u|^p, \tag{4}$$

is considered by many authors, we refer the reader to eg. [9], [10] where they showed that the critical exponent in the $L^1 \cap L^2$ theory for Cauchy problem of (4) is the Fujita exponent

$$p_{Fuj} = 1 + \frac{n}{2}. \tag{5}$$

In the recent paper [11], we proved that some class of effective damping influence the critical exponent in Fujita sense. In this paper we will show that the dissipative nonlinear memory dominates the damping term u_t by considering the following Cauchy problem:

$$\begin{array}{c} u_{tt} - \Delta u + u_t = \int_0^t (t-\tau)^{-\gamma} |u_t(\tau,x)|^p \, d\tau, \;\; (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \;\; x \in \mathbb{R}^n, \end{array} \tag{6}$$

where $\gamma \in (0,1)$ and $p > 1$. Here, the dissipative memory has to be seen as Riemann-Liouville fractional integral to a constant up of $|u_t|^p$. For this reason the non linear term in (6) is interpreted as a fractional power type non linearity.

Finally, we finish this discussion by the connexion between the semilinar damped wave equation and the damped wave equation with non linear memory by noting that

$$\int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds \longrightarrow \Gamma(1-\gamma)|u(t,\cdot)|^p \quad \text{as} \;\; \gamma \longrightarrow 1, \tag{7}$$

in distribution sense, where Γ is the Euler gamma function. For this reason we have

$$\lim_{\gamma\to 1} p^*(n,\gamma) = p_{Fuj}, \tag{8}$$

2.2. Representation of the solution

The presence of the nonlinear term on the right-hand side suggests to us the apply the Duhamel's principle where the study of the Cauchy problem (6) reduces to the study of the following Cauchy problem:

$$\begin{array}{c} u_{tt} - \Delta u + u_t = 0, \;\; (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \;\; x \in \mathbb{R}^n, \end{array} \tag{9}$$

and the family of parameter-dependent Cauchy problems

$$\begin{array}{c} v_{tt} - \Delta v + v_t = 0, \;\; (t,x) \in (\tau,\infty)\times\mathbb{R}^n, \\ v(\tau,x) = 0, \;\; x \in \mathbb{R}^n, \;\; \tau \geq 0, \\ v_t(\tau,x) = h(\tau,x) := \int_0^\tau (\tau-s)^{-\gamma} |u_t(s,x)|^p \, ds, \;\; x \in \mathbb{R}^n, \;\; \tau \geq 0. \end{array} \tag{10}$$

Denoting $v = v(t, \tau, x)$ the solution of Cauchy problem (10), and $u^{lin} = u^{lin}(t, x)$ the solution of Cauchy problem (9), then the solution $u = u(t, x)$ of the Cauchy problem (6) is formally given by

$$u(t, x) = u^{lin}(t, x) + u^{nl}(t, x), \tag{11}$$

where

$$u^{nl}(t, x) := \int_0^t v(t, \tau, x)\, d\tau.$$

By using Fourier transform argument, the solution $u^{lin} = u^{lin}(t, x)$ of Cauchy problem (9) is given by

$$u^{lin}(t, x) = E_0(t, 0, x) *_x u_0(x) + E_1(t, 0, x) *_x u_1(x), \tag{12}$$

where $E_0 = E_0(t, 0, x)$ and $E_1 = E_1(t, 0, x)$ are the fundamental solutions for the Cauchy problem (9), that is, E_0 corresponds to the initial data $(u_0, u_1) = (\delta_0, 0)$ and $E_1 = E_1(t, 0, x)$ corresponds to the initial data $(u_0, u_1) = (0, \delta_0)$, where δ_0 denotes the Dirac Delta-distribution supported in 0. Here and in the sequel, $*_x$ denotes for the convolution with respect to the $x-$variables.

Similarly, the solution $v = v(t, \tau, x)$ of the Cauchy problem (10) is given by

$$\begin{gathered} v(t, \tau, x) = E_1(t, \tau, x) *_x h(\tau, u), \\ \text{where } h(\tau, u) = \int_0^\tau (\tau - s)^{-\gamma} |u_t(s, x)|^p\, ds, \quad t > \tau \geq 0, \end{gathered} \tag{13}$$

since $v(\tau, x) = 0$ for all $x \in \mathbb{R}^n$ and $\tau \geq 0$. By using (11), (12) and (13) the solution of the Cauchy problem (6) is formally given as a solution of the fixed point equation

$$u(t, x) = E_0(t, 0, x) *_x u_0(x) + E_1(t, 0, x) *_x u_1(x) + \int_0^t E_1(t, \tau, x) *_x h(\tau, u)\, d\tau.$$

Since the coefficients of the linear problem (9) associated to the cauchy problem (6) are constants then the considered model (6) is invariant by translation. For this reason we can make a shift of the solution in the non linear part. In this way, the solution of Cauchy problem (6) is formally represented as

$$\begin{aligned} u(t, x) = {} & E_0(t, 0, x) *_x u_0(x) + E_1(t, 0, x) *_x u_1(x) \\ & + \int_0^t E_1(t - \tau, 0, x) *_x h(\tau, u)\, d\tau. \end{aligned} \tag{14}$$

Now, let us introduce some notations that will be used in the sequel. For all $T > 0$ we denote by $X(T)$ the space of solutions to the Cauchy problem (6). For all $u \in X(T)$ we define the mapping N as follows:

$$\begin{aligned} N : u \in X(T) \to {} & N(u) \\ & = E_0(t, 0, x) * u_0(x) + E_1(t, 0, x) * u_1(x) + \int_0^t E_1(t - \tau, 0, x) *_x h(\tau, u)\, d\tau. \end{aligned}$$

Our main strategy is to prove well-posedness results for solutions to (6) as solutions of the fixed point equation $u = N(u)$ by proving the following inequalities for all $u, v \in X(T)$:

$$\|Nu\|_{X(T)} \leq C\|(u_0, u_1)\|_{\mathcal{A}_{\sigma,m}} + C\|u\|^p_{X(T)}, \tag{15}$$

$$\|Nu - Nv\|_{X(T)} \leq C\|u - v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big), \tag{16}$$

where the norm $\|\cdot\|_{X(T)}$ in $X(T)$ will be defined later in a suitable way and makes $X(T)$ as a Banach space. For the further considerations we introduce the scale $\{\mathcal{A}_{m,\sigma}\}$ of Banach spaces with $\sigma \geq 1$ and $m \in [1, 2)$, where

$$\mathcal{A}_{m,\sigma} = \mathcal{A}_{m,\sigma}(\mathbb{R}^n) = \big(H^\sigma(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(H^{\sigma-1}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big). \tag{17}$$

2.3. Outline of this paper

This paper is structured to help the reader clearly understand its objectives. It consists of six sections and an appendix. Section 1 provides the work and outlines its purpose. Section 2 presents the background and preliminaries, including the strategy of proof. In Section 3, we states the main results, which are proved in Section 4. For brevity, the proof of Theorem 4 is omitted and breviary discussed since it follows the same arguments as the proof of Theorem 3. Section 5 concludes the paper, highlighting in particular its novelty and main contributions. Section 6 contains our acknowledgements, followed by a bibliography arranged in alphabetical order. The paper ends with an appendix, which gathers the main tools used, especially in Section 4, together with additional remarks.

Throughout the present paper we write $f \lesssim g$ when there exists a constant $C > 0$ such that $f \leq Cg$, and $f \approx g$ when $g \lesssim f \lesssim g$. Nonnegative constants C or C_j, $j \in \mathbb{N}$, are always supposed to be independent of $T > 0$. For the sake of brevity we sometimes put for all $n \in \mathbb{N}^*$ and $(k, j) \in \mathbb{N}^2$

$$J_n^{(k,j)}(t) = \int_0^t (1 + t - \tau)^{-\frac{n}{4}-\frac{k}{2}-j} \int_0^\tau (\tau - s)^{-\gamma}(1 + s)^{-\beta} ds d\tau. \tag{18}$$

3. Main result

3.1. Global existence of energy solution in low dimension

In term of comparison, the dissipative term u_t does not influence the critical exponent since the two models (3) and (1) have the same critical exponent in Fujita sense. In the other hand, one can note that the non linear memory influence the critical exponent by comparing the critical exponent of (1) and (4) and there is continuity with respect to γ (see (8)). In this section we study the influence of the damping term in the memory on the critical exponent by comparing the critical exponent of Cauchy problem (6) with the one obtained for (1). For this reason, we begin by taking $m = \sigma = 1$ in (17). In Section 3.3, we will see that the range of admissible p for global existence can not be improved even by adding an additional regularity. So, let us fix in this section $m = \sigma = 1$ in (17), that is the data are from the energy space without additional regularity (ie. $m = 1$ in (17)), and set for all $n \geq 1$ and $\gamma \in (0,1)$

$$p_\gamma = p_\gamma(n) := \frac{1}{\gamma}. \tag{19}$$

Then we have

Theorem 1. *Assume that $\gamma \in (1/2, 1)$ and $p > p_\gamma$ for $n = 1$ or $\gamma \in (0,1)$ and $p > p_\gamma$ for $n = 2$. Then there exists $\varepsilon > 0$ such that for any initial data*

$$(u_0, u_1) \in \mathcal{A}_{1,1} := H^1 \cap L^1 \times L^2 \cap L^1 \quad \text{with } \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} \leq \varepsilon,$$

there exists a unique global (in time) solution to Cauchy problem (6)

$$u \in \mathcal{C}\big([0,\infty), H^1\big) \cap \mathcal{C}^1\big([0,\infty), L^2\big).$$

Moreover, the solution satisfies the following Matsumura type decay estimates

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}-\gamma+1} \|(u_0,u_1)\|_{\mathcal{A}_{1,1}},$$

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{\frac{1}{4}-\gamma} \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} & \text{if } n = 1, \\ (1+t)^{-\gamma} \log(2+t) \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} & \text{if } n = 2, \end{cases}$$

$$\|\partial_t^j \nabla^k u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} \quad \text{for all } j \geq 1 \text{ and } k \geq 2.$$

3.2. Global existence in high dimension

In Section 3.1 we presented global existence for low dimension ($n = 1$ and $n = 2$). In that case the admissible range for p is superiorly unbounded. In space dimensional $n \geq 3$, there appears an upper bound for the admissible values for p. This upper bound is caused by the application of Gagliardo-Nirenberg inequality (see condition 108). So, let us, first, state the global (in time) existence results

Theorem 2. *Let $n \geq 3$. Assume that $\gamma \in (\frac{n-2}{n}, 1)$ and $p_\gamma < p \leq p_{GN} := \frac{n}{n-2}$. Then there exists $\varepsilon > 0$ such that for any initial data*

$$(u_0, u_1) \in \mathcal{A}_{1,1} := H^1 \cap L^1 \times L^2 \cap L^1 \quad \text{with } \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} \leq \varepsilon,$$

there exists a unique global (in time) solution to Cauchy problem (6)

$$u \in \mathcal{C}\big([0,\infty), H^1\big) \cap \mathcal{C}^1\big([0,\infty), L^2\big).$$

Moreover, the solution satisfies the following Matsumura type decay estimates

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{\frac{1}{4}-\gamma} \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} & \text{if } n = 3, \\ (1+t)^{-\gamma} \log(2+t) \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} & \text{if } n = 4, \\ (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} & \text{if } n \geq 5, \end{cases} \tag{20}$$

$$\|u(t,\cdot)\|_{\dot{H}^{1/2}} \lesssim \begin{cases} (1+t)^{-\gamma} \log(2+t) \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} & \text{if } n = 3, \\ (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}_{1,1}} & \text{if } n \geq 4, \end{cases} \tag{21}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}_{1,1}}, \tag{22}$$

$$\|\partial_t u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}_{1,1}}. \tag{23}$$

3.3. Global existence of high regular solution with additional regularity $m \in [1, 2)$

In this section we require an additional regularity on the data. However, we choose $m \in (1, 2)$ and $\sigma > 1$ in the data space (17). The purpose of this section is to show that the regularity of the data does not influence the critical exponent, although the additional regularity improve somehow the range of admissible p as it is well known. First, let us set for all $n \geq 3$, $m \in (1, 2)$, $\gamma \in (0, 1)$ and $\sigma > 1$

$$p_{\gamma,\,m}(n) = \frac{1}{\gamma}. \tag{24}$$

Then we have

Theorem 3. *Assume that $n \geq 3$, $\sigma \in (1, \frac{n}{2})$, $\gamma \in (0, 1)$ and $m \in (1, 2)$ such that*

$$\frac{n}{2}\Big(\frac{1}{m} - \frac{1}{2}\Big) < 1, \qquad \frac{n}{2}\Big(\frac{1}{m} - \frac{1}{2}\Big) + \frac{\sigma}{2} > 1, \tag{25}$$

and p satisfies the condition

$$p > \max\big\{\lceil\sigma\rceil;\, p_{\gamma,m}(n)\big\}, \tag{26}$$

Then, there exists a positive constant ε_0 such that for any initial data

$$(u_0, u_1) \in \mathcal{A}_{m,\sigma}(\mathbb{R}^n) \quad \textit{satisfying} \quad \|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}} \leq \varepsilon \textit{ for all } \varepsilon \leq \varepsilon_0,$$

there is a uniquely determined global (in time) energy solution

$$u \in \mathcal{C}\big([0, \infty), H^{\sigma}\big) \cap \mathcal{C}^1\big([0, \infty), H^{\sigma-1}\big)$$

to the Cauchy problem (6). Moreover, the solution satisfies the following estimates:

$$\begin{aligned}
&\|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{1-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\gamma}\|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}},\\
&\||D|^{\sigma}u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\gamma}\|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}},\\
&\|u_t(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\gamma}\|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}},\\
&\||D|^{\sigma-1}u_t(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\gamma}\|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}}.
\end{aligned}$$

The second result we have is

Theorem 4. *Let us assume $n \geq 3$, $\sigma \in (1, \frac{n}{2})$, $\gamma \in (0, 1)$ and $m \in [1, 2)$ such that*

$$\frac{n}{2}\Big(\frac{1}{m} - \frac{1}{2}\Big) \geq 1, \tag{27}$$

and p satisfies the condition

$$\max\big\{\lceil\sigma\rceil;\, p_{\gamma,m}(n)\big\} < p \leq p_{GN} := \frac{n}{n-2}, \tag{28}$$

Then, there exists a positive constant ε_0 such that for any initial data

$$(u_0, u_1) \in \mathcal{A}_{m,\sigma}(\mathbb{R}^n) \textit{satisfying} \quad \|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}} \leq \varepsilon \textit{ for all } \varepsilon \leq \varepsilon_0,$$

there is a uniquely determined global (in time) energy solution

$$u \in \mathcal{C}\big([0, \infty), H^\sigma\big) \cap \mathcal{C}^1\big([0, \infty), H^{\sigma-1}\big)$$

to the Cauchy problem (6). Moreover, the solution satisfies the following estimates:

$$\|u(t, \cdot)\|_{L^2} \lesssim \left\{ \begin{array}{ll} (1+t)^{-\gamma} \log(2+t) \|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}} & \textit{if } n \geq 4 \textit{ and } m = \frac{2n}{n+4}, \\ (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}} & \textit{else} \end{array} \right. ,$$
$$\||D|^\sigma u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}},$$
$$\|u_t(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}},$$
$$\||D|^{\sigma-1} u_t(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}_{m,\sigma}}.$$

4. Proof of the results

4.1. Proof of Theorems 1 and 2

4.1.1. Proof of Theorem 1

Let us introduce, for $T > 0$, the space of energy solutions

$$X(T) = \mathcal{C}\big([0, T], H^1(\mathbb{R})\big) \cap \mathcal{C}^1\big([0, T], L^2(\mathbb{R})\big) \tag{29}$$

with the norm

$$\|u\|_{X(T)} = \sup_{0 \leq t \leq T} \Big\{ (1+t)^{\frac{n}{4}+\gamma-1} \|u(t, \cdot)\|_{L^2} + \ell_n(t)^{-1} (1+t)^\gamma \|\nabla u(t, \cdot)\|_{L^2}$$
$$+ (1+t)^\gamma \|u_t(t, \cdot)\|_{L^2} + (1+t)^\gamma \|\nabla u_t(t, \cdot)\|_{L^2} \Big\}, \tag{30}$$

where

$$\ell_n(t) = \left\{ \begin{array}{ll} (1+t)^{\frac{1}{4}} & \text{if } n = 1, \\ \log(2+t) & \text{if } n = 2. \end{array} \right. \tag{31}$$

First, let us prove the inequality (15).
The inequality

$$\|u^{lin}\|_{X(T)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}_{1,1}}$$

is an immediate consequence of Proposition 1 and Corollary 1 for $\tau = 0$ and $h(0, u(x)) = u_1(x)$. It remains to show the inequality

$$\|u^{nl}\|_{X(T)} \lesssim \|u\|^p_{X(T)}. \tag{32}$$

Taking into account the results of Corollary 1 we have

$$\|u^{nl}(t, \cdot)\|_{L^2} \lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4}} \int_0^\tau (\tau - s)^{-\gamma} \||u_t(s, \cdot)|^p\|_{L^1 \cap L^2} \, ds d\tau. \tag{33}$$

Since

$$\||u_t(s,\cdot)|^p\|_{L^1\cap L^2} = \|u_t(s,\cdot)\|^p_{L^p\cap L^{2p}} = \|u_t(s,\cdot)\|^p_{L^p} + \|u_t(s,\cdot)\|^p_{L^{2p}},$$

then, it is obvious that one has to estimate the norms:

$$\||u_t(s,\cdot)\|^p_{L^p} \quad \text{and} \quad \||u_t(s,\cdot)\|^p_{L^{2p}}.$$

For this purpose, we apply the classical Gagliardo-Nirenberg inequality (107). In this way we obtain for $j = 1,2$ the chain of inequalities

$$\begin{aligned}\|u_t(s,\cdot)\|^p_{L^{jp}} &\lesssim \|u_t(s,\cdot)\|^{p(1-\theta_j(p))}_{L^2} \|\nabla u_t(s,\cdot)\|^{p\theta_j(p)}_{L^2} \\ &\lesssim (1+s)^{-\gamma p}\|u\|^p_{X(T)} \\ &\lesssim (1+s)^{-\beta}\|u\|^p_{X(T)},\end{aligned} \tag{34}$$

where

$$\theta_j(p) = n\Big(\frac{1}{2} - \frac{1}{jp}\Big), \tag{35}$$

$\theta_j(p) \in [0,1]$ if and only if $p \geq 2$ since $n \leq 2$, and

$$\beta = \gamma p. \tag{36}$$

Taking $j = 1$ and $j = 2$ in (34), respectively, we find

$$\|u(s,\cdot)\|^p_{L^p\cap L^{2p}} \lesssim (1+s)^{-\beta}\|u\|^p_{X(T)}. \tag{37}$$

Noting that $\beta > 1$ if and only if $p > \dfrac{1}{\gamma}$. Including (37) in (33) we find

$$\|u^{nl}(t,\cdot)\|_{L^2} \lesssim J_n^{(0,0)}(t)\|u\|^p_{X(T)}, \tag{38}$$

where $J_n^{(0,0)}(t)$ is defined by (18) (case: $k = j = 0$). Since $\beta > 1$ we estimate $J_n^{(0,0)}(t)$ by using Lemma 1 as follows:

$$J_n^{(0,0)}(t) \lesssim (1+t)^{-\frac{n}{4}+1-\gamma}. \tag{39}$$

Then, introducing the estimate (39) into (38) we find

$$\|u^{nl}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}+1-\gamma}\|u\|^p_{X(T)}. \tag{40}$$

Now we deal with $\|\partial_t^j \nabla^k u^{nl}(t,\cdot)\|_{L^2}$ for all $k, j \in \mathbb{N}$ such that $1 \leq k + j \leq 2$. Again, taking into account the results of Corollary 1 we have the estimate

$$\|\partial_t^j \nabla^k u^{nl}(t,\cdot)\|_{L^2} \lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}-j} \int_0^\tau (\tau-s)^{-\gamma} \||u_t(s,\cdot)|^p\|_{L^1\cap L^2}\, ds d\tau.$$

Using the estimate (37) we get

$$\|\nabla^k \partial_t^j u^{nl}(t,\cdot)\|_{L^2} \lesssim J_n^{(k,j)}(t)\|u\|^p_{X(T)}, \tag{41}$$

where $J_n^{(k,j)}(t)$ is defined by (18). Since $\beta > 1$ we may estimate $J_n^{(k,j)}(t)$, after using Lemma 1 as follows:

$$J_n^{(k,j)}(t) \lesssim \begin{cases} (1+t)^{-\frac{3}{4}} & \text{if } n = k = 1 \text{ and } j = 0, \\ (1+t)^{-\gamma}\log(2+t) & \text{if } n = 2,\ k = 1 \text{ and } j = 0, \\ (1+t)^{-\gamma} & \text{if } n = 1, 2 \text{ and } k + j \geq 1. \end{cases} \tag{42}$$

Putting (42) into (41) we derive the estimates

$$\|\nabla u^{nl}(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{3}{4}}\|u\|_{X(T)}^p & \text{if } n = k = 1 \text{ and } j = 0, \\ (1+t)^{-\gamma}\log(2+t)\|u\|_{X(T)}^p & \text{if } n = 2,\ k = 1 \text{ and } j = 0, \\ (1+t)^{-\gamma}\|u\|_{X(T)}^p & \text{if } n = 1, 2 \text{ and } 1 \leq k + j \leq 2. \end{cases} \tag{43}$$

Then, the inequality (32) is concluded from (43), (40) and the definition (30) of the norm in $X(T)$.

Now, let us turn to the inequality (16).

By definition of the operator N and the results of Corollary 1, we have

$$\begin{aligned} \|(Nu - Nv)(t,\cdot)\|_{L^2} \lesssim &\int_0^t (1+t-\tau)^{-\frac{n}{4}} \\ &\times \int_0^\tau (\tau - s)^{-\gamma} \big\| |u_t(s,\cdot)|^p - |v_t(s,\cdot)|^p \big\|_{L^1 \cap L^2}\, ds d\tau. \end{aligned} \tag{44}$$

Then we have to estimate

$$\||u_t(s,\cdot)|^p - |v_t(s,\cdot)|^p\|_{L^1} \quad \text{and} \quad \||u_t(s,\cdot)|^p - |v_t(s,\cdot)|^p\|_{L^2}.$$

First, we have by Hölder's inequality for $j = 1, 2$,

$$\||u_t|^p - |v_t|^p\|_{L^j} \lesssim \|u_t - v_t\|_{L^{jp}} \big(\|u_t\|_{L^{jp}}^{p-1} + \|v_t\|_{L^{jp}}^{p-1}\big). \tag{45}$$

Then, the aim is to estimate the following terms for $j = 1$ and $j = 2$:

$$\|u_t - v_t\|_{L^{jp}}, \quad \|u_t\|_{L^{jp}}^{p-1} \quad \text{and} \quad \|v_t\|_{L^{jp}}^{p-1}.$$

Let us begin with the estimate of the term $\|u_t(s,\cdot) - v_t(s,\cdot)\|_{L^{jp}}$. By using Gagliardo-Nirenberg inequality (107) with $k = 1$ and $q = p$ we estimate

$$\begin{aligned} \|u_t(s,\cdot) - v_t(s,\cdot)\|_{L^{jp}} &\lesssim \|u_t(s,\cdot) - v_t(s,\cdot)\|_{L^2}^{1-\theta_j(p)} \big\| |\nabla^k (u_t - v_t)(s,\cdot) \big\|_{L^2}^{\theta_j(p)} \\ &\lesssim (1+s)^{-\gamma(1-\theta_j(p))}(1+s)^{-\gamma\theta_j(p)}\|u - v\|_{X(T)} \\ &\lesssim (1+s)^{-\gamma}\|u - v\|_{X(T)}, \end{aligned} \tag{46}$$

where $\theta_j(p)$ is defined (35). We use the same tools to estimate $\|u_t(s,\cdot)\|_{L^{jp}}^{p-1}$. We get

$$\|u_t(s,\cdot)\|_{L^{jp}}^{p-1} \lesssim \|u_t(s,\cdot)\|_{L^2}^{(p-1)(1-\theta_j(p))} \|\nabla u_t(s,\cdot)\|_{L^2}^{(p-1)\theta_j(p)}$$

$$\lesssim (1+s)^{-\gamma(p-1)(1-\theta_j(p))}(1+s)^{-\gamma(p-1)\theta_j(p)}\|u\|^{p-1}_{X(T)}$$
$$\lesssim (1+s)^{-\gamma(p-1)}\|u\|^{p-1}_{X(T)}. \tag{47}$$

In the same way we derive

$$\|v_t(s,\cdot)\|^{p-1}_{L^{jp}} \lesssim (1+s)^{-\gamma(p-1)}\|v\|^{p-1}_{X(T)}. \tag{48}$$

Finally, by (48), (47) and (46) we get from (45), for $j=1,\,2$ the estimates

$$\||u_t(s,\cdot)|^p - |v_t(s,\cdot)|^p\|_{L^j} \lesssim (1+s)^{-\beta}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big), \tag{49}$$

where β is given by (36). As a consequence, it follows from (49) that

$$\||u_t(s,\cdot)|^p - |v_t(s,\cdot)|^p\|_{L^1\cap L^2} \lesssim (1+s)^{-\beta}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{50}$$

Plugging the estimate (50) in (44) we find

$$\|(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim J_n^{(0,0)}(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big),$$

where $J_n^{(0,0)}(t)$ is defined by (18) (case: $k=j=0$). Then, using the estimate (39) we may conclude that

$$\|(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}+1-\gamma}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{51}$$

Now, let us turn to estimate the term $\|\partial_t^j\nabla^k(Nu-Nv)(t,\cdot)\|_{L^2}$ for $1\le k+j\le 2$. As we did above we have after using the results of Corollary 2 we have

$$\begin{aligned}\big\|\partial_t^j\nabla^k(Nu-Nv)(t,\cdot)\big\|_{L^2} &\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}-j}\\ &\times \int_0^\tau (\tau-s)^{-\gamma}\||u_t(s,\cdot)|^p - v_t(s,\cdot)|^p\|_{L^1\cap L^2}\,ds d\tau.\end{aligned} \tag{52}$$

Including the estimate (50) into (52) we find

$$\big\|\nabla^k\partial_t^j(Nu-Nv)(t,\cdot)\big\|_{L^2} \lesssim J_n^{(k,j)}(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big), \tag{53}$$

where $J_n^{(k,j)}(t)$ is defined by (18). Using the estimate (42) for $J_n^{(k,j)}(t)$ we find from (53) the estimate

$$\begin{aligned}&\big\|\nabla^k\partial_t^j(Nu-Nv)(t,\cdot)\big\|_{L^2}\\ &\lesssim \begin{cases} (1+t)^{-\frac{3}{4}}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) & \text{if } n=k=1 \text{ and } j=0,\\ \frac{\log(2+t)}{(1+t)^\gamma}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) & \text{if } n=2,\ k=1 \text{ and } j=0,\\ (1+t)^{-\gamma}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) & \text{if } n=1,2 \text{ and } 1\le k+j\le 2.\end{cases}\end{aligned} \tag{54}$$

Finally, the desired inequality (16) is concluded from the estimates (54), (51) and the definition (30) for the norm in $X(T)$. This ends the proof of Theorem 1.

4.1.2. Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1, except, in Theorem 2 an upper bound appears for the range of admissible values of p. This upper bound is caused by the application of Gagliardo-Nirenberg inequality. More precisely, in formula (35), the condition $\theta_j(p) \in [0,1]$ implies

$$\frac{2}{j} \leq p \leq \frac{2n}{(n-2)j}, \tag{55}$$

according to (108). Since condition (55) must hold simultaneously for $j = 1, 2$ we conclude that $2 \leq p \leq \frac{n}{n-2}$. The parameter $\frac{n}{n-2}$ is called Gagliardo-Nirenberg exponent and, usually, denoted by p_{GN}. Since the proofs are essentially the same we omit the details of proof of Theorem 2.

4.2. Proof of Theorems 3 and 4

4.2.1. Proof of Theorem 3

Let us introduce, for $T > 0$, the space of energy solutions

$$X(T) = \mathcal{C}\big([0,T], H^s(\mathbb{R})\big) \cap \mathcal{C}^1\big([0,T], H^{s-1}(\mathbb{R})\big) \tag{56}$$

with the norm

$$\begin{aligned} \|u\|_{X(T)} = \sup_{0\leq t\leq T} \Big\{ &(1+t)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\gamma-1}\big(\|u(t,\cdot)\|_{L^2} + (1+t)^{\gamma}\||D|^{\sigma}u(t,\cdot)\|_{L^2}\big) \\ &+ (1+t)^{\gamma}\|u_t(t,\cdot)\|_{L^2} + (1+t)^{\gamma}\||D|^{\sigma-1}u_t(t,\cdot)\|_{L^2}\Big\}. \end{aligned} \tag{57}$$

As usual, the aim is to prove the inequalities (15) and (16). Let us start with the first one. The inequality

$$\|u^{lin}\|_{X(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^1_{1,0}}$$

is concluded directly by Proposition 2 from [11] and Corollary 2 for $\tau = 0$ and $h(0,u(x)) = u_1(x)$. It remains to show the inequality

$$\|u^{nl}\|_{X(T)} \lesssim \|u\|^p_{X(T)}. \tag{58}$$

Thanks to the results of Proposition 2 we have for $\kappa = 0, \sigma$ (with the convention $|D|^{\kappa} = \mathrm{Id}$ if $\kappa = 0$),

$$\begin{aligned} \||D|^{\kappa}u^{nl}(t,\cdot)\|_{L^2} \lesssim &\int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{\kappa}{2}} \\ &\times \int_0^{\tau} (\tau-s)^{-\gamma}\big\||u(s,\cdot)|^p\big\|_{L^m\cap L^2\cap \dot{H}^{\sigma-1}}\, ds d\tau. \end{aligned} \tag{59}$$

For this reason we shall estimate the norms

$$\||u(s,\cdot)|^p\|_{L^m}, \quad \||u(s,\cdot)|^p\|_{L^2}, \quad \text{and} \quad \||u(s,\cdot)|^p\|_{\dot{H}^{\sigma-1}}.$$

First, we apply fractional Gagliardo-Nirenberg inequality we get for $j = m$ and $j = 2$ the estimates

$$\begin{aligned} \||u_t(s,\cdot)|^p\|_{L^j} &= \|u_t(s,\cdot)\|^p_{L^{jp}} \lesssim \|u_t(s,\cdot)\|^{p(1-\theta_{\sigma,j}(p))}_{L^2} \||D|^\sigma u_t(s,\cdot)\|^{p\theta_{\sigma,j}(p)}_{L^2} \\ &\lesssim (1+s)^{-\beta}\|u\|^p_{X(T)}, \end{aligned} \tag{60}$$

where

$$\beta = \gamma p, \tag{61}$$

and $\theta_{\sigma,j}(p) = \frac{n}{\sigma}(\frac{1}{2} - \frac{1}{jp}) \in [0,1]$ if and only if

$$\begin{cases} \frac{2}{j} \le p & \text{if } n \le 2\sigma, \\ \frac{2}{j} \le p \le \frac{2n}{(n-2\sigma)j} & \text{if } n > 2\sigma. \end{cases}$$

Summarizing, we have

$$\||u(s,\cdot)|^p\|_{L^m\cap L^2} \lesssim (1+s)^{-\beta}\|u\|^p_{X(T)}, \tag{62}$$

where β is given by (61) and

$$\begin{cases} 2 \le p & \text{if } n \le 2\sigma, \\ 2 \le p \le \frac{n}{n-2\sigma} & \text{if } n > 2\sigma. \end{cases} \tag{63}$$

To estimate the norm $\||u_t(s,\cdot)|^p\|_{\dot{H}^{\sigma-1}}$ we apply the fractional chain rule (110) as follows:

$$\begin{aligned} \||u_t(s,\cdot)|^p\|_{\dot{H}^{\sigma-1}} &= \||D|^{\sigma-1}|u_t(s,\cdot)|^p\|_{L^2} \\ &\lesssim \|u_t(s,\cdot)\|^{p-1}_{L^{q_1}}\||D|^{\sigma-1}u_t(s,\cdot)\|_{L^{q_2}} \quad \text{for} \quad p > \lceil \sigma - 1 \rceil, \end{aligned} \tag{64}$$

where

$$\frac{p-1}{q_1} + \frac{1}{q_2} = \frac{1}{2}. \tag{65}$$

A possible choice is, for example, $q_1 = n(p-1)$ and $q_2 = \frac{2n}{n-2}$ for $n \ge 3$. The norm $\|u_t(s,\cdot)\|^{p-1}_{L^{q_1}}$ can be estimated by using classical Gagliardo-Nirenberg inequality (107). In this way we may conclude

$$\begin{aligned} \|u_t(s,\cdot)\|^{p-1}_{L^{q_1}} &\lesssim \|u_t(s,\cdot)\|^{(p-1)(1-\theta_{\sigma,3}(q_1))}_{L^2}\||D|^\sigma u_t(s,\cdot)\|^{(p-1)\theta_{\sigma,3}(q_1)}_{L^2} \\ &\lesssim (1+s)^{-\gamma(p-1)}\|u\|^{p-1}_{X(T)}, \end{aligned} \tag{66}$$

where $\theta_{\sigma,3}(q_1) = \frac{n}{\sigma}(\frac{1}{2} - \frac{1}{q_1}) \in [0,1]$ if and only if

$$2 \le q_1 \le \frac{2n}{n-2\sigma} \qquad \text{if } n > 2\sigma.$$

On the other hand, applying the fractional Gagliardo-Nirenberg inequality (106) gives

$$\||D|^{\sigma-1}u_t(s,\cdot)\|_{L^{q_2}} \lesssim \|u_t(s,\cdot)\|^{1-\theta_{\sigma,\sigma-1}(q_2)}_{L^2}\||D|^\sigma u_t(s,\cdot)\|^{\theta_{\sigma,\sigma-1}(q_2)}_{L^2}$$

$$\lesssim (1+s)^{-\gamma}\|u\|_{X(T)}, \tag{67}$$

where $\theta_{\sigma,\sigma-1}(q_2) = \frac{n}{\sigma}(\frac{1}{2}-\frac{1}{q_2}) + \frac{\sigma-1}{\sigma} \in [\frac{\sigma-1}{\sigma}, 1]$ if and only if

$$2 \leq q_2 \leq \frac{2n}{n-2} \qquad \text{if } n \geq 3.$$

Substituting the estimates (67) and (66) into (64) we obtain the estimate

$$\||u(s,\cdot)|^p\|_{\dot{H}^{\sigma-1}} \lesssim (1+s)^{-\gamma p}\|u\|^p_{X(T)} = (1+s)^{-\beta}\|u\|^p_{X(T)}, \tag{68}$$

where β is defined by (61). Consequently, from the estimates (68) and (62) we conclude the estimate

$$\||u(s,\cdot)|^p\|_{L^m\cap L^2\cap \dot{H}^{\sigma-1}} \lesssim (1+s)^{-\beta}\|u\|^p_{X(T)}, \tag{69}$$

where β is defined by (61). Notice that $\beta > 1$ if and only if $p > p_{\gamma,m}(n)$. Then, plugging (69) into (59) we get

$$\begin{aligned}\||D|^\kappa u^{nl}(t,\cdot)\|_{L^2} &\lesssim \|u\|^p_{X(T)} \int_0^t (1+t-\tau))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{\kappa}{2}} \\ &\times \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\beta}\, ds d\tau \lesssim J_n^{\kappa,0}(t)\|u\|^p_{X(T)},\end{aligned} \tag{70}$$

where $J_n^{\kappa,0}(t)$ is defined by (18) (case $(k,j) = (\kappa,0)$). So, due to Lemma 1 and the assumptions of Theorem 3 we may estimate $J_n^{\kappa,0}(t)$ for $\kappa = 0$ or $\kappa = \sigma$ as follows:

$$J_n^{\kappa,0}(t) \lesssim \begin{cases} (1+t)^{1-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\gamma} & \text{if } \kappa = 0, \\ (1+t)^{-\gamma} & \text{if } \kappa = \sigma. \end{cases} \tag{71}$$

The estimates (71) lead to the following estimates:

$$\|u^{nl}(t,\cdot)\|_{L^2} \lesssim (1+t)^{1-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\gamma}\|u\|^p_{X(T)}, \tag{72}$$

and

$$\big\||D|^\sigma u^{nl}(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\gamma}\|u\|^p_{X(T)}. \tag{73}$$

Now let us turn to estimate the norms $\|u_t^{nl}(t,\cdot)\|_{L^2}$ and $\||D|^{\sigma-1}u_t^{nl}(t,\cdot)\|_{L^2}$. Again, by Proposition 2, we have for $\kappa = 1, \sigma$ the estimates

$$\begin{aligned}\||D|^{\kappa-1}u_t^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{\kappa-1}{2}-1} \\ &\times \int_0^\tau (\tau-s)^{-\gamma}\big\||u(s,\cdot)|^p\big\|_{L^m\cap L^2\cap \dot{H}^{\sigma-1}}\, ds d\tau.\end{aligned} \tag{74}$$

Taking into account the estimate (69) we get from (74) the estimates

$$\||D|^{\kappa-1}u_t^{nl}(t,\cdot)\|_{L^2} \lesssim J_n^{(\kappa-1,1)}(t)\|u\|^p_{X(T)}, \tag{75}$$

where $J_n^{(\kappa-1,1)}(t)$ is defined by (18) (case $(k,j)=(\kappa-1,1)$). Thanks to Lemma 1, the integral $J_n^{(\kappa-1,1)}(t)$ is estimated In both cases $\kappa=1$ and $\kappa=\sigma$, as follows:

$$J_n^{(\kappa-1,1)}(t) \lesssim (1+t)^{-\gamma}. \tag{76}$$

Including the estimate (76) into (75) we find for $\kappa=1$

$$\|u_t^{nl}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma}\|u\|^p_{X(T)}, \tag{77}$$

and for $\kappa=\sigma$

$$\||D|^{\sigma-1}u_t^{nl}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma}\|u\|^p_{X(T)}. \tag{78}$$

Finally, the desired inequality (58) is concluded after using the estimates (72), (73) (77), (78) and the definition (57) of the norm in $X(T)$. Summarizing we obtained (15).
Now, let us turn to the inequality (16). Taking into account the definition of the aaplication N, we have by Proposition 2 for $\kappa=0$ and σ the estimates

$$\begin{aligned}\||D|^{\kappa}(Nu-Nv)(t,\cdot)\|_{L^2} &\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{\kappa}{2}} \\ &\times \int_0^\tau (\tau-s)^{-\gamma}\big\||u_t(s,\cdot)|^p-|v_t(s,\cdot)|^p\big\|_{L^m\cap L^2\cap \dot H^{\sigma-1}}\,dsd\tau.\end{aligned} \tag{79}$$

Then, we have to estimate the norms

$$\big\||u_t(s,\cdot)|^p-|v_t(s,\cdot)|^p\big\|_{L^m}, \quad \big\||u_t(s,\cdot)|^p-|v_t(s,\cdot)|^p\big\|_{L^2}, \quad \text{and } \big\||u_t(s,\cdot)|^p-|v_t(s,\cdot)|^p\big\|_{\dot H^{\sigma-1}}.$$

Following the steps of proof of Theorem 1, we show immediately that

$$\||u_t(s,\cdot)|^p-|v_t(s,\cdot)|^p\|_{L^m\cap L^2} \lesssim (1+s)^{-\beta}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big), \tag{80}$$

where β is as in (61). Then, it remains to estimate the norm $\||u_t(s,\cdot)|^p-|v_t(s,\cdot)|^p\|_{\dot H^{\sigma-1}}$. First, applying Leibniz formula from Proposition 3 allows us to conclude for $p>\lceil\sigma\rceil$ the estimate

$$\begin{aligned}&\||u_t(s,\cdot)|^p-|v_t(s,\cdot)|^p\|_{\dot H^{\sigma-1}} = \||D|^{\sigma-1}\big(|u_t|^p-|v_t|^p\big)(s,\cdot)\|_{L^2}\\
&\lesssim \int_0^1 \big\||D|^{\sigma-1}\left[(u_t-v_t)(u_t-w(u_t-v_t))|u_t-w(u_t-v_t)|^{p-2}\right](s,\cdot)\big\|_{L^2}\,dw\\
&\lesssim \int_0^1 \||D|^{\sigma-1}(u_t-v_t)(s,\cdot)\|_{L^{r_1}}\big\|(u_t-w(u_t-v_t))|u_t-w(u_t-v_t)|^{p-2}(s,\cdot)\big\|_{L^{r_2}}\,dw\\
&\qquad+\int_0^1 \|(u_t-v_t)(s,\cdot)\|_{L^{r_3}}\big\||D|^{\sigma-1}\left[(u_t-w(u_t-v_t))|u_t-w(u_t-v_t)|^{p-2}\right](s,\cdot)\big\|_{L^{r_4}}\,dw\end{aligned}$$

(81)

with
$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4}=\frac{1}{2}. \tag{82}$$

The term $\||D|^{\sigma-1}(u_t-v_t)(s,\cdot)\|_{L^{r_1}}$ can be estimated by using the fractional Gagliardo-Nirenberg inequality (106) in the form

$$\begin{aligned}\||D|^{\sigma-1}(u_t-v_t)(s,\cdot)\|_{L^{r_1}} &\lesssim \|u_t(s,\cdot)-v_t(s,\cdot)\|_{L^2}^{1-\theta_{31}(r_1)}\||D|^{\sigma}(u_t-v_t)(s,\cdot)\|_{L^2}^{\theta_{31}(r_1)}\\ &\lesssim (1+s)^{-\gamma}\|u-v\|_{X(T)}\end{aligned} \tag{83}$$

with $\theta_{31}(r_1)=\frac{n}{\sigma}\left(\frac{1}{2}-\frac{1}{r_1}\right)+\frac{\sigma-1}{\sigma}\in[\frac{\sigma-1}{\sigma},1]\in[0,1]$ if and only if

$$2\le r_1\le\frac{2n}{n-2}\qquad \text{for } n\ge 3.$$

In the same way, by using the classical Gagliardo-Norenberg inequality, we estimate the norm $\big\|(u-w(u-v))|u_t-w(u_t-v_t)|^{p-2}(s,\cdot)\big\|_{L^{r_2}}$ as follows:

$$\begin{aligned}\big\|(u_t-w(u_t-v_t))|u_t-w(u_t-v_t)|^{p-2}(s,\cdot)\big\|_{L^{r_2}} &\lesssim \|\big(u_t-w(u_t-v_t)\big)(s,\cdot)\|_{L^{(p-1)r_2}}^{p-1}\\ &\lesssim \|u_t-w(u_t-v_t)\|_{L^2}^{(p-1)(1-\theta_{32}(r_2))}\||D|^{\sigma}\big(u_t-w(u_t-v_t)\big)\|_{L^2}^{(p-1)\theta_{32}(r_2)}\\ &\lesssim (1+s)^{-\gamma(p-1)}\|u-w(u-v)\|_{X(T)}^{p-1},\end{aligned} \tag{84}$$

where $\theta_{32}(r_2)=\frac{n}{\sigma}\left(\frac{1}{2}-\frac{1}{(p-1)r_2}\right)\in[0,1]$ if and only if

$$\frac{2}{p-1}\le r_2\le\frac{2n}{(p-1)(n-2\sigma)}\qquad \text{if } n>2\sigma.$$

Due to the estimates (84) and (83), we may conclude the estimate

$$\begin{aligned}\||D|^{\sigma-1}(u_t-v_t)(s,\cdot)\|_{L^{r_1}}&\big\|(u_t-w(u_t-v_t))|u_t-w(u_t-v_t)|^{p-2}(s,\cdot)\big\|_{L^{r_2}}\\ &\lesssim (1+s)^{-\beta-\frac{1}{\sigma}(1-a_m)(1-r)(2a_m+\sigma-1)}\|u-v\|_{X(T)}\|u-w(u-v)\|_{X(T)}^{p-1}\\ &\lesssim (1+s)^{-\beta}\|u-v\|_{X(T)}\|u-w(u-v)\|_{X(T)}^{p-1},\end{aligned} \tag{85}$$

where β is as in (61). In order to estimate the norm $\|(u_t-v_t)(s,\cdot)\|_{L^{r_3}}$ we apply the classical Gagliardo-Nirenberg inequality (107) to obtain

$$\begin{aligned}\|(u_t-v_t)(s,\cdot)\|_{L^{r_3}} &\lesssim \|(u_t-v_t)(s,\cdot)\|_{L^2}^{1-\theta_{33}(r_3)}\||D|^{\sigma}(u_t-v_t)(s,\cdot)\|_{L^2}^{\theta_{33}(r_3)}\\ &\lesssim (1+s)^{-\gamma}\|u-v\|_{X(T)},\end{aligned} \tag{86}$$

where $\theta_{33}(r_3)=\frac{n}{\sigma}\left(\frac{1}{2}-\frac{1}{r_3}\right)\in[0,1]$ if and only if

$$2\le r_3\le\frac{2n}{n-2\sigma}\qquad \text{if } n>2\sigma.$$

Now, we turn to estimate the norm $\big\||D|^{\sigma-1}\big((u_t-w(u_t-v_t))|u_t-w(u_t-v_t)|^{p-2}\big)\big\|_{L^{r_4}}$. We can do this by applying the fractional chain rule (110). In this way we get

$$\begin{aligned}&\big\||D|^{\sigma-1}\big((u_t-w(u_t-v_t))|u_t-w(u_t-v_t)|^{p-2}\big)\big\|_{L^{r_4}}\\&\quad\lesssim \big\|u_t-w(u_t-v_t)\|^{p-2}_{L^{r_5}}\||D|^{\sigma-1}(u_t-w(u_t-v_t))\big\|_{L^{r_6}}\end{aligned} \tag{87}$$

with

$$\frac{p-2}{r_5}+\frac{1}{r_6}=\frac{1}{r_4} \text{ and } p>\lceil\sigma\rceil. \tag{88}$$

In one hand the classical Gagliardo-Nirenberg inequality (107) allows us to get

$$\begin{aligned}&\|(u_t-w(u_t-v_t))(s,\cdot)\|^{p-2}_{L^{r_5}}\\&\lesssim \|(u_t-w(u_t-v_t))(s,\cdot)\|^{(p-2)(1-\theta_5(r_5))}_{L^2}\||D|^{\sigma}\big(u_t-w(u_t-v_t)\big)(s,\cdot)\|^{(p-2)\theta_5(r_5)}_{L^2}\\&\lesssim (1+s)^{-\gamma(p-2)}\|u-w(u-v)\|^{p-2}_{X(T)}\end{aligned} \tag{89}$$

with $\theta_5(r_5)=\frac{n}{\sigma}\big(\frac{1}{2}-\frac{1}{r_5}\big)\in[0,1]$ if and only if

$$2\le r_5\le\frac{2n}{n-2\sigma}\qquad \text{if } n>2\sigma.$$

In the other hand, by applying the fractional chain rule (110) we may estimate the norm $\big\||D|^{\sigma-1}(u_t-r(u_t-v_t))\big\|_{L^{r_6}}$ as follows:

$$\begin{aligned}&\big\||D|^{\sigma-1}(u_t-w(u_t-v_t))\big\|_{L^{r_6}}\lesssim\|u_t-w(u_t-v_t)\|^{1-\theta_6(r_6)}_{L^2}\big\||D|^{\sigma}(u_t-w(u_t-v_t))\big\|^{\theta_6(r_6)}_{L^2}\\&\lesssim(1+s)^{-\gamma}\|u-w(u-v)\|_{X(T)}\end{aligned} \tag{90}$$

with $\theta_6=\frac{n}{\sigma}\big(\frac{1}{2}-\frac{1}{r_6}\big)+\frac{\sigma-1}{\sigma}\in[\frac{\sigma-1}{\sigma},1]$ if and only if

$$2\le r_6\le\frac{2n}{n-2}\qquad \text{for } n\ge 3.$$

For r_1 and r_2 we may choose $r_1=\frac{2n}{n-2}$ and $r_2=n$. For $r_3,\cdots,r_6$, we may choose $r_3=r_5=n(p-1)$, $r_6=\frac{2n}{n-2}$ and, consequently, we find $r_4=\frac{2n(p-1)}{n(p-1)-2}$.
Therefore, by using the estimates (89) and (90) we get from (87) the estimate

$$\big\||D|^{\sigma-1}\big[(u-w(u-v))|u-w(u-v)|^{p-2}\big]\big\|_{L^{r_4}}\lesssim(1+s)^{-\gamma(p-1)}\|u-w(u-v)\|^{p-1}_{X(T)}. \tag{91}$$

Hence, both estimates (91) and (86) imply the estimate

$$\begin{aligned}&\|u-v\|_{L^{r_3}}\big\||D|^{\sigma-1}[(u-w(u-v))|u-w(u-v)|^{p-2}]\big\|_{L^{r_4}}\\&\lesssim(1+s)^{-\beta-\frac{1}{\sigma}(2a_m+\sigma-1)(1-a)(1-r)}\|u-v\|_{X(T)}\|u-w(u-v)\|^{p-1}_{X(T)}\\&\lesssim(1+s)^{-\beta}\|u-v\|_{X(T)}\|u-w(u-v)\|^{p-1}_{X(T)},\end{aligned} \tag{92}$$

where β is defined by (61). Then, plugging (85) and (92) into (81) we get

$$\begin{aligned}
&\big\||u(s,\cdot)|^p - |v(s,\cdot)|^p)(s,\cdot)\big\|_{\dot{H}^{\sigma-1}} \\
&\quad\lesssim (1+s)^{-\beta}\|u-v\|_{X(T)}\int_0^1 \|u-w(u-v)\|_{X(T)}^{p-1}\,dw \\
&\quad\lesssim (1+s)^{-\beta}\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big). \qquad (93)
\end{aligned}$$

Finally, thanks to the estimates (93) and (80) we conclude that

$$\begin{aligned}
&\big\||u(s,\cdot)|^p - |v(s,\cdot)|^p\big\|_{L^m\cap L^2\cap \dot{H}^{\sigma-1}} \\
&\qquad\lesssim (1+s)^{-\beta}\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big), \qquad (94)
\end{aligned}$$

where β is as (61). Next, including the estimate (94) into (79) we get

$$\||D|^{\kappa}(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim J_n^{\kappa,0}(t)\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big),$$

where $J_n^{\kappa,0}(t)$ is given by (18). Hence, due to the estimates (71) we find

$$\|(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim (1+t)^{1-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\gamma}\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big), \qquad (95)$$

and

$$\||D|^{\sigma}(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma}\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big), \qquad (96)$$

respectively. Now, we turn to estimate the norms

$$\|\partial_t(Nu-Nv)(t,\cdot)\|_{L^2} \quad\text{and}\quad \||D|^{\sigma-1}\partial_t(Nu-Nv)(t,\cdot)\|_{L^2}.$$

Again by using Proposition 2 we have for $\kappa=1$ and $\kappa=\sigma$ the estimates

$$\begin{aligned}
\||D|^{\kappa-1}\partial_t(Nu-Nv)(t,\cdot)\|_{L^2} &\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{\kappa-1}{2}-1} \\
&\quad\times \int_0^{\tau}(\tau-s)^{-\gamma}\big\||u_t(s,\cdot)|^p-|v_t(s,\cdot)|^p\big\|_{L^m\cap L^2\cap\dot{H}^{\sigma-1}}\,dsd\tau. \qquad (97)
\end{aligned}$$

Including the estimate (94) into (97) we get

$$\||D|^{\kappa-1}\partial_t(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim J_n^{(\kappa-1,1)}(t)\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big), \qquad (98)$$

where $J_n^{(\kappa-1,1)}(t)$ is defined by (18). Due to (76) we obtain for $\kappa=0$ the estimate

$$\|\partial_t(Nu-Nv)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma}\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big), \qquad (99)$$

and for $\kappa=\sigma$ the estimate

$$\big\||D|^{\sigma-1}\partial_t(Nu-Nv)(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\gamma}\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big). \qquad (100)$$

Finally, the estimates (100), (99), (96), (95) and the definition of the norm in $X(T)$ yield the desired inequality (16). This ends the proof of Theorem 3.

Remark 1. *Let us explain how to estimate the norm of u in $\dot{H}^{1/2}$ in the case $n = 3$ and why the logarithmic term does appear. One can estimate $\|u\|_{\dot{H}^{1/2}}$ by using interpolation argument. So, taking in (109)) $\sigma = \frac{1}{2}$, $k_1 = 0$, $k_2 = 1$ and $\theta = \frac{1}{2}$ we get*

$$\begin{aligned}\|u(t,\cdot)\|_{\dot{H}^{1/2}} &\lesssim \|u(t,\cdot)\|_{L^2}^{\frac{1}{2}}\|\nabla u(t,\cdot)\|_{L^2}^{\frac{1}{2}} \\ &\lesssim (1+t)^{\frac{1}{2}(\frac{1}{4}-\gamma)-\frac{\gamma}{2}}\|(u_0,u_1)\|_{\mathcal{A}^1_{1,0}} \\ &\lesssim (1+t)^{\frac{1}{8}-\gamma}\|(u_0,u_1)\|_{\mathcal{A}^1_{1,0}}.\end{aligned} \tag{101}$$

In the other hand, if we estimate the norm of u in $\dot{H}^{1/2}$ by using Corollary 1 and Lemma 1 we find

$$\|u^{nl}(t,\cdot)\|_{\dot{H}^{1/2}} \lesssim \Big(\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{1}{4}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\beta}dsd\tau\Big)\|u\|^p_{X(T)},$$

for some $\beta > 1$. Since $\beta > 1$ and $\frac{3}{4}+\frac{1}{4} = 1$ we get the estimate

$$\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{1}{4}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\beta}dsd\tau \lesssim (1+t)^{-\gamma}\log(2+t).$$

As a consequence, we get

$$\|u(t,\cdot)\|_{\dot{H}^{1/2}} \lesssim (1+t)^{-\gamma}\log(2+t)\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}. \tag{102}$$

In term of comparison, the estimate (102) is more precise than the estimate (101). For this reason, we consider the estimate (102) in Theorem 3.

4.2.2. Proof of Theorem 4

The proof of Theorem 4 is similar to the proof of Theorem 3. Expect in the case of Theorem 4 we have

$$\frac{n}{2}\Big(\frac{1}{m}-\frac{1}{2}\Big) = 1 \quad \text{if and only if} \quad n \geq 4 \text{ and } m = \frac{2n}{n+4}.$$

In this case, the integral $J_n^{(k,j)}(t)$ is estimated for $k = j = 0$ as follows:

$$J_n^{(0,0)}(t) = \begin{cases} (1+t)^{-\gamma}\log(2+t) & \text{if } n \geq 4 \text{ and } m = \frac{2n}{n+4}, \\ (1+t)^{-\gamma} & \text{else .}\end{cases}$$

Remark 2. *We refer the curious reader asking for the existence of the parameters q_j, $j = 1,2$ and r_i, $i = 1,\cdots,6$ which appears in the proof of Theorem 3 and Theorem 4 to check [11] and some of the references therein.*

5. Conclusion

We showed in Theorems 1, 3 and 4 the influence of the dissipative memory by its influence on the critical exponent and the estimates of the norms of the solution and its derivatives together since $\gamma > 0$. Indeed, it appears a loss of decay in the rate of the estimate of these norms. In the other hand, let us consider the following Cauchy problem of semi-linear dissipative wave equation:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u_t|^p & t > 0,\, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{103}$$

Due to (7), it appears that Cauchy problem (103) has no critical exponent in Fujita sense since

$$\lim_{\gamma \to 1} \frac{1}{\gamma} = 1.$$

One can interpret this result by concluding that the solutions of Cauchy problem (103), if they exist, then they are global in time. Consequently, any solution of Cauchy problem (103) blows-up in finite time.

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Appendix

5.1. Some auxiliary estimates for integrals

The following lemma is the key to estimate certain integrals that appear in the proof of the global existence of small data solutions (Section 4), in particular those arising from the nonlinear terms.

Lemma 1. *Assume that* $0 < \gamma < 1$, $a \geq 0$ *and* $b > 1$. *Then we have*

$$\int_0^t (1+t-s)^{-a} \int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-b}\, d\tau ds$$
$$\leq C \begin{cases} (1+t)^{-\gamma} & \text{if } a > 1, \\ (1+t)^{-\gamma}\log(2+t) & \text{if } a = 1, \\ (1+t)^{1-a-\gamma} & \text{if } a < 1. \end{cases}$$

Proof. Thanks to Lemma 4.1 from [12] and the fact that $\gamma < 1$, the integral with respect to τ on $[0, s]$ is estimated as follows:

$$\int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-b}\, d\tau \lesssim (1+s)^{-\gamma}.$$

Finally, the statements of Lemma 1 are concluded after applying Lemma 4.1 from [12] again.

5.2. What about the condition $\beta > 1$ in Lemma 1?

The reader may ask for a corresponding result in the case $b \in (0, 1]$ and how about the condition $\beta > 1$. We confirm that the condition $\beta > 1$ allows to get optimal estimate for the integral. The curious peoples are advised to check the discussion of this point in [11].

5.3. Decay estimates for solutions to auxiliary Cauchy problems

Let us announce some results on the decay estimates for Cauchy problems (9) and (10). First, for problem (9), A. Matsumura proved in [9] the following proposition

Proposition 1. *[9] If u is solution to Cauchy problem (9), then u and its derivatives $\partial_t^j \nabla^\kappa u$ satisfy for all $j \in \mathbb{N}$ and $\kappa > 0$ the following estimates:*

$$\|u(t,\cdot)\| \le C(1+t)^{-\frac{n}{4}-\frac{\kappa}{2}-j}(\|(u_0,u_1)\|_{L^1} + \|u_0\|_{H^{\kappa+j}} + \|u_1\|_{H^{[\kappa+j-1]_+}}), \tag{104}$$

for some constant $C > 0$.

The following result is an immediate consequence of Proposition 1 and the fact that equation $u_{tt} - \Delta u + u_t = 0$ is invariant by translation.

Corollary 1. *If v is solution to Cauchy problem (10), then v and its derivatives $\partial_t^j \nabla^\kappa v$ satisfy for all $j \in \mathbb{N}$ and $\kappa > 0$ the following estimates:*

$$\|v(t,\cdot)\| \le C(1+t-\tau)^{-\frac{n}{4}-\frac{\kappa}{2}-j}(\|h(\tau,\cdot)\|_{L^1} + \|h(\tau,\cdot)\|_{H^{[\kappa+j-1]_+}}), \tag{105}$$

for some constant $C > 0$.

The following result is a particular case of Proposition 5.2 from [11]. See also Theorem 3 in [13].

Corollary 2. [Proposition 5.1, [11]] *Let us consider the Cauchy problem (10) with $h = h(\tau,\cdot) \in L^m \cap H^{\sigma-1}$ for some $m \in [1,2)$ and $\sigma > 1$. Then the energy solution belongs to*

$$\mathcal{C}([\tau,+\infty), H^\sigma) \cap \mathcal{C}^1([\tau,+\infty), H^{\sigma-1}),$$

and satisfies the following Matsumura type decay estimates:

$$\begin{aligned}
&\|v(t,\cdot)\|_{L^2} \le C(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|h(\tau,\cdot)\|_{L^m\cap H^{\sigma-1}},\\
&\||D|^\sigma v(t,\cdot)\|_{L^2} \le C(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{\sigma}{2}}\|h(\tau,\cdot)\|_{L^m\cap H^{\sigma-1}},\\
&\|v_t(t,\cdot)\|_{L^2} \le C(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\|h(\tau,\cdot)\|_{L^m\cap H^{\sigma-1}},\\
&\||D|^{\sigma-1} v_t(t,\cdot)\|_{L^2} \le C(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{\sigma-1}{2}-1}\|h(\tau,\cdot)\|_{L^m\cap H^{\sigma-1}}.
\end{aligned}$$

5.4. Main inequalities-tools from Harmonic Analysis

The following results can be found among other things in [14] or [15].

5.4.1. Fractional Gagliardo-Nirenberg inequality

Proposition 2. *[11, 14, 15] Let $1 < p, p_0, p_1 < \infty$ and $\kappa \in [0,\sigma)$. Then the following fractional Gagliardo-Nirenberg inequality holds for all $u \in L^{p_0} \cap \dot{H}^\sigma_{p_1}$:*

$$\|u\|_{\dot{H}^\kappa_p} \lesssim \|u\|_{L^{p_0}}^{1-\theta}\|u\|_{\dot{H}^\sigma_{p_1}}^{\theta} \quad \textit{for } \frac{\kappa}{\sigma} \le \theta \le 1, \tag{106}$$

where

$$\theta = \frac{\frac{1}{p_0} - \frac{1}{p} + \frac{\kappa}{p}}{\frac{1}{p_0} - \frac{1}{p_1} + \frac{\sigma}{n}}.$$

The following corollary is a particular case of Proposition 2.

Corollary 3. *[7] Let $u \in L^2 \cap \dot{H}^k_2$. Then the following inequality holds:*

$$\|u\|_{L^q} \lesssim \|u\|_{L^2}^{1-\theta_k(q)} \|u\|_{\dot{H}^k_2}^{\theta_k(q)}, \quad \theta_k(q) = \frac{n}{k}\Big(\frac{1}{2} - \frac{1}{q}\Big) \tag{107}$$

for any $k \in (0, \frac{n}{2})$ and any q such that

$$2 \le q \le \frac{2n}{n-2k}. \tag{108}$$

The case $q = \frac{2n}{n-2k}$ reduces the inequality (107) to a well-known statement in the frame of Sobolev embeddings.

Corollary 4. *[11, 14, 15] For $u \in \dot{H}^k_2$, where $q \in [2, \infty)$ and $k = n(\frac{1}{2} - \frac{1}{q})$, the following inequality holds:*

$$\|u\|_{L^q} \lesssim \|u\|_{\dot{H}^k_2}.$$

Interpolation formulas are sometimes used to obtain suitable estimates. Here we recall the relation

$$\|u\|_{\dot{H}^\sigma_2} \le \|u\|_{\dot{H}^{k_1}_2}^{1-\theta} \|u\|_{\dot{H}^{k_2}_2}^{\theta} \quad \text{for some } \theta \in [0,1] \quad \text{with } k_1(1-\theta) + k_2\theta = \sigma. \tag{109}$$

5.4.2. Fractional Leibniz rule

Proposition 3. *[11, 14, 15] Let $\sigma > 0$, $1 \le r \le \infty$ and $1 < p_1, p_2, q_1, q_2 \le \infty$ satisfying*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then it holds the following fractional Leibniz rule:

$$\||D|^\sigma (fg)\|_{L^r} \lesssim \||D|^\sigma f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \||D|^\sigma g\|_{L^{q_2}}$$

for any $f \in \dot{H}^\sigma_{p_1} \cap L^{q_1}$ and $g \in \dot{H}^\sigma_{q_2} \cap L^{p_2}$.

5.4.3. Fractional chain rule

Proposition 4. *[11, 14, 15] Let $\sigma \in (0,1)$, $1 < r, r_1, r_2 < \infty$ and F a $\mathcal{C}^1$ function satisfying for any $\tau \in [0,1]$ and $u, v \in \mathbb{R}$ the inequality*

$$|F'(\tau u + (1-\tau)v)| \le \mu(\tau)(G(u) + G(v))),$$

for some continuous nonnegative function G and $\mu \in L^1[0,1]$. Then,

$$\|F(u)\|_{\dot{H}^\sigma_r} \lesssim \|G(u)\|_{L^{r_1}} \|u\|_{\dot{H}^\sigma_{r_2}},$$

for any $u \in \dot{H}^\sigma_{r_2}$ such that $G(u) \in L^{r_1}$, provided that

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.$$

In particular, to estimate norms like $\||u|^p\|_{\dot{H}^{s-1}_r}$ or $\|\pm u|u|^{p-1}\|_{\dot{H}^{s-1}_r}$ we use the fractional chain rule and the Gagliardo-Nirenberg inequality. In this way we may may conclude at first for $s \in (1,2)$ and then by a straight-forward step to $s \geq 2$ the estimate

$$\| \pm u|u|^{p-1}\|_{\dot{H}^{s-1}_r} + \||u|^p\|_{\dot{H}^{s-1}_r} \lesssim \|u\|^{p-1}_{L^{q_1}} \||D|^{s-1}u\|_{L^{q_2}}, \tag{110}$$

where

$$\frac{p-1}{q_1} + \frac{1}{q_2} = \frac{1}{r}, \;\; s > 1.$$

5.4.4. Fractional powers

The following tool is useful to estimate the p-power of a given function and the product of two functions in $H^s(\mathbb{R}^n)$. This tool is meaningful in the case in which we have the embedding

$$L^\infty(\mathbb{R}^n) \hookrightarrow H^s_r(\mathbb{R}^n),$$

that is, when $s > \frac{n}{r}$.

Proposition 5. *[11, 14, 15] Let $r \in (1,\infty)$, $p > 1$ and $s \in (0,p)$. Let $F(u)$ denote one of the functions $|u|^p$ or $\pm u|u|^{p-1}$. Then, it holds the following inequality:*

$$\|F(u)\|_{H^s_r} \lesssim \|u\|_{H^s_r} \|u\|^{p-1}_{L^\infty}$$

for any $u \in H^s_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

The next result is a direct consequence of the previous one for the case of homogeneous Sobolev spaces.

Corollary 5. *Let $r \in (1,\infty)$, $p > 1$ and $s \in (0,p)$. Let $F(u)$ denote one of the functions $|u|^p$ or $\pm u|u|^{p-1}$. Then, it holds the following inequality:*

$$\|F(u)\|_{\dot{H}^s_r} \lesssim \|u\|_{\dot{H}^s_r} \|u\|^{p-1}_{L^\infty}$$

for any $u \in \dot{H}^s_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.