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Hybrid Ideals of a Near Algebra

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Abstract. This study aims to explore hybrid ideals in a near algebra. It concludes precise definitions and theorems regarding the hybrid ideal, near algebra homomorphism, the Cartesian product of hybrid ideals, and the coset of the hybrid ideal within the context of near algebras. It is demonstrated that an onto homomorphic image, and the Cartesian product of a hybrid ideal of a near algebra, is the hybrid ideal of a near algebra.

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1. Introduction

A near ring is an algebraic system equipped with two binary operations that satisfy all ring axioms, except possibly one of the distributive laws. The concept of a near ring was first presented in a monograph by Pilz [1]. A near algebra is defined as a near ring in which the right scalar domain is a field, and Brown [2] studied its foundational properties. According to Jordan, within the formalism of quantum mechanics, the set of operators forms only a near algebra, making the study of such structures relevant not only for purely axiomatic reasons but also due to their physical applications.

In recent years, various generalizations of classical algebraic structures have emerged, especially in the context of non-associative algebras. These include alternative rings, Jordan algebras and Γ-rings serve as fertile ground for exploring additive and multiplicative mappings. The behavior of such mappings under weakened structural assumptions has

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been the subject of extensive research. Notable contributions include the investigation of n-multiplicative mappings on Γ-rings [3], the additivity of multiplicative maps on alternative rings [4], and additive maps preserving generalized inverses on alternative division algebras [5]. Additionally, Breš ar's monograph on zero product determined algebras [6] offers a comprehensive treatment of how structural constraints influence the nature of linear preservers and related mappings. In particular, Srinivas and Narasimha Swamy [7] introduced fuzzy near algebras over fuzzy fields, defining fuzzy ideals and examining their algebraic behavior under homomorphisms and direct sums.

Parallel to these algebraic developments, fuzzy set theory—pioneered by Zadeh [8]—has evolved into a robust framework for dealing with uncertainty in mathematical modeling. This theory has found widespread applications in engineering, robotics, computer science, and decision theory. Later generalizations, such as hesitant fuzzy sets [9], vague sets, interval-valued sets, and rough sets, have extended this capacity. To address the limitations found in these frameworks, Molodtsov [10] introduced soft set theory, a flexible tool for modeling uncertainty, which has since been applied in diverse fields, including game theory, integration theory, and operations research.

To unify the strengths of fuzzy and soft set theories, Jun et al. [11] introduced the notion of hybrid structures. These frameworks combine multiple uncertainty paradigms through parameterization over a universe set, allowing for more nuanced representations. Based on these ideas, hybrid ideals, hybrid fields, and hybrid subalgebras have been defined and studied in [12–14]. Building on this foundation, Bhurgula et al. [15] introduced the concept of hybrid near algebras, establishing key structural properties, including closure under homomorphisms and Cartesian products.

Motivated by these developments, this work introduces the concept of hybrid ideals in near algebras over hybrid fields. Hybrid structures are then employed to examine structural aspects of near algebras. Throughout this paper, Y denotes a (right) near algebra over a field L.

2. Preliminaries

This section provides the necessary background and notational framework for developing hybrid ideals in near algebras. We begin by recalling essential definitions related to near algebras and hybrid structures, including hybrid sets, hybrid subalgebras, and related homomorphisms. These foundational concepts serve as the basis for formalizing the hybrid ideal structure introduced in subsequent sections.

Definition 1. *[1] Let U be a universal set, $P(U)$ be the power set, L be the set of parameters, and I be the unit interval. A mapping $\tilde{\xi}_\lambda := (\tilde{\xi}, \lambda) : L \to P(U) \times I; q \mapsto (\tilde{\xi}(q), \lambda(q))$, i.e., the image of q is signified by $(\tilde{\xi}(q), \lambda(q))$ is named a hybrid structure (HS) in L upon U, where $\tilde{\xi} : L \to P(U)$ and $\lambda : L \to I$ are the mappings.*

Definition 2. *[1] Let $\tilde{\xi}_\lambda$ be an HS in L upon U. Then the sets*

$$\tilde{\xi}_\lambda[\alpha, t] = \{q \in L \mid \tilde{\xi}(q) \supseteq \alpha, \lambda(q) \leq t\},$$

$$\tilde{\xi}_\lambda(\alpha, t] = \{q \in L \mid \tilde{\xi}(q) \supset \alpha, \lambda(q) \leq t\},$$
$$\tilde{\xi}_\lambda[\alpha, t) = \{q \in L \mid \tilde{\xi}(q) \supseteq \alpha, \lambda(q) < t\},$$
$$\tilde{\xi}_\lambda(\alpha, t) = \{q \in L \mid \tilde{\xi}(q) \supset \alpha, \lambda(q) < t\},$$

are called the $[\alpha, t]$*-hybrid cut (HC),* $(\alpha, t]$*-HC,* $[\alpha, T)$*-HC, and* (α, t)*-HC of* $\tilde{\xi}_\lambda$ *correspondingly, where* $\alpha \in P(U)$ *and* $t \in I$*. Apparently,* $\tilde{\xi}_\lambda(\alpha, t) \subseteq \tilde{\xi}_\lambda(\alpha, t] \subseteq \tilde{\xi}_\lambda[\alpha, t]$ *and* $\tilde{\xi}_\lambda(\alpha, t) \subseteq \tilde{\xi}_\lambda[\alpha, t) \subseteq \tilde{\xi}_\lambda[\alpha, t]$*.*

Definition 3. *[13] Let* L *be a field. An HS* $\tilde{\xi}_\lambda$ *in* L *upon* U *is called a hybrid field (HF) of* L *upon* U *if the following conditions hold:*

(i) $\tilde{\xi}(s + a) \supseteq \tilde{\xi}(s) \cap \tilde{\xi}(a), \lambda(s + a) \leq \bigvee\{\lambda(s), \lambda(a)\}, \forall s, a \in L$

(ii) $\tilde{\xi}(-s) \supseteq \tilde{\xi}(s), \lambda(-s) \leq \lambda(s), \forall s \in L$

(iii) $\tilde{\xi}(sa) \supseteq \tilde{\xi}(s) \cap \tilde{\xi}(a), \lambda(sa) \leq \bigvee\{\lambda(s), \lambda(a)\}, \forall s, a \in L$

(iv) $s \neq 0 \Rightarrow \tilde{\xi}(s^{-1}) \supseteq \tilde{\xi}(s), \lambda(s^{-1}) \leq \lambda(s), \forall s \in L.$

Definition 4. *[3] Let* $\tilde{\xi}_\lambda$ *be an HF of a field* L *upon* U *and* Y *be an NA over* L*. An HS* $\tilde{\varrho}_\gamma$ *in* Y *upon* U *is called a hybrid near algebra (HNA) upon HF* $(\tilde{\xi}_\lambda, L)$ *if the resulting conditions hold:*

(i) $\tilde{\varrho}(q + \varsigma) \supseteq \tilde{\varrho}(q) \cap \tilde{\varrho}(\varsigma), \gamma(q + \varsigma) \leq \bigvee\{\gamma(q), \gamma(\varsigma)\}, \forall q, \varsigma \in Y$

(ii) $\tilde{\varrho}(sq) \supseteq \tilde{\xi}(s) \cap \tilde{\varrho}(q), \lambda(sq) \leq \bigvee\{\lambda(s), \gamma(q)\}, \forall s \in L, q \in Y$

(iii) $\tilde{\varrho}(q\varsigma) \supseteq \tilde{\varrho}(q) \cap \tilde{\varrho}(\varsigma), \gamma(q\varsigma) \leq \bigvee\{\gamma(q), \gamma(\varsigma)\}, \forall q, \varsigma \in Y$

(iv) $\tilde{\xi}(1) \supseteq \tilde{\varrho}(q), \lambda(1) \leq \gamma(q), \forall q \in Y.$

Definition 5. *[3] Let* $\tilde{\varrho}_\gamma$ *and* $\tilde{h}_\mu$ *be two HSs in* L *upon* U*. Then the hybrid intersection of* $\tilde{\varrho}_\gamma$ *and* $\tilde{h}_\mu$ *is an HS* $\tilde{\varrho}_\gamma \Cap \tilde{h}_\mu : L \to P(U) \times I; q \mapsto ((\tilde{\varrho} \tilde{\cap} \tilde{h})(q), (\gamma \vee \mu)(q)), \forall q \in L$*, where* $\tilde{\varrho} \tilde{\cap} \tilde{h} : L \to P(U); q \mapsto \tilde{\varrho}(q) \cap \tilde{h}(q)$ *and* $\gamma \vee \mu : L \to I; q \mapsto \vee\{\gamma(q), \mu(q)\}$*.*

3. Hybrid ideal of a near algebra

The hybrid ideal of a near algebra is introduced in this section, along with some of its features over the hybrid field.

Definition 6. *Let* $\tilde{\xi}_\lambda$ *be an HF of a field* L *upon* U *and* Y *be a NA over* L*. An HS* $\tilde{\varrho}_\gamma$ *in* Y *upon* U *is called a hybrid ideal of a near algebra (HINA) upon the HF* $(\tilde{\xi}_\lambda, L)$ *if*

(i) $\tilde{\varrho}(q + \varsigma) \supseteq \tilde{\varrho}(q) \cap \tilde{\varrho}(\varsigma), \gamma(q + \varsigma) \leq \bigvee\{\gamma(q), \gamma(\varsigma)\}, \forall q, \varsigma \in Y$

(ii) $\tilde{\varrho}(sq) \supseteq \tilde{\xi}(s) \cap \tilde{\varrho}(q), \lambda(sq) \leq \bigvee\{\lambda(s), \gamma(q)\}, \forall s \in L, q \in Y$

(iii) $\tilde{\xi}(1) \supseteq \tilde{\varrho}(q), \lambda(1) \leq \gamma(q), \forall q \in Y$

(iv) $\tilde{\varrho}(q\varsigma) \supseteq \tilde{\varrho}(q), \gamma(q\varsigma) \leq \gamma(q), \forall q, \varsigma \in Y$

(v) $\tilde{\varrho}(\varsigma(q+i) - \varsigma q) \supseteq \tilde{\varrho}(i), \gamma(\varsigma(q+i) - \varsigma q) \leq \gamma(i), \forall q, \varsigma, i \in Y, 1$ *is the unity in* L.

If $\tilde{\varrho}_\gamma$ *satisfies* $(i), (ii), (iii)$ *and* (iv) *then* $\tilde{\varrho}_\gamma$ *is called a right HINA of* Y. *If* $\tilde{\varrho}_\gamma$ *satisfies* $(i), (ii), (iii)$ *and* (v) *then* $\tilde{\varrho}_\gamma$ *is called a left HINA of* Y.

Example 1. *Let* $L = Z_2 = \{0, 1\}_{\oplus_2, \otimes_2}$ *be a field. The HS* $\tilde{\xi}_\lambda$ *in* L *upon* $U = \{u_1, u_2, u_3, u_4, u_5\}$ *is given by*

L	$\tilde{\xi}$	λ
0	$\{u_1, u_2, u_3, u_4\}$	0.4
1	$\{u_3, u_4, u_5\}$	0.5

Then $\tilde{\xi}_\lambda$ *is an HF in* L *upon* U. *Let* $Y = \{0, a_3, b_3, c_3\}$ *be a set with two binary operations* $+$ *by*

$+$	0	a_3	b_3	c_3
0	0	a_3	b_3	c_3
a_3	a_3	0	c_3	b_3
b_3	b_3	c_3	0	a_3
c_3	c_3	b_3	a_3	0

.	0	a_3	b_3	c_3
0	0	0	0	0
a_3	a_3	a_3	a_3	a_3
b_3	b_3	b_3	b_3	b_3
c_3	c_3	c_3	c_3	c_3

Clearly, Y *forms an NA over* L. *The HS* $\tilde{\varrho}_\gamma$ *in* Y *upon* $U = \{u_1, u_2, u_3, u_4, u_5\}$ *is given as follows:*

Y	$\tilde{\varrho}$	γ
0	$\{u_1, u_4, u_5\}$	0.5
a_3	$\{u_1, u_2\}$	0.6
b_3	$\{u_1, u_3, u_4\}$	0.8
c_3	$\{u_1, u_4\}$	0.9

Therefore, $(\tilde{\varrho}_\gamma, Y)$ *is an HINA upon* $(\tilde{\xi}_\lambda, L)$.

Theorem 1. $(\tilde{\varrho}_\gamma, Y)$ *is an HINA upon an HF* $(\tilde{\xi}_\lambda, L)$ *if and only if nonempty set* $\tilde{\varrho}_\gamma[\alpha, t]$ *is an ideal of* Y *upon the field* $\tilde{\xi}_\lambda[\alpha, t], \forall t \in [0, 1], \alpha \in P(U)$.

Proof. Let $t \in [0, 1]$ be such that $\tilde{\varrho}_\gamma[\alpha, t] \neq \emptyset$ and $q, \varsigma \in \tilde{\varrho}_\gamma[\alpha, t], s \in \tilde{\xi}_\lambda[\alpha, t]$. Then $q, \varsigma \in Y, s \in L$ and $\tilde{\varrho}(q) \supseteq \alpha, \gamma(q) \leq t, \tilde{\varrho}(\varsigma) \supseteq \alpha, \gamma(\varsigma) \leq t, \tilde{\xi}(s) \supseteq \alpha, \lambda(s) \leq t$, so that $q - \varsigma \in Y$ and $sq \in Y$. Also, $\tilde{\varrho}(q - \varsigma) \supseteq \tilde{\varrho}(q) \cap \tilde{\varrho}(\varsigma) \supseteq \alpha \cap \alpha = \alpha$ and $\gamma(q - \varsigma) \leq \bigvee\{\gamma(q), \gamma(\varsigma)\} \leq \bigvee\{t, t\} = t$.

Also, $\tilde{\varrho}(sq) \supseteq \tilde{\xi}(s) \cap \tilde{\varrho}(q) \supseteq \alpha \cap \alpha = \alpha$ and $\gamma(sq) \leq \bigvee\{\lambda(s), \gamma(q)\} \leq \bigvee\{t, t\} = t$. Thus, $q - \varsigma \in \tilde{\varrho}_\gamma[\alpha, t]$ and $sq \in \tilde{\varrho}_\gamma[\alpha, t]$. Hence, $\tilde{\varrho}_\gamma[\alpha, t]$ is a subspace of the linear space Y over a field $\tilde{\xi}_\lambda[\alpha, t]$. Suppose that $\tilde{\xi}_\lambda[\alpha, t] \neq \emptyset$. We know that $\tilde{\xi}_\lambda[\alpha, t]$ is a subfield of L. Let $q \in Y, i \in \tilde{\varrho}_\gamma[\alpha, t]$. Then $i \in Y$ and $\tilde{\varrho}(i) \supseteq \alpha, \gamma(i) \leq t$. Thus, $i, q \in Y, iq \in Y$ and $\tilde{\varrho}(iq) \supseteq \tilde{\varrho}(i) \supseteq \alpha, \gamma(iq) \leq \gamma(i) \leq t$. Thus, $iq \in \tilde{\varrho}_\gamma[\alpha, t]$. Let $q, \varsigma \in Y, i \in \tilde{\varrho}_\gamma[\alpha, t]$. Then $i \in Y$ and $\tilde{\varrho}(i) \supseteq \alpha, \gamma(i) \leq t$. Thus, $i, q, \varsigma \in Y, \varsigma(q + i) - \varsigma q \in Y$ and $\tilde{\varrho}(\varsigma(q + i) - \varsigma q) \supseteq \tilde{\varrho}(i) \supseteq \alpha, \gamma(\varsigma(q + i) - \varsigma q) \leq \gamma(i) \leq t$. Thus, $\varsigma(q + i) - \varsigma q \in \tilde{\varrho}_\gamma[\alpha, t]$. Hence, $\tilde{\varrho}_\gamma[\alpha, t]$ is an ideal of Y over a field $\tilde{\xi}_\lambda[\alpha, t]$.

Conversely, suppose that $\tilde{\varrho}_\gamma[\alpha, t] \neq \emptyset$ is an ideal of Y. Since $(\tilde{\varrho}_\gamma, Y)$ is an HNA upon $(\tilde{\xi}_\lambda, L)$, the first three conditions of HINA holds directly. If possible, suppose that there exists $q, \varsigma \in Y$ such that $\tilde{\varrho}(q\varsigma) \subseteq \tilde{\varrho}(q), \gamma(q\varsigma) \geq \gamma(q)$. Put $u_1 = \frac{1}{2}\{\tilde{\varrho}(q\varsigma) + \tilde{\varrho}(q)\}, v_1 = \frac{1}{2}\{\gamma(q\varsigma) + \gamma(q)\}$. Then $\tilde{\varrho}(q\varsigma) \subset u_1 \subset \tilde{\varrho}(q)$ and $\gamma(q\varsigma) > v_1 > \gamma(q)$. Hence, $\tilde{\varrho}(q\varsigma) \subset u_1, \tilde{\varrho}(q) \supset u_1$ and $\gamma(q\varsigma) > v_1, \gamma(q) < v_1$. Since $q, \varsigma \in Y$, we have $q\varsigma \in Y$. Thus, $\tilde{\varrho}(q\varsigma) \subset u_1, \tilde{\varrho}(q) \supset u_1$ and $\gamma(q\varsigma) > v_1, \gamma(q) < v_1$. Therefore, $q\varsigma \notin \tilde{\varrho}_\gamma[u_1, t] \cap \tilde{\varrho}_\gamma[v_1, t], q \in \tilde{\varrho}_\gamma[u_1, t] \cap \tilde{\varrho}_\gamma[v_1, t], \forall \varsigma \in Y$, which is a contradiction. Hence, $\tilde{\varrho}(q\varsigma) \supseteq \tilde{\varrho}(q), \gamma(q\varsigma) \leq \gamma(q)$. If possible, presume that there exists $i, q, \varsigma \in Y$ such that $\tilde{\varrho}(\varsigma(q + i) - \varsigma q) \subseteq \tilde{\varrho}(i), \gamma(\varsigma(q + i) - \varsigma q) \geq \gamma(i)$. Put $u_2 = \frac{1}{2}\{\tilde{\varrho}(\varsigma(q + i) - \varsigma q) + \tilde{\varrho}(i)\}, v_2 = \frac{1}{2}\{\gamma(\varsigma(q + i) - \varsigma q) + \gamma(i)\}$. Then $\tilde{\varrho}(\varsigma(q + i) - \varsigma q) \subset u_2 \subset \tilde{\varrho}(i)$ and $\gamma(\varsigma(q + i) - \varsigma q) > v_2 > \gamma(i)$. Hence, $\tilde{\varrho}(\varsigma(q + i) - \varsigma q) \subset u_2, \tilde{\varrho}(i) \supset u_2$ and $\gamma(\varsigma(q + i) - \varsigma q) > v_2, \gamma(i) < v_2$. Since $i, q, \varsigma \in Y$, $\varsigma(q + i) - \varsigma q \in Y$. Thus, $\tilde{\varrho}(\varsigma(q + i) - \varsigma q) \subset u_2, \tilde{\varrho}(i) \supset u_2$ and $\gamma(\varsigma(q + i) - \varsigma q) > v_2, \gamma(i) < v_2$. Therefore, $\varsigma(q + i) - \varsigma q \notin \tilde{\varrho}_\gamma[u_2, t] \cap \tilde{\varrho}_\gamma[v_2, t], i \in \tilde{\varrho}_\gamma[u_2, t] \cap \tilde{\varrho}_\gamma[v_2, t]$, which is a contradiction. Thus, $\tilde{\varrho}(\varsigma(q + i) - \varsigma q) \supseteq \tilde{\varrho}(i), \gamma(\varsigma(q + i) - \varsigma q) \leq \gamma(i)$. Hence, $(\tilde{\varrho}_\gamma, Y)$ is an HINA upon $(\tilde{\xi}_\lambda, L)$.

Theorem 2. *Let $\tilde{\varrho}_\gamma$ and $\tilde{h}_\mu$ be two HINAs of Y upon an HF $(\tilde{\xi}_\lambda, L)$. Then $\tilde{\varrho}_\gamma \Cap \tilde{h}_\mu$ is an HINA of Y upon an HF $(\tilde{\xi}_\lambda, L)$.*

Proof. We know that $\tilde{\varrho}_\gamma \Cap \tilde{h}_\mu$ is an HNA of Y upon an HF $\tilde{\xi}$ of L. Let $q, \varsigma, i \in Y$. Then

$$\begin{aligned}
(\tilde{\varrho}_\gamma \Cap \tilde{h}_\mu)(q\varsigma) &= (\tilde{\varrho} \tilde{\cap} \tilde{h})(q\varsigma) \\
&= \tilde{\varrho}(q\varsigma) \tilde{\cap} \tilde{h}(q\varsigma) \\
&\supseteq (\tilde{\varrho}(q) \tilde{\cap} (\tilde{h}(q)) \\
&= (\tilde{\varrho} \cap \tilde{h})(q),
\end{aligned}$$

$$\begin{aligned}
(\gamma \vee \mu)(q\varsigma) &= \vee\{\gamma(q\varsigma), \mu(q\varsigma)\} \\
&\leq \vee\{\gamma(q), \mu(q)\} \\
&= (\gamma \vee \mu)(q),
\end{aligned}$$

$$\begin{aligned}
(\tilde{\varrho}_\gamma \Cap \tilde{h}_\mu)(\varsigma(q + i) - \varsigma q) &= (\tilde{\varrho} \tilde{\cap} \tilde{h})(\varsigma(q + i) - \varsigma q) \\
&= \tilde{\varrho}(\varsigma(q + i) - \varsigma q) \tilde{\cap} \tilde{h}(\varsigma(q + i) - \varsigma q) \\
&\supseteq (\tilde{\varrho}(i) \tilde{\cap} (\tilde{h}(i))
\end{aligned}$$

$$= (\tilde{\varrho} \cap \tilde{h})(i),$$

and

$$\begin{aligned}(\gamma \vee \mu)(\varsigma(q+i) - \varsigma q) &= \vee\{\gamma(\varsigma(q+i) - \varsigma q), \mu(\varsigma(q+i) - \varsigma q)\} \\ &\leq \vee\{\gamma(i), \mu(i)\} \\ &= (\gamma \vee \mu)(i).\end{aligned}$$

Definition 7. *Let Y and Y' be two NAs and $\omega : Y \to Y'$ be a mapping. Then:*

(*i*) *If $\tilde{h}_\mu$ is an HS of Y', then the preimage of $\tilde{h}_\mu$ under ω is the HS in Y upon U demarcated by $\omega^{-1}(\tilde{h}_\mu)(q) = (\omega^{-1}(\tilde{h})(q), \omega^{-1}(\mu)(q)) = (\tilde{h}(\omega)(q), \mu(\omega)(q)), \forall q \in Y$.*

(*ii*) *If $\tilde{\varrho}_\gamma$ is an HS of Y, then the image of $\tilde{\varrho}_\gamma$ under ω is the HS in Y' upon U demarcated by*

$$\omega(\tilde{\varrho})(\varsigma) = \begin{cases} \bigcup_{q \in \omega^{-1}(\varsigma)} \tilde{\varrho}(q), & \text{if } \omega^{-1}(\varsigma) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\omega(\gamma)(\varsigma) = \begin{cases} \bigwedge_{q \in \omega^{-1}(\varsigma)} \gamma(q), & \text{if } \omega^{-1}(\varsigma) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

for every $\varsigma \in Y$.

Theorem 3. *Let Y and Y' be two near algebras upon a field L and $\omega : Y \to Y'$ be an onto near algebra homomorphism. If $(\tilde{\varrho}_\gamma, Y)$ is an HINA upon an HF $(\tilde{\xi}_\lambda, L)$, then $(\omega(\tilde{\varrho}_\gamma), Y')$ is an HINA upon $(\tilde{\xi}_\lambda, L)$.*

Proof. Let $q, \varsigma \in Y'$. Then $\{r \mid r \in \omega^{-1}(q+\varsigma)\} \supseteq \{d + t \mid d \in \omega^{-1}(q) \text{ and } t \in \omega^{-1}(\varsigma)\}$ and $\{r \mid r \in \omega^{-1}(q\varsigma)\} \supseteq \{dt \mid d \in \omega^{-1}(q) \text{ and } t \in \omega^{-1}(\varsigma)\}$. If $\omega^{-1}(q) \neq \emptyset$ and $\omega^{-1}(\varsigma) \neq \emptyset$, then $\omega^{-1}(q\varsigma) \neq \emptyset$.

(*i*) For all $q, \varsigma \in Y' \exists\, d, t \in Y$ such that $q = \omega(d), \varsigma = \omega(t)$,

$$\begin{aligned}\omega(\tilde{\varrho})(q+\varsigma) &= \bigcup_{r \in \omega^{-1}(q+\varsigma)} \tilde{\varrho}(r) \\ &\supseteq \bigcup_{d \in \omega^{-1}(q), t \in \omega^{-1}(\varsigma)} \tilde{\varrho}(d+t) \\ &\supseteq \bigcup_{d \in \omega^{-1}(q), t \in \omega^{-1}(\varsigma)} \tilde{\varrho}(d) \cap \tilde{\varrho}(t) \\ &= (\bigcup_{d \in \omega^{-1}(q)} \tilde{\varrho}(d)) \cap (\bigcup_{t \in \omega^{-1}(\varsigma)} \tilde{\varrho}(t))\end{aligned}$$

$$= (\omega(\tilde{\varrho})(q)) \cap (\omega(\tilde{\varrho})(\varsigma))$$

and

$$\begin{aligned}
\omega(\gamma)(q+\varsigma) &= \bigwedge_{r\in\omega^{-1}(q+\varsigma)} \gamma(r) \\
&\leq \bigwedge_{d\in\omega^{-1}(q), t\in\omega^{-1}(\varsigma)} \gamma(d+t) \\
&\leq \bigwedge_{d\in\omega^{-1}(q), t\in\omega^{-1}(\varsigma)} (\vee\{\gamma(d), \gamma(d)\}) \\
&= \vee\{\bigwedge_{d\in\omega^{-1}(q)} \gamma(d), \bigwedge_{t\in\omega^{-1}(\varsigma)} \gamma(t)\} \\
&= \vee\{\omega(\gamma)(q), \omega(\gamma)(\varsigma)\}.
\end{aligned}$$

(ii) Let $s \in L$. Then

$$\begin{aligned}
\omega(\tilde{\varrho})(sq) &= \bigcup_{r\in\omega^{-1}(sq)} \tilde{\varrho}(r) \\
&\supseteq \bigcup_{d\in\omega^{-1}(q)} \tilde{\varrho}(sd) \\
&\supseteq \bigcup_{d\in\omega^{-1}(q)} \tilde{\xi}(s) \cap \tilde{\varrho}(d) \\
&= \tilde{\xi}(s) \cap (\bigcup_{d\in\omega^{-1}(q)} \tilde{\varrho}(q)) \\
&= \tilde{\xi}(s) \cap \omega(\tilde{\varrho})(q)
\end{aligned}$$

and

$$\begin{aligned}
\omega(\gamma)(sq) &= \bigwedge_{r\in\omega^{-1}(sq)} \gamma(r) \\
&\leq \bigwedge_{d\in\omega^{-1}(q)} \gamma(sd) \\
&\leq \bigwedge_{d\in\omega^{-1}(q)} (\bigvee\{\lambda(s), \gamma(d)\}) \\
&= \bigvee\{\lambda(s), \bigwedge_{d\in\omega^{-1}(q)} \gamma(d)\} \\
&= \bigvee\{\lambda(s), \omega(\gamma)(q)\}.
\end{aligned}$$

(iii) $\tilde{\xi}(1) \supseteq \omega(\tilde{\varrho})(q)$ and $\lambda(1) \leq \omega(\gamma)(q), \forall q \in Y^{'}$.

(iv)

$$\begin{aligned}\omega(\tilde{\varrho})(q\varsigma) &= \bigcup_{r\in\omega^{-1}(q\varsigma)} \tilde{\varrho}(r)\\ &\supseteq \bigcup_{d\in\omega^{-1}(q),t\in\omega^{-1}(\varsigma)} \tilde{\varrho}(dt)\\ &\supseteq \bigcup_{d\in\omega^{-1}(q)} \tilde{\varrho}(d)\\ &= (\omega(\tilde{\varrho})(q))\end{aligned}$$

and

$$\begin{aligned}\omega(\gamma)(q\varsigma) &= \bigwedge_{r\in\omega^{-1}(q\varsigma)} \gamma(r)\\ &\leq \bigwedge_{d\in\omega^{-1}(q),t\in\omega^{-1}(\varsigma)} \gamma(dt)\\ &\leq \bigwedge_{d\in\omega^{-1}(q)} \gamma(d)\\ &= \omega(\gamma)(q).\end{aligned}$$

(v) Let $q, \varsigma, i \in Y^{'}$. Then there exist $d, t, m \in Y$ such that $q = \omega(d), \varsigma = \omega(t)$, and $i = \omega(m)$. Thus,

$$\begin{aligned}\omega(\tilde{\varrho})(\varsigma(q+i)-\varsigma q) &= \bigcup_{r\in\omega^{-1}(\varsigma(q+i)-\varsigma q)} \tilde{\varrho}(r)\\ &\supseteq \bigcup_{d\in\omega^{-1}(q),t\in\omega^{-1}(\varsigma),m\in\omega^{-1}(i)} \tilde{\varrho}(t(d+m)-td)\\ &\supseteq \bigcup_{m\in\omega^{-1}(i)} \tilde{\varrho}(i)\\ &= (\omega(\tilde{\varrho})(i))\end{aligned}$$

and

$$\begin{aligned}\omega(\gamma)(\varsigma(q+i)-\varsigma q) &= \bigwedge_{r\in\omega^{-1}(\varsigma(q+i)-\varsigma q)} \gamma(r)\\ &\leq \bigwedge_{d\in\omega^{-1}(q),t\in\omega^{-1}(\varsigma),m\in\omega^{-1}(i)} \gamma(t(d+m)-td)\\ &\leq \bigwedge_{m\in\omega^{-1}(i)} \gamma(i)\end{aligned}$$

$$= \omega(\gamma)(i).$$

Hence, $(\omega(\tilde{\varrho}_\gamma), Y^{'})$ is an HINA upon $(\tilde{\xi}_\lambda, L)$.

Theorem 4. *Let Y and $Y^{'}$ be two near algebras upon a field L and $\omega : Y \to Y^{'}$ be an onto near algebra homomorphism. If $\tilde{h}_\mu$ is an HINA in $Y^{'}$ upon $(\tilde{\xi}_\lambda, L)$, then $\omega^{-1}(\tilde{h}_\mu) = (\omega^{-1}(\tilde{h}), \omega^{-1}(\mu))$ is an HINA in Y upon $(\tilde{\xi}_\lambda, L)$.*

Proof. Let $q, \varsigma, i \in Y$. Then
(i) $\omega^{-1}(\tilde{h})(q\varsigma) = \tilde{h}(\omega(q\varsigma)) = \tilde{h}(\omega(q)\omega(\varsigma)) \supseteq \tilde{h}(\omega(q)) = \omega^{-1}(\tilde{h})(q)$ and $\omega^{-1}(\mu)(q\varsigma) = \mu(\omega(q\varsigma)) = \mu(\omega(q)\omega(\varsigma)) \leq \mu(\omega(q)) = \omega^{-1}(\mu)(q)$.
(ii) $\omega^{-1}(\tilde{h})(\varsigma(q+i) - \varsigma q) = \tilde{h}(\omega(\varsigma(q+i) - \varsigma q)) = \tilde{h}(\omega(\varsigma)(\omega(q) + \omega(i)) - \omega(q)\omega(\varsigma)) \supseteq \tilde{h}(\omega(i)) = \omega^{-1}(\tilde{h})(i)$ and $\omega^{-1}(\mu)(\varsigma(q+i) - \varsigma q) = \mu(\omega(\varsigma(q+i) - \varsigma q)) = \mu(\omega(\varsigma)(\omega(q) + \omega(i)) - \omega(q)\omega(\varsigma)) \leq \mu(\omega(i)) = \omega^{-1}(\mu)(i)$.
Therefore, $\omega^{-1}(\tilde{h}_\mu) = (\omega^{-1}(\tilde{h}), \omega^{-1}(\mu))$ is an HINA in Y upon $(\tilde{\xi}_\lambda, L)$.

Theorem 5. *Let Y and $Y^{'}$ be two near algebras upon a field L and $\omega : Y \to Y^{'}$ be an onto near algebra homomorphism. If $\tilde{h}_\mu$ is an HINA in $Y^{'}$ upon $(\tilde{\xi}_\lambda, L)$, then $\omega^{-1}(\tilde{h}_\mu) = (\omega^{-1}(\tilde{h}), \omega^{-1}(\mu))$ is an HINA in Y upon $(\tilde{\xi}_\lambda, L)$.*

Proof. Let $q, \varsigma, i \in Y$. Then
(i) $\omega^{-1}(\tilde{h})(q\varsigma) = \tilde{h}(\omega(q\varsigma)) = \tilde{h}(\omega(q)\omega(\varsigma)) \supseteq \tilde{h}(\omega(q)) = \omega^{-1}(\tilde{h})(q)$ and $\omega^{-1}(\mu)(q\varsigma) = \mu(\omega(q\varsigma)) = \mu(\omega(q)\omega(\varsigma)) \leq \mu(\omega(q)) = \omega^{-1}(\mu)(q)$.
(ii) $\omega^{-1}(\tilde{h})(\varsigma(q+i) - \varsigma q) = \tilde{h}(\omega(\varsigma(q+i) - \varsigma q)) = \tilde{h}(\omega(\varsigma)(\omega(q) + \omega(i)) - \omega(q)\omega(\varsigma)) \supseteq \tilde{h}(\omega(i)) = \omega^{-1}(\tilde{h})(i)$ and $\omega^{-1}(\mu)(\varsigma(q+i) - \varsigma q) = \mu(\omega(\varsigma(q+i) - \varsigma q)) = \mu(\omega(\varsigma)(\omega(q) + \omega(i)) - \omega(q)\omega(\varsigma)) \leq \mu(\omega(i)) = \omega^{-1}(\mu)(i)$.
Therefore, $\omega^{-1}(\tilde{h}_\mu) = (\omega^{-1}(\tilde{h}), \omega^{-1}(\mu))$ is an HINA in Y upon $(\tilde{\xi}_\lambda, L)$.

Definition 8. *Let $\tilde{\varrho}_\gamma$ and $\tilde{h}_\mu$ be two hybrid structures of near algebras Y and $Y^{'}$ over L, respectively. Then the Cartesian product of $\tilde{\varrho}_\gamma$ and $\tilde{h}_\mu$ is denoted by $\tilde{\varrho}_\gamma \times \tilde{h}_\mu$, is defined to be a hybrid structure $\tilde{\varrho}_\gamma \times \tilde{h}_\mu : Y \times Y^{'} \to P(U) \times I; (q, q^{'}) \mapsto ((\tilde{\varrho} \times \tilde{h})(q, q^{'}), (\gamma \times \mu)(q, q^{'})), \forall (q, q^{'}) \in Y \times Y^{'}$, where $\tilde{\varrho} \times \tilde{h} : Y \times Y^{'} \to P(U); (q, q^{'}) \mapsto \tilde{\varrho}(q) \cap \tilde{h}(q^{'})$ and $\gamma \times \mu : Y \times Y^{'} \to I; (q, q^{'}) \mapsto \bigvee\{\gamma(q), \mu(q^{'})\}$.*

Theorem 6. *Let $\tilde{\varrho}_\gamma$ and $\tilde{h}_\mu$ be two HINAs of Y and $Y^{'}$ upon an HF $(\tilde{\xi}_\lambda, L)$. Then $\tilde{\varrho}_\gamma \times \tilde{h}_\mu$ is an HINA of $Y \times Y^{'}$ upon $(\tilde{\xi}_\lambda, L)$.*

Proof. Let $(q, q^{'}), (\varsigma, \varsigma^{'}), (i, i^{'}) \in Y \times Y^{'}$ and $s \in L$. Then
(i)

$$\begin{aligned}(\tilde{\varrho} \times \tilde{h})[(q, q^{'}) + (\varsigma, \varsigma^{'})] &= (\tilde{\varrho} \times \tilde{h})(q + \varsigma, q^{'} + \varsigma^{'}) \\ &= \tilde{\varrho}(q + \varsigma) \cap \tilde{h}(q^{'} + \varsigma^{'}) \\ &\supseteq [\tilde{\varrho}(q) \cap \tilde{\varrho}(\varsigma)] \cap [\tilde{h}(q^{'}) \cap \tilde{h}(\varsigma^{'})] \\ &= [\tilde{\varrho}(q) \cap \tilde{h}(q^{'})] \cap [\tilde{\varrho}(\varsigma) \cap \tilde{h}(\varsigma^{'})]\end{aligned}$$

$$= (\tilde{\varrho} \times \tilde{h})(q, q^{'}) \cap (\tilde{\varrho} \times \tilde{h})(\varsigma, \varsigma^{'})$$

and

$$\begin{aligned}(\gamma \times \mu)[(q, q^{'}) + (\varsigma, \varsigma^{'})] &= (\gamma \times \mu)(q + \varsigma, q^{'} + \varsigma^{'}) \\ &= \bigvee\{\gamma(q + \varsigma), \mu(q^{'} + \varsigma^{'})\} \\ &\leq \bigvee\{\bigvee\{\gamma(q), \gamma(\varsigma)\}, \bigvee\{\mu(q^{'}), \mu(\varsigma^{'})\}\} \\ &\leq \bigvee\{\bigvee\{\gamma(q), \gamma(q^{'})\}, \bigvee\{\mu(\varsigma), \mu(\varsigma^{'})\}\} \\ &= \bigvee\{(\gamma \times \mu)(q, q^{'}), (\gamma \times \mu)(\varsigma, \varsigma^{'})\}.\end{aligned}$$

(ii)

$$\begin{aligned}(\tilde{\varrho} \times \tilde{h})(s(q, q^{'})) &= (\tilde{\varrho} \times \tilde{h})(sq, sq^{'}) \\ &= \tilde{\varrho}(sq) \cap \tilde{h}(sq^{'}) \\ &\supseteq [\tilde{\xi}(s) \cap \tilde{\varrho}(q)] \cap [\tilde{\xi}(s) \cap \tilde{h}(q^{'})] \\ &= \tilde{\xi}(s) \cap (\tilde{\varrho}(q) \cap \tilde{h}(q^{'})) \\ &= \tilde{\xi}(s) \cap (\tilde{\varrho} \times \tilde{h})(q, q^{'})\end{aligned}$$

and

$$\begin{aligned}(\gamma \times \mu)[s(q, q^{'})] &= (\gamma \times \mu)(sq, sq^{'}) \\ &= \bigvee\{\gamma(sq), \mu(sq^{'})\}\} \\ &\leq \bigvee\{\bigvee\{\lambda(s), \gamma(q)\}, \bigvee\{\lambda(s), \mu(q^{'})\}\} \\ &\leq \bigvee\{\lambda(s), \bigvee\{\gamma(q), \mu(q^{'})\}\} \\ &= \bigvee\{\lambda(s), (\gamma \times \mu)(q, q^{'})\}.\end{aligned}$$

(iii) Let 1 be the unity in L. Since $(\tilde{\varrho}_\gamma, Y)$ and $(\tilde{h}_\mu, Y^{'})$ are HINAs over HF $(\tilde{\xi}_\lambda, L)$, we have $\tilde{\xi}(1) \supseteq \tilde{\varrho}(q), \forall q \in Y, \lambda(1) \leq \gamma(q)$ and $\tilde{\xi}(1) \supseteq \tilde{h}(q^{'}), \forall q^{'} \in Y^{'}, \lambda(1) \leq \mu(q^{'})$. Then $\tilde{\xi}(1) \supseteq \tilde{\varrho}(q) \cap \tilde{h}(q^{'}) = (\tilde{\varrho} \times \tilde{h})(q, q^{'})$ and $\lambda(1) \leq \bigvee\{\gamma(q), \mu(q^{'})\} = (\gamma \times \mu)(q, q^{'})$.

(iv)

$$\begin{aligned}(\tilde{\varrho} \times \tilde{h})[(q, q^{'})(\varsigma, \varsigma^{'})] &= (\tilde{\varrho} \times \tilde{h})(q\varsigma, q^{'}\varsigma^{'}) \\ &= \tilde{\varrho}(q\varsigma) \cap \tilde{h}(q^{'}\varsigma^{'}) \\ &\supseteq \tilde{\varrho}(q) \cap \tilde{h}(q^{'}) \\ &= (\tilde{\varrho} \times \tilde{h})(q, q^{'})\end{aligned}$$

and

$$(\gamma \times \mu)[(q, q^{'})(\varsigma, \varsigma^{'})] = (\gamma \times \mu)(q\varsigma, q^{'}\varsigma^{'})$$

$$
\begin{aligned}
&= \bigvee\{\gamma(q\varsigma), \mu(q^{'}\varsigma^{'})\} \\
&\leq \bigvee\{\gamma(q), \mu(q^{'})\} \\
&= (\gamma \times \mu)(q, q^{'}).
\end{aligned}
$$

(v)

$$
\begin{aligned}
(\tilde{\varrho} \times \tilde{h})[(\varsigma, \varsigma^{'})((q, q^{'}) + (i, i^{'})) - (\varsigma, \varsigma^{'})(q, q^{'}] &= (\tilde{\varrho} \times \tilde{h})[(\varsigma, \varsigma^{'})(q + i, q^{'} + i^{'}) - (\varsigma q, \varsigma^{'} q^{'})] \\
&= (\tilde{\varrho} \times \tilde{h})[(\varsigma(q + i), \varsigma^{'}(q^{'} + i^{'})) - (\varsigma q, \varsigma^{'} q^{'})] \\
&= (\tilde{\varrho} \times \tilde{h})[(\varsigma(q + i) - \varsigma q, \varsigma^{'}(q^{'} + i^{'})) - \varsigma^{'} q^{'})] \\
&= \tilde{\varrho}(\varsigma(q + i) - \varsigma q) \cap \tilde{h}(\varsigma^{'}(q^{'} + i^{'})) - \varsigma^{'} q^{'}) \\
&\supseteq \tilde{\varrho}(i) \cap \tilde{h}(i^{'}) = (\tilde{\varrho} \times \tilde{h})(i, i^{'})
\end{aligned}
$$

and

$$
\begin{aligned}
(\gamma \times \mu)[(\varsigma, \varsigma^{'})((q, q^{'}) + (i, i^{'})) - (\varsigma, \varsigma^{'})(q, q^{'}] &= (\gamma \times \mu)[(\varsigma, \varsigma^{'})(q + i, q^{'} + i^{'}) - (\varsigma q, \varsigma^{'} q^{'})] \\
&= (\gamma \times \mu)[(\varsigma(q + i), \varsigma^{'}(q^{'} + i^{'})) - (\varsigma q, \varsigma^{'} q^{'})] \\
&= (\gamma \times \mu)[(\varsigma(q + i) - \varsigma q, \varsigma^{'}(q^{'} + i^{'})) - \varsigma^{'} q^{'})] \\
&= \gamma(\varsigma(q + i) - \varsigma q) \cap \mu(\varsigma^{'}(q^{'} + i^{'})) - \varsigma^{'} q^{'}) \\
&\leq \gamma(i) \cap \mu(i^{'}) = (\gamma \times \mu)(i, i^{'}).
\end{aligned}
$$

Hence, $\tilde{\varrho}_\gamma \times \tilde{h}_\mu$ is an HINA of $Y \times Y^{'}$ upon $(\tilde{\xi_\lambda}, L)$.

Definition 9. *Let $\tilde{\varrho}_\gamma$ be an HINA of Y upon U and $\varsigma \in Y$. Then the hybrid coset (or coset) of $\tilde{\varrho}_\gamma$ is denoted by $\varsigma + \tilde{\varrho}_\gamma$ and is defined by $(\varsigma + \tilde{\varrho})(q) = \tilde{\varrho}(q - \varsigma)$ and $(\varsigma + \gamma)(q) = \gamma(q - \varsigma), \forall q \in Y$.*

Theorem 7. *Let $\tilde{\varrho}_\gamma$ be an HINA of Y upon U and $q, \varsigma \in Y$. Then $q + \tilde{\varrho}_\gamma = \varsigma + \tilde{\varrho}_\gamma$ if and only if $\tilde{\varrho}(q - \varsigma) = \tilde{\varrho}(0)$ and $\gamma(q - \varsigma) = \gamma(0)$.*

Proof. Let $q, \varsigma \in Y$. Suppose that $q + \tilde{\varrho}_\gamma = \varsigma + \tilde{\varrho}_\gamma$. Then $\tilde{\varrho}(q - \varsigma) = (\varsigma + \tilde{\varrho})(q) = (q + \tilde{\varrho})(q) = \tilde{\varrho}(q - q) = \tilde{\varrho}(0)$ and $\gamma(q - \varsigma) = (\varsigma + \gamma)(q) = (q + \gamma)(q) = \gamma(q - q) = \gamma(0)$.

Conversely, suppose that $\tilde{\varrho}(q - \varsigma) = \tilde{\varrho}(0)$ and $\gamma(q - \varsigma) = \gamma(0)$. For every $\kappa \in Y$, we have

$$
\begin{aligned}
(q + \tilde{\varrho})(\kappa) &= \tilde{\varrho}(\kappa - q) \\
&= \tilde{\varrho}(\kappa - \varsigma + \varsigma - q) \\
&= \tilde{\varrho}[(\kappa - \varsigma) + (\varsigma - q)] \\
&\supseteq \tilde{\varrho}(\kappa - \varsigma) \cap \tilde{\varrho}(\varsigma - q) \\
&= \tilde{\varrho}(\kappa - \varsigma) \cap \tilde{\varrho}(q - \varsigma) \\
&= \tilde{\varrho}(\kappa - \varsigma) \cap \tilde{\varrho}(0)
\end{aligned}
$$

$$\begin{aligned} &= \tilde{\varrho}(\kappa - \varsigma) \\ &= (\varsigma + \tilde{\varrho})(\kappa) \end{aligned}$$

and

$$\begin{aligned} (q+\gamma)(\kappa) &= \gamma(\kappa - q) \\ &= \gamma(\kappa - \varsigma + \varsigma - q) \\ &= \gamma[(\kappa - \varsigma) + (\varsigma - q)] \\ &\leq \bigvee\{\gamma(\kappa - \varsigma), \gamma(\varsigma - q)\} \\ &= \bigvee\{\gamma(\kappa - \varsigma), \gamma(q - \varsigma)\} \\ &= \bigvee\{\gamma(\kappa - \varsigma), \gamma(0)\} \\ &= \gamma(\kappa - \varsigma) \\ &= (\varsigma + \gamma)(\kappa). \end{aligned}$$

Thus, $q + \tilde{\varrho} \supseteq \varsigma + \tilde{\varrho}$ and $q + \gamma = \varsigma + \gamma$. Now,

$$\begin{aligned} (\varsigma + \tilde{\varrho})(\kappa) &= \tilde{\varrho}(\kappa - \varsigma) \\ &= \tilde{\varrho}(\kappa - q + q - \varsigma) \\ &= \tilde{\varrho}[(\kappa - q) + (q - \varsigma)] \\ &\supseteq \tilde{\varrho}(\kappa - q) \cap \tilde{\varrho}(q - \varsigma) \\ &= \tilde{\varrho}(\kappa - q) \cap \tilde{\varrho}(0) \\ &= \tilde{\varrho}(\kappa - q) \\ &= (q + \tilde{\varrho})(\kappa) \end{aligned}$$

and

$$\begin{aligned} (\varsigma+\gamma)(\kappa) &= \gamma(\kappa - \varsigma) \\ &= \gamma(\kappa - q + q - \varsigma) \\ &= \gamma[(\kappa - q) + (q - \varsigma)] \\ &\leq \bigvee\{\gamma(\kappa - q), \gamma(q - \varsigma)\} \\ &= \bigvee\{\gamma(\kappa - q), \gamma(0)\} \\ &= \gamma(\kappa - q) \\ &= (q + \gamma)(\kappa). \end{aligned}$$

Thus, $\varsigma + \tilde{\varrho} \supseteq q + \tilde{\varrho}$ and $\varsigma + \gamma = q + \gamma$. Hence, $q + \tilde{\varrho} = \varsigma + \tilde{\varrho}$ and $q + \gamma = \varsigma + \gamma$.

Theorem 8. *Let $\tilde{\varrho}_\gamma$ be an HINA of Y upon U. Then the following two statements hold:*

(i) If $q + \tilde{\varrho} = p + \tilde{\varrho}, \varsigma + \tilde{\varrho} = m + \tilde{\varrho}$, then $(q + \varsigma) + \tilde{\varrho} = (p + m) + \tilde{\varrho}, q\varsigma + \tilde{\varrho} = pm + \tilde{\varrho}$, and if $q + \gamma = p + \gamma, \varsigma + \gamma = m + \gamma$, then $(q + \varsigma) + \gamma = (p + m) + \gamma, q\varsigma + \gamma = pm + \gamma$

(ii) If $q + \tilde{\varrho} = p + \tilde{\varrho}$, then $sq + \tilde{\varrho} = sp + \tilde{\varrho}$, and if $q + \gamma = p + \gamma$, then $sq + \gamma = sp + \gamma$

for all $q, \varsigma, p, m \in Y$ and $s \in L$.

Proof. (i) Suppose that $q+\tilde{\varrho} = p+\tilde{\varrho}, \varsigma+\tilde{\varrho} = m+\tilde{\varrho}$ and $q+\gamma = p+\gamma, \varsigma+\gamma = m+\gamma$. Then $\tilde{\varrho}(q-p) = \tilde{\varrho}(0), \tilde{\varrho}(\varsigma-m) = \tilde{\varrho}(0)$ and $\gamma(q-p) = \gamma(0), \gamma(\varsigma-m) = \gamma(0)$. Consider, $\tilde{\varrho}[(q+\varsigma)-(p+m)] = \tilde{\varrho}[(q-p)+(\varsigma-m)] \supseteq \tilde{\varrho}(q-p) \cap \tilde{\varrho}(\varsigma-m) = \tilde{\varrho}(0) \cap \tilde{\varrho}(0) = \tilde{\varrho}(0)$ and $\gamma[(q+\varsigma)-(p+m)] = \gamma[(q-p)+(\varsigma-m)] \leq \bigvee\{\gamma(q-p), \gamma(\varsigma-m)\} = \bigvee\{\gamma(0), \gamma(0)\} = \gamma(0)$. But $\tilde{\varrho}(0) \supseteq \tilde{\varrho}[(q+\varsigma)-(p+m)]$ and $\gamma(0) \leq \gamma[(q+\varsigma)-(p+m)]$. Therefore, $\tilde{\varrho}[(q+\varsigma)-(p+m)] = \tilde{\varrho}(0)$ and $\gamma[(q+\varsigma)-(p+m)] = \gamma(0)$. Thus, $(q+\varsigma)+\tilde{\varrho} = (p+m)+\tilde{\varrho}$ and $(q+\varsigma)+\gamma = (p+m)+\gamma$. Again,

$$\begin{aligned}
\tilde{\varrho}[q\varsigma - pm] &= \tilde{\varrho}[pm - q\varsigma] \\
&= \tilde{\varrho}(pm - qm + qm - q\varsigma) \\
&= \tilde{\varrho}[(p-q)m + q(\varsigma + (-\varsigma + m)) - q\varsigma)] \\
&\supseteq \tilde{\varrho}((p-q)m) \cap \tilde{\varrho}(q(\varsigma + (-\varsigma + m)) - q\varsigma) \\
&= \tilde{\varrho}(p-q) \cap \tilde{\varrho}(\varsigma - m) \\
&= \tilde{\varrho}(0) \cap \tilde{\varrho}(0) \\
&= \tilde{\varrho}(0)
\end{aligned}$$

and

$$\begin{aligned}
\gamma[q\varsigma - pm] &= \gamma[pm - q\varsigma] \\
&= \gamma(pm - qm + qm - q\varsigma) \\
&= \gamma[(p-q)m + q(\varsigma + (-\varsigma + m)) - q\varsigma] \\
&\leq \bigvee\{\gamma((p-q)m), \gamma(q(\varsigma + (-\varsigma + m)) - q\varsigma)\} \\
&\leq \bigvee\{\gamma(p-q), \gamma(\varsigma - m)\} \\
&= \bigvee\{\gamma(0), \gamma(0)\} \\
&= \gamma(0).
\end{aligned}$$

But $\tilde{\varrho}(0) \supseteq \tilde{\varrho}[(q\varsigma)-(pm)]$ and $\gamma(0) \leq \gamma[(q\varsigma)-(pm)]$. Therefore, $\tilde{\varrho}[q\varsigma - pm] = \tilde{\varrho}(0)$ and $\gamma[q\varsigma - pm] = \gamma(0)$. Thus, $q\varsigma + \tilde{\varrho} = pm + \tilde{\varrho}$ and $q\varsigma + \gamma = pm + \gamma$.

(ii) Suppose that $q + \tilde{\varrho} = p + \tilde{\varrho}$ and $q + \gamma = p + \gamma$. Then $\tilde{\varrho}(q-p) = \tilde{\varrho}(0)$ and $\gamma(q-p) = \gamma(0)$. Now, $\tilde{\varrho}(sq - sp) = \tilde{\varrho}(s(q-p)) \supseteq \tilde{\varrho}(q-p) = \tilde{\varrho}(0)$ and $\gamma(sq - sp) = \gamma(s(q-p)) \leq \gamma(q-p) = \gamma(0)$. But $\tilde{\varrho}(0) \supseteq \tilde{\varrho}(sq - sp)$ and $\gamma(0) \leq \gamma(sq - sp)$. Therefore, $\tilde{\varrho}(sq - sp) = \tilde{\varrho}(0)$ and $\gamma(sq - sp) = \gamma(0)$. Thus, $sq + \tilde{\varrho} = sp + \tilde{\varrho}$ and $sq + \gamma = sp + \gamma$.

Definition 10. *Let $\tilde{\varrho}_\gamma$ be an HINA of Y upon U. Then the set of all cosets of $\tilde{\varrho}_\gamma$ is $Y/\tilde{\varrho}_\gamma = \{\varsigma + \tilde{\varrho}_\gamma \mid \varsigma \in Y\}$, where $Y/\tilde{\varrho} = \{\varsigma + \tilde{\varrho} \mid \varsigma \in Y\}$ and $Y/\gamma = \{\varsigma + \gamma \mid \varsigma \in Y\}$.*

Theorem 9. *Let $\tilde{\varrho}_\gamma$ be an HINA of Y upon U. Then $Y/\tilde{\varrho}_\gamma$ is a near algebra with respect to the operations defined by*

$$\begin{aligned}(q+\tilde{\varrho})+(\varsigma+\tilde{\varrho}) &= (q+\varsigma)+\tilde{\varrho}\\ (q+\tilde{\varrho})(\varsigma+\tilde{\varrho}) &= q\varsigma+\tilde{\varrho}\\ s(q+\tilde{\varrho}) &= sq+\tilde{\varrho}\\ (q+\gamma)+(\varsigma+\gamma) &= (q+\varsigma)+\gamma\\ (q+\gamma)(\varsigma+\gamma) &= q\varsigma+\gamma\\ s(q+\gamma) &= sq+\gamma\end{aligned}$$

for all $q, \varsigma \in Y$ and $s \in L$.

Proof. A direct verification shows that $Y/\tilde{\varrho}_\gamma$ is a linear space. Let $q+\tilde{\varrho}, \varsigma+\tilde{\varrho}, j+\tilde{\varrho} \in Y/\tilde{\varrho}$ and $q+\gamma, \varsigma+\gamma, j+\gamma \in Y/\gamma$, where $q, \varsigma, j \in Y$. Then

$$\begin{aligned}[(q+\tilde{\varrho})(\varsigma+\tilde{\varrho})](j+\tilde{\varrho}) &= (q\varsigma+\tilde{\varrho})(j+\tilde{\varrho})\\ &= (q\varsigma)j+\tilde{\varrho}\\ &= (q+\tilde{\varrho})[(\varsigma+\tilde{\varrho})(j+\tilde{\varrho})]\end{aligned}$$

and

$$\begin{aligned}[(q+\gamma)(\varsigma+\gamma)](j+\gamma) &= (q\varsigma+\gamma)(j+\gamma)\\ &= (q\varsigma)j+\gamma\\ &= (q+\gamma)[(\varsigma+\gamma)(j+\gamma)].\end{aligned}$$

This shows that $Y/\tilde{\varrho}_\gamma$ is a semigroup under multiplication. Consider,

$$\begin{aligned}[(q+\tilde{\varrho})+(\varsigma+\tilde{\varrho})](j+\tilde{\varrho}) &= ((q+\varsigma)+\tilde{\varrho})(j+\tilde{\varrho})\\ &= (q+\varsigma)j+\tilde{\varrho}\\ &= (qj+\varsigma j)+\tilde{\varrho}\\ &= (qj+\tilde{\varrho})+(\varsigma j+\tilde{\varrho})\\ &= (q+\tilde{\varrho})(j+\tilde{\varrho})+(\varsigma+\tilde{\varrho})(j+\tilde{\varrho})\end{aligned}$$

and

$$\begin{aligned}[(q+\gamma)+(\varsigma+\gamma)](j+\gamma) &= ((q+\varsigma)+\gamma)(j+\gamma)\\ &= (q+\varsigma)j+\gamma\\ &= (qj+\varsigma j)+\gamma\\ &= (qj+\gamma)+(\varsigma j+\gamma)\\ &= (q+\gamma)(j+\gamma)+(\varsigma+\gamma)(j+\gamma).\end{aligned}$$

Let $s \in L$. Then

$$\begin{aligned}(s(q+\tilde{\varrho}))(\varsigma+\tilde{\varrho}) &= (sq+\tilde{\varrho})(\varsigma+\tilde{\varrho}) \\ &= (sq)\varsigma+\tilde{\varrho} \\ &= s(q\varsigma)+\tilde{\varrho} \\ &= s(q\varsigma+\tilde{\varrho}) \\ &= s((q+\tilde{\varrho})(\varsigma+\tilde{\varrho}))\end{aligned}$$

and

$$\begin{aligned}(s(q+\gamma))(\varsigma+\gamma) &= (sq+\gamma)(\varsigma+\gamma) \\ &= (sq)\varsigma+\gamma \\ &= s(q\varsigma)+\gamma \\ &= s(q\varsigma+\gamma) \\ &= s((q+\gamma)(\varsigma+\gamma)).\end{aligned}$$

Hence, $Y/\tilde{\varrho}_\gamma$ is a near algebra over L.

4. Conclusion

In this work, we combined soft sets and fuzzy sets in near algebras, thereby developing a novel structure of hybrid ideals. This manuscript provided an in-depth exploration of hybrid ideals within a near algebra, elucidating their unique properties through a systematic investigation. Theoretical findings were substantiated with illustrative examples. We also introduced the notions of HINA homomorphism, the Cartesian product of HINA, and the coset of HINA. Future research can be extended to investigate the hybrid gamma near algebra, the hybrid ideal of a gamma near algebra, and explore the concept of hybrid near algebra on anti-fuzzy sets.

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