EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6270
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A Topological Structure on D-Algebras

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Abstract. The primary aim of this paper is applying the concept of b-open sets in topological spaces to explore the idea of b-topological d- algebra ($T_b d$-algebra), which is a d-algebra equipped with a specific type of topology that ensured the binary operation that is defined on them to be d-topologically continuous. This idea generalizes the notion of topological d-algebras. In addition, we present some relations between open sets and b-open sets in a $T_b d$-algebra. We also construct some relations between T_i and b-T_i-spaces for (i=0,1,2). Finally, we use left maps on positive implicative d-algebras to establish some $T_b d$-algebras.

2020 Mathematics Subject Classifications: 03G25, 22A30, 54A05

Key Words and Phrases: b-open set, d-algebra, positive implicative d-algebra, edge d-algebra, b-T_2 space, $T_b d$-algebra

1. Introduction

Topology and Algebra are two significant areas of pure mathematics. Topology focuses on concepts such as continuity and convergence, while Algebra explores various operations, forming the foundation for calculations and algorithms. A key principle that links algebraic operations and topology is the requirement that these operations be continuous topologically, either jointly continuous or in the first or second variable, this field is known as Topological Algebra. In recent years, numerous researchers have advanced this area of study. Since the early twentieth century, many mathematicians have made substantial contributions to progress of these interrelated subjects. Generalized open sets are fundamental to general topology and are acknowledged as significant research areas by topologists around the world. These sets are central to major themes in both general topology and real analysis, particularly in connection with various modified forms of continuity, separation axioms, and other related concepts. One of the forms of generalized open sets in topological spaces, known as b-open sets, was introduced by Andrijevic [1] in 1996. These sets were presented by Al-Etik [2] under the name λ-open sets. Furthermore, Caldas and Jafari [3] used open sets b to define separation axioms b-T_i $(i \in 0, 1, 2)$

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6270

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in topological spaces. The concept of d-algebra was introduced by Neggers and Kim in [4]. The topological BCK-algebra and the topological d-algebras were defined in [5] and [6] respectively. Recently, Khalaf in [7–9] introduced some topological notions defined on BCK-algebras. We can notice that the concept of b-open was included in many topics and here we review some of them, in [10] the supra-b limit points and supra-b separation axioms were investigated, also in [11] b-open sets via infra soft topological spaces were studied. Limit points and separation axioms with respect to supra semi-open sets were established in [12].
By $(\mathcal{W}, \Omega)$ we mean a topological space and if M is any subset of a topological space $(\mathcal{W}, \Omega)$, then the interior and closure of M are denoted by $Int(M)$ and $Cl(M)$, respectively.

2. Preliminaries

In this section, we recall some definitions and results that are needed in the next section.

Definition 1. In a topological space $\mathcal{W}$, a subset M is called b-open [1] (resp., regular open [13], semi-open [14], pre-open [15]) if $M \subseteq Int(Cl(M)) \cup Cl(Int(M))$ $\Big($resp., $M = Int(Cl(M))$, $M \subseteq Cl(Int(M))$, $M \subseteq Int(Cl(M))\Big)$.

Lemma 1. [1] The intersection of an open and a b-open set is a b-open set.

Lemma 2. [1] In a topological space $\mathcal{W}$, a subset M is b-open if and only if $M = \big(M \cap Int(Cl(M))\big) \cup \big(M \cap Cl(Int(M))\big)$

Definition 2. [16] A topological space $(\mathcal{W}, \Omega)$ is called:

(i) locally indiscrete if every open set is closed.

(ii) extremally disconnected if $Cl(U) \in \Omega$ for every $U \in \Omega$.

(iii) submaximal if every dense subset of $\mathcal{W}$ is open.

Lemma 3. [16]

(i) In a locally indiscrete space, every subset of $\mathcal{W}$ is pre-open.

(ii) In a submaximal space, every pre-open set is open.

Lemma 4. [2] If $(\mathcal{W}, \Omega)$ is an extremally disconnected, then

(i) every b-open set is pre-open.

(ii) a subset M of $\mathcal{W}$ is b-open if and only if $\mathcal{W} \setminus M$ is dense.

Lemma 5. [2]

(i) If M is both open and b-closed subset in $\mathcal{W}$, then it is semi-closed.

(ii) If M is a b-closed subset of a b-compact space $\mathcal{W}$, then it is b-compact.

Definition 3. [17] A topological space $\mathcal{W}$ is semi-disconnected if it can be partitioned into two non-empty open sets U and V such that $Cl(U) \cap V = \emptyset$ and $Cl(V) \cap U = \emptyset$.

Definition 4. [3, 13] A topological space $(\mathcal{W}, \Omega)$ is said to be:

(i) b-T_0 (resp., T_0) $\forall\ \zeta, \eta \in \mathcal{W}$, $\exists$ a b-open (open) set containing one of them but not the other.

(ii) b-T_1 (resp., T_1) if $\forall\ \zeta, \eta \in \mathcal{W}$, $\exists$ b-open (open) sets G, H such that $\zeta \in G$, $\eta \notin G$ and $\eta \in H$, $\zeta \notin H$.

(iii) b-T_2 (resp., T_2) if $\forall\ \zeta, \eta \in \mathcal{W}$, $\exists$ b-open (open) sets G, H such that $\zeta \in G$, $\eta \in H$, $G \cap H = \phi$.

Definition 5. [4] d-algebra is an algebra $(\mathcal{W}, \odot, 0)$ of type $(2, 0)$ such that $\odot$ is a binary operation and 0 is a fixed element, satisfy the axioms listed below: for each $\zeta, \eta, \theta \in \mathcal{W}$,

(i) $\zeta \odot \zeta = 0$,

(ii) $\zeta \odot \eta = 0$ and $\eta \odot \zeta = 0 \Rightarrow \zeta = \eta$,

(iii) $0 \odot \zeta = 0$.

Moreover, if a d-algebra satisfies the following axioms

(i) (B1) $((\zeta \odot \eta) \odot (\zeta \odot \theta)) \odot (\theta \odot \eta) = 0$,

(ii) (B2) $(\zeta \odot (\zeta \odot \eta)) \odot \eta = 0$.

Then it is a BCK-algebra.
We indicate a partial order relation $(\leq)$ by $\zeta \leq \eta \iff \zeta \odot \eta = 0$.

Definition 6. [18] A d-algebra $(\mathcal{W}, \odot, 0)$ is said to be edge d-algebra, if $\zeta \odot \mathcal{W} = \{\zeta, 0\}$ for all $\zeta \in \mathcal{W}$

Lemma 6. [4] If $(\mathcal{W}, \odot, 0)$ is an edge d-algebra, then it satisfies condition (B2).

Definition 7. [6] A nonempty subset $\mathcal{I}$ of a d-algebra $(W, \odot, 0)$ is said to be an ideal of W if both of the following conditions are met:

(i) $0 \in \mathcal{I}$,

(ii) $\forall\ \zeta \in \mathcal{W}$, $\forall \eta \in \mathcal{I}$, if $\zeta \odot \eta \in \mathcal{I}$, then $\zeta \in \mathcal{I}$.

Definition 8. [19] Let α be an element of a d-algebra $\mathcal{W}$. α is said to be an atom in $\mathcal{W}$ if, for any $\zeta \in \mathcal{W}$, $\alpha \odot \zeta = 0$ implies $\alpha = \zeta$.

Definition 9. [4] Let $\mathcal{Z}$ be a subset of a d-algebra $\mathcal{W}$, $\mathcal{Z}$ is called a d-subalgebra if it is also a d-algebra.

Definition 10. [6] Let $\mathcal{W}$ be a d-algebra , $\alpha \in \mathcal{W}$. A left map $\mathcal{L}_\alpha : \mathcal{W} \to \mathcal{W}$ defined by, $\mathcal{L}_\alpha(\zeta) = \alpha \odot \zeta, \forall \zeta \in \mathcal{W}$ and a right map $\mathcal{R}_\alpha : \mathcal{W} \to \mathcal{W}$ by $\mathcal{R}_\alpha(\zeta) = \zeta \odot \alpha\ \forall \zeta \in \mathcal{W}$.
$\mathcal{L}(\mathcal{W})$ represents the family of all left maps on $\mathcal{W}$, while $\mathcal{R}(\mathcal{W})$ represents the family of all right maps. $\mathcal{W}$ is denoted by and the family of all right maps If $S \subseteq \mathcal{W}$ then $\mathcal{L}_\alpha(S) = \alpha \odot S$ and $\mathcal{R}_\alpha(S) = S \odot \alpha$.

Definition 11. [20] A d-algebra $\mathcal{W}$ is known as a positive implicative d-algebra , if $(\eta \odot \zeta) \odot (\theta \odot \zeta) = (\eta \odot \theta) \odot \zeta$ for all $\zeta, \eta, \theta \in \mathcal{W}$.

Definition 12. [5] A BCK-algebra $\mathcal{W}$ with a topology Ω is a TBCK-algebra if the function $f : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ known as $f(\zeta, \eta) = \zeta \odot \eta$ possesses the attribute that for every open set O having $\zeta \odot \eta$, there exist open sets U, V having ζ, η correspondingly such that $f(U, V) = U \odot V \subseteq O\ \forall\ \zeta, \eta \in \mathcal{W}$.

Definition 13. [6] A d-algebra $\mathcal{W}$ with a topology Ω is called an Td-algebra if the function $f : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ known as $f(\zeta, \eta) = \zeta \odot \eta$ possesses the attribute that for every open set O having $\zeta \odot \eta$, there exist open sets U, V having ζ, η correspondingly such that $f(U, V) = U \odot V \subseteq O, \forall\ \zeta, \eta \in \mathcal{W}$.

3. b-topological d-algebras

This section presents the idea of b-topological d-algebras and covers some of its properties.

Definition 14. A d-algebra $\mathcal{W}$ equipped with a topology Ω is known as b-topological d-algebra ($T_b d$-algebra) if $f : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ defined by $f(\zeta, \eta) = \zeta \odot \eta$ has the property that for each open set O having $\zeta \odot \eta,$, there are b-open sets U, V having ζ, η respectively, such that $f(U, V) = U \odot V \subseteq O\ \forall\ \zeta, \eta \in \mathcal{W}$.

By the definitions of TBCK-algebra and Td-algebra, and since every BCK-algebra is a d-algebra, we deduce that every TBCK-algebra is a Td-algebra and every Td-algebra is a $T_b d$-algebra. The following example shows that the implication is not reversible in general.

Example 1. Let $\mathcal{W} = \{0, \alpha, \beta, \gamma\}$ and $\odot$ be given as in the Cayley table as follows:

It is easy to verify that From Table 2. Next think about the topology Ω on $\mathcal{W}$ given as:
$\Omega = \{\phi, \{\alpha\}, \{\gamma\}, \{\alpha, \gamma\}, \mathcal{W}\}$. Then $\mathcal{W}$ is not a Td-algebra because $\gamma \odot \beta = \gamma$, and the only open set having β is $\mathcal{W}$ and $\{\gamma\} \times \mathcal{W} \nsubseteq \{\gamma\}$. It is easy to confirm that the b-open sets in $(\mathcal{W}, \Omega)$ are $P(\mathcal{W}) \setminus (\{0\}, \{\beta\}, \{0, \beta\})$ and we can show by basic computation that $(\mathcal{W}, \Omega)$ is a $T_b d$-algebra.

$\odot$	0	α	β	γ
0	0	0	0	0
α	α	0	0	α
β	β	β	0	0
γ	γ	γ	γ	0

Table 1: A T_bd-algebra which is not Td-algebra

Proposition 1. For each subset M of a T_bd-algebra $\mathcal{W}$ and any element $\zeta \in \mathcal{W}$, the following statements are true:

(i) $Cl_b(M) \odot \zeta \subseteq Cl(M \odot \zeta)$.

(ii) If $Cl_b(M) \odot \zeta$ is closed, then $Cl_b(M) \odot \zeta = Cl(M \odot \zeta)$.

Proof.

(i) Suppose that $\eta = \alpha \odot \zeta \in Cl_b(M) \odot \zeta$ where $\alpha \in Cl_b(M)$ and U is any open set having η. Since $\mathcal{W}$ is a T_bd-algebra, so there exist b-open sets V that have α and G having ζ such that $V \odot G \subseteq U$. Also, we have $\alpha \in Cl_b(M)$ implies that $M \cap V \neq \phi$. Now assume that $\theta \in M \cap V$, so $\theta \odot \zeta \in M \odot \zeta$ and $\theta \odot \zeta \in V \odot \zeta \subseteq V \odot G \subseteq U$. Hence $\theta \odot \zeta \in U \cap (M \odot \zeta)$ implies that $\eta \in Cl(M \odot \zeta)$. Thus, $Cl_b(M) \odot \zeta \subseteq Cl(M \odot \zeta)$.

(ii) Suppose that $Cl_b(M) \odot \zeta$ is closed, we have $M \odot \zeta \subseteq Cl_b(M) \odot \zeta$ and hence $Cl(M \odot \zeta) \subseteq Cl_b(M) \odot \zeta$. Therefore, by (i) we get the equality.

In general, the inclusion of (i) can not replaced by equality as it can be seen in Example 1, if we take $M = \{o, \alpha\}$, then $Cl_b(M) \odot \beta = \{0\}$ and $Cl(M \odot \beta) = \{0, \beta\}$, so $Cl_b(M) \odot \beta \neq Cl(M \odot \beta)$.

Proposition 2. If M is any subset of a T_bd-algebra $\mathcal{W}$ and any element $\zeta \in \mathcal{W}$, , the statements listed below are accurate:

(i) $\zeta \odot Cl_b(M) \subseteq Cl(\zeta \odot M)$.

(ii) If $\zeta \odot Cl_b(M)$ is closed, then $\zeta \odot Cl_b(M) = Cl(\zeta \odot M)$.

Proof.

(i) Suppose that $\eta \in \mathcal{W} \odot Cl_b(M)$ and U be any open set having η. So $\eta = \zeta \odot \alpha$ where $\alpha \in Cl_b(M)$. Since $\mathcal{W}$ is a T_bd-algebra, there exist open sets V having ζ and G having α such that $V \odot G \subseteq U$. Also, we have $\alpha \in Cl_b(M)$ implies that $M \cap G \neq \phi$. Now assume that $\theta \in M \cap G$, so $\zeta \odot \theta \in \mathcal{W} \odot M$ and $\zeta \odot \theta \in \mathcal{W} \odot G \subseteq V \odot G \subseteq U$. Hence we obtain that $\eta \in Cl(M \odot \zeta)$.

(ii) Assume that $\zeta \odot Cl_b(M)$ is closed and let $\zeta \odot M \subseteq \zeta \odot Cl_b(M)$. Therefore, $Cl(\zeta \odot M) \subseteq \zeta \odot Cl_b(M)$. From (1), we get $\zeta \odot Cl_b(M) = Cl(\zeta \odot M)$.

In general the inclusion in (i) can not be replaced by equality, as shown in Example 1, if we take $M = \{o, \alpha\}$, then $\beta \odot Cl_b(M) = \{\beta\}$ and $Cl(\beta \odot M) = \{0, \beta\}$, so $\beta \odot Cl_b(M) \neq Cl(\beta \odot M)$.

Proposition 3. For each subsets M and N of a $T_b d$-algebra $\mathcal{W}$, the statements listed below are accurate.:

(i) $Cl_b(M) \odot Cl_b(N) \subseteq Cl(M \odot N)$.

(ii) If $Cl_b(M) \odot Cl_b(N)$ is closed, then $Cl_b(M) \odot Cl_b(N) = Cl(M \odot N)$.

Proof.

(i) Let $\zeta = \alpha \odot \beta \in Cl_b(M) \odot Cl_b(N)$ and U be any open set having ζ. Since $\mathcal{W}$ is a $T_b d$-algebra, so there exist open sets V having α and G having β such that $V \odot G \subseteq U$. Also, we have $\alpha \in Cl_b(M)$ and $\beta \in Cl_b(N)$, implies that $M \cap V \neq \phi$ and $N \cap G \neq \phi$. assume that $\alpha_1 \in M \cap V$ and $\beta_1 \in N \cap G$, so $\alpha_1 \odot \beta_1 \in M \odot N$ and $\alpha_1 \odot \beta_1 \in V \odot G \subseteq U$. Hence we get $\zeta \in Cl(M \odot N)$.

(ii) Assume that $Cl_b(M) \odot Cl_b(N)$ is closed. We have $M \subseteq Cl_b(M)$ and $N \subseteq Cl_b(N)$ and hence $M \odot N \subseteq Cl_b(M) \odot Cl_b(N)$. Therefore, by hypothesis, $Cl(M \odot N) \subseteq Cl_b(M) \odot Cl_b(N)$. Hence, from (i), the equality holds.

From Proposition 1 and Proposition 2, we obtain the following result:

Corollary 1. For a subset M of a $T_b d$-algebra $\mathcal{W}$ and an element $\zeta \in \mathcal{W}$, the statements listed below are accurate:

(i) If $M \odot \zeta$ is closed, then $Cl_b(M) \odot \zeta = M \odot \zeta$.

(ii) If $\zeta \odot M$ is closed, then $\zeta \odot Cl_b(M) = \zeta \odot M$.

Proof.

(i) by Proposition 1, we have $Cl_b(M) \odot \zeta \subseteq Cl(M \odot \zeta) = M \odot \zeta$ and $M \odot \zeta \subseteq Cl_b(M) \odot \zeta$. Hence, the proof.

(ii) is similar.

Definition 15. Let $\mathcal{W}$ be a d-algebra, U be a non-empty subset of $\mathcal{W}$ and $\alpha \in \mathcal{W}$. We define the following subsets. U_α and ${}_\alpha U$: $U_\alpha = \{\zeta \in \mathcal{W} : \zeta \odot \alpha \in U\}$ and ${}_\alpha U = \{\zeta \in \mathcal{W} : \alpha \odot \zeta \in U\}$.
Also if $K \subseteq \mathcal{W}$ we define

$$_K U = \bigcup_{\alpha \in K} {}_\alpha U \qquad \& \qquad U_K = \bigcup_{\alpha \in K} U_\alpha$$

.

Proposition 4. Let $\mathcal{W}$ be an d-algebra and M, N, W, K are subsets of $\mathcal{W}$ then:

(i) ${}_M W \subseteq {}_N W$, if $M \subseteq N$.

(ii) ${}_M W \subseteq {}_M K$, if $W \subseteq K$.

(iii) $(F_\alpha)^c = (F^c)_\alpha$ and $({}_\alpha F)^c =_\alpha (F^c)$ for each $\alpha \in \mathcal{W}$, If $F \subseteq \mathcal{W}$.

Proof. These results follow from Definition 15.

Proposition 5. Let $\mathcal{W}$ be a $T_b d$-algebra and $\alpha \in \mathcal{W}$ and U be any nonempty subset of $\mathcal{W}$, , then the statements listed below are accurate:

(i) U_α and ${}_\alpha U$ are b-open sets, if U is an open set.

(ii) F_α and ${}_\alpha F$ are b-closed sets, if F is closed set.

(iii) If U is open, then ${}_K U$ and U_K are b-open sets for every subset K of $\mathcal{W}$.

Proof.

(i) Let $\zeta \in U_\alpha$, so $\zeta \odot \alpha \in U$. Since $\mathcal{W}$ is $T_b d$-algebra and U is open, so there exists a b-open set G having ζ such that $G \odot \alpha \subseteq U$, $\zeta \odot \alpha \in G \odot \alpha \subseteq U$. Therefore, for every $\theta \in G$, we have $\theta \odot \alpha \subseteq U$ implies that $\theta \in U_\alpha$. Hence, $G \subseteq U_\alpha$ implies that U_α is b-open.

To show that ${}_\alpha U$ is b-open, let $\zeta \in {}_\alpha U$ implies that $\alpha \odot \zeta \in U$. Since $\mathcal{W}$ is $T_b d$-algebra, then there exists a b-open set H that has ζ such that $\alpha \odot H \subseteq U$, so for each $\theta \in H$, we have $\alpha \odot \theta \in U$. Hence, $\theta \in H \subseteq {}_\alpha U$. Therefore, ${}_\alpha U$ is a b-open set.

(ii) Assuming F be a closed set, then F^c is open. Hence, according (1), $(F^c)_\alpha$ and ${}_\alpha(F^c)$ are b-open. using Proposition 4, $(F_\alpha)^c = (F^c)_\alpha$ and $({}_\alpha F)^c =_\alpha (F^c)$. Hence, $(F_\alpha)^c$ and $({}_\alpha F)^c$ are b-open. Consequently, F_α and ${}_\alpha F$ are b-closed.

(iii) Follows from (i) and the fact that arbitrary union of b-open sets is b-open.

Corollary 2. Let $\mathcal{W}$ be T_bd–algebra, U and M be two non–empty subset of $\mathcal{W}$, then the statements listed below are accurate:
i) The sets $_MU$ and U_M are b-open sets if U is open.
ii)The sets $_MU$ and U_M are closed sets if U is closed set and M is finite.

Example 2. In Example 1, we have $U = \{\alpha, \gamma\} \in \Omega$ and by simple calculation we can see that $_MU$ and U_M are b-open sets for any subset M of $\mathcal{W}$.

Proposition 6. Let $\mathcal{W}$ be T_bd–algebra and $\mathcal{W}$ be T_2 b-compact space. If S is a compact subset of $\mathcal{W}$, then S_α and $_\alpha S$ are b-compact sets for all $\alpha \in \mathcal{W}$.

Proof. Let S be a compact subset of $\mathcal{W}$. Since $\mathcal{W}$ is T_2, then S is closed set in $\mathcal{W}$. Thus by Proposition 5 S_α and $_\alpha S$ are b-closed sets in $\mathcal{W}$ for all $\zeta \in \mathcal{W}$. Then, by Lemma 5, S_α and $_\alpha S$ are b-compact sets in $\mathcal{W}$ for all $\zeta \in \mathcal{W}$.

In the following example, we see that S_α and $_\alpha S$ are b-compact sets while S is no compact.

Example 3. Consider the set of real numbers $\mathbb{R}$ and an operation $\odot$ defined as:

$$x \odot y = \begin{cases} 0 & \text{if } x \leq y \\ 1 & otherwise. \end{cases}$$

Then, $(\mathbb{R}, \odot, \tau)$ is a T_bd–algebra T_2-space where τ is the discrete topology. Suppose that $S = (1, 5)$, then obviously, S is not compact, but we have S_α and $_\alpha S$ are empty sets for every $\alpha \in \mathbb{R}$ and hence they are b-compact.

Proposition 7. If $\{0\}$ is open in a T_bd-algebra $\mathcal{W}$, then it is b-T_1.

Proof. Assume that $\{0\}$ is open and $\zeta, \eta \in \mathcal{W}$ be any two distinct points. Since $\zeta \odot \zeta = 0$ for all $\zeta \in \mathcal{W}$ and $\mathcal{W}$ is T_bd-algebra, then there exist b-open sets H and G containing ζ such that $H \odot G \subseteq \{0\}$. Hence, either $\eta \notin H$ or $\eta \notin G$. Also, we have $\eta \odot \eta = 0$, so the exist two b-open sets U, V containing η such that $U \odot V \subseteq \{0\}$. Hence, either $\zeta \notin U$ or $\zeta \notin V$.Therefore, we obtain that a b-open set containing ζ but not η and a b-open set containing η but not ζ. Therefore, $\mathcal{W}$ is a b-T_1-space.

Remark 1. The space $(\mathcal{W}, \Omega)$ in Example 1 is b-T_1 but $\{0\}$ is not b-open.

Corollary 3. If $\{0\}$ is an open set in a T_bd-algebra $\mathcal{W}$, then every open singular set is regular open.

Proof. From Proposition 7, we have $\mathcal{W}$ is b-T_1. Hence, every singleton set is b-closed. If $\{\zeta\}$ is an open set, so we have $Int(Cl(\{\zeta\} \subseteq \{\zeta\} \subseteq Int(Cl(\{\zeta\}$ which implies that $\{\zeta\}$ is regular open.

Corollary 4. Suppose that $\{0\}$ is an open set in a T_bd-algebra $\mathcal{W}$. If $\mathcal{W}$ is a door space, then every singular set is either closed or regular open.

Proof. Follows from Corollary 3 and the definition of door space.

Proposition 8. In a $T_b d$-algebra $\mathcal{W}$ which is both submaximal and extremally disconnected space. If $\{0\}$ is open, then the space is discrete.

Proof. Assume that $\{0\}$ is open and let ζ be any point in $\mathcal{W}$. Since $\zeta \odot \zeta = 0$ for all $\zeta \in \mathcal{W}$ and $\mathcal{W}$ is $T_b d$-algebra, so there exist b-open sets U and V having ζ such that $U \odot V \subseteq \{0\}$. Since $\mathcal{W}$ is extremally disconnected, so by using Lemma 4, U, V are pre-open sets. Also, $\mathcal{W}$ is submaximal, so by Lemma 3, U, V are open. Hence $W = U \cap V$ is an open set having ζ. Now if W having another point η, then we obtain $\zeta \odot \eta = 0$ and $\eta \odot \zeta = 0$ which is a contradiction. Hence W is an open set having ζ only. therefore, $\{\zeta\}$ is open for each $\zeta \in \mathcal{W}$. Thus, the space $\mathcal{W}$ is discrete.

Proposition 9. In a $T_b d$-algebra $(\mathcal{W}, \odot, \Omega)$. If W is an open set not containing 0, then for each distinct elements $\zeta, \eta \in \mathcal{W}$ with $\zeta \odot \eta \in W$ there exist two disjoint b-open sets containing ζ and η.

Proof. Let $\zeta \odot \eta \in W$, Since $(\mathcal{W}, \odot, \Omega)$ is a $T_b d$-algebra, then there exist b-open sets U and V of containing ζ and η such that $U \odot V \subseteq W$. If $U \cap V \neq \phi$, then there is a point $\theta \in U \cap V$ which implies $0 = \theta \odot \theta \in U \odot V \subseteq W$ which contradicts itself.therefore, $U \cap V = \phi$.

Proposition 10. In a $T_b d$-algebra $\mathcal{W}$, if $\{0\}$ is closed, then $\mathcal{W}$ is b-T_2.

Proof. Let $\{0\}$ is closed and let ζ and η be any two distinct points in $\mathcal{W}$, then either $\zeta \odot \eta \neq 0$ or $\eta \odot \zeta \neq 0$ without loss of generality assume that $\zeta \odot \eta \neq 0$. Hence, there exist b-open sets U and V having ζ and η respectively such that $U \odot V \subseteq \zeta \setminus \{0\}$ and hence $U \cap V = \phi$. Therefore, $\mathcal{W}$ is b-T_2.

Example 4. Consider any infinite d-algebra $(\mathcal{W}, \odot, \Omega)$ where Ω is the indescrete topology on $\mathcal{W}$. Then, $bO(\mathcal{W}, \Omega) = P(\mathcal{W})$ and hence $(\mathcal{W}, \Omega)$ is a b-T_2-space but $\{0\}$ is not closed.

Corollary 5. If a $T_b d$-algebra $(\mathcal{W}, \odot, \Omega)$ is T_1, then it is b-T_2.

Proof. Since $(\mathcal{W}, \odot, \Omega)$ is T_1, so $\{0\}$ is closed. Hence, by Proposition 10, $(\mathcal{W}, \odot, \Omega)$ is $b - T_2$.

Proposition 11. If a $T_b d$-algebra $(\mathcal{W}, \odot, \Omega)$ is T_0, then it is b-T_1.

Proof. Suppose that $\zeta, \eta \in \mathcal{W}$ and $\zeta \neq \eta$. Then either $\zeta \odot \eta \neq 0$ or $\eta \odot \zeta \neq 0$. Now let $\zeta \odot \eta \neq 0$. Since $\mathcal{W}$ is T_0 space, then there is an open set W that contains one of them but not the other.
Case 1. Assume that W contains $\zeta \odot \eta$ and $0 \notin W$.
Since $(\mathcal{W}, \odot, \Omega)$ is a $T_b d$-algebra, then there exist b-open sets U of ζ and V of η such that

$U \odot V \subseteq W$. Then U is a b-open set and V is a b-open set having η . If $U \cap V \neq \phi$, that is mean there is a point $\theta \in U \cap V$. Thus $0 = \eta \odot \theta \in U \odot V \subseteq W$ that is a contradiction. Case 2. now if $0 \in W$ and $\zeta \odot \eta \notin W$. Then we have, $\zeta \odot \zeta = 0 \in W$, so there exist b-open sets U_1, U_2 having ζ such that $U_1 \odot U_2 \in W$. Obviously, U_2 is a b-open set containing ζ and does not contains η. Again $\eta \odot \eta = 0 \in W$, so there exist b-open sets U_η and V_η containing η such that $U_\eta \odot V_\eta \subseteq W$. Hence, U_η is a b-open set containing η but not ζ. Therefore, $(\mathcal{W}, \odot, \Omega)$ is a b-T_1 space.

Remark 2. The space $(\mathcal{W}, \odot, \Omega)$ in Example 4 is a b-T_1-space which is not T_0.

Definition 16. We say that a topological space $(\mathcal{W}, \Omega)$ is B_ζ-space if U_1, U_2 are any b-open subsets of $\mathcal{W}$ containing ζ, then there exists a b-open set U such that $\zeta \in U \subseteq U_1 \cap U_2$.

Proposition 12. In a B_ζ $T_b d$-algebra $(\mathcal{W}, \odot, \Omega)$. If W is an open set having 0, then for all $\zeta \in \mathcal{W}$ there exists a b-open set U having ζ such that $U \odot U \subseteq W$.

Proof. Let W be an open set having 0. We have $\zeta \odot \zeta = 0$ for all $\zeta \in \mathcal{W}$ and $\mathcal{W}$ is $T_b d$-algebra, so there exist b-open sets U_1, U_2 having ζ such that $U_1 \odot U_2 \subseteq W$. Since $\mathcal{W}$ is a B_ζ space, so there exists a b-open set $U \subseteq U_1 \cap U_2$. Therefore, we obtain that $U \odot U \subseteq W$.

Proposition 13. In a B_ζ $T_b d$-algebra, if $(\mathcal{W}, \odot, \Omega)$ is an extremally disconnected submaximal T_0 space, then it is b-T_2.

Proof. Let $\zeta, \eta \in \mathcal{W}$, so either $\zeta \odot \eta \neq 0$ or $\eta \odot \zeta \neq 0$. Assume that $\zeta \odot y \neq 0$. Since $\mathcal{W}$ is T_0, so there exists an open set W having either $\zeta \odot \eta$ and not having 0 or conversely. If $\zeta \odot \eta \in W$ and $0 \notin W$, so by Proposition 9, there exist two disjoint b-open sets containing ζ and η. If $\zeta \odot \eta \notin W$ and $0 \in W$, then we have, $\zeta \odot \zeta = 0 \in W$, so there exist b-open sets U_1, U_2 containing ζ such that $U_1 \odot U_2 \in W$. Since $\mathcal{W}$ is a B_ζ space, so there exists a b-open set U such that $\zeta \in U \subseteq U_1 \cap U_2$ implies that U is a b-open set containing ζ such that $U \odot U \subseteq W$. Again $\eta \odot \eta = 0 \in W$, so there exists a b-open set$V$ containing η such that $V \odot V \subseteq W$. Hence, we have $\eta \notin U$ and $\zeta \notin V$ because we get a contradiction. Since $\mathcal{W}$ is extremally disconnected, so by Lemma 4, $\mathcal{W} \setminus U$ and $\mathcal{W} \setminus V$ are dense in $\mathcal{W}$. Since $\mathcal{W}$ is submaximal, so by Definition 2, $\mathcal{W} \setminus U$ and $\mathcal{W} \setminus V$ are open. Thus, $\zeta \in U \cap (\mathcal{W} \setminus V)$ and $\eta \in V \cap (\mathcal{W} \setminus U)$. Obviously, $U \cap (\mathcal{W} \setminus V)$ and $V \cap (\mathcal{W} \setminus U)$ are disjoint b-open sets in $\mathcal{W}$. Hence, $\mathcal{W}$ is b-T_2.

Proposition 14. If $\mathcal{Z}$ is an open d-subalgebra of a $T_b d$-algebra $\mathcal{W}$, then $\mathcal{Z}$ is also a $T_b d$-algebra.

Proof. Suppose that $\zeta, \eta \in \mathcal{Z}$ and let U be an open set in the subspace $\mathcal{Z}$ containing $\zeta \odot \eta$. Since $\mathcal{Z}$ is open in $\mathcal{W}$, so U is open in $\mathcal{W}$. Since $\mathcal{W}$ is a $T_b d$-algebra, so there exist b-open sets H, G in $\mathcal{W}$ having ζ and η respectively such that $H \odot G \subseteq U$. Then, by Lemma 1, we have $O_1 = H \cap \mathcal{Z}$ and $O_2 = G \cap \mathcal{Z}$ are b-open sets in $\mathcal{Z}$ having ζ and η respectively and obviously, $O_1 \odot O_2 \subseteq H \odot G \subseteq U$.given the proof.

Proposition 15. If $\mathcal{I}$ is an ideal in a T_bd-algebra $\mathcal{W}$ with $0 \in int(\mathcal{I})$, then $\mathcal{I}$ is b-open.

Proof. Suppose that $\zeta \in \mathcal{I}$. Since $0 \in int(\mathcal{I})$, then there is an open set U such that $0 \in U \subseteq \mathcal{I}$. Since $\mathcal{W}$ is a T_bd-algebra, then there exists a b-open set V having ζ such that $V \odot \zeta \subseteq U$. If there is a point $\eta \in V \cap (\mathcal{W} \setminus \mathcal{I})$, so we get $\eta \odot \zeta \in \mathcal{I}$. Since $\zeta \in \mathcal{I}$ and $\mathcal{I}$ is an ideal, so $\eta \in \mathcal{I}$ that is a contradiction. therefore $\zeta \in V \subseteq \mathcal{I}$ means that $\mathcal{I}$ is b-open.

Example 5. Consider the d-algebra $(\mathcal{W}, \odot, 0)$ in Example 1. we define a topology Ω on $\mathcal{W}$ as follows: $\Omega = \{\phi, \{\alpha, \beta\}, \mathcal{W}\}$. Then, $\mathcal{I} = \{0, \alpha, \beta\}$ is an ideal which is b-open but $0 \notin int(\mathcal{I})$.

Proposition 16. If $\mathcal{I}$ is an open proper ideal in a T_bd-algebra $\mathcal{W}$, then $\mathcal{I}$ is b-closed and regular open.

Proof. Suppose that $\zeta \in \mathcal{W} \setminus \mathcal{I}$. Since $\mathcal{I}$ is an ideal so $\zeta \odot \zeta = 0 \in \mathcal{I}$. Since $\mathcal{W}$ is a T_bd-algebra, so there exist b-open sets V and U having ζ such that $V \odot U \subseteq \mathcal{I}$. If there exists $\eta \in (U \cap \mathcal{I})$, then we have $\zeta \odot \eta \in \mathcal{I}$ and $\eta \in \mathcal{I}$. Since $\mathcal{I}$ is an ideal, then we get $\zeta \in \mathcal{I}$ which is contradiction. Hence, $\zeta \in U \subseteq \mathcal{W} \setminus \mathcal{I}$ and so $\zeta \setminus \mathcal{I}$ is b-open. Therefore, $\mathcal{I}$ is b-closed. Since $\mathcal{I}$ is open, so by Lemma 5, we find $\mathcal{I}$ is semi-closed which implies that $\mathcal{I}$ is regular open.

Corollary 6. In a T_bd-algebra $(\mathcal{W}, \odot, 0)$, if $\{0\}$ is open, then $\mathcal{W}$ is semi-disconnected.

Proof. Since $\{0\}$ is an ideal in $\mathcal{W}$, by using Proposition 16, $\{0\}$ is regular open. thus, $\mathcal{W}$ is semi-disconnected so it is having a proper non-empty set which is open and semi-closed.

Proposition 17. In a T_bd-algebra $(\mathcal{W}, \odot, 0)$. If $\mathcal{I}$ is an ideal such that every sequence in $\mathcal{I}$ converging to 0 contains 0, then $\mathcal{I}$ is b-closed.

Proof. Let $\zeta \in Cl_b(\mathcal{I})$, so there exists a sequence $\ll \zeta_n \gg$ in $\mathcal{I}$ which is b-convergent to ζ. Now, we claim that the sequence $\ll \zeta \odot \zeta_n \gg$ converges to 0. For this, Assume that U is any open set that contains 0 and we have $\zeta \odot \zeta = 0$. Since, $\mathcal{W}$ is a T_bd-algebra, so there exists a b-open set V containing ζ such that $\zeta \odot V \subseteq U$. Again, we have $\ll \zeta_n \gg$ b-converges to ζ, so there is $K \in \mathbb{N}$ such that $\zeta_n \in V$ for all $n \gg K$. Therefore, $\zeta \odot \zeta_n \in \mathcal{W} \odot V \subseteq U$ for all $n \gg K$. Hence, $\ll \zeta \odot \zeta_n \gg$ converges to 0 and by hypothesis the sequence $\ll \zeta \odot \zeta_n \gg$ contains 0. Thus, there exists $n \in \mathbb{N}$ such that $\zeta \odot \zeta_n = 0$. Since $\zeta_n \in \mathcal{I}$ and $\mathcal{I}$ is an ideal, so $\zeta \in \mathcal{I}$. Hence, $\mathcal{I}$ is b-closed.

Proposition 18. In an edge T_bd-algebra $(\mathcal{W}, \odot, 0)$. If all elements of $\mathcal{W}$ are atoms and $\phi \neq S \subseteq \mathcal{W}$ not containing 0, then $0 \in Cl(S)$.

Proof. Suppose that $0 \neq \zeta \in Cl_b(S)$, then there exists a sequence $\ll s_n \gg$ in S which b-converges to ζ. Hence, as in Proposition 17, we obtain that $\ll s_n \odot \zeta \gg$ converges to 0. Since all elements of $\mathcal{W}$ are atoms and $0 \neq \zeta \neq s_n$, so $s_n \odot \zeta \neq 0$ implies that $s_n \odot \zeta = s_n$. Therefore, the sequence, $\ll s_n \gg$ in S converges to 0 implies that $0 \in Cl(S)$.

Proposition 19. In an edge T_bd-algebra $(\mathcal{W}, \odot, 0)$. If $S \subseteq \mathcal{W}$ and there is a sequence of atom elements in S b-converges to an element $\zeta \notin S$, then the sequence converges to 0 and $0 \in Cl(S)$.

Proof. Let $\ll \zeta_n \gg$ be a sequence of atom elements in S which is b-convergent to $\zeta \notin S$, so as in Proposition 16, we obtain the sequence $\ll \zeta_n \odot \zeta \gg$ which converges to 0. Since $\mathcal{W}$ is an edge d-algebra, so $\zeta_n \odot \zeta = \{0, \zeta_n\}$ but ζ_n are atoms and $\zeta \notin S$, so $\zeta_n \odot \zeta = \zeta_n$. Therefore, $\ll \zeta_n \gg$ converges to 0 and hence $0 \in Cl(S)$.

Definition 17. Let $\mathcal{W}$ be a d-algebra. We can define the binary operation $\circ$ on $\mathcal{L}(\mathcal{W})$ by $(\mathcal{L}_\alpha \circ \mathcal{L}_\beta)(\zeta) = \mathcal{L}_\alpha(\zeta) \odot \mathcal{L}_\beta(\zeta)$ for each $\zeta \in \mathcal{W}$.

Theorem 1. Let $\mathcal{W}$ be a positive implicative d-algebra, then $(\mathcal{L}(\mathcal{W}), \circ, \mathcal{L}_0)$ is a d-algebra.

Proof. Assume that$\mathcal{L}_\alpha$, $\mathcal{L}_\beta \in \mathcal{L}(\mathcal{W})$. Then, by using the definition of the binary operation $\circ$ on $\mathcal{L}(\mathcal{W})$ we get $(\mathcal{L}_\alpha \circ \mathcal{L}_\beta)(\zeta) = \mathcal{L}_\alpha(\zeta) \odot \mathcal{L}_\beta(\zeta) = (\alpha \odot \zeta) \odot (\beta \odot \zeta)$. Since $\mathcal{W}$ is a positive implication d-algebra, then $(\alpha \odot \zeta) \odot (\beta \odot \zeta) = (\alpha \odot \beta) \odot \zeta$. therefore, $(\mathcal{L}_\alpha \circ \mathcal{L}_\beta)(\zeta) = L_{\alpha\odot\beta}(\zeta)$ which implies that $\mathcal{L}_\alpha \circ \mathcal{L}_\beta = L_{\alpha\odot\beta}$ $\forall$ $\alpha, \beta \in \mathcal{W}$. Furthermore, the following statements are correct.

(i) $\mathcal{L}_\zeta \circ \mathcal{L}_\zeta = \mathcal{L}_{\zeta\odot\zeta} = \mathcal{L}_0$,

(ii) $\mathcal{L}_\zeta \circ \mathcal{L}_\eta = \mathcal{L}_0$ and $\mathcal{L}_\eta \circ \mathcal{L}_\zeta = \mathcal{L}_0$, then $\mathcal{L}_{\zeta\odot\eta} = \mathcal{L}_0$ and $\mathcal{L}_{\eta\odot\zeta} = \mathcal{L}_0$ which implies that $\zeta \odot \eta = 0$ and $\eta \odot \zeta = 0 \Rightarrow \zeta = \eta$ and hence, $\mathcal{L}_\zeta = \mathcal{L}_\eta$,

(iii) $\mathcal{L}_0 \circ \mathcal{L}_\zeta = \mathcal{L}_{0\odot\zeta} = \mathcal{L}_0$.

Therefore, $\mathcal{L}(\mathcal{W})$ is a d-algebra.

Proposition 20. Let $\mathcal{W}$ be a T_bd-algebra, then every left map on $\mathcal{W}$ is b-continuous.

Proof. Let $\zeta \in \mathcal{W}$ and W be any open set having $\mathcal{L}_\alpha(\zeta) = \alpha \odot \zeta$. Since $\mathcal{W}$ is a T_bd-algebra, so there exists a b-open set V having ζ such that $a \odot V \subseteq W$. Hence, $\mathcal{L}_\alpha(V) \subseteq W$. Thus, we get $\mathcal{L}_\alpha$ is b-continuous.

Proposition 21. Let $\mathcal{W}$ be an edge T_bd-algebra. If $\alpha \in \mathcal{W}$ such that $\alpha \odot (\alpha \odot \zeta)$ is an atom for every $\zeta \in \mathcal{W}$, then the left map $\mathcal{L}_\alpha$ on $\mathcal{W}$ is b-open.

Proof. Let W be any open set in $\mathcal{W}$ and $\alpha \in \mathcal{W}$. We have to prove that $\mathcal{L}_\alpha(W)$ is b-open. Let $\zeta \in \mathcal{L}_\alpha(W)$, so there exists $\eta \in W$ such that $\zeta = \mathcal{L}_\alpha(\eta) = \alpha \odot \eta$. Since $\mathcal{W}$, is an edge d-algebra, so it satisfies condition (B2), that is $(\alpha \odot (\alpha \odot \eta)) \odot \eta = 0$ for each $\eta \in \mathcal{W}$. Also, by hypothesis, we have $\alpha \odot (\alpha \odot \eta)$ is an atom. Hence, we get $\alpha \odot \zeta = \alpha \odot (\alpha \odot \eta) = \eta$. Therefore, $\alpha \odot \zeta \in W$. Since $\mathcal{W}$ is a T_bd-algebra, so there exists a b-open set V containing ζ such that $a \odot V \subseteq W$. Since $\mathcal{W}$, satisfies (B2), so we get $V = \alpha \odot (\alpha \odot V) \subseteq \alpha \odot W = \mathcal{L}_\alpha(W)$. This implies that $\mathcal{L}_\alpha(W)$ is b-open.

Proposition 22. Let $\mathcal{W}$ be a $T_b d$-algebra, then every right map on $\mathcal{W}$ is b-continuous.

Proof. The proof is similar to the proof of Proposition 20.

Proposition 23. If $\mathcal{W}$ is an edge d-algebra in which condition (B1) does not hold for all $\zeta, \eta \in \mathcal{W}$, then $(\zeta \odot \eta) \odot y = \zeta$ for all $\zeta, \eta \in \mathcal{W}$.

Proof. Since $\mathcal{W}$ is an edge d-algebra, so $\zeta \odot y = \{\zeta, 0\}$. Now if $\zeta \odot y = 0$, and by hypothesis, we have $(\zeta \odot \eta) \odot (\zeta \odot \theta)) \odot (\theta \odot \eta) \neq 0$ for all $\zeta, \eta, \theta \in \mathcal{W}$. Hence, we get $0 \neq (\zeta \odot \eta) \odot (\zeta \odot \theta)) \odot (\theta \odot \eta) = 0 \odot (\theta \odot \eta) = 0$ which is contradiction. Therefore, $\zeta \odot \eta = \zeta$ and thus $(\zeta \odot \eta) \odot \eta = \zeta$.

Corollary 7. If $\mathcal{W}$ is an edge d-algebra in which all elements of $\mathcal{W}$ are atoms, then $(\zeta \odot \eta) \odot \eta = \zeta$ for all $\zeta, \eta \in \mathcal{W}$.

Proof. Since each element of $\mathcal{W}$ is an atom, so if $\zeta, y \in \mathcal{W}$, then $\zeta \odot y \neq 0$. Since $\mathcal{W}$ is an edge d-algebra, so $\zeta \odot \eta = \zeta$. Hence, $(\zeta \odot \eta) \odot \eta = \zeta$ for all $\zeta, \eta \in \mathcal{W}$.

Proposition 24. Let $\mathcal{W}$ be an edge $T_b d$-algebra not satisfying condition (B1) for all $\zeta, \eta, \theta \in \mathcal{W}$, then every right map on $\mathcal{W}$ is b-open.

Proof. Let W be any open set in $\mathcal{W}$ and $\alpha \in \mathcal{W}$. We have to prove that $\mathcal{R}_\alpha(W)$ is b-open. Let $\zeta \in \mathcal{R}_\alpha(W)$, then $\exists\ \eta \in W$ such that $\zeta = \mathcal{R}_\alpha(\eta) = \eta \odot \alpha$. Hence, we have $\zeta \odot \alpha = (\eta \odot \alpha) \odot \alpha$ and by Proposition 23, we get $\zeta \odot \alpha = (\eta \odot \alpha) \odot \alpha = \eta$. Hence, $\zeta \odot \alpha \in W$. Since $\mathcal{W}$ is a $T_b d$-algebra, so $\exists$ a b-open set V having ζ such that $V \odot \alpha \subseteq W$. Again by Proposition 23, we get $V = (V \odot \alpha) \odot \alpha \subseteq W \odot \alpha = \mathcal{R}_\alpha(W)$. Thus, $\mathcal{R}_\alpha(W)$ is b-open.

Corollary 8. Let $\mathcal{W}$ be an edge $T_b d$-algebra such that every element of $\mathcal{W}$ is an atom, then every right map on $\mathcal{W}$ is b-open.

Proof. Follows from Proposition 24 and Corollary 7.

Definition 18. Let $\mathcal{W}$ be a d-algebra, we define a map $\mathcal{F} : \zeta \to \mathcal{L}(\mathcal{W})$ by $\mathcal{F}(\mathcal{W}) = \mathcal{L}_\zeta$ for all $\zeta \in \mathcal{W}$. If $M \subseteq \mathcal{W}$, then $\mathcal{F}(M) = \{\mathcal{L}_\zeta : \zeta \in M\}$.

Proposition 25. If $\mathcal{W}$ is a d-algebra, then the following statements are true:

(i) If $M \subseteq N$, then $\mathcal{F}(M) \subseteq \mathcal{F}(N)$.

(ii) $\mathcal{F}(M^c) = (\mathcal{F}(M))^c$.

(iii) If $\{M_\lambda : \lambda \in \Lambda\}$ is any family of subsets of $\mathcal{W}$, then
$\mathcal{F}(\bigcup_{\lambda \in \Lambda} M_\lambda) = \bigcup_{\lambda \in \Lambda} \mathcal{F}(M_\lambda)$ and $\mathcal{F}(\bigcap_{\lambda \in \Lambda} M_\lambda) = \bigcap_{\lambda \in \Lambda} \mathcal{F}(M_\lambda)$.

Proof.

(i) Let $M \subseteq N$ and let $\mathcal{L}_\zeta \in \mathcal{F}(M)$. Hence, $\zeta \in M \subseteq N$ implies $\mathcal{L}_\zeta \in \mathcal{F}(N)$. Thus, $\mathcal{F}(M) \subseteq \mathcal{F}(N)$.

(ii) $\mathcal{L}_\zeta \in \mathcal{F}(M^c)$ if, and only if $\zeta \in M^c$ implies $\zeta \notin M$ if, and only if $\mathcal{L}_\zeta \notin \mathcal{F}(M)$ if, and only if $\mathcal{L}_\zeta \in (\mathcal{F}(M))^c$.

(iii) We shall prove for the union, the other proof is similar. Let $\mathcal{L}_\zeta \in \mathcal{F}(\bigcup_{\lambda\in\Lambda} M_\lambda)$, then $\zeta \in \bigcup_{\lambda\in\Lambda} M_\lambda$ that is there is $\lambda \in \Lambda$ such that $\zeta \in M_\lambda$, so $\mathcal{L}_\zeta \in \mathcal{F}(M_\lambda)$ for some $\lambda \in \Lambda$. Hence, $\mathcal{L}_\zeta \in \bigcup_{\lambda\in\Lambda} \mathcal{F}(M_\lambda)$.

Proposition 26. Let $\mathcal{W}$ be a positive implicative d-algebra, then the map $\mathcal{F} : \zeta \to \mathcal{L}(\mathcal{W})$ is a d-isomorphism.

Proof. It is obvious that $\mathcal{F}$ is a bijection. We have $\mathcal{F}(\zeta \odot \eta) = \mathcal{L}_{\zeta\odot\eta}$ and $\mathcal{L}_{\zeta\odot\eta}(\theta) = (\zeta \odot \eta) \odot \theta$. Since $\mathcal{W}$ is positive implicative, we have $(\zeta \odot \eta) \odot \theta = (\zeta \odot \theta) \odot (\eta \odot \theta)$. Therefore, $\mathcal{L}_{\zeta\odot\eta}(\theta) = \mathcal{L}_\zeta(\theta) \circ \mathcal{L}_\eta(\theta) = (\mathcal{L}_\zeta \circ \mathcal{L}_\eta)(\theta)$. Hence, $\mathcal{F}(\zeta \odot \eta) = \mathcal{F}(\zeta) \circ \mathcal{F}(\eta)$ for all $\zeta, \eta \in \mathcal{W}$, so $\mathcal{F}$ is a d-isomorphism.

Proposition 27. Let $\mathcal{W}$ be a positive implicative d-algebra and Ω be a topology on $\mathcal{W}$, Consequently, the following statements are correct:

(i) The family $\sigma = \{\mathcal{F}(G) \subseteq \mathcal{L}(\mathcal{W}) : G \in \Omega\}$ is a topology on $\mathcal{L}(\mathcal{W})$.

(ii) For every subset M of $\mathcal{W}$, $\mathcal{L}_{Cl(M)} = \sigma Cl(\mathcal{L}_M)$.

(iii) For any subset M of $\mathcal{W}$, $\mathcal{F}(Int(M)) = \sigma Int(\mathcal{F}(M))$.

(iv) If M is any pre-open set in $(\mathcal{W}, \Omega)$, then $\mathcal{F}(M)$ is a pre-open set in $(\mathcal{L}(\mathcal{W}), \sigma)$.

(v) If M is any semi-open set in $(\mathcal{W}, \Omega)$, then $\mathcal{F}(M)$ is a semi-open set in $(\mathcal{L}(\mathcal{W}), \sigma)$.

Proof.

(i) The proof of σ being a topology is clear.

(ii) For every subset M of $\mathcal{W}$, we have $M \subseteq Cl(M)$. Hence, $\mathcal{L}_M \subseteq \mathcal{L}_{Cl(M)}$ and $Cl(M)$ is closed in $\mathcal{W}$, so by definition of σ, we have $\mathcal{L}_{Cl(M)}$ is σ-closed in $\mathcal{L}(\mathcal{W})$. Therefore, we obtain $\sigma Cl(\mathcal{L}_M) \subseteq \sigma Cl(\mathcal{L}_{Cl(M)}) = \mathcal{L}_{Cl(M)}$. To prove $\mathcal{L}_{Cl(M)} \subseteq \sigma Cl(\mathcal{L}_M)$, let $\mathcal{L}_\zeta \in \mathcal{L}_{Cl(M)}$, then $\zeta \in Cl(M)$ and let $\mathcal{F}(G)$ be any σ-open set containing $\mathcal{L}_\zeta$. Hence G is an open set that contains ζ, so $M \cap G \neq \phi$ and by Proposition 25, $\mathcal{F}(M \cap G) = \mathcal{F}(M) \cap \mathcal{F}(G)$. Therefore, $\mathcal{F}(M) \cap \mathcal{F}(G) \neq \phi$. Implies that $\mathcal{L}_\zeta \in \sigma Cl(\mathcal{L}_M)$, so $\mathcal{L}_{Cl(M)} \subseteq \sigma Cl(\mathcal{L}_M)$ and hence $\mathcal{L}_{Cl(M)} = \sigma Cl(\mathcal{L}_M)$.

(iii) We have $Int(M) \subseteq M$, so $\mathcal{F}(Int(M)) \in \sigma$ and $\mathcal{F}(Int(M)) \subseteq \mathcal{F}(M)$. Hence $\mathcal{F}(Int(M)) \subseteq \sigma Int(\mathcal{F}(M))$. Now if $\mathcal{L}_\zeta \in \sigma Int(\mathcal{F}(M))$, so there exists $\mathcal{F}(G) \in \sigma$ such that $\mathcal{L}_\zeta \in \mathcal{F}(G) \subset \mathcal{F}(M)$. Hence, $\zeta \in G \subseteq M$ implies that $\zeta \in Int(M)$. Therefore, $\mathcal{L}_\zeta \in \mathcal{F}(Int(M))$. Thus, $\mathcal{F}(Int(M)) = \sigma Int(\mathcal{F}(M))$.

(iv) Let M be any pre-open set in $\mathcal{W}$, so there exists an open set V in $\mathcal{W}$ such that $M \subseteq V \subseteq Cl(M)$. Hence $\mathcal{F}(M) \subseteq \mathcal{F}(V) \subseteq \mathcal{F}(Cl(M))$ and by (2, 3), we have $\mathcal{F}(M) \subseteq \mathcal{F}(V) \subseteq \sigma Cl(\mathcal{F}(M))$ and $\mathcal{F}(V)$ is σ-open. Hence, $\mathcal{F}(M)$ is pre-open in $(\mathcal{L}(\mathcal{W}), \sigma)$.

(v) The proof is similar to the proof of (iv).

Corollary 9. If M is any subset of $(\mathcal{W}, \Omega)$, then $\mathcal{F}(Int(Cl(M))) = \sigma Int(\sigma Cl(\mathcal{F}(M)))$ and $\mathcal{F}(Cl(Int(M))) = \sigma Cl(\sigma Int(\mathcal{F}(M)))$ where $(\mathcal{L}(\mathcal{W}), \sigma)$ is defined as in Proposition 27.

Proof. The proof follows from (ii, iii) of Proposition 27.

Proposition 28. A subset M is b-open in $(\mathcal{W}, \Omega)$ if and only if $\mathcal{F}(M)$ is a b-open set in $(\mathcal{L}(\mathcal{W}), \sigma)$.

Proof. Suppose that M is a b-open set, then by Lemma 2, $M = M \cap (Int(Cl(M)) \cap Cl(Int(M))$. From Proposition 25, we get $\mathcal{F}(M) = \mathcal{F}(M) \cap \big(\mathcal{F}(Int(Cl(M))) \cap \big(\mathcal{F}Cl(Int(M))\big)\big)$. By Corollary 9, we get $\mathcal{F}(M) = \mathcal{F}(M) \cap \big(\sigma Int(\sigma Cl(\mathcal{F}(M))\big) \cap \big(\sigma Cl(\sigma Int(\mathcal{F}(M))\big)$. Hence, $\mathcal{F}(M)$ is a b-open set in $(\mathcal{L}(\mathcal{W}), \sigma)$. Reversing the statement, we get the result.

Proof. Suppose that M is a b-open set, then by Lemma 2, $M = M \cap (Int(Cl(M)) \cap Cl(Int(M))$. From Proposition 25, we get $\mathcal{F}(M) = \mathcal{F}(M) \cap \big(\mathcal{F}(Int(Cl(M))) \cap \big(\mathcal{F}Cl(Int(M))\big)\big)$. By Corollary 9, we get $\mathcal{F}(M) = \mathcal{F}(M) \cap \big(\sigma Int(\sigma Cl(\mathcal{F}(M))\big) \cap \big(\sigma Cl(\sigma Int(\mathcal{F}(M))\big)$. Hence, $\mathcal{F}(M)$ is a b-open set in $(\mathcal{L}(\mathcal{W}), \sigma)$.

Proposition 29. Let $\mathcal{W}$ be a positive implicative $T_b d$-algebra. Then $(\mathcal{L}(\mathcal{W}), \circ, \sigma)$ is a $T_b d$-algebra.

Proof. Suppose that $\mathcal{L}_\zeta$, $\mathcal{L}_\eta$ are any elements in $\mathcal{L}(\mathcal{W})$ and $\mathcal{F}(W)$ is a σ-open set having $\mathcal{L}_\zeta \circ \mathcal{L}_\eta = \mathcal{L}_{\zeta \odot \eta}$. Then, W is an open set having $\zeta \odot \eta$ in $\mathcal{W}$, since $\mathcal{W}$ is a $T_b d$-algebra, so there exist b-open sets U and V containing ζ and η respectively such that $U \odot V \subseteq W$. Therefore, $\mathcal{F}(U \odot V) \subseteq \mathcal{F}(W)$. Since $\mathcal{W}$ is positive implicative, and by Proposition 26, $\mathcal{F}(U \odot V) = \mathcal{F}(U) \circ \mathcal{F}(V) \subseteq \mathcal{F}(W)$. By Proposition 27, $\mathcal{F}(U)$ and $\mathcal{F}(V)$ are b-open sets in $(\mathcal{L}(\mathcal{W}), \circ, \sigma)$ containing $\mathcal{L}_\zeta$ and $\mathcal{L}_\eta$ respectively, hence the proof.

Example 6. Consider the d-algebra $(\mathcal{W}, \odot, \Omega)$ in Example 1, the operation $(\circ)$ defined on $\mathcal{L}(\mathcal{W})$ is given as follows:

$\circ$	$\mathcal{L}_0$	$\mathcal{L}_\alpha$	$\mathcal{L}_\beta$	$\mathcal{L}_\gamma$
$\mathcal{L}_0$	$\mathcal{L}_0$	$\mathcal{L}_0$	$\mathcal{L}_0$	$\mathcal{L}_0$
$\mathcal{L}_\alpha$	$\mathcal{L}_\alpha$	$\mathcal{L}_0$	$\mathcal{L}_0$	$\mathcal{L}_\alpha$
$\mathcal{L}_\beta$	$\mathcal{L}_\beta$	$\mathcal{L}_\beta$	$\mathcal{L}_0$	$\mathcal{L}_0$
$\mathcal{L}_\gamma$	$\mathcal{L}_\gamma$	$\mathcal{L}_\gamma$	$\mathcal{L}_\gamma$	$\mathcal{L}_0$

Table 2: The d-algebra $(\mathcal{L}(\mathcal{W}), \circ, \sigma)$

From Proposition 27 and the definition of Ω, we obtain that $\sigma = \{\Phi, \mathcal{L}_\alpha, \mathcal{L}_\gamma, \mathcal{L}_{\{\alpha,\gamma\}}, \mathcal{L}(\mathcal{W})\}$. By routine calculation, we can show that $(\mathcal{L}(\mathcal{W}), \circ, \sigma)$ is a $T_b d$-algebra but $(\mathcal{W}, \odot, \Omega)$ is not positive implicative because $(\alpha \odot \gamma) \odot (\beta \odot \gamma) = \alpha \neq (\alpha \odot \beta) \odot \gamma = 0$.

4. Conclusions

In this paper, we applied the family of b-open sets in topological spaces to define the concept of b-topological d-algebra. Through this work, we reviewed the basics necessary for studying this new topological algebra. We also provided proofs for several issues related to continuity, separation axioms, and other concepts. This paves the way for future studies related to topological d-algebras by applying various types of nearly open sets. Moreover, d-algebras can be studied in supratopological spaces.

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