EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6274
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Grand Variable Herz-Morrey Type Besov Spaces and Triebel-Lizorkin Spaces

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Abstract. In the article, the boundedness of vector-valued sublinear operators in grand variable Herz-Morrey spaces $M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)$ are obtained. Then grand variable Herz-Morrey type Besov and Triebel-Lizorkin spaces are defined. We will also prove the equivalent quasi-norms by Peetre's maximal operators in these spaces.

2020 Mathematics Subject Classifications: 46E35, 42B25, 42B35

Key Words and Phrases: Grand variable Herz-Morrry space, Besov space, Triebel Lizorkin space, maximal operator

1. Introduction

In variable exponent spaces, the Hardy-Littlewood maximal operator's boundedness is crucial. For instance, it is well known that if the Hardy-Littlewood maximal operator is bounded in variable exponent Lebesgue space, then numerous conclusions from classical harmonic analysis and function theory also apply for the variable exponent case; see [1–4]. Moreover, a variety of variable exponent spaces are presented, including: Bessel potential spaces, Besov and Trieble-Lizorkin spaces, Hardy spaces, Herz spaces, grand variable Herz spaces, grand variable weighted Herz spaces, Herz-Morrey spaces, grand variable Herz-Morrey spaces, Morrey spaces, Morrey type Besov and Trieble-Lizorkin spaces, Trieble-Lizorkin-Morrey spaces, Trieble-Lizorkin-Morrey spaces, and so forth; see [5–13, 13, 14,

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6274

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14, 14–35] and references therein. As you can see from [36–40], numerous conclusions about the boundedness of sublinear operators in these spaces have been established.

A sublinear operator T satisfies the size condition

$$|Tg(x)| \leqslant C \int_{\mathbb{R}^n} |x-y|^{-n} |g(y)| \mathrm{d}y$$

for all $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ with compact support and a.e. $x \notin \operatorname{supp} g$. Then, T is bounded on the grand variable Herz-Morrey spaces and the homogeneous and non-homogeneous Herz spaces (see the monographs [41, 42]). L. Tang and D. Yang then expand these conclusions for the weighted vector-valued situation in [43]. Herz type Besov and Triebel-Lizorkin spaces with variable exponent $\dot{K}^{\eta,q}_{p(\cdot)}(\mathbb{R}^n)$ were introduced by C. Shi and the second author in [18]. . For the Besov and Triebel-Lizorkin spaces of constant exponent Herz type, we direct the reader to [44–47]. M. Izuki [48] used variable exponent $M\dot{K}^{\eta,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ to get the vector-valued boundedness for certain sublinear operators that meet the size requirement on Herz-Morrey spaces. Sultan et al. [49] presented the concept of grand variable Herz-Morrey spaces; see [50–52] for additional findings on these spaces. The boundedness of vector-valued Hardy-Littlewood maximal operators in Herz spaces with variable exponents was established by the authors in [53]. Peetre's maximal operators were used to characterize Herz type Besov and Triebel-Lizorkin spaces with variable exponents.

The current research examines the boundedness of vector-valued Hardy-Littlewood maximal operators on grand variable Herz-Morrey type Besov and Triebel-Lizorkin spaces, drawing inspiration from the aforementioned publications. The paper is organised as follows. We provide some conventions in the remainder of the section. Our primary findings, which are a generalization of related findings in [48] and [53, 54], will be presented in Section 2. We provide evidence for our findings in Section 3.

2. Main results

The n-dimensional real Euclidean space is denoted as $\mathbb{R}^n$, as is customary. Let $\mathcal{A}$ be a measurable subset of $\mathbb{R}^n$. Throughout this paper, C and c are always positive constants which may vary from line to line. For a measurable set S, $|S|$ denote its Lebesgue measure and $\mathbf{1}_S$ denote its characteristic function. We write $u \leqslant v$, if $u \leq cv$, and if $u \leqslant v$ and $v \leqslant u$, then $u \sim v$. By variable exponents we mean a measurable function on $\mathbb{R}^n$. For every $l \in \mathbb{Z}$, we denote $B_t = B(0, 2^t) = \{x \in \mathbb{R}^n : |x| < 2^t\}$. After deducting B_{t-1} from R_t, $\mathbf{1}_{R_t} = \mathbf{1}_t$. By $\mathcal{P}(\mathbb{R}^n)$ we denote the subset of variable exponents with range $[1, \infty]$. Let $q_- := \operatorname*{ess\,inf}_{y \in \mathcal{A}} q(y) > 1$ and $q_+ := \operatorname*{ess\,sup}_{y \in \mathcal{A}} q(y) < \infty$, then we have

$$1 \leq q_-(\mathcal{A}) \leq q(y) \leq q_+(\mathcal{A}) < \infty. \tag{2.1}$$

The notation B is the ball such that $B(z,r) := \{y \in \mathcal{A} : |z-y| < r\}$. Now variable Lebesgue space $L^{q(\cdot)}(\mathcal{A})$ is given as

$$L^{p(\cdot)}(\mathcal{A}) = \left\{ g \textit{ is measurable} : \int_{\mathcal{A}} \left(\frac{|g(z)|}{\lambda} \right)^{q(z)} dz < \infty \textit{ where } \lambda \textit{ is a constant} \right\}.$$

Lebesgue space is equipped with the norm

$$\|g\|_{L^{q(\cdot)}(\mathcal{A})} = \inf \left\{ \lambda > 0 : \int_{\mathcal{A}} \left(\frac{|g(z)|}{\lambda} \right)^{q(z)} dz \leq 1 \right\}.$$

The Hardy-Littlewood maximal operator M for $g \in L^1_{\text{loc}}(\mathcal{A})$ is defined as

$$Mg(z) := \sup_{0<r} \frac{1}{r^n} \int\limits_{B(z,r)} |g(z)| dz \quad (z \in \mathcal{A}).$$

The notion of $\mathcal{B}(\mathcal{A})$ denote the collection of $q(\cdot) \in \mathcal{P}(\mathcal{A})$ such that M is bounded on $L^{q(\cdot)}(\mathcal{A})$. Let $x, y \in \mathcal{A}$ with $|x-y| \leq \frac{1}{2}$ and $q : \mathcal{A} \mapsto (0, \infty)$. Now log-Hölder continuity condition (or Dini-Lipschitz condition) is given as

$$\frac{C}{-\ln|x-y|} \geq |q(x) - q(y)|, \tag{2.2}$$

where C is called log-Hölder continuity constant.

Let log-Dini-Lipschitz constant (or decay constant) $C_\infty > 0$, $q(\cdot)$ satisfiy the decay condition if $\lim\limits_{|x|\to\infty} q(x) = q_\infty := q(\infty)$ such that

$$\frac{C_\infty}{\ln(e+|h|)} \geq |q(h) - q_\infty|. \tag{2.3}$$

Let $C_0 > 0$, $q(\cdot)$ satisfy the log Hölder continuity condition at 0 for $|h| \leq \frac{1}{2}$, such that

$$\frac{C_0}{\ln|h|} \geq |q(h) - q(0)|. \tag{2.4}$$

$\mathcal{P}^{\log} = \mathcal{P}^{\log}(\mathcal{A})$ consists of all functions $q(\cdot) \in \mathcal{P}(\mathcal{A})$ satisfying (2.1) and (2.2).

$\mathcal{P}_\infty(\mathcal{A})$ (resp. $\mathcal{P}_{0,\infty}(\mathcal{A})$) is the subset of $\mathcal{P}(\mathcal{A})$ consisting of functions which satisfy condition (2.3) (resp. both conditions (2.3) and (2.4)). The set of positive integers is denoted by $\mathbb{Z}_+$. For each $t_0 \in \mathbb{Z}$, $\mathbf{1}_{A_{t_0}} = \mathbf{1}_{t_0}$; for $t_0 \in \mathbb{Z}_+$, $\mathbf{1}_{B_0} = \tilde{\mathbf{1}}_0$; and so on. $\mathbf{1}_{A_{t_0}}$ denote the characteristic function of A_{t_0}. By $a \leqslant b$ we denote $a \leq Cb$.

Moreover, we define $\mathcal{P}^0(\mathbb{R}^n)$ to be the set of measurable functions p on $\mathbb{R}^n$ with the range in $(0, \infty)$ such that $p_- > 0$ and $p_+ < \infty$. Given $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, one can define the space $L^{p(\cdot)}(\mathbb{R}^n)$. This is equivalent to defining it to be the set of all functions g such that

$$|g|^{p_0} \in L^{p(\cdot)/p_0}(\mathbb{R}^n)$$

where

$$0 < p_0 < p_-, \quad \frac{p(\cdot)}{p_0} \in \mathcal{P}(\mathbb{R}^n).$$

Then the quasi-norm is given as

$$\|g\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \||g|^{p_0}\|^{1/p_0}_{L^{p(\cdot)/p_0}(\mathbb{R}^n)}.$$

Lemma 1. *[55]*

Let f belong to the Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and g belong to $L^{p'(\cdot)}(\mathbb{R}^n)$, where $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, the product function fg is integrable over $\mathbb{R}^n$, and the following inequality holds:

$$\int_{\mathbb{R}^n} \left|f(x)g(x)\right| dx \leq r_p \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)},$$

where r_p is defined as $r_p = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

Lemma 2. *[56] Let $\tilde{q}(\cdot)$ be a variable exponent defined as $\frac{1}{p(z)} - \frac{1}{q} = \frac{1}{\tilde{q}(z)}$ where $(z \in \mathbb{R}^n)$. Then for all measurable functions f and g we have*

$$\|fg\|_{p(\cdot)} \leq C \|g\|_{\tilde{q}(\cdot)} \|g\|_q.$$

Lemma 3. *[57]*

Let $p(\cdot)$ be a function within the class $\mathcal{B}(\mathbb{R}^n)$. For any ball B in $\mathbb{R}^n$, we get

$$\frac{1}{|B|} \|\mathbf{1}_B\|_{p'(\cdot)} \|\mathbf{1}_B\|_{p(\cdot)} \leqslant C,$$

where $C > 0$.

Lemma 4. *[57]*

Assuming that $p(\cdot)$ is a function in the class $\mathcal{B}(\mathbb{R}^n)$, there exists a positive constant C such that, for every ball B in $\mathbb{R}^n$ and every measurable subset S within B, the following inequalities hold:

$$\frac{\|\mathbf{1}_B\|_{p(\cdot)}}{\|\mathbf{1}_S\|_{p(\cdot)}} \leqslant \frac{|B|}{|S|}, \quad \frac{\|\mathbf{1}_S\|_{p(\cdot)}}{\|\mathbf{1}_B\|_{p(\cdot)}} \leqslant \left(\frac{|S|}{|B|}\right)^{\omega_1}, \quad \frac{\|\mathbf{1}_S\|_{p'(\cdot)}}{\|\mathbf{1}_B\|_{p'(\cdot)}} \leqslant \left(\frac{|S|}{|B|}\right)^{\omega_2},$$

here $0 < \omega_1, \omega_2 < 1$.

Lemma 5. *[1]*

Let $\{g_j\}_{j=1}^{\infty}$ be the sequences of locally integrable functions, $1 < r < \infty$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then

$$\left\| \left(\sum_{j=1}^{\infty} |Mg_j|^r \right)^{1/r} \right\|_{p(\cdot)} \leqslant C \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{1/r} \right\|_{p(\cdot)}.$$

Definition 6. Let $0 < q < \infty, p(\cdot) \in \mathcal{P}(\mathbb{R}^n), 0 \leq \lambda < \infty$, $\theta > 0$ and $\eta(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\eta \in L^{\infty}(\mathbb{R}^n)$.

(i) The homogeneous grand variable Herz-Morrey space $M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n) := \left\{ g \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|g\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)} < \infty \right\}$$

where

$$\|g\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)} := \sup_{\epsilon>0}\sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{L}\left\|2^{k\eta(\cdot)}g\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}.$$

(ii) The non-homogeneous grand variable Herz-Morrey space $MK^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)$ is defined by

$$MK^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n) := \left\{g\in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}) : \|g\|_{MK^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)} < \infty\right\}$$

where

$$\|g\|_{MK^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)} := \sup_{\epsilon>0}\sup_{L\in\mathbb{N}_0} 2^{-L\lambda}\left(\epsilon^\theta\sum_{k=0}^{L}\left\|2^{k\eta(\cdot)}g\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}.$$

Lemma 7. *[17] Let $D>1$ and $q\in\mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{c_0}r^{\frac{n}{q(0)}} \leq \|\mathbf{1}_{B(0,Dr)\backslash B(0,r)}\|_{L^{q(\cdot)}} \leq c_0 r^{\frac{n}{q(0)}},\ \textit{for } 0<r\leq 1 \tag{2.5}$$

and

$$\frac{1}{c_\infty}r^{\frac{n}{q_\infty}} \leq \|\mathbf{1}_{B(0,Dr)\backslash B(0,r)}\|_{L^{q(\cdot)}} \leq c_\infty r^{\frac{n}{q_\infty}},\ \textit{for } r\geq 1, \tag{2.6}$$

respectively, where $c_0\geq 1$ and $c_\infty\geq 1$ depend on D but not depend on r.

Theorem 8. *If $1<r<\infty$ and $\eta(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $\eta(0),\eta_\infty\in(-n\omega_1,n\omega_2)$, where $\omega_1,\omega_2\in(0,1)$ are constants appearing in (2.1). Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $0<q<\infty$, and*

$$0\leqslant\lambda<\min\left\{(n\omega_1+\eta(0))/2,(n\omega_1+\eta_\infty)/2\right\}.$$

Suppose that T is a sublinear operator satisfying vector-valued inequality on $L^{p(\cdot)}(\mathbb{R}^n)$

$$\left\|\left(\sum_{j=1}^\infty|Tg_j|^r\right)^{\frac{1}{r}}\right\|_{p(\cdot)} \leqslant C\left\|\left(\sum_{j=1}^\infty|g_j|^r\right)^{\frac{1}{r}}\right\|_{p(\cdot)}, \tag{2.7}$$

for all sequences $\{g_j\}_{j=1}^\infty$ of locally integrable functions on $\mathbb{R}^n$. Then we have the vector-valued inequality on $M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)$

$$\left\|\left(\sum_{k=1}^\infty|Tg_k|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)} \leqslant C\left\|\left(\sum_{k=1}^\infty|g_k|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)}.$$

Remark 9. Here and below, we only declare our main results in the homogeneous grand variable Herz-Morrey space because the proof for the non-homogeneous case can be treated by the similar way and is much more easier.

Lemma 10. *(see [1]) Let $p(\cdot), r$, and $\{g_j\}_{j=1}^{\infty}$ are given in Theorem 2.8, then*

$$\left\| \left(\sum_{j=1}^{\infty} |\mathcal{M} g_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

From Theorem 2.5 and Lemma 2.7, we obtain the following result for the Hardy-Littlewood maximal operator.

Corollary 1. *Let η, r, q, p, are given in Theorem 2.8 and*

$$0 \leqslant \lambda < \min\left\{ \left(n\omega_1 + \eta(0)\right)/2, \left(n\omega_1 + \eta_\infty\right)/2 \right\},$$

then

$$\left\| \left(\sum_{k=1}^{\infty} |\mathcal{M} g_k|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}_{\lambda, p(\cdot)}^{\eta(\cdot), q), \theta}(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_{k=1}^{\infty} |g_k|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}_{\lambda, p(\cdot)}^{\eta(\cdot), q), \theta}(\mathbb{R}^n)}.$$

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz functions and $\mathcal{S}'(\mathbb{R}^n)$ the set of all tempered distributions. We define the Fourier transform of a function $g \in \mathcal{S}(\mathbb{R}^n)$ by

$$\widehat{\varphi}(g)(y) = 2\pi^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot y} g(x) dx, \ y \in \mathbb{R}^n,$$

while $\varphi^{\vee}$ is the inverse Fourier transform. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(y) \geqslant 0$ then we have

$$\varphi_0(y) = \begin{cases} 1, & |y| \leqslant 1 \\ 0, & |y| \geqslant 2. \end{cases}$$

Let

$$\varphi(y) = \varphi_0(y) - \varphi_0(2y)$$

and define

$$\varphi_\ell(y) = \varphi\left(2^{-\ell} y\right), \quad \ell \in \mathbb{N}.$$

Then $\{\varphi_\ell\}_{\ell \in \mathbb{N}_0}$ be the resolution of unity, such that

$$\sum_{\ell=0}^{\infty} \varphi_\ell(x) = 1, \qquad x \in \mathbb{R}^n.$$

Definition 11. Let $\{\varphi_j\}_{j \in \mathbb{N}_0}$ be a resolution of unity as above, $s \in \mathbb{R}, 0 < \kappa, q \leq \infty, p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\eta(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\eta(\cdot) \in L^{\infty}(\mathbb{R}^n)$.

(i) Then the grand variable Herz-Morrey type Besov space is defined by

$$M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}B^s_\kappa(\mathbb{R}^n) := \left\{g \in \mathcal{S}'(\mathbb{R}^n) : \|g\|_{\ell_\kappa\left(M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)\right)} < \infty\right\},$$

where

$$\|g\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}B^s_\kappa} := \left\|\left\{2^{sj}\varphi_j^\vee * g\right\}_{j=0}^\infty\right\|_{\ell_\kappa\left(M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}\right)}.$$

(ii) For $p_+ < \infty$, the grand variable Herz-Morrey type Triebel-Lizorkin space is defined by

$$M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa(\mathbb{R}^n) := \left\{g \in \mathcal{S}'(\mathbb{R}^n) : \|g\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\ell_\kappa)} < \infty\right\},$$

where

$$\|g\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa} := \left\|\left\{2^{sj}\varphi_j^\vee * g\right\}_{j=0}^\infty\right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\ell_\kappa)}.$$

Here we denote respectively by $\ell_\kappa\left(M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}\right)$ and $M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\ell_\kappa)$ the spaces of all sequences $\{g_j\}$ of measurable functions on $\mathbb{R}^n$ with finite quasi-norms

$$\left\|\{g_j\}_{j=0}^\infty\right\|_{\ell_\kappa\left(M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}\right)} := \left(\sum_{j=0}^\infty \|g_j\|^\kappa_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}}\right)^{\frac{1}{\kappa}},$$

and

$$\left\|\{g_j\}_{j=0}^\infty\right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\ell_\kappa)} := \left\|\left(\sum_{j=0}^\infty |g_j|^\kappa\right)^{1/\kappa}\right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}}.$$

Let $S \geqslant 0$, $\varepsilon > 0$ and $\Psi_0, \Psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\left|\widehat{\Psi}_0(\varrho)\right| > 0 \quad \text{on } \{|\varrho| < 2\varepsilon\} \tag{2.8}$$

$$|\widehat{\Psi}(\varrho)| > 0 \quad \text{on } \left\{\frac{\varepsilon}{2} < |\varrho| < 2\varepsilon\right\} \tag{2.9}$$

and

$$D^\tau\widehat{\Psi}(0) = 0, \; \textit{for all } |\tau| \leqslant S. \tag{2.10}$$

Here, (2.8) and (2.9) are Tauberian conditions, while (2.10) expresses the vanishing moment conditions in Ψ.

In [58], J. Peetre introduced the classical Peetre's maximal operator:

Let a tempered distribution $g \in \mathcal{S}'(\mathbb{R}^n)$, $a > 0$, and $\{\Xi_\ell\}_{\ell\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$. Then system of maximal functions are given as

$$(\Xi_\ell^*)_a g(x) := \sup_{y\in\mathbb{R}^n} \frac{|\Xi_\ell * g(x+y)|}{(1+2^k|y|)^a}, \quad x \in \mathbb{R}^n, \ell \in \mathbb{Z}.$$

Since $\Xi_\ell * g(y)$ makes sense pointwise, everything is well defined. We will often use dilates

$$\Xi_\ell'(x) = 2^{kn}\Xi\left(2^k x\right)$$

of a fixed function $\Xi \in \mathcal{S}(\mathbb{R}^n)$, where $\Xi_0(x)$ might be given by a separate function. Continuous dilates are also needed. If

$$\Xi_t := t^{-n}\Xi\left(t^{-1}\cdot\right).$$

Then

$$\Psi_{t,a}^* g(x) := \sup_{y\in\mathbb{R}^n} \frac{|\Xi_t * g(x+y)|}{\left(1+\frac{|y|}{t}\right)^a} \quad x \in \mathbb{R}^n, t > 0.$$

Theorem 12. *If $\kappa, q \in (0,\infty], \omega > 0$, $s \in \mathbb{R}$ with $s < S+1$, and η, q, p, are the same as given in Lemma 2.9. Let $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$ with $p_0 < \min(p_-, 1)$. Let $\Theta_0, \Theta \in \mathcal{S}(\mathbb{R}^n)$ be given by (2.8) and (2.9), respectively. Then*

(i) For $a > n/p_0$, then the space $M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta} B_\kappa^s(\mathbb{R}^n)$ can be characterized by

$$M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta} B_\kappa^s(\mathbb{R}^n) = \left\{ g \in \mathcal{S}'(\mathbb{R}^n) : \|g\|^{(i)}_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta} B_\kappa^s} < \infty \right\}, \quad i = 1, \cdots, 4$$

where

$$\|g\|^{(1)}_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta} B_\kappa^s} := \|\Phi_0 * g\|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}} + \left(\int_0^1 t^{-s\kappa} \|\Phi_t * g\|^\kappa_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}} \frac{\mathrm{d}t}{t}\right)^{1/\kappa}$$

$$\|g\|^{(2)}_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta} B_\kappa^s} := \|(\Phi_0^* g)_a\|_{M\dot{K}_{q,p}^{\eta(\cdot),\lambda}} + \left(\int_0^1 t^{-s\kappa} \|(\Phi_t^* g)_a\|^\kappa_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}} \frac{\mathrm{d}t}{t}\right)^{1/\kappa}$$

$$\|g\|^{(3)}_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta} B_\kappa^s} := \left(\sum_{k=0}^\infty 2^{sk\kappa} \|(\Phi_k^* g)_a\|^\kappa_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}}\right)^{1/\kappa}$$

$$\|g\|^{(4)}_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta} B_\kappa^s} := \left(\sum_{k=0}^\infty 2^{sk\kappa} \|\Phi_k * g\|^\kappa_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}}\right)^{1/\kappa}.$$

Then, $\left\{ \|\cdot\|^{(i)}_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta} B_\kappa^s} \right\}_{i=1}^4$ are equivalent.

(ii) If $p_0 < \kappa$, then for $a > n/p_0$ the space $M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa(\mathbb{R}^n)$ can be characterized by

$$M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa(\mathbb{R}^n) = \left\{ g \in \mathcal{S}'(\mathbb{R}^n) : \|g\|^{(i)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa} < \infty \right\}, i = 1, \ldots, 5$$

where

$$\begin{aligned} \|g\|^{(1)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa} &:= \|\Phi_0 * g\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}} \\ &\quad + \left\| \left(\int_0^1 t^{-s\kappa} |\Phi_t * g|^\kappa \frac{\mathrm{d}t}{t} \right)^{1/\kappa} \right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}} \end{aligned} \tag{2.11}$$

$$\begin{aligned} \|g\|^{(2)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa} &:= \|(\Phi_0^* g)_a\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}} \\ &\quad + \left\| \left(\int_0^1 \left[t^{-s} (\Phi_t^* g)_a \right]^\kappa \frac{\mathrm{d}t}{t} \right)^{1/\kappa} \right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}} \end{aligned} \tag{2.12}$$

$$\begin{aligned} \|g\|^{(3)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa} &:= \|\Phi_0 * g\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}} \Bigg\| \Bigg(\int_0^1 t^{-s\kappa} \\ &\quad \times \int_{|z|<t} |(\Phi_t * g)(\cdot + z)|^\kappa \,\mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}} \Bigg)^{1/\kappa} \Bigg\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}} \end{aligned} \tag{2.13}$$

$$\|g\|^{(4)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa} := \left\| \left(\sum_{k=0}^\infty \left[2^{ks\kappa} (\Phi_k^* g)_a \right]^\kappa \right)^{1/\kappa} \right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}} \tag{2.14}$$

$$\|g\|^{(5)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa} := \left\| \left(\sum_{k=0}^\infty 2^{ks\kappa} |\Phi_k * g|^\kappa \right)^{1/\kappa} \right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}}. \tag{2.15}$$

Then, $\left\{ \|\cdot\|^{(i)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa} \right\}_{i=1}^5$ are equivalent.

3. Proofs of the main results

We need the following lemmas to prove our main results.

Lemma 13. *Let $p, q, \lambda, \eta,$ are given in Theorem 2.8 and $\theta > 0$, then*

$$\|g\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}} \approx \max \left\{ \sup_{\epsilon>0} \sup_{L \le 0, L \in \mathbb{Z}} 2^{-L\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{L} 2^{k\eta(0)q(1+\epsilon)} \|g \mathbf{1}_k\|^{q(1+\epsilon)}_{p(\cdot)} \right)^{\frac{1}{q(1+\epsilon)}} \right.$$

$$\sup_{L>0,L\in\mathbb{Z}}\sup_{\epsilon>0}\left[2^{-L\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\eta(0)q(1+\epsilon)}\left\|g\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}+2^{-L\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{L}2^{k\eta_\infty q(1+\epsilon)}\left\|g\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\right]\Bigg\}$$

Lemma 3.1 is similar to Proposition 3.8 in [5]. Indeed, when $\eta(\cdot)\in L^{\infty}(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, there exist positive constants C_1,C_2 such that if $k\leq 0$ and $x\in D_k$ then $C_1 2^{k\eta(0)}\leq 2^{k\eta(x)}\leq C_2 2^{k\eta(0)}$; if $k>1$ and $x\in D_k$ then $C_1 2^{k\eta_\infty}\leq 2^{k\eta(x)}\leq C_2 2^{k\eta_\infty}$, where $D_k:=B_k\setminus B_{k-1}$. Thus, we obtain Lemma 3.1.

Proof.

Now we will give the proof of Theorem 2.8**.** Let $(\sum_{k=1}^{\infty}|g_k|^r)^{\frac{1}{r}}\in M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}(\mathbb{R}^n)$. Using the Lemma 3.1 we get

$$\begin{aligned}
&\left\|\left(\sum_{j=1}^{\infty}|Tg_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}}\\
&\approx\max\left\{\sup_{\epsilon>0}\sup_{L\leq 0,L\in\mathbb{Z}}2^{-L\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{L}2^{k\eta(0)q(1+\epsilon)}\left\|\left(\sum_{j=1}^{\infty}|Tg_j|^r\right)^{\frac{1}{r}}\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\right.\\
&\sup_{\epsilon>0}\sup_{L>0,L\in\mathbb{Z}}\left[2^{-L\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\eta(0)q(1+\epsilon)}\left\|\left(\sum_{j=1}^{\infty}|Tg_j|^r\right)^{\frac{1}{r}}\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\right.\\
&\left.\left.+2^{-L\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{L}2^{k\eta_\infty q(1+\epsilon)}\left\|\left(\sum_{j=1}^{\infty}|Tg_j|^r\right)^{\frac{1}{r}}\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\right]\right\}\\
&=:\max\left\{E_T,F_T\right\}.
\end{aligned}$$

We also denote F_T by $F_T:=\sup\limits_{\epsilon>0}\sup\limits_{L>0,L\in\mathbb{Z}}[G_T+H_T]$ with

$$G_T:=2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\eta(0)q(1+\epsilon)}\left\|\left(\sum_{j=1}^{\infty}|Tg_j|^r\right)^{\frac{1}{r}}\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}}$$

$$H_T:=2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=0}^{L}2^{k\eta_\infty q(1+\epsilon)}\left\|\left(\sum_{j=1}^{\infty}|Tg_j|^r\right)^{\frac{1}{r}}\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}}.$$

We need to prove that $E_T \lesssim E_f, G_T \lesssim G_f$ and $H_T \lesssim H_f$ respectively, where

$$E_f := \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_k \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$G_f := 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\eta(0)q(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_k \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$H_f := 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=0}^{L} 2^{k\eta_\infty q(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_k \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}.$$

Hence we get $E_T \lesssim E_f$ and $F_T \lesssim F_g$ where F_g denote $\sup\limits_{\epsilon>0} \sup\limits_{L>0, L\in\mathbb{Z}} [G_f + H_f]$. From above all and using Lemma 3.1 again, we have

$$\begin{aligned}
\left\| \left(\sum_{j=1}^{\infty} |Tg_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}} &\approx \max\{E_T, F_T\} \\
&\lesssim \max\{E_g, F_g\} \\
&\approx \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}}.
\end{aligned}$$

Estimate of $G_T \lesssim G_f$ is similar to $E_{\mathcal{M}} \lesssim E_f$ so omit the details.

By using the size condition and Minkowski's inequality, for E_T we get

$$\begin{aligned}
E_T = &\sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_k \left(\sum_{j=1}^{\infty} |Tg_j|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
= &\sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_k \left(\sum_{j=1}^{\infty} \left| T \sum_{i=-\infty}^{\infty} g_j^i \right|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}
\end{aligned}$$

$$\leqslant \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_k \sum_{i=-\infty}^{\infty} \left(\sum_{j=1}^{\infty} \left| T g_j^i \right|^r \right)^{\frac{1}{r}} \right\|^{q(1+\epsilon)} \right\}_{p(\cdot)}^{\frac{1}{q(1+\epsilon)}}$$

$$\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_k \sum_{i=-\infty}^{k-2} \left(\sum_{j=1}^{\infty} \left| T g_j^i \right|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$+ \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_k \sum_{i=k-1}^{k+1} \left(\sum_{j=1}^{\infty} \left| T g_j^i \right|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$+ \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_k \sum_{i=k+2}^{\infty} \left(\sum_{j=1}^{\infty} \left| T g_j^i \right|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$=: E_T^1 + E_T^2 + E_T^3.$$

By the same way we consider H_T.

$$H_T \lesssim 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=0}^{L} 2^{\eta_\infty q(1+\epsilon)k} \left\| \mathbf{1}_k \sum_{i=-\infty}^{k-2} \left(\sum_{j=1}^{\infty} \left| T g_j^i \right|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$+ 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=0}^{L} 2^{\eta_\infty q(1+\epsilon)k} \left\| \mathbf{1}_k \sum_{i=k-1}^{k+1} \left(\sum_{j=1}^{\infty} \left| T g_j^i \right|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$+ 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=0}^{L} 2^{\eta_\infty q(1+\epsilon)k} \left\| \mathbf{1}_k \sum_{i=k+2}^{\infty} \left(\sum_{j=1}^{\infty} \left| T g_j^i \right|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$=: H_T^1 + H_T^2 + H_T^3.$$

Secondly, we will prove E_T^i and $H_T^i, i = 1, 2, 3$.

Step 1. For E_T^2, we have

$$
\begin{aligned}
E_T^2 &\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \sum_{i=k-1}^{k+1} \left\| \left(\sum_{j=1}^{\infty} |g_j^i|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \sum_{i=k-1}^{k+1} \left\| \mathbf{1}_i \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_{k-1} \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&+ \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_{k} \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&+ \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_{k+1} \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_{k} \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&=: E_f.
\end{aligned}
$$

Then turn to H_T^2, similarly, we have

$$
\begin{aligned}
H_T^2 \lesssim 2^{-L\lambda} &\left\{ \epsilon^\theta \sum_{k=0}^{L} 2^{\eta_\infty q(1+\epsilon)k} \left\| \mathbf{1}_{k-1} \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
+ 2^{-L\lambda} &\left\{ \epsilon^\theta \sum_{k=0}^{L} 2^{\eta_\infty q(1+\epsilon)k} \left\| \mathbf{1}_{k} \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}
\end{aligned}
$$

$$
\begin{aligned}
&+2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=0}^{L}2^{\eta_{\infty}q(1+\epsilon)k}\left\|\mathbf{1}_{k+1}\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\right\|_{p(\cdot)}^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}}\\
&\lesssim H_f.
\end{aligned}
$$

Step 2.

Let $\forall i\leqslant k-2, x\in R_k, 1<r<\infty$, by the size condition and the generalized Minkowski's inequality, we obtain

$$
\begin{aligned}
\left(\sum_{j=1}^{\infty}T^r\left(g_j^i\right)(x)\right)^{\frac{1}{r}} &\lesssim \left[\sum_{j=1}^{\infty}\left(2^{-kn}\int_{\mathbb{R}^n}\left|g_j^i(y)\right|\mathrm{d}y\right)^r\right]^{\frac{1}{r}}\\
&\lesssim 2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}\left|g_j^i\right|^r\right)^{\frac{1}{r}}\mathrm{d}y.
\end{aligned}
$$

By Hölder's inequality we get

$$
\begin{aligned}
E_T^1 &\lesssim \sup_{\epsilon>0}\sup_{L\leqslant 0, L\in\mathbb{Z}}2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=-\infty}^{L}2^{\eta(0)q(1+\epsilon)k}\left\|\mathbf{1}_k\sum_{i=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}\left|g_j^i\right|^r\right)^{\frac{1}{r}}\mathrm{d}y\right\|_{p(\cdot)}^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}}\\
&= \sup_{\epsilon>0}\sup_{L\leqslant 0, L\in\mathbb{Z}}2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=-\infty}^{L}2^{\eta(0)q(1+\epsilon)k}\left\|\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\left(\sum_{i=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}\left|g_j^i\right|^r\right)^{\frac{1}{r}}\mathrm{d}y\right)^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}}\\
&\lesssim \sup_{\epsilon>0}\sup_{L\leqslant 0, L\in\mathbb{Z}}2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=-\infty}^{L}2^{\eta(0)q(1+\epsilon)k}\left\|\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)}\left(\sum_{i=-\infty}^{k-2}2^{-kn}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}\left\|\mathbf{1}_i\right\|_{L^{p'(\cdot)}}\right)^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}}
\end{aligned} \tag{3.1}
$$

On the other hand, by using Lemma 2.4, we have

$$
2^{-kn}\left\|\mathbf{1}_k\right\|_{p(\cdot)}\left\|\mathbf{1}_i\right\|_{L^{p'(\cdot)}}\lesssim 2^{-kn}2^{\frac{kn}{p(0)}}2^{\frac{in}{p'(0)}}\lesssim 2^{\frac{(i-k)n}{p'(0)}}. \tag{3.2}
$$

We put (3.2) into (3.1) and get

$$
\begin{aligned}
E_T^1 &\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)kq(1+\epsilon)} \left(\sum_{i=-\infty}^{k-2} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)} 2^{\frac{(i-k)n}{p'(0)}} \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} \left(\sum_{i=-\infty}^{k-2} 2^{\eta(0)k} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)} 2^{\frac{(i-k)n}{p'(0)}} \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} \left(\sum_{i=-\infty}^{k-2} 2^{\eta(0)i} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)} 2^{b(i-k)} \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}},
\end{aligned} \tag{3.3}
$$

here $b := \frac{n}{p'(0)} - \eta(0) > 0$.

Let $1 < q(1+\epsilon) < \infty$, then the Hölder's inequality yields

$$
\begin{aligned}
E_T^1 &\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} \left(\sum_{i=-\infty}^{k-2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{bq(1+\epsilon)(i-k)}{2}} \right) \right. \\
&\quad \left. \times \left(\sum_{i=-\infty}^{k-2} 2^{\frac{b(q(1+\epsilon))'(i-k)}{2}} \right)^{\frac{q(1+\epsilon)}{(q(1+\epsilon))'}} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} \sum_{i=-\infty}^{k-2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{bq(1+\epsilon)(i-k)}{2}} \right\}^{\frac{1}{q}} \\
&= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{i=-\infty}^{L-2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} \sum_{k=i+2}^{L} 2^{\frac{bq(1+\epsilon)(i-k)}{2}} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{i=-\infty}^{L-2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim E_f.
\end{aligned}
$$

If $0 < q(1+\epsilon) \leqslant 1$, then we have

$$\left(\sum_{i=1}^{\infty} a_i\right)^{q(1+\epsilon)} \leqslant \sum_{i=1}^{\infty} a_i^{q(1+\epsilon)}, \quad i \in \mathbb{N}, a_i \geqslant 0, \tag{3.4}$$

and obtain

$$\begin{aligned}
E_T^1 &\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} \sum_{i=-\infty}^{k-2} 2^{\eta(0)q(1+\epsilon)i} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{bq(1+\epsilon)(i-k)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{i=-\infty}^{L-2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} \sum_{k=i+2}^{L} 2^{bq(1+\epsilon)(i-k)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{i=-\infty}^{L-2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim E_f.
\end{aligned}$$

Similarly, by using Lemmas 2.5 and 3.2, we have

$$2^{-kn} \|\mathbf{1}_k\|_{p(\cdot)} \|\mathbf{1}_i\|_{L^{p'(\cdot)}} \lesssim 2^{-kn} \|\mathbf{1}_{B_i}\|_{L^{p'(\cdot)}} |B_k| \|\mathbf{1}_{B_k}\|_{L^{p'(\cdot)}}^{-1} \lesssim 2^{n\omega_2(i-k)}.$$

$$H_T^1 \lesssim 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=0}^{\infty} \left(\sum_{i=-\infty}^{k-2} 2^{\eta_\infty i} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)} 2^{b_1(i-k)} \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

here $b_1 = n\omega_2 - \eta_\infty > 0$.

For $1 < q(1+\epsilon) < \infty$, using Hölder's inequality, we have

$$
\begin{aligned}
H_T^1 \lesssim\ & 2^{-L\lambda}\left\{\epsilon^\theta \sum_{k=0}^{L}\sum_{i=-\infty}^{k-2} 2^{\eta_\infty iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{b_q(i-k)}{2}} \left(\sum_{i=-\infty}^{k-2} 2^{\frac{b_1(q(1+\epsilon))'(i-k)}{2}}\right)^{\frac{q(1+\epsilon)}{q(1+\epsilon)'}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
\lesssim\ & 2^{-L\lambda}\left\{\epsilon^\theta \sum_{k=0}^{L}\sum_{i=-\infty}^{k-2} 2^{\eta_\infty iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{b_1 q(1+\epsilon)(i-k)}{2}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
\leqslant\ & 2^{-L\lambda}\left\{\epsilon^\theta \sum_{k=0}^{L}\sum_{i=-2}^{k-2} 2^{\eta_\infty iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q} 2^{\frac{b_1 q(1+\epsilon)(i-k)}{2}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
& + 2^{-L\lambda}\left\{\epsilon^\theta \sum_{k=0}^{L}\sum_{i=-\infty}^{-3} 2^{\eta_\infty iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{b_1 q(1+\epsilon)(i-k)}{2}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
=\ & I_1 + I_2.
\end{aligned}
$$

Now we consider I_1 and I_2 respectively. Due to $b_1 > 0$, we have

$$
\begin{aligned}
I_1 =\ & 2^{-L\lambda}\left\{\epsilon^\theta \sum_{i=-2}^{L-2} 2^{\eta_\infty iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} \sum_{k=i+2}^{L} 2^{\frac{b_1 q(1+\epsilon)(i-k)}{2}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
\lesssim\ & 2^{-L\lambda}\left\{\epsilon^\theta \sum_{i=-2}^{L-2} 2^{\eta_\infty iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}} \\
\lesssim\ & H_f.
\end{aligned}
$$

Because $L > 0, b_1 > 0$ and $\lambda \geqslant 0$, we obtain

$$
\begin{aligned}
I_2 \lesssim\ & 2^{-L\lambda}\left\{\epsilon^\theta \sum_{k=0}^{L}\sum_{i=-\infty}^{-3} 2^{\frac{b_1 q(1+\epsilon)(i-k)}{2}} \left[\sum_{m=\infty}^{i} 2^{\eta_\infty mq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}} \mathbf{1}_m \right\|_{p(\cdot)}^{q(1+\epsilon)}\right]\right\}^{\frac{1}{q(1+\epsilon)}} \\
=\ & 2^{-L\lambda}\left\{\epsilon^\theta \sum_{k=0}^{L}\sum_{i=-\infty}^{-3} 2^{\frac{b_1 q(1+\epsilon)(i-k)}{2}} \cdot 2^{iq(1+\epsilon)\lambda} H_f^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}}
\end{aligned}
$$

$$
\begin{aligned}
&= 2^{-L\lambda}\left\{\epsilon^{\theta}\left(\sum_{k=0}^{L} 2^{-kb_1 q(1+\epsilon)/2}\right)\left(\sum_{i=-\infty}^{-3} 2^{(b_1/2+\lambda)q(1+\epsilon)i}\right) H_f^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}}\\
&\lesssim 2^{-L\lambda}\left\{\epsilon^{\theta}\left(\sum_{k=0}^{L} 2^{-kb_1 q(1+\epsilon)/2}\right)\left(\sum_{i=-\infty}^{L} 2^{(b_1/2+\lambda)q(1+\epsilon)i}\right) H_f^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}}\\
&\lesssim 2^{-L\lambda} 2^{-Lb_1/2} 2^{(b_1/2+\lambda)L} H_f\\
&= H_f.
\end{aligned}
$$

For $0 < q(1+\epsilon) \leqslant 1$, using (3.4)

$$
\begin{aligned}
H_T^1 &\lesssim 2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=0}^{\infty}\sum_{i=-\infty}^{k-2} 2^{\eta_\infty i q(1+\epsilon)}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{b_1(i-k)}\right\}^{\frac{1}{q(1+\epsilon)}}\\
&\lesssim 2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=0}^{L}\sum_{i=-2}^{k-2} 2^{\eta_\infty i q(1+\epsilon)}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{b_1(i-k)}\right\}^{\frac{1}{q(1+\epsilon)}}\\
&\quad + 2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=0}^{L}\sum_{i=-\infty}^{-3} 2^{\eta_\infty i q(1+\epsilon)}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{b_1(i-k)}\right\}^{\frac{1}{q(1+\epsilon)}}\\
&=: J_1 + J_2.
\end{aligned}
$$

Similar to Step 2 $H_T^2 \lesssim H_f$ is also true for $0 < q(1+\epsilon) \leqslant 1$.
Step 3. Let $\forall i \geq k+2, x \in R_k$, then we get

$$
\begin{aligned}
\left(\sum_{j=1}^{\infty} T^r\left(g_j^i\right)(x)\right)^{\frac{1}{r}} &\lesssim \left[\sum_{j=1}^{\infty}\left(2^{-in}\int_{\mathbb{R}^n}\left|g_j^i(y)\right|\mathrm{d}y\right)^r\right]^{\frac{1}{r}}\\
&= 2^{-in}\left(\sum_{j=1}^{\infty}\left(\int_{\mathbb{R}^n}\left|g_j^i(y)\right|\mathrm{d}y\right)^r\right)^{\frac{1}{r}}\\
&\lesssim 2^{-in}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}\left|g_j^i\right|^r\right)^{\frac{1}{r}}\mathrm{d}y.
\end{aligned}
$$

By Hölder's inequality we get

$$E_T^3 \lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\| \mathbf{1}_k \sum_{i=k+2}^{\infty} 2^{-in} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \left|g_j^i\right|^r \right)^{\frac{1}{r}} \mathrm{d}y \right\|_{L^{p(\cdot)}}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\|\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)} \left(\sum_{i=k+2}^{\infty} 2^{-in} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \left|g_j^i\right|^r \right)^{\frac{1}{r}} \mathrm{d}y \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)q(1+\epsilon)k} \left\|\mathbf{1}_k\right\|_{p(\cdot)}^{q(1+\epsilon)} \left(\sum_{i=k+2}^{\infty} 2^{-in} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)} \left\|\mathbf{1}_i\right\|_{L^{p'(\cdot)}} \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \tag{3.5}$$

Using Lemmas 2.5 and 3.2 again, we obtain

$$\begin{aligned} 2^{-in} \left\|\mathbf{1}_k\right\|_{p(\cdot)} \left\|\mathbf{1}_i\right\|_{L^{p'(\cdot)}} &\leqslant 2^{-in} \left\|\mathbf{1}_{B_k}\right\|_{p(\cdot)} \left\|\mathbf{1}_{B_i}\right\|_{L^{p'(\cdot)}} \\ &\lesssim 2^{-in} \left\|\mathbf{1}_{B_k}\right\|_{p(\cdot)} |B_i| \left\|\mathbf{1}_{B_i}\right\|_{p(\cdot)}^{-1} \\ &\lesssim 2^{n\omega_1(k-i)}. \end{aligned} \tag{3.6}$$

We put (3.6) into (3.5) and get

$$\begin{aligned} E_T^3 &\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} 2^{\eta(0)kq(1+\epsilon)} \left(\sum_{i=k+2}^{\infty} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)} 2^{n\omega_1(k-i)} \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\ &= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} \left(\sum_{i=k+2}^{\infty} 2^{\eta(0)k} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)} 2^{n\omega_1(k-i)} \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\ &= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=-\infty}^{L} \left(\sum_{i=k+2}^{\infty} 2^{\eta(0)i} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)} 2^{d(k-i)} \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}. \end{aligned} \tag{3.7}$$

where $d := n\omega_1 + \eta(0) > 0$.

Let $1 < q(1+\epsilon) < \infty$, then we get

$$
\begin{aligned}
E_T^3 &\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} \left(\sum_{i=k+2}^{\infty} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{dq(1+\epsilon)(k-i)}{2}} \right) \left(\sum_{i=k+2}^{\infty} 2^{\frac{d(q(1+\epsilon))'(k-i)}{2}} \right)\right. \\
&\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} \sum_{i=k+2}^{\infty} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{dq(1+\epsilon)(k-i)}{2}} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} \sum_{i=k+2}^{L+2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{dq(1+\epsilon)(k-i)}{2}} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\quad + \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} \sum_{i=L+3}^{\infty} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{dq(1+\epsilon)(k-i)}{2}} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&=: I_3 + I_4.
\end{aligned}
$$

Now we consider I_3 and I_4 respectively. For $d>0$, we get

$$
\begin{aligned}
I_3 &= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{i=-\infty}^{L+2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} \sum_{k=-\infty}^{i-2} 2^{\frac{dq(1+\epsilon)(k-i)}{2}} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{i=-\infty}^{L+2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim E_f.
\end{aligned}
$$

If $d>0$ and $\lambda - d/2 < 0$, then we get

$$
\begin{aligned}
I_4 &\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} \sum_{i=L+3}^{\infty} 2^{\frac{dq(1+\epsilon)(k-i)}{2}} \left[\sum_{m=-\infty}^{i} 2^{\eta(0)mq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_m \right\|_{p(\cdot)}^{q(1+\epsilon)} \right] \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=-\infty}^{L} \sum_{i=L+3}^{\infty} 2^{\frac{dq(1+\epsilon)(k-i)}{2}} \cdot 2^{iq(1+\epsilon)\lambda} E_f^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}
\end{aligned}
$$

$$= \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \epsilon^{\theta} \left(\sum_{k=-\infty}^{L} 2^{dq(1+\epsilon)k/2} \right) \left(\sum_{i=L+3}^{\infty} 2^{(\lambda-d/2)q(1+\epsilon)i} \right) E_f^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} \left\{ 2^{-Lq(1+\epsilon)\lambda} 2^{dq(1+\epsilon)L/2} 2^{(\lambda-d/2)q(1+\epsilon)L} E_f^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$= E_f.$$

Hence we get $E_T^3 \lesssim E_f$.

For $0 < q(1+\epsilon) \leqslant 1$, then using (3.4) in (3.7) we get

$$E_T^3 \lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^{L} \sum_{i=k+2}^{\infty} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{dq(k-i)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$\lesssim \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^{L} \sum_{i=k+2}^{L+2} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{dq(1+\epsilon)(k-i)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$+ \sup_{\epsilon>0} \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^{L} \sum_{i=L+3}^{\infty} 2^{\eta(0)iq(1+\epsilon)} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{dq(1+\epsilon)(k-i)} \right\}^{\frac{1}{q(1+\epsilon)}}$$

$$=: J_3 + J_4.$$

Similarly we conclude that $E_T^3 \lesssim E_f$ holds for $0 < q(1+\epsilon) \leqslant 1$.

Then we consider H_T^3. Similarly, we have

$$H_T^3 \lesssim 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=0}^{L} \left(\sum_{i=k+2}^{\infty} 2^{\eta_\infty i} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)} 2^{d_1(k-i)} \right)^{q(1+\epsilon)} \right\}^{\frac{1}{q(1+\epsilon)}} \tag{3.8}$$

here $d_1 = n\omega_1 + \eta_\infty > 0$.

If $1 < q(1+\epsilon) < \infty$, then Hölder's inequality yields

$$H_T^3 \lesssim 2^{-L\lambda} \left\{ \epsilon^{\theta} \sum_{k=0}^{L} \left(\sum_{i=k+2}^{\infty} 2^{\eta_\infty q(1+\epsilon)i} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{d_1 q(1+\epsilon)(k-i)}{2}} \right) \right.$$

$$
\begin{aligned}
&\times\left(\sum_{i=k+2}^{\infty} 2^{\frac{d_1(q(1+\epsilon))'(k-i)}{2}}\right)^{\frac{q(1+\epsilon)}{(q(1+\epsilon))'}}\Bigg\}^{\frac{1}{q(1+\epsilon)}} \\
\lesssim & 2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=0}^{\frac{1}{q}}\sum_{i=k+2}^{\infty} 2^{\eta_\infty q(1+\epsilon)i}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{d_1 q(1+\epsilon)(k-i)}{2}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
\leqslant & 2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=0}^{L}\sum_{i=k+2}^{L+2} 2^{\eta_\infty q(1+\epsilon)i}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{d_1 q(1+\epsilon)(k-i)}{2}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
&+ 2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=0}^{L}\sum_{i=L+3}^{\infty} 2^{\eta_\infty q(1+\epsilon)i}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{d_1 q(1+\epsilon)(k-i)}{2}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
=: & I_5 + I_6.
\end{aligned}
$$

Because $d_1 > 0$, we have

$$
\begin{aligned}
I_5 &= 2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{i=2}^{L+2} 2^{\eta_\infty q(1+\epsilon)i}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}^{q(1+\epsilon)} \sum_{k=0}^{i-2} 2^{\frac{d_1 q(k-i)}{2}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim 2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{i=2}^{L+2} 2^{\eta_\infty q(1+\epsilon)i}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}^{q(1+\epsilon)}\right\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim H_f.
\end{aligned}
$$

Since $d_1 > 0$ and $\lambda - d_1/2 < 0$, we obtain

$$
\begin{aligned}
I_6 &= 2^{-L\lambda}\left\{\epsilon^{\theta}\sum_{k=0}^{L}\sum_{i=L+3}^{\infty} 2^{\eta_\infty q(1+\epsilon)i}\left\|\left(\sum_{j=1}^{\infty}|g_j|^r\right)^{\frac{1}{r}}\mathbf{1}_i\right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{\frac{d_1 q(1+\epsilon)(k-i)}{2}}\right\}^{\frac{1}{q(1+\epsilon)}} \\
&\leqslant 2^{-L\lambda}\Bigg\{\epsilon^{\theta}\left(\sum_{k=0}^{L} 2^{\frac{d_1 q(1+\epsilon)k}{2}}\right)\left(\sum_{i=L+3}^{\infty} 2^{(\lambda-d_1/2)q(1+\epsilon)i}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\sum_{m=0}^{i} 2^{\eta_\infty q(1+\epsilon)m} \left\| 2^{-i\lambda} \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_m \right\|_{p(\cdot)}^{q(1+\epsilon)} \right] \Bigg\}^{\frac{1}{q(1+\epsilon)}} \\
&\lesssim 2^{-L\lambda} 2^{d_1/2L} 2^{(\lambda - d_1/2)L} H_f \\
&= H_f.
\end{aligned}
$$

For $0 < q(1+\epsilon) \leqslant 1$, we use (3.4) in (3.8) and have

$$
\begin{aligned}
H_T^3 &\lesssim 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=0}^{L} \sum_{i=k+2}^{\infty} 2^{\eta_\infty q(1+\epsilon)i} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{d_1 q(1+\epsilon)(k-i)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\leqslant 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=0}^{L} \sum_{i=k+2}^{L+2} 2^{\eta_\infty q(1+\epsilon)i} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{d_1 q(1+\epsilon)(k-i)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&\quad + 2^{-L\lambda} \left\{ \epsilon^\theta \sum_{k=0}^{L} \sum_{i=L+3}^{\infty} 2^{\eta_\infty q(1+\epsilon)i} \left\| \left(\sum_{j=1}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \mathbf{1}_i \right\|_{p(\cdot)}^{q(1+\epsilon)} 2^{d_1 q(1+\epsilon)(k-i)} \right\}^{\frac{1}{q(1+\epsilon)}} \\
&=: I_J + J_6.
\end{aligned}
$$

Similarly we can get $H_T^3 \lesssim H_f$ where $0 < q(1+\epsilon) \leqslant 1$.
Hence we completes our proof.

Now we turn to prove Theorem 2.13. Because the proofs of B-parts and F-parts are similar, we only prove F-parts below. Our proof will use the idea that comes from [59]. To continue, we recall some lemmas.

Lemma 14. *[60]*
Let $\mu, \nu \in \mathcal{S}(\mathbb{R}^n), -1 \leqslant M \in \mathbb{Z}$,

$$D^\tau \widehat{\mu}(0) = 0 \quad \textit{for all} \quad |\tau| \leqslant M.$$

Then for any $N > 0$ *there is a constant* C_N *such that*

$$\sup_{z \in \mathbb{R}^n} |\mu_t * \nu(z)| (1+|z|)^N \leqslant C_N t^{M+1},$$

where $\mu_t(x) = t^{-n} \mu\left(\frac{x}{t}\right)$ *for all* $0 < t \leqslant 2$.

Lemma 15. *[60]* If $\omega > 0$ and $q \in (0, \infty]$. Then for a sequence $\{g_j\}_0^\infty$, we have

$$G_j = \sum_{\ell=0}^{\infty} 2^{-|\ell-j|\omega} g_\ell.$$

Then

$$\left\| \{G_j\}_0^\infty \right\|_{\ell_q} \leqslant C \left\| \{g_j\}_0^\infty \right\|_{\ell_q}. \tag{3.9}$$

Lemma 16. *If $\kappa, q \in (0, \infty], \omega > 0$, $s \in \mathbb{R}$, and η, q, p, are same as given in Theorem 2.8. For a sequence $\{g_j\}_0^\infty$, we have*

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|\omega} g_k(x), \quad x \in \mathbb{R}^n.$$

Then there are some constants $C_1 = C_1(q, \omega)$ and $C_2 = C_2(p(\cdot), q, \omega)$ such that

$$\left\| \{G_j\}_{j=0}^\infty \right\|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}(\ell_\kappa)} \leqslant C_1 \left\| \{g_j\}_{j=0}^\infty \right\|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}(\ell_\kappa)} \tag{3.10}$$

and

$$\left\| \{G_j\}_{j=0}^\infty \right\|_{\ell_\kappa\left(M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}\right)} \leqslant C_2 \left\| \{g_j\}_{j=0}^\infty \right\|_{\ell_\kappa\left(M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}\right)}. \tag{3.11}$$

Proof.

Firstly, (3.10) follows immediately from Lemma 3.3. Next we prove (3.11) for $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ and we separate it into two cases.

Case 1. $p_- \geqslant 1, q \geqslant 1$. Because $\| \cdot \|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}}$ is a norm, we have

$$\|G_j\|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}} \leqslant \sum_{k=0}^{\infty} 2^{-|k-j|\omega} \|g_k\|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}}.$$

Using Lemma 3.4, we get (3.11).

Case 2. If $q < 1$, let $p_0 < \min(p_-, q)$ then we get

$$\begin{aligned}
\|G_j\|_{M\dot{K}_{\lambda,p(\cdot)}^{\eta(\cdot),q),\theta}}^{p_0} &= \left\| |G_j|^{p_0} \right\|_{M\dot{K}_{p_0\lambda,p(\cdot)/p_0}^{p_0\eta(\cdot),\,q/p_0),p_0\theta}} \\
&\leqslant \left\| \sum_{k=0}^{\infty} 2^{-|k-j|p_0\omega} |g_k|^{p_0} \right\|_{M\dot{K}_{p_0\lambda,p(\cdot)/p_0}^{p_0\eta(\cdot),\,q/p_0),p_0\theta}} \\
&\leqslant \sum_{k=0}^{\infty} 2^{-|k-j|p_0\omega} \left\| |g_k|^{p_0} \right\|_{M\dot{K}_{p_0\lambda,p(\cdot)/p_0}^{p_0\eta(\cdot),\,q/p_0),p_0\theta}}.
\end{aligned}$$

Consequently we get

$$
\begin{aligned}
\|\{G_j\}\|^{p_0}_{\ell_\kappa\left(M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}\right)} &= \|\{|G_j|^{p_0}\}\|_{\ell_{\kappa/p_0}\left(M\dot{K}^{p_0\eta(\cdot),\, q/p_0),p_0\theta}_{p_0\lambda,p(\cdot)/p_0}\right)} \\
&\lesssim \|\{|g_k|^{p_0}\}\|_{\ell_{\kappa/p_0}\left(M\dot{K}^{p_0\eta(\cdot),\, q/p_0),p_0\theta}_{p_0\lambda,p(\cdot)/p_0}\right)} \\
&= \|\{g_k\}\|^{p_0}_{\ell_\kappa\left(M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}\right)}.
\end{aligned}
$$

By using the power $1/p_0$, we get (3.11).

Lemma 17 ([61], Theorem 6)**.** Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is the resolution of unity, $R \in \mathbb{N}$. Then there exists functions $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ which satisfy

$$
\begin{gathered}
\operatorname{supp}\theta, \operatorname{supp}\theta_0 \subseteq \{y \in \mathbb{R}^n : |y| \leqslant 1\}, \\
\left|\widehat{\theta}_0(\varrho)\right| > 0 \quad \text{ on } \{|\varrho| < 2\varepsilon\}, \\
|\widehat{\theta}(\varrho)| > 0 \quad \text{ on } \left\{\frac{\varepsilon}{2} < |\varrho| < 2\varepsilon\right\}, \\
\int_{\mathbb{R}^n} y^\gamma \theta(x)\mathrm{d}y = 0, \quad \forall \gamma, 0 < |\gamma| \leqslant R,
\end{gathered}
$$

such that

$$
\widehat{\theta}_0(\varrho)\widehat{\Psi}_0(\varrho) + \sum_{j=1}^{\infty} \widehat{\theta}\left(p^{-j}\varrho\right)\widehat{\Psi}\left(p^{-j}\varrho\right) = 1, \quad \forall \varrho \in \mathbb{R}^n,
$$

and $\Psi_0, \Psi \in \mathcal{S}(\mathbb{R}^n)$ are given as

$$
\widehat{\Psi}_0(\varrho) = \frac{\varphi_0(\varrho)}{\widehat{\theta}_0(\varrho)}, \quad \widehat{\Psi}(\varrho) = \frac{\varphi_1(2\varrho)}{\widehat{\theta}(\varrho)}.
$$

Proof. Now we will give the proof of Theorem 2.13.

Step 1. Let $g \in \mathcal{S}'(\mathbb{R}^n)$, then

$$
\|g\|^{(2)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa} \lesssim \|g\|^{(1)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa} \lesssim \|g\|^{(2)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa}.
$$

Using Lemmas 3.2 and 3.5, and the fact $r < \min\{p_-, \kappa\}$ for $N \in \mathbb{N}$. Then for $g \in \mathcal{S}(\mathbb{R}^n)$, then

$$
\begin{aligned}
\left(\int_1^2 \left|2^{ls}\left(\Phi^*_{2^{-l}t}g\right)_a(x)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa} &\lesssim \sum_{k\in l+\mathbb{N}_0} 2^{(l-k)(Nr-n+rs)}2^{krs} \\
&\quad \times \mathcal{M}\left[\left(\int_1^2 |((\Phi_k)_t * g)(\cdot)|^\kappa \tfrac{\mathrm{d}t}{t}\right)^{r/\kappa}\right](x).
\end{aligned}
$$

If $l \in \mathbb{N}$, $p_0 = r \in (n/a, < \min\{p_-, \kappa\})$, $N > \max\{0, -s\} + a$ and $\omega := N + s - d/r > 0$, then we get

$$\left(\int_1^2 \left|2^{ls}\left(\Phi^*_{2^{-l}} g\right)_a(x)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa}$$
$$\lesssim \sum_{k \in l+\mathbb{N}_0} 2^{-\omega r|l-k|} 2^{krs} \mathcal{M}\left[\left(\int_1^2 |((\Phi_k)_t * g)(\cdot)|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa}\right](x).$$

Using Lemma 3.5 in $M\dot{K}^{r\eta(\cdot), q/r), r\theta}_{r\lambda, p(\cdot)/r}(\ell_{\kappa/r})$, we obtain

$$\left\|\left\{\left(\int_1^2 \left|2^{ls}\left(\Phi^*_{2^{-l}t} g\right)_a(x)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa}\right\}_{l \in \mathbb{N}}\right\|_{M\dot{K}^{r\eta(\cdot), q/r), r\theta}_{r\lambda, p(\cdot)/r}(\ell_{\kappa/r})}$$
$$\lesssim \left\|\left\{\mathcal{M}\left[\left(\int_1^2 \left|2^{ks}((\Phi_l)_t * g)(\cdot)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa}\right]\right\}_{l \in \mathbb{N}}\right\|_{M\dot{K}^{r\eta(\cdot), q/r), r\theta}_{r\lambda, p(\cdot)/r}(\ell_{\kappa/r})}$$

Theorem 2.8 yields

$$\left\|\left\{\left(\int_1^2 \left|2^{ls}\left(\Phi^*_{2^{-l}} g\right)_a(x)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa}\right\}_{l \in \mathbb{N}}\right\|_{M\dot{K}^{r\eta(\cdot), q/r), r\theta}_{r\lambda, p(\cdot)/r}(\ell_{\kappa/r})}$$
$$\lesssim \left\|\left\{\left(\int_1^2 \left|2^{ks}((\Phi_l)_t * g)(\cdot)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa}\right\}_{l \in \mathbb{N}}\right\|_{M\dot{K}^{r\eta(\cdot), q/r), r\theta}_{r\lambda, p(\cdot)/r}(\ell_{\kappa/r})}$$
$$= \left\|\left\{\left(\int_1^2 \left|2^{ks}((\Phi_l)_t * g)(\cdot)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{1/\kappa}\right\}_{l \in \mathbb{N}}\right\|^r_{M\dot{K}^{\eta(\cdot), q), \theta}_{\lambda, p(\cdot)}(\ell_\kappa)}.$$

Hence, we have

$$\left\|\left(\int_0^1 \left|\lambda^{-s}(\Phi^*_\lambda g)_a(\cdot)\right|^\kappa \frac{d\lambda}{\lambda}\right)^{1/\kappa}\right\|_{M\dot{K}^{\eta(\cdot), q), \theta}_{\lambda, p(\cdot)}}$$
$$\approx \left\|\left(\sum_{l=1}^\infty \int_1^2 \left|2^{ls}\left(\Phi^*_{2^{-l}t} g\right)_a(\cdot)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{1/\kappa}\right\|_{M\dot{K}^{\eta(\cdot), q), \theta}_{\lambda, p(\cdot)}}$$
$$\lesssim \left\|\left\{\left(\int_1^2 \left|2^{ls}\Phi_{2^{-l}t} * g(\cdot)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{1/\kappa}\right\}_{l \in \mathbb{N}}\right\|_{M\dot{K}^{\eta(\cdot), q), \theta}_{\lambda, p(\cdot)}(\ell_\kappa)}$$

$$\approx \left\| \left(\int_0^1 \left| \lambda^{-s} \Phi_\lambda * g(\cdot) \right|^\kappa \frac{d\lambda}{\lambda} \right)^{1/\kappa} \right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}}.$$

This proves $\|g\|^{(2)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa} \lesssim \|g\|^{(1)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa}$.

Step 2. Suppose that $\Psi_0, \Psi \in \mathcal{S}'(\mathbb{R}^n)$ and $g \in \mathcal{S}'(\mathbb{R}^n)$.

$$\|g\|^{(4)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa(\mathbb{R}^n,\Psi)} \lesssim \|g\|^{(2)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)} F^s_\kappa(\mathbb{R}^n,\Phi)}. \tag{3.11}$$

By applying Lemmas 3.5 and 3.2, and the fact $\omega = \min\{1, S+1-s\}$,then for $g \in \mathcal{S}$, we get

$$2^{ls} \left(\Psi_l^* g \right)_a (x) \leqslant C \sum_{k \in \mathbb{N}_0} 2^{-|k-l|\omega} 2^{ks} \left(\Phi^*_{2^{-k}t} g \right)_a (x), x \in \mathbb{R}^n \text{ and } t \in [1,2]. \tag{3.12}$$

If $\kappa \geqslant 1$. By using the $\left(\int_1^2 |\cdot|^\kappa \mathrm{d}t/t \right)^{1/\kappa}$, we get

$$2^{ls} \left(\Psi_l^* g \right)_a (x) \lesssim \sum_{k \in \mathbb{N}_0} 2^{-|k-l|\omega} 2^{ks} \left(\int_1^2 \left| \left(\Phi^*_{2^{-k}t} g \right)_a (x) \right|^\kappa \frac{\mathrm{d}t}{t} \right)^{1/\kappa}.$$

Applying Lemma 3.5, we obtain

$$\left\| \left\{ 2^{ls} \left(\Psi_l^* g \right)_a \right\}_{l \in \mathbb{N}} \right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}(\ell_\kappa)} \lesssim \left\| \left(\sum_{k=1}^\infty 2^{ks\kappa} \int_1^2 \left| \left(\Phi^*_{2^{-k}t} g \right)_a (x) \right|^\kappa \frac{\mathrm{d}t}{t} \right)^{1/\kappa} \right\|_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}}.$$

Hence we obtain the required result.

If $\kappa < 1$, then $\left(\int_1^2 |\cdot|^\kappa \mathrm{d}t/t \right)^{1/\kappa}$ is not the norm. Thus we get

$$\left(2^{ls} \left(\Psi_l^* g \right)_a (x) \right)^\kappa \lesssim \sum_{k \in \mathbb{N}_0} 2^{-\kappa|k-l|\omega} 2^{ks\kappa} \int_1^2 \left| \left(\Phi^*_{2^{-k}t} g \right)_a (x) \right|^\kappa \frac{\mathrm{d}t}{t}.$$

Convolution $(\gamma * \eta)_\ell$ of the sequences yields

$$\gamma_k = 2^{-|k|\omega\kappa} \quad \text{and} \quad \tau_k = 2^{ks\kappa} \int_1^2 \left| \left(\Phi^*_{2-kt} g \right)_a (x) \right|^\kappa \frac{\mathrm{d}t}{t}$$

For $x \in \mathbb{R}^n$, using ℓ_1-norm gives as

$$\begin{aligned} \left\| 2^{ls} \left(\Psi_l^* g \right)_a (x) \right\|^\kappa_{\ell_\kappa} &\leqslant \|\gamma\|_{\ell_1} \cdot \|\tau\|_{\ell_1} \\ &\lesssim \sum_{k=1}^\infty 2^{ks\kappa} \int_1^2 \left| \left(\Phi^*_{2^{-k}t} g \right)_a (x) \right|^\kappa \frac{\mathrm{d}t}{t}. \end{aligned}$$

By taking $(\cdots)^{1/\kappa}$ and using $\dot{K}_{p(\cdot)}^{\eta(\cdot),q),\theta}$-norm. We obtain desired result (3.11).
Similarly, for any $g \in \mathcal{S}'(\mathbb{R}^n)$, we obtain

$$\|g\|^{(2)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa(\mathbb{R}^n,\Phi)} \lesssim \|g\|^{(4)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa(\mathbb{R}^n,\Psi)}.$$

Step 3. Using $t=1$ in **Step 1**, we get

$$\|g\|^{(5)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa} \lesssim \|g\|^{(4)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa} \lesssim \|g\|^{(5)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa}.$$

Step 4. We show (2.15) is equivalent to the rest.
First, we will show that for any $g \in \mathcal{S}'(\mathbb{R}^n)$

$$\|g\|^{(2)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa(\mathbb{R}^n)} \lesssim \|g\|^{(3)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa}. \tag{3.13}$$

For $0<r<\min\{p_-,\kappa\}$, see [59], there exists a positive constant C such that for any $g \in \mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned}
&\left(\int_1^2 \left|\left(\Psi^*_{2^{-l}}g\right)_a(x)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa} \\
&\leqslant C\sum_{k\in\mathbb{N}_0} 2^{-kNs}2^{(k+l)n}\int_{\mathbb{R}^n}\frac{\left(\int_1^2\int_{|z|<2^{-(k+l)}t}\left|\left(\left(\Phi_{k+l}\right)_t * g\right)(z+y)\right|^\kappa \mathrm{d}z\frac{\mathrm{d}t}{t^{n+1}}\right)^{r/\kappa}}{\left(1+2^l|x-y|\right)^{ar}}\,\mathrm{d}y.
\end{aligned}$$

Let $ar>n$, we get

$$g_l(y) := \frac{2^{nl}}{\left(1+2^l|y|\right)^{ar}}, \forall y \in \mathbb{R}^n.$$

Hence we get

$$\begin{aligned}
&\left(\int_1^2 \left|2^{ls}\left(\Phi^*_{2^{-l}t}g\right)_a(x)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa} \\
&\lesssim \sum_{k\in\mathbb{N}_0} 2^{-kNr}2^{kn}2^{lsr}\left[g_l * \left(\int_1^2\int_{|z|<2^{-(k+l)t}}\left|\left(\left(\Phi_{k+l}\right)_t * g\right)(z+\cdot)\right|^\kappa \mathrm{d}z\frac{\mathrm{d}t}{t^{n+1}}\right)^{r/\kappa}\right](x).
\end{aligned}$$

By applying the majorant property see [62] to obtain

$$\left(\int_1^2 \left|2^{ls}\left(\Phi^*_{2^{-l}t}g\right)_a(x)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa}$$

$$\lesssim \sum_{k\in\mathbb{N}_0} 2^{lsr}2^{k(-Nr+n)}\mathcal{M}\left[\left(\int_1^2 \int_{|z|<2^{-(k+l)t}} \left|\left(\left(\Phi_{k+l}\right)_t * g\right)(z+\cdot)\right|^\kappa \mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/\kappa}\right](x).$$

An index shift on the right-hand side gives

$$\begin{aligned}
&\left(\int_1^2 \left|2^{ls}\left(\Phi^*_{2^{-l}}g\right)_a(x)\right|^\kappa \frac{\mathrm{d}t}{t}\right)^{r/\kappa} \\
&\lesssim \sum_{k\in l+\mathbb{N}_0} 2^{lsr}2^{(k-l)(-Nr+n)}\mathcal{M}\left[\left(\int_1^2 \int_{|z|<2^{-k}t} \left|\left(\left(\Phi_{k}\right)_t * g\right)(z+\cdot)\right|^\kappa \mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/\kappa}\right](x) \\
&= \sum_{k\in l+\mathbb{N}_0} 2^{(l-k)(Nr-n+rs)}2^{krs}\mathcal{M}\left[\left(\int_1^2 \int_{|z|<2^{-k}t} \left|\left(\left(\Phi_{k}\right)_t * g\right)(z+\cdot)\right|^\kappa \mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/\kappa}\right](x).
\end{aligned}$$

It is simple to note that $\|g\|^{(3)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa} \lesssim \|g\|^{(2)}_{M\dot{K}^{\eta(\cdot),q),\theta}_{\lambda,p(\cdot)}F^s_\kappa}$, since for any $t>0$

$$\frac{1}{t^n}\int_{|z|<t} \left|\left(\Phi_t * g\right)(x+z)\right| \mathrm{d}z \lesssim \sup_{|z|<t} \frac{\left|\left(\Phi_t * g\right)(x+z)\right|}{(1+1/t|z|)^a} \lesssim \left(\Phi^*_t g\right)_a(x).$$

Hence, we complete the proof.

4. Conclusion

In this paper, we proved the boundedness results for vector-valued sublinear operators on grand variable Herz-Morrey spaces. Then we define the idea of grand variable Herz-Morrey type Besov and Triebel-Lizorkin spaces and proved equivalent quasi-norms by Peetre's maximal operators in these spaces under some proper assumptions.

5. Ethics declarations

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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