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Nonlinear Advanced Differential Equations: Improving Some Properties of Positive Solutions and Their Applications

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Abstract. This study examines the oscillatory performance of solutions of functional differential equations with an advanced argument. Equations of the advanced type have not received as much study as equations with delay. We deduce some new monotonic properties of the positive solutions of the studied equation. Then, we use these properties to obtain criteria that test the oscillatory nature of the solutions. We apply the results to some special cases, compare them with previous results, and analyze the results to demonstrate the novelty and importance.

2020 Mathematics Subject Classifications: 34C10, 34K11

Key Words and Phrases: Differential equations, advanced argument, oscillatory performance, noncanonical case

1. Introduction

Advanced differential equations (ADEs) are necessary for defining a wide range of systems that rely on current values and future predictions. These equations are used in a variety of phenomena, including population dynamics, economic models, and mechanical control systems; see [1–3]. Obtaining a closed solution to these equations is difficult, so we use oscillation theory, among the most important subfields of qualitative theory; see [4, 5].

The primary goal of this study is to examine how second-order ADEs exhibit oscillatory behavior. Here, we consider the ADE

$$\left(\rho\left(s\right)\left[x'\left(s\right)\right]^{\kappa}\right)'+q\left(s\right)x^{\kappa}\left(h\left(s\right)\right)=0, \tag{1}$$

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where $s \geq s_0$, $\kappa \geq 1$ is a ratio of two integers, $\rho,\ q \in \mathbf{C}([s_0,\infty),[0,\infty))$, $\rho(s) > 0$, $h \in \mathbf{C}([s_0,\infty),\mathbb{R})$, $h(s) \geq s$, $h'(s) \geq 0$, and

$$\int_{s_0}^{\infty} \rho^{-1/\kappa}(\varrho)\,\mathrm{d}\varrho < \infty.$$

For convenience, we define the integral operator

$$\Lambda[f(\cdot);\ u,v] := \int_u^v f(\varrho)\,\mathrm{d}\varrho,$$

for $v \geq u \geq s_0$ and $f \in \mathbf{C}([s_0,\infty))$, and the function

$$\mathcal{P}(s) := \Lambda\left[\rho^{1/\kappa};\ s,\infty\right].$$

A function $x \in \mathbf{C}^1([s_x,\infty))$, $s_x \geq s_0$, is called a proper solution of (1) if it has the features $\sup\{|x(s)| : s \geq s_1\} > 0$ for $s_1 \geq s_x$, $\rho[x']^\kappa \in \mathbf{C}^1([s_x,\infty))$, and satisfies (1) for all $s_x \geq s_0$. In this study, x is said to be an oscillatory solution if it has a sequence of zeros $\{s_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} s_n = \infty$. An equation is said to be oscillatory if all of its solutions exhibit oscillatory behavior.

While the oscillation of equation (1) in the context of delay, i.e., $h(s) \leq s$, has been extensively studied (see [6–10]), research on the oscillation of ADEs, where $h(s) \geq s$, remains relatively limited. However, interest in this area has grown significantly in recent decades due to its increasing relevance; see [11–15]. For related results in fractional models, see [16, 17] and references therein.

In 1981, Kusano and [18] demonstrated the oscillation of the equation

$$\left(\rho(s)\,x'(s)\right)' + q(s)\,x(h(s)) = 0 \tag{2}$$

can be inferred from the oscillation of the corresponding ODE

$$\left(\rho(s)\,x'(s)\right)' + q(s)\,x(s) = 0. \tag{3}$$

Equation (3) is oscillatory if

$$\rho(s)\Lambda[q;\ s,\infty] \geq \delta_0 > \frac{1}{4}. \tag{4}$$

where δ_0 is a constant; however, this condition is better suited for ODEs since it loses information on the value of advanced argument $h(s)$.

In [19], Džurina established oscillation criteria for equation (2), based on the related ODE (3), and he attempted to modify and improve the condition 4, which fails if ($\delta_0 \leq \frac{1}{4}$). He used the condition

$$\rho(s)\Lambda[q;\ s,\infty] \geq \delta_0 > 0, \tag{5}$$

which ensures that the equation

$$(\rho(s)x'(s))' + \left(\frac{\rho(h(s))}{\rho(s)}\right)^{\delta} q(s)x(s) = 0 \tag{6}$$

is oscillate.

In 2020, Bohner et al. [12] developed the previous works and investigated the oscillation of equation (2) in the noncanonical case. They transformed it into an equation in canonical form.

2. Main Results

As usual in oscillation theory, we focus on studying the properties of the positive solutions of equation (1). We directly exclude the negative solutions due to their symmetry with the positive solutions. We indicate that the solution belongs to the class of eventually positive solutions of (1) by the expression $x \in \mathbb{X}^+$.

2.1. Auxiliary Lemmas

We begin the main results with the following lemmas, which deduce asymptotic and monotonic properties for $x \in \mathbb{X}^+$ and transform the equation (1) into a linear equivalent form (in oscillatory features).

Beginning by the following lemma which provides a criterion that rules out the existence of positive increasing solutions under certain conditions. This insight plays a crucial role in the subsequent analysis of oscillatory behavior.

Lemma 1. *Assume that $x \in \mathbb{X}^+$. Then x is decreasing and converging to zero if*

$$\Lambda\left[\frac{1}{\rho^{1/\kappa}}\left(\Lambda\left[q;\ s_0, s\right]\right)^{1/\kappa};\ s_0, \infty\right] = \infty. \tag{7}$$

Proof. Assume that $x \in \mathbb{X}^+$. Therefore, there is a $s_1 \geq s_0$ such that $x\left(h\left(s\right)\right) > 0$, and so

$$\left(\rho\left(s\right)\left[x'\left(s\right)\right]^{\kappa}\right)' = -q\left(s\right)x^{\kappa}\left(h\left(s\right)\right) \leq 0. \tag{8}$$

Hence, x' is of fixed sign.

Now, let x' is positive for $s \geq s_1$. Then, $x \to x_0$ as $s \to \infty$, where x_0 is a positive constant. We also conclude that $(x \circ h) \geq x_0$, for $s \geq s_1$. Before proceeding to prove that x is decreasing, we note that (7), with the fact that $\mathcal{P}'\left(s\right) \leq 0$, guarantees this

$$\Lambda\left[q;\ s_0, \infty\right] = \infty. \tag{9}$$

Applying $\Lambda\left[(\cdot); s_1, \infty\right]$ on (8), we obtain

$$\begin{aligned}\rho\left(s_1\right)\left[x'\left(s_1\right)\right]^{\kappa} &\geq \Lambda\left[q \cdot \left(x^{\kappa} \circ h\right);\ s_1, \infty\right] \\ &\geq x_0^{\kappa}\Lambda\left[q;\ s_1, \infty\right].\end{aligned}$$

which contradicts to (9). So, x' is negative for $s \geq s_1$.

The positivity and decreasing of x ensures its convergence to $x_1 \geq 0$. Suppose that $x_1 > 0$. Applying $\Lambda\left[(\cdot); s_1, s\right]$ on (8), we find

$$\rho\left(s\right)\left[x'\left(s\right)\right]^{\kappa} \leq -\Lambda\left[q \cdot \left(x^{\kappa} \circ h\right);\ s_1, s\right]$$

$$\leq \quad -x_1^{\kappa}\Lambda\left[q;\ s_1,s\right],$$

or

$$x'(s) \leq -x_1\frac{1}{\rho^{1/\kappa}(s)}\left(\Lambda\left[q;\ s_1,s\right]\right)^{1/\kappa}. \tag{10}$$

Applying $\Lambda\left[(\cdot);s_1,\infty\right]$ on (10), we arrive at

$$x(s_1) \geq x_1\Lambda\left[\frac{1}{\rho^{1/\kappa}}\left(\Lambda\left[q;\ s_1,s\right]\right)^{1/\kappa};\ s_1,\infty\right],$$

which contradicts to (7). Therefore, $x_1 = 0$.

The proof is complete.

Lemma 2. *Assume that (7) holds. Then*

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x}{\mathcal{P}}\right) \geq 0 \tag{11}$$

and

$$\left(\rho^{1/\kappa}\cdot x'\right)' + \frac{1}{\kappa}q\cdot(\mathcal{P}\circ h)^{\kappa-1}\cdot(x\circ h) \leq 0. \tag{12}$$

Proof. Assume that $x \in \mathbb{X}^{+}$. We have

$$-x(s) = \Lambda\left[x';\ s,\infty\right] = \Lambda\left[\frac{\rho^{1/\kappa}\cdot x'}{\rho^{1/\kappa}};\ s,\infty\right] \leq \rho^{1/\kappa}(s)\,x'(s)\,\mathcal{P}(s).$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x}{\mathcal{P}}\right) = \frac{\mathcal{P}\cdot\rho^{1/\kappa}\cdot x' + x}{\rho^{1/\kappa}\cdot\mathcal{P}^2} \geq 0.$$

Thus, from the fact that $h(s) \geq s$, we have

$$\frac{(x\circ h)}{(\mathcal{P}\circ h)} \geq \frac{x}{\mathcal{P}} \geq \rho^{1/\kappa}\cdot\left(-x'\right), \tag{13}$$

and so

$$\left(\frac{(x\circ h)}{(\mathcal{P}\circ h)}\right)^{1-\kappa} \leq \left[\rho\cdot\left(-x'\right)^{\kappa}\right]^{1/\kappa-1}. \tag{14}$$

Now, it follows from (1) and (14) that

$$\begin{aligned}
\left(\rho^{1/\kappa}\cdot\left(-x'\right)\right)' &= \left(\left[\rho\cdot\left(-x'\right)^{\kappa}\right]^{1/\kappa}\right)' \\
&= \frac{1}{\kappa}\left[\rho\cdot\left(-x'\right)^{\kappa}\right]^{1/\kappa-1}\left(\rho\cdot\left(-x'\right)^{\kappa}\right)' \\
&\geq \frac{1}{\kappa}\left(\frac{(x\circ h)}{(\mathcal{P}\circ h)}\right)^{1-\kappa} q\cdot\left(x^{\kappa}\circ h\right) \\
&= \frac{1}{\kappa}q\cdot(\mathcal{P}\circ h)^{\kappa-1}\cdot(x\circ h).
\end{aligned}$$

The proof is complete.

Lemma 3. *Assume that (7) holds. Then*

$$\left(\rho^{1/\kappa}\cdot\mathcal{P}^2\cdot\left(\frac{x}{\mathcal{P}}\right)'\right)'+\frac{1}{\kappa}q\cdot\mathcal{P}\cdot(\mathcal{P}\circ h)^{\kappa-1}\cdot(x\circ h)\leq 0. \tag{15}$$

Proof. Assume that $x\in\mathbb{X}^+$. It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(\rho^{1/\kappa}\cdot\mathcal{P}^2\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x}{\mathcal{P}}\right)\right) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\rho^{1/\kappa}\cdot\mathcal{P}^2\cdot\left(\frac{\mathcal{P}\cdot x'+\rho^{-1/\kappa}x}{\mathcal{P}^2}\right)\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{P}\cdot\rho^{1/\kappa}\cdot x'+x\right)\\ &= \mathcal{P}\cdot\left(\rho^{1/\kappa}\cdot x'\right)'-\rho^{-1/\kappa}\cdot\rho^{1/\kappa}\cdot x'+x'\\ &= \mathcal{P}\cdot\left(\rho^{1/\kappa}\cdot x'\right)'.\end{aligned}$$

So, equation (1) reduces to

$$\left(\rho^{1/\kappa}\cdot\mathcal{P}^2\cdot\left(\frac{x}{\mathcal{P}}\right)'\right)'\leq-\frac{1}{\kappa}q\cdot\mathcal{P}\cdot(\mathcal{P}\circ h)^{\kappa-1}\cdot(x\circ h).$$

The proof is complete.

2.2. Oscillatory performance of solutions

The following theorem tests the oscillatory performance of solutions of equation (1) using the comparison technique with first-order equations.

Theorem 1. *Assume that (7) holds. Then, equation (1) is oscillatory if*

$$\liminf_{s\to\infty}\Lambda\left[\frac{\Lambda\left[q\cdot\mathcal{P}\cdot(\mathcal{P}\circ h)^{\kappa};\ s,\infty\right]}{\rho^{1/\kappa}\cdot\mathcal{P}^2};\ s,h(s)\right]>\frac{\kappa}{\mathrm{e}}. \tag{16}$$

Proof. Assume the contrary that $x\in\mathbb{X}^+$. From Lemma 2 and 3, we obtain that (11) and (15) hold for $s\geq s_1\geq s_0$. Applying $\Lambda[(\cdot);s,\infty]$ on (15), we get

$$\rho^{1/\kappa}(s)\,\mathcal{P}^2(s)\left(\frac{x(s)}{\mathcal{P}(s)}\right)'\geq\frac{1}{\kappa}\Lambda\left[q\cdot\mathcal{P}\cdot(\mathcal{P}\circ h)^{\kappa}\cdot\frac{(x\circ h)}{(\mathcal{P}\circ h)};\ s,\infty\right],$$

which with (11) gives

$$\rho^{1/\kappa}(s)\,\mathcal{P}^2(s)\left(\frac{x(s)}{\mathcal{P}(s)}\right)'\geq\frac{1}{\kappa}\frac{x(h(s))}{\mathcal{P}(h(s))}\Lambda\left[q\cdot\mathcal{P}\cdot(\mathcal{P}\circ h)^{\kappa};\ s,\infty\right],$$

Setting $w:=x/\mathcal{P}$, we have that w is a positive solution of

$$w'(s)\geq\frac{1}{\kappa}\frac{\Lambda\left[q\cdot\mathcal{P}\cdot(\mathcal{P}\circ h)^{\kappa};\ s,\infty\right]}{\rho^{1/\kappa}(s)\,\mathcal{P}^2(s)}w(h(s)).$$

Using Lemma 2.3 in [20], the equation

$$w'(s) - \frac{1}{\kappa}\frac{\Lambda\left[q \cdot \mathcal{P} \cdot (\mathcal{P} \circ h)^{\kappa};\ s, \infty\right]}{\rho^{1/\kappa}(s)\,\mathcal{P}^2(s)} w(h(s)) = 0$$

has also a positive solution, which contradicts to (16); see Theorem 1 in [21].

The proof is complete.

Based on Riccati's approach, we test the oscillatory behavior of equation (1) in the following theorem:

Theorem 2. *Assume that (7) holds. Then, equation (1) is oscillatory if there is a* $\psi \in \mathbf{C}([s_0, \infty), (0, \infty))$ *such that*

$$\limsup_{s\to\infty} \frac{\mathcal{P}(s)}{\psi(s)} \Lambda\left[\frac{1}{\kappa}\psi \cdot q \cdot \frac{(\mathcal{P} \circ h)^{\kappa}}{\mathcal{P}} - \frac{1}{4}\frac{\rho^{1/\kappa} \cdot (\psi')^2}{\psi}; s_1, s\right] > 1. \tag{17}$$

Proof. Assume the contrary that $x \in \mathbb{X}^+$. From Lemma 2, we obtain that (11) and (12) hold for $s \geq s_1 \geq s_0$.

Now, we define

$$H := \psi \cdot \left(\frac{\rho^{1/\kappa} \cdot x'}{x} + \frac{1}{\mathcal{P}}\right) > 0;\ \text{see (13)}. \tag{18}$$

Then,

$$H' = \frac{\psi'}{\psi} \cdot H + \psi \cdot \left(\frac{\left(\rho^{1/\kappa} \cdot x'\right)'}{x} - \rho^{1/\kappa} \cdot \frac{(x')^2}{x^2} + \frac{1}{\rho^{1/\kappa} \cdot \mathcal{P}^2}\right),$$

which with (12) and (18) yields

$$\begin{aligned} H' \ \leq\ & \frac{\psi'}{\psi} \cdot H - \frac{1}{\kappa}\psi \cdot q \cdot (\mathcal{P} \circ h)^{\kappa-1} \cdot \frac{(x \circ h)}{x} \\ & - \frac{1}{\psi \cdot \rho^{1/\kappa}}\left(H - \frac{\psi}{\mathcal{P}}\right)^2 + \frac{\psi}{\rho^{1/\kappa} \cdot \mathcal{P}^2}. \end{aligned}$$

From (11), we obtain

$$H' \leq -\frac{1}{\kappa}\psi \cdot q \cdot \frac{(\mathcal{P} \circ h)^{\kappa}}{\mathcal{P}} + \frac{\psi'}{\psi} \cdot H - \frac{1}{\psi \cdot \rho^{1/\kappa}}\left(H - \frac{\psi}{\mathcal{P}}\right)^2 + \frac{\psi}{\rho^{1/\kappa} \cdot \mathcal{P}^2}. \tag{19}$$

We define

$$F := \frac{\psi'}{\psi} \cdot H - \frac{1}{\psi \cdot \rho^{1/\kappa}}\left(H - \frac{\psi}{\mathcal{P}}\right)^2.$$

So, F gattains its maximum value at

$$\frac{\psi}{\mathcal{P}} + \frac{1}{2}\psi' \rho^{1/\kappa},$$

and

$$F \leq \frac{\psi'}{\mathcal{P}} + \frac{1}{4}\rho^{1/\kappa} \cdot \frac{(\psi')^2}{\psi}.$$

Therefore, (19) becomes

$$\begin{aligned} H' &\leq -\frac{1}{\kappa}\psi \cdot q \cdot \frac{(\mathcal{P} \circ h)^{\kappa}}{\mathcal{P}} + \frac{1}{4}\rho^{1/\kappa} \cdot \frac{(\psi')^2}{\psi} + \frac{\psi'}{\mathcal{P}} + \frac{\psi}{\rho^{1/\kappa} \cdot \mathcal{P}^2} \\ &= -\frac{1}{\kappa}\psi \cdot q \cdot \frac{(\mathcal{P} \circ h)^{\kappa}}{\mathcal{P}} + \frac{1}{4}\rho^{1/\kappa} \cdot \frac{(\psi')^2}{\psi} + \left(\frac{\psi}{\mathcal{P}}\right)'. \end{aligned} \tag{20}$$

Applying $\Lambda [(\cdot) ; s_1, s]$ on (20), we get

$$\begin{aligned} &\Lambda \left[\frac{1}{\kappa}\psi \cdot q \cdot \frac{(\mathcal{P} \circ h)^{\kappa}}{\mathcal{P}} - \frac{1}{4}\frac{\rho^{1/\kappa} \cdot (\psi')^2}{\psi}; s_1, s\right] \\ &\leq \left(H (s_1) - \frac{\psi (s_1)}{\mathcal{P} (s_1)}\right) - \left(H (s) - \frac{\psi (s)}{\mathcal{P} (s)}\right). \end{aligned}$$

This implies that

$$\Lambda \left[\frac{1}{\kappa}\psi \cdot q \cdot \frac{(\mathcal{P} \circ h)^{\kappa}}{\mathcal{P}} - \frac{1}{4}\frac{\rho^{1/\kappa} \cdot (\psi')^2}{\psi}; s_1, s\right] \leq -\psi (s) \frac{\rho^{1/\kappa} (s) x' (s)}{x (s)} \leq \frac{\psi (s)}{\mathcal{P} (s)},$$

or

$$\frac{\mathcal{P} (s)}{\psi (s)}\Lambda \left[\frac{1}{\kappa}\psi \cdot q \cdot \frac{(\mathcal{P} \circ h)^{\kappa}}{\mathcal{P}} - \frac{1}{4}\frac{\rho^{1/\kappa} \cdot (\psi')^2}{\psi}; s_1, s\right] \leq 1. \tag{21}$$

Taking the lim sup of (21), we get at contradiction with (17).

This completes the proof.

3. Discussion and Applications

This section tests the oscillatory performance for some special cases of the studied equation and compares our results with those previously reported in the literature.

Example 1. *Consider the ADE*

$$\left(s^2 x' (s)\right)' + q_0 x (h_0 s) = 0, \tag{22}$$

where $s > 0$, $h_0 \geq 1$ *and* $q_0 > 0$. *Then* $\mathcal{P} (s) = 1/s$, *and so*

$$\Lambda \left[\frac{1}{s^2}\Lambda [q_0;\ s_0, s];\ s_0, \infty\right] = q_0\Lambda \left[\frac{s - s_0}{s^2};\ s_0, \infty\right] = \infty,$$

Table 1: Comparison of the oscillation criteria of (22) for special cases of parameters q_0 and h_0

	Our results		Previous results	
Criterion	(23)	(24)	(25)	(26)
(a) $q_0 = 3$	$h_0 > 26.824$	$h_0 > 0.0833$	$h_0 > 19.078$	$h_0 > 31.005$
(b) $h_0 = 2$	$q_0 > 1.0615$	$q_0 > 0.1250$	$q_0 > 0.4307$	$q_0 > 0.6934$

which means that condition (7) is fulfilled.
Now, condition (16) becomes

$$\begin{aligned}\liminf_{s\to\infty}\Lambda\left[\Lambda\left[q_0\frac{1}{s}\frac{1}{h_0 s};\ s,\infty\right];\ s,h_0 s\right] &= \frac{q_0}{h_0}\liminf_{s\to\infty}\Lambda\left[\frac{1}{s};\ s,h_0 s\right]\\ &= \frac{q_0}{h_0}\ln\left(h_0\right)\\ &> \frac{1}{\mathrm{e}}.\end{aligned}$$

From Theorem 1, equation (22) is oscillatory if

$$q_0 > \frac{h_0}{\mathrm{e}\ln\left(h_0\right)}. \tag{23}$$

On the other hand, by choosing $\psi(s) = 1/s$, condition (17) reduces to

$$\limsup_{s\to\infty}\Lambda\left[q_0 h_0\frac{1}{s} - \frac{1}{4}\frac{1}{s}; s_1, s\right] = \left(q_0 h_0 - \frac{1}{4}\right)\limsup_{s\to\infty}\Lambda\left[\frac{1}{s}; s_1, s\right] > 1.$$

From Theorem 2, equation (22) is oscillatory if

$$q_0 > \frac{1}{4h_0}. \tag{24}$$

Remark 1. *According to Theorem 10 in [11] and Theorem 3.4 in [12], equation (22) oscillates when*

$$q_0 h_0^{\frac{q_0}{h_0}-1} > \frac{1}{4}. \tag{25}$$

The results in [13] also confirm the oscillation of equation (22) under the condition

$$\frac{q_0}{h_0}\ln\left(h_0\right) > \frac{1}{\mathrm{e}}\left(1 - \frac{q_0}{h_0}\right). \tag{26}$$

The following table, Table 1, illustrates the comparison mentioned in Remark 1 for different values of q_0 and h_0. It is clear from this comparison that our results provide more efficient criteria for testing the oscillatory performance of solutions.

Example 2. *Consider the ADE*

$$\left(\mathrm{e}^s x'(s)\right)' + q_0\mathrm{e}^s x\left(s+h_0\right) = 0 \tag{27}$$

where $s > 0$, $h_0 > 0$ and $q_0 > 0$. Then $\mathcal{P}(s) = \mathrm{e}^{-s}$, and so

$$\Lambda\left[\frac{1}{\mathrm{e}^{s}}\Lambda\left[q_0\mathrm{e}^{s};\ s_0, s\right];\ s_0, \infty\right] = q_0\Lambda\left[\frac{\mathrm{e}^{s}-\mathrm{e}^{s_0}}{\mathrm{e}^{s}};\ s_0, \infty\right] = \infty,$$

which means that condition (7) is fulfilled.
Now, condition (16) becomes

$$\begin{aligned}\liminf_{s\to\infty}\Lambda\left[\frac{1}{\mathrm{e}^{-s}}\Lambda\left[q_0\mathrm{e}^{-s-h_0};\ s, \infty\right];\ s, s+h_0\right] &= \mathrm{e}^{-h_0}q_0\liminf_{s\to\infty}\Lambda\left[1;\ s, s+h_0\right] \\ &= h_0\mathrm{e}^{-h_0}q_0 \\ &> \frac{1}{\mathrm{e}}.\end{aligned}$$

It follows from Theorem 1 that equation (27) is oscillatory if

$$q_0 > \frac{\mathrm{e}^{h_0}}{\mathrm{e}h_0}. \tag{28}$$

On the other hand, by choosing $\psi(s) = \mathrm{e}^{-s}$, condition (17) reduces to

$$\limsup_{s\to\infty}\Lambda\left[q_0\mathrm{e}^{-h_0} - \frac{1}{4}; s_1, s\right] > 1$$

From Theorem 2, equation (27) is oscillatory if

$$q_0 > \frac{\mathrm{e}^{h_0}}{4}. \tag{29}$$

Figure 1 shows the minimum values of q_0 at which oscillation occurs according to criteria (28) and (29). We notice that criterion (28) is more efficient during $h_0 \geq \frac{4}{e}$, while criterion (29) is superior for $h_0 \in \left[1, \frac{4}{e}\right]$.

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Conflict of interest

The authors declare there is no conflicts of interest.

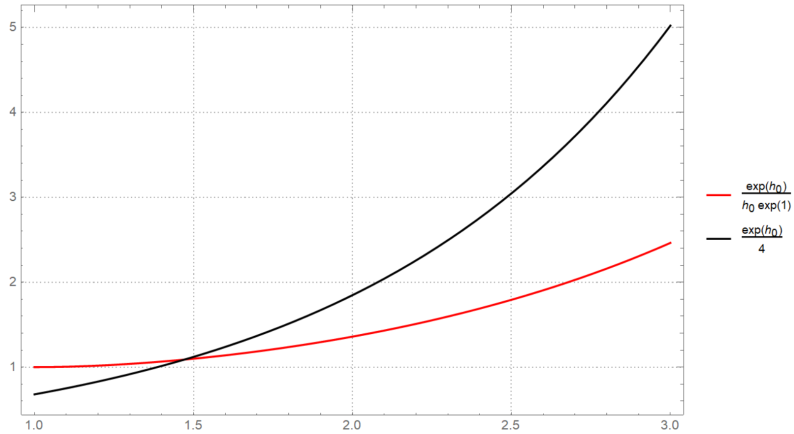

Figure 1: Comparison of minimum values of q_0 for criteria (28) and (29)

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