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Generalized Extended Confluent, Whittaker k-Functions and Their Properties

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Abstract. The main objective of this research paper is to explore further generalization of confluent hypergeometric and Whittaker functions by introducing a new parameter $k > 0$, in generalized extended confluent hypergeometric and Whittaker functions defined by Khan *et al.* [1]. We also investigate the Mellin transformations, inverse Mellin transformations, Hankel transformations, Laplace transformations, and derivative of the newly defined generalized extended confluent hypergeometric and Whittaker k-functions. We also obtain Riemann-Liouville fractional integral and Riemann-Liouville k-fractional integral of these new generalized extended Whittaker k-function.

2020 Mathematics Subject Classifications: 33C60, 33C20, 33C05, 33C15

Key Words and Phrases: Gamma k-function

1. Introduction and Preliminaries

Special functions play a compelling role in different fields such as statistics, physics, mathematics and engineering etc. The hypergeometric, beta, gamma functions and Legendre polynomial play a remarkable role to solve complex mathematical problems. In the solutions of partial differential equation to control the physical phenomena such as wave propagation and heat flow, the special functions are contiguously arrive. The hypergeometric functions are used in the calculation of Feymann integrals. The confluent hypergeometric has many applications in probability, statistics, approximation theory, solution of differential equation and physics. The confluent hypergeometric function helps

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in the solution of Schrödinger equation in quantum mechanics and the heat conduction equation in the thermodynamics. It also helps in the study of random process, stochastic model statistical distributions, in the calculation of Laplace transform and Fourier transform. Nasa uses generalized Whittaker functions to model spacecraft re-entry plasma sheaths. Due to generalization of such hypergeometric and Whittaker functions we can solve closed-form solutions to differential equations with non-polynomial coefficients. In quantum field theory and fractional calculus models higher-order differential equations can be solved by using such generalizations.

The generalization and extension of special functions such as hypergeometric, gamma, beta, Bessel functions etc. play a central role in mathematical modeling, economics, physics and mathematical analysis. In mathematical analysis such as integral transforms, and special function theory generalizes the Mellin and Barnes integral representations for functions with branch cuts. Various researches introduced the generalizations, extensions, integral representations and properties of various special functions for parameter $k > 0$, (see [2–7] and [8–14]). In [15], Diaz and Pariguan investigated gamma, beta, hypergeometric k-functions and Pochhammer's k-symbol. Nasar *et al.*[16], introduced some inequalities involving extended gamma and confluent hypergeometric k-functions. In [17], Mubeen *et al.* investigated the extensions of beta, gamma, and beta distribution. In [18], Qayyumm *et al.* introduced extended conformable k- beta and hypergeometric functions by using Mittag-Leffer k-function. In [19], Kokologiannaki proved some properties and inequalities of gamma, beta and zeta k- functions. In [20], Mubeen *et al.* introduced integral representations of k-hypergeometric functions. Rahman *et al.* [21] investigated inequalities involving extended gamma and beta k-functions.

Many other researchers introduced the generalizations, extensions, integral representations and properties of various special functions without $k > 0$ parameter, (see [1, 4, 10–12, 22]). In 2004, Chaudhary *et al.* [23] extended the Gauss, confluent hypergeometric functions by using the extended beta functions. Further, Parmar [24] introduced a new generalization of extended Gauss, confluent hypergeometric functions by using generalized extended beta functions. In [25], Ozregion *et al.* gave the extension of gamma, beta, and hypergeometric functions. Sarivastava *et al.* [26] introduced a new generalized extended Gauss hypergeometric functions. Issenova *et al.* [27] gave some generalizations of Whittaker, Horn, Bessel, Legendre functions, discussed some related properties and examples as applications. Paula *et al.* [28] investigated the generalization of some special functions and discussed related application in probability distributions in the field of statistics. The Whittaker function which was introduced by Whittaker in (1903) is a unique solution of Whittaker equation. It is a modified form of confluent hypergeometric function. It has many applications in physics, engineering and mathematics. Whittaker function helps in the signal processing, and to solve the differential equations. The Whittaker function introduced by whittaker in [29]. After that various researchers introduced the generalizations and extensions of Whittaker function in terms of $k > 0$ parameter and without k parameter. In [30], Nagar *et al.* introduced extended Whittaker function and its properties. Khan *et al.* [31] generalized extended whittaker function. In [32], Khan *et al.* investigated the analysis of extended Whittaker function. In [33], Khan *et al.* also introduced multi-index Whittaker function. In [34], Panwar and Rai introduced Whittaker k-function and investigated the fractional integral of Whittaker k-function. In [1], Khan *et al.* investigated generalized extended Whittaker function introducing an extra parameter.

The main objective of this research paper is to provide a further generalization of confluent hypergeometric and Whittaker functions by introducing $k > 0$ parameter in generalized extended confluent hypergeometric and Whittaker functions defined by Khan *et al.* [1]. We also investigate some properties such as integral representations, Mellin transform, inverse Mellin, Hankel, laplace transformations and derivative of these new generalized extended confluent hypergeometric and Whittaker k-functions. We also obtain Riemann-Liouville fractional integral and k-Riemann-Liouville fractional integral of these new generalized extended Whittaker k-function. The laplace transformations helps us in solving complex mathematical problems, design and analyze the control systems, analyzing and optimizing communications signals in telecommunications and used in financial modeling. Mellin and Hankel transformations are important mathematical tools in the field of integral transforms. For the simplifying complex problems, dealing with special functions, in number theory and probability theory Mellin and Hankel transformations play a key role.

In [15], Diaz and Pariguan investigated gamma, beta, hypergeometric k-functions and Pochhammer's k-symbol as follows:
Let $w \in \mathbb{C}$ ($\mathbb{C}$ is a set of complex numbers), then

$$\Gamma_k(w) = \int_0^\infty v^{w-1} e^{-\frac{v^k}{k}} dv. \tag{1}$$

If $\Re(s_1) > 0, \ \Re(s_2) > 0, \ k > 0$, then

$$\beta_k(s_1, s_2) = \frac{\Gamma_k(s_1)\Gamma_k(s_2)}{\Gamma_k(s_1 + s_2)} \tag{2}$$

$$= \frac{1}{k} \int_0^1 s^{\frac{s_1}{k}-1} (1-s)^{\frac{s_2}{k}-1} ds. \tag{3}$$

If $\tau, u \in \mathbb{C}; k > 0$, then

$$(\lambda)_{u,k} = \frac{\Gamma_k(\tau + uk)}{\Gamma_k(\tau)} \quad (\tau \in \mathbb{C}\backslash\{0\}) = \begin{cases} 1 & (u = 0), \\ \tau(\tau + k) \cdots (\tau + (l-1)k) & (u = l \in \mathbb{N}). \end{cases} \tag{4}$$

The Gauss hypergeometric k-function is defined as

$${}_2\,\mathrm{F}_{1,\mathrm{k}}(\lambda_1, \lambda_2; \lambda_3; z) = \sum_{n=0}^{\infty} \frac{(\lambda_1)_{n,k}(\lambda_2)_{n,k}}{(\lambda_3)_{n,k}} \frac{z^n}{n!}, \left(\lambda_3 \in \mathbb{C}\backslash \digamma_0^-; |z| < 1; k \in \mathbb{R}^+\right). \tag{5}$$

In [16], confluent hypergeometric k-function is defined as

$$ {}_1\psi_{1,k}(\sigma_1, \sigma_2; t) = \sum_{m=0}^{\infty} \frac{(\sigma_1)_{m,k}}{(\sigma_2)_{m,k}} \frac{t^m}{m!}, \tag{6}$$

where $|t| < \frac{1}{k}$, $\Re(\sigma_1) > \Re(\sigma_2) > 0$, $k > 0$.

In [34], Savita Panwar and Prakriti Rai introduced a new form of confluent hypergeometric k-function as

$$ {}_1F_{1,k}(s_1, s_2; v) = \sum_{m=0}^{\infty} \frac{\beta_k(s_1 + mk, s_2 - s_1)}{\beta_k(s_1, s_2 - s_1)} \frac{v^m}{m!}. \tag{7}$$

In [2], Mubeen introduced the following k-analague of Kummer's first formula

$$ {}_1F_{1,k}(s_1, s_2; v) = \exp(v)\, {}_1F_{1,k}(s_2 - s_1, s_2; -v). \tag{8}$$

In [17], Mubeen *et al.* defined extended gamma k-function as follows:

$$ \Gamma_{w,k}^{(p,q)}(v) = \int_0^{\infty} t^{v-1}\, {}_1F_{1,k}(p; q; -\frac{t^k}{k} - \frac{w^k}{kt^k}) dt, \tag{9}$$

where $\Re(w) > 0$, $\Re(q) > 0$, $\Re(v) > 0$, $\Re(p) > 0$, $k > 0$.

In the same paper [17], extended beta k-function is defined as

$$ \beta_{\xi,k}^{(\delta_1,\delta_2)}(p, q) = \frac{1}{k} \int_0^1 s^{\frac{p}{k}-1}(1-s)^{\frac{q}{k}-1}\, {}_1F_{1,k}(\delta_1; \delta_2; \frac{-\xi^k}{ks(1-s)}) ds, \tag{10}$$

where $\Re(\delta_1) > 0$, $\Re(\delta_2) > 0$, $\Re(\xi) \geq 0$, $\Re(p) \geq 0$, $\Re(q) \geq 0$, $k > 0$.

Let $k > 0, \alpha \in (0, 1)$, then extended (α, k)-beta function defined in [18] as

$$ \beta_{k,p_1,p_2}^{\alpha,Q}(p, q) = \frac{1}{\alpha k} \int_0^1 s^{\frac{p}{\alpha k}-1}(1-s)^{\frac{q}{\alpha k}-1} E_{(k,p_1,p_2)} \left(\frac{-Q^k}{ks(1-s)} \right) d_\alpha s, \tag{11}$$

where $\Re(p), \Re(q) > 0$, $s \in C$, $Q \geq 0$.

In the same paper [18], (α, k)- hypergeometric and confluent hypergeometric functions respectively, defined as

$$ F_{k,p_1,p_2}^{\alpha,Q}(v_1, v_2, v_3; x^\alpha) = \sum_{m=0}^{\infty} \frac{(v_1)_{m,k} \beta_{k,p_1,p_2}^{\alpha,Q}(v_2 + mk\alpha, v_3 - v_2)}{\beta_k^\alpha(v_2, v_3 - v_2)} \frac{x^{\alpha m}}{m!}, \tag{12}$$

where $\Re(v_1, \Re(v_2), \Re(v_3 > 0)$, $\alpha \in (0,1)$, $k > 0, |x^\alpha| < 1, Q \geq 0$
and

$$\phi^{\alpha,Q}_{k,p_1,p_2}(v_2, v_3; x^\alpha) = \sum_{m=0}^{\infty} \frac{\beta^{\alpha,Q}_{k,p_1,p_2}(v_2 + mk\alpha, v_3 - v_2)}{\beta^\alpha_k(v_2, v_3 - v_2)} \frac{x^{\alpha m}}{m!}, \tag{13}$$

where $\Re(v_2), \Re(v_3) > 0$, $\alpha \in (0,1)$, $k > 0, |v^\alpha| < 1, Q \geq 0$.

By using equation (11) in (12) and (13), we obtain following integral representations

$$\begin{aligned} F^{\alpha,Q}_{k,p_1,p_2}(v_1, v_2, v_3; x^\alpha) \quad &= \frac{1}{\alpha k \beta^\alpha_k(v_2, v_3 - v_2)} \int\limits_0^1 s^{\frac{v_2}{\alpha k}-1}(1-s)^{\frac{v_3-v_2}{\alpha k}-1}(1 - kx^\alpha s)^{\frac{-v_1}{k}} \\ &\times E_{(k,p_1,p_2)}\left(\frac{-Q^k}{ks(1-s)}\right) d_\alpha s, \end{aligned} \tag{14}$$

where $\Re(v_1) > 0, \Re(v_2), \Re(v_3) > 0$, $\alpha \in (0,1)$, $k > 0, |x^\alpha| < 1, Q \geq 0$.

$$\begin{aligned} \phi^{\alpha,Q}_{k,p_1,p_2}(v_2, v_3; x^\alpha) \quad &= \frac{1}{\alpha k \beta^\alpha_k(v_2, v_3 - v_2)} \int\limits_0^1 s^{\frac{v_2}{\alpha k}-1}(1-s)^{\frac{v_3-v_2}{\alpha k}-1} e^{x^\alpha s} \\ &\times E_{(k,p_1,p_2)}\left(\frac{-Q^k}{ks(1-s)}\right) d_\alpha s, \end{aligned} \tag{15}$$

where $\Re(v_2), \Re(v_3) > 0$, $\alpha \in (0,1)$, $k > 0, |x^\alpha| < 1, Q \geq 0$.

Khan *et al.* [1], introduced the generalized extended confluent and beta functions as

$$\psi^{(\delta_1,\delta_2,l_1:l_2)}_\xi(\sigma_2, \sigma_3; t) = \sum_{m=0}^{\infty} \frac{\beta^{(\delta_1,\delta_2,l_1,l_2)}_\xi(\sigma_2 + m, \sigma_3 - \sigma_2)}{\beta(\sigma_2, \sigma_3 - \sigma_2)} \frac{t^m}{m!}, \tag{16}$$

where $|t| < 1, \min\{\Re(\delta_1), \Re(\delta_2), \Re(l_1), \Re(l_2)\} > 0, \Re(\sigma_3) > \Re(\sigma_2) > 0, \Re(\xi) \geq 0$ $l_1, l_2 \geq 1$,
and

$$\beta^{(\delta_1,\delta_2,l_1,l_2)}_\xi(u, v) = \int\limits_0^1 z^{u-1}(1-z)^{v-1}{}_1F_1(\delta_1; \delta_2; \frac{-\xi}{z^{l_1}(1-z)^{l_2}})dz. \tag{17}$$

By using equation (17) into (16), we obtain the following integral representation of generalized extended confluent hypergeometric function

$$\begin{aligned} \psi^{(\delta_1,\delta_2,l_1:l_2)}_\xi(\sigma_1, \sigma_2, \sigma_3; t) \quad &= \frac{1}{\beta(\sigma_2, \sigma_3 - \sigma_2)} \int_0^1 \mathrm{t}^{\sigma_2-1}(1-\mathrm{t})^{\sigma_3-\sigma_2-1} e^{zt} \\ &\times {}_1F_1(\delta_1; \delta_2; \frac{-\xi}{t^{l_1}(1-t)^{l_2}})dt. \end{aligned} \tag{18}$$

Srivastava *et al.* [26], introduced the generalized extended Gauss hypergeometric as follow:

$$F_{\xi}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_1,\sigma_2,\sigma_3;t)=\sum_{m=0}^{\infty}(\sigma_1)_m\frac{\beta_{\xi}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+m,\sigma_3-\sigma_2)}{\beta(\sigma_2,\sigma_3-\sigma_2)}\frac{t^m}{m!},\tag{19}$$

where $|t|<1,\min\{\Re(\delta_1),\Re(\delta_2),\Re(l_1),\Re(l_2)\}>0,\Re(\sigma_3)>\Re(\sigma_2)>0,\Re(\sigma_1)>0,\omega\geq 0,l_1,l_2\geq 1$.

By using equation (17) into (19), we obtain the following integral representation of generalized extended confluent hypergeometric function

$$\begin{aligned}F_{\xi}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_1,\sigma_2,\sigma_3;t) \quad &= \frac{1}{\beta(\sigma_2,\sigma_3-\sigma_2)}\int_0^1 \mathrm{t}^{\sigma_2-1}(1-\mathrm{t})^{\sigma_3-\sigma_2-1}\\ &\times(1-z\mathrm{t})^{-\sigma_1}{}_1F_1(\delta_1;\delta_2;\frac{-\xi}{t^{l_1}(1-t)^{l_2}})dt.\end{aligned}\tag{20}$$

Khan *et al.* [1] introduced the extension of Kummer's first formula as follows:

$$\psi_{\xi}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;t)=\exp(t)\psi_{\xi}^{(\delta_1,\delta_2,l_1;l_2)}(\sigma_3-\sigma_2,\sigma_3;-t).\tag{21}$$

Remark 1. *If we take* $l_1=l_2$ *into* (16), (17), (18), (19), (20), *we get the extended Gauss, confluent and beta function and their integral representations respectively which introduced by Parmar [24]. Further if we take* $\sigma_1=\sigma_2$ *and* $l_1=l_2=1$ *into* (16), (17), (18), (19), (20), *we get the extended Gauss, confluent, beta functions and their integral representations respectively which introduced by Chaudhry* et al. *[23]. Further if we take* $l_1=l_2=1$ *into* (16), (17), (18), (19), (20), *we get extension of gauss, confluent beta functions and their integral representations respectively which was defined by Özergin* et al. *[25].*

For some $p>0$, Mellin transform introduced by Mellin in 1897 (see [35]) as

$$M[f(s);p]=F(p)=\int\limits_0^{\infty}s^{p-1}f(s)ds.\tag{22}$$

The Laplace trasformation of $f(v)$ for $\Re(s)>0$ is defined (see [35]) as

$$F(s)=L\{f(v)\}=\int_0^{\infty}e^{-sv}f(v)dv.\tag{23}$$

The Riemann-Liouville k-fractional integral of order $-\mu$ is defined as follows (see [3])

$${}_k\Im_v^{-\mu}(f(v))=\frac{1}{k\Gamma_k(-\mu)}\int_0^v(v-t)^{\frac{-\mu}{k}-1}f(t)dt,\quad\left(k\in\mathbb{R}^+;\Re(\mu)<0\right).\tag{24}$$

Remark 2. *If we take* $k=1$ *into* (24), *then* k*-Riemann-Liouville fractional integral of order* $-\mu$ *is reduced to Riemann-Liouville fractional integral of order* $-\mu$ *which defined in [36].*

The classical Whitttaker function introduced by Whittaker [29] is

$$M_{\tau,p}(z) = z^{p+1/2} \exp\left(-\frac{z}{2}\right) \phi\left(p - \tau + \frac{1}{2}; 2p+1; z\right), \tag{25}$$

where $\Re(p) > -\frac{1}{2}; \Re(p \pm \tau) > -\frac{1}{2}$.

Nagar *et al.* [30], investigated the extended whittaker function as follows

$$M_{\lambda,\tau,p}(z) = z^{p+1/2} \exp\left(-\frac{z}{2}\right) \phi_{\lambda}\left(p - \tau + \frac{1}{2} : 2p+1 : z\right), \tag{26}$$

where $\lambda \geq 0, \Re(p) > -\frac{1}{2}; \Re(p \pm \tau) > -\frac{1}{2}$.

The generalized extended Whittaker function is defined as in [1]

$$M_{\xi,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z) = z^{\mu+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) \phi_{\xi}^{(\delta_1,\delta_2,l_1:l_2)}(\mu - p + \frac{1}{2} : 2\mu+1 : z), \tag{27}$$

where $l_1, l_2 \geq 1, \xi \in R_0^+, \Re(\mu) > -\frac{1}{2}, \Re(\mu \pm p) > -1/2, z \in \mathbb{C}|_{(-\infty,0]}, \Re(\delta_1) > 0, \Re(\delta_2) > 0$.

In [34], Savita Panwar and Prakriti Rai introduced Whittaker k-function as

$$M_{\tau,p,k}(z) = z^{p+1/2} \exp\left(-\frac{z}{2}\right) {}_1F_{1,k}\left(p - \tau + \frac{1}{2} : 2p+1 : z\right), \tag{28}$$

where $\Re(p) > -\frac{1}{2}$; $\Re(p \pm \tau) > -\frac{1}{2}$, $z \in \mathbb{C}|_{(-\infty,0]}$.

Remark 3. *If we take k=1 into* (28) *then* (28)*, reduces to* (25)*.*

By keeping in view the direction of the researchers in the field of special functions, we generalize some known functions like confluent hypergeometric, and Whittaker functions, prove some of their related properties as follows:

2. Generalized Extended Confluent Hypergeometric And Beta k-Function

In this section, first we generalize the beta function in terms of new parameter $k > 0$, then we use definition of these generalized extended beta k-function to generalize the confluent hypergeometric in term of $k > 0$. We also investigate some properties like as integral representation, Mellin transforms, inverse Mellin transforms, Laplace transformation and derivative of these new generalized extended confluent hypergeometric k-functions.

Definition 1. *If* $k > 0$, $\min\{\Re(\delta_1), \Re(\delta_2), \Re(l_1), \Re(l_2)\} > 0, \Re(\xi) \geq 0, \Re(p), \Re(q) > 0, l_1, l_2 \geq 1$, *then we define the generalized extended beta k-function as*

$$\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(p,q) \qquad = \frac{1}{k}\int_0^1 s^{\frac{p}{k}-1}(1-s)^{\frac{q}{k}-1}$$

$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{ks^{l_1}(1-s)^{l_2}})ds. \tag{29}$$

Definition 2. *By using above definition* (29)*, we extend the confluent hypergeometric function in terms of new parameter $k>0$ as follows:*
If $\min\{\Re(\delta_1),\Re(\delta_2),\Re(l_1),\Re(l_2)\}>0,\Re(\xi)\geq 0,\Re(p),\Re(q)>0,l_1,l_2\geq 1$*, then*

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;t)=\sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+mk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{t^m}{m!}. \tag{30}$$

Remark 4. *If we take $k=1$, then* (29)*,* (30) *reduces to generalize extended beta and confluent hypergeometric functions which defined by Khan* et al. *[1]. If we take $k=1$ and $l_1=l_2$ into* (29)*,* (30)*, we get extended beta and confluent hypergeometric functions which introduced by Parmar [24]. If we take $\sigma_1=\sigma_2$ and $l_1=l_2=1$ and $k=1$ into* (29)*,* (30)*, we get extended beta and confluent hypergeometric functions which introduced by Chaudhry* et al. *[22, 23]. Further if we take k=1 and $l_1=l_2=1$ into* (29)*,* (30)*, we get extension of beta and confluent hypergeometric functions which defined by Özergin* et al. *[25].*

3. Integral Representations of Generalized Extended Confluent Hypergeometric k-Function

Theorem 1. *If $k>0$, $\min\{\Re(\delta_1),\Re(\delta_2),\Re(l_1),\Re(l_2)\}>0,\omega\geq 0,\Re(\sigma_3)>\Re(\sigma_2)>0$, then following integral representations hold true*

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z) = \frac{1}{k\beta_k(\sigma_2,\sigma_3-\sigma_2)}\int_0^1 s^{\frac{\sigma_2}{k}-1}(1-s)^{\frac{\sigma_3-\sigma_2}{k}-1}\exp(zs) \times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{ks^{l_1}(1-s)^{l_2}})ds \tag{31}$$

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z) = \frac{1}{k\beta_k(\sigma_2,\sigma_3-\sigma_2)}\int_0^\infty \frac{(u)^{\frac{\sigma_2}{k}-1}}{(1+u)^{\frac{\sigma_3}{k}}}\exp(\frac{zu}{1+u}) \times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k(1+u)^{l_1+l_2}}{ku^{l_1}})ds \tag{32}$$

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z) = \frac{2}{k\beta_k(\sigma_2,\sigma_3-\sigma_2)}\int_0^{\frac{\pi}{2}} \cos^{\frac{2\sigma_2}{k}-1}\theta\sin^{\frac{2(\sigma_3-\sigma_2)}{k}-1}\theta\exp(z\cos^2\theta) \times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k\sec^{2l_1}\theta\csc^{2l_2}\theta}{k})d\theta \tag{33}$$

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z) = \frac{2}{k\beta_k(\sigma_2,\sigma_3-\sigma_2)}\int_0^{\frac{\pi}{4}} \tanh^{\frac{2\sigma_2}{k}-1}\theta sech^{\frac{2(\sigma_3-\sigma_2)}{k}}\theta\exp(z\tanh^2\theta)$$

$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k \coth^{2l_1}\theta \cosh^{2l_2}\theta}{k})d\theta \tag{34}$$

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z) = \frac{(q-p)^{1-\frac{\sigma_3}{k}}}{k\beta_k(\sigma_2,\sigma_3-\sigma_2)}\int\limits_p^q (u-p)^{\frac{\sigma_2}{k}-1}(q-u)^{\frac{\sigma_3-\sigma_2}{k}-1}\exp\left[\frac{z(u-p)}{q-p}\right]$$
$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k(q-p)^{l_1+l_2}}{k(u-p)^{l_1}(q-u)^{l_2}})du \tag{35}$$

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z) = \frac{2^{1-\frac{\sigma_3}{k}}}{k\beta_k(\sigma_2,\sigma_3-\sigma_2)}\int\limits_{-1}^1 (1+u)^{\frac{\sigma_2}{k}-1}(1-u)^{\frac{\sigma_3-\sigma_2}{k}-1}\exp\left[\frac{z(u+1)}{2}\right]$$
$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k(2)^{l_1+l_2}}{k(u+1)^{l_1}(1-u)^{l_2}})du. \tag{36}$$

Proof. From equation (29), we have

$$\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(r_2+mk,r_3-r_2) = \frac{1}{k}\int\limits_0^1 s^{\frac{r_2}{k}+m-1}(1-s)^{\frac{r_3-r_2}{k}-1}$$
$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{ks^{l_1}(1-s)^{l_2}})ds. \tag{37}$$

By using equation (37) into (30) and by changing the order of integration and summation, we get

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z) = \sum_{m=0}^{\infty} k\beta_k(\sigma_2,\sigma_3-\sigma_2)\int\limits_0^1 s^{\frac{r_2}{k}+m-1}(1-s)^{\frac{r_3-r_2}{k}-1}$$
$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{ks^{l_1}(1-s)^{l_2}})\frac{z^m}{m!}ds$$
$$= \frac{1}{k\beta_k(\sigma_2,\sigma_3-\sigma_2)}\int\limits_0^1 s^{\frac{\sigma_2}{k}-1}(1-s)^{\frac{\sigma_3-\sigma_2}{k}-1}\sum_{m=0}^{\infty}\frac{(zs)^m}{m!}$$
$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{ks^{l_1}(1-s)^{l_2}})ds$$
$$= \frac{1}{k\beta_k(\sigma_2,\sigma_3-\sigma_2)}\int\limits_0^1 s^{\frac{\sigma_2}{k}-1}(1+s)^{\frac{\sigma_3-\sigma_2}{k}-1}\exp(zs)$$
$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{ks^{l_1}(1-s)^{l_2}})ds.$$

which is (31) further by putting $s=\frac{u}{1+u}$, $s=\cos^2\theta$, $s=\tanh^2\theta$, $s=\frac{u-p}{q-p}$ and $s=\frac{u+1}{2}$ into equation (31), we get equations (32), (33), (34), (35), and (36)

Remark 5. *If we put $k=1$ into equations* (31), (32), (33), (34), (35), (36), *we get integral representations of generalized extended confluent hypergeometric functions.*

4. Mellin Transform And Transformation Formula of Generalized Extended Confluent Hypergeometric k-Function

Theorem 2. *If $k>0$, $\Re(r)>0$, $\Re(\delta_1+r)>0$, $\Re(\delta_2+r)>0$, $\Re(\xi)\geq 0$, $\Re(\delta_1)>0$, $\Re(\delta_2)>0$, $l_1,l_2\geq 1$, then following Mellin tranformations holds true*

$$\int_0^\infty \xi^{r-1}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z)d\xi = \frac{\Gamma_k^{(\delta_1,\delta_2)}(\lambda)\beta_k(\sigma_2+l_1r,\sigma_3-\sigma_2+l_2r)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)} \times {}_1F_{1,k}(\sigma_2+l_1r,\sigma_3+(l_1+l_2)r;z). \tag{38}$$

Proof. Multiplying equation (30) by ξ^{r-1} on both sides and integrate w.r.t ξ from $\xi=0$ to $\xi=\infty$ and by changing the order of integraton and summation, we get

$$\int_0^\infty \xi^{r-1}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z)d\xi = \frac{1}{k\beta_k(\sigma_2,\sigma_3-\sigma_3)}\int_0^1 s^{\frac{\sigma_2}{k}-1}(1-s)^{\frac{\sigma_3-\sigma_2}{k}-1}\exp(zs)ds \times \int_0^\infty \xi^{r-1}{}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{ks^{l_1}(1-s)^{l_2}})d\xi. \tag{39}$$

By substituting $\lambda=\frac{\xi}{s^{\frac{l_1}{k}}(1-s)^{\frac{l_2}{k}}}$ into (39), we get

$$\int_0^\infty \xi^{r-1}{}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{ks^{l_1}(1-s)^{l_2}})d\xi = s^{\frac{rm_1}{k}}(1-s)^{\frac{n_1r}{k}}\int_0^\infty \lambda^{r-1}{}_1F_{1,k}(\sigma_1,\sigma_2;\frac{-\lambda^k}{k})d\lambda = s^{\frac{l_1r}{k}}(1-s)^{\frac{n_1r}{k}}\Gamma_k^{(\sigma_1,\sigma_2)}(\lambda). \tag{40}$$

By substituting equation (40) into equation (39), we get

$$= \frac{\Gamma_k^{(p_1,q_1)}(\lambda)}{k\beta_k(\sigma_2,\sigma_3-\sigma_3)}\int_0^1 s^{\frac{\sigma_2+rl_1}{k}-1}(1-s)^{\frac{(\sigma_3-\sigma_2)+rl_2}{k}-1}\exp(zs)ds$$
$$= \frac{\Gamma_k^{(\delta_1,\delta_2)}(\lambda)\beta_k(\sigma_2+l_1r,\sigma_3-\sigma_2+l_2r)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}{}_1F_{1,k}(\sigma_2+l_1r,\sigma_3+(l_1+l_2)r;z).$$

Corollary 1. *By the Mellin inversion formula, we have following integral of genetalized extended confluent hypergeometric k-function*

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;z) = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{\Gamma_k^{(\delta_1,\delta_2)}(\lambda)\beta_k(\sigma_2+l_1r,\sigma_3-\sigma_2+l_2r)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}$$

$$\times {}_1F_{1,k}(\sigma_2 + l_1 r, \sigma_3 + (l_1 + l_2)r; z)\xi^{-r} d\xi. \tag{41}$$

Proof. By taking Mellin inverse of equation (38) on both sides, we get the required result.

Remark 6. *If we take $k = 1$ into equation (38), (41), then we get Mellin and inverse Mellin transforms of generalized extended confluent hypergeometric function which introduced by Khan et al. [1].*

Theorem 3. *If $k > 0$, $\min\{\Re(\delta_1), \Re(\delta_2), \Re(l_1), \Re(l_2)\} > 0, \Re(\xi) \geq 0, \Re(\delta_2), \Re(\delta_3) > 0, l_1, l_2 \geq 1$, then following transformation formula holds for generalized extended confluent hypergeometric k-function*

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2, \sigma_3; t) = \exp(t)\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_3 - \sigma_2, \sigma_3; -t). \tag{42}$$

Proof. By using integral representation of generalized extended confluent hypergeometric k-function, we have

$$\begin{aligned}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2, \sigma_3; z) \quad &= \frac{1}{k\beta_k(\sigma_2, \sigma_3 - \sigma_2)} \int_0^1 s^{\frac{\sigma_2}{k}-1}(1-s)^{\frac{\sigma_3-\sigma_2}{k}-1} \exp(zs) \\ &\times {}_1F_{1,k}(\delta_1; \delta_2; \frac{-\xi^k}{ks^{l_1}(1-s)^{l_2}})ds.\end{aligned}$$

Replace s by $s - 1$, we get the required result.

Remark 7. *If we put $k = 1$ into equation (42), we get generalized extended Kummar's first fomula defined by Khan et al. [1]. Further if we take m=n, then (42) reduces to extended Kummar's first fomula defined by Parmar in [24]. If we take $\xi = 0$, then we get classical Kummar's first fomula.*

5. Laplace Transformation of Generalized Extended Confluent Hypergeometric k-Function

Theorem 4. *If $k > 0, \min\{\Re(\delta_1), \Re(\delta_2), \Re(l_1), \Re(l_2)\} > 0, \omega \geq 0, \Re(\sigma_3) > \Re(\sigma_2) > 0, \Re(d) > 0$, then*

$$\begin{aligned}\int_0^\infty e^{-st} t^{d-1} \psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2, \sigma_3; vt)dt \quad &= \sum_{m=0}^\infty \frac{\Gamma(m+d)}{s^{m+d}} \\ &\times \frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2 + mk, \sigma_3 - \sigma_2)}{\beta_k(\sigma_2, \sigma_3 - \sigma_2)} \frac{v^m}{m!}.\end{aligned} \tag{43}$$

Proof. Consider left hand side of equation (43), we have

$$\int_0^\infty e^{-st}t^{d-1}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;vt)dt = \int_0^\infty e^{-st}t^{d-1}\sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+mk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)} \times\frac{v^m t^m}{m!}dt.$$

By changing the order of integration and summation, we have

$$\begin{aligned}\int_0^\infty e^{-st}t^{d-1}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;vt)dt &= \sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+mk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{v^m}{m!}\int_0^\infty e^{-st}t^{m+d-1}dt\\ &= \sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+mk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{v^m}{m!}\frac{\Gamma(m+d)}{s^{m+d}}\\ &= \sum_{m=0}^{\infty}\frac{\Gamma(m+d)}{s^{m+d}}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+mk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{v^m}{m!}.\end{aligned}$$

Theorem 5. *If* $k>0, \min\{\Re(\delta_1),\Re(\delta_2),\Re(l_1),\Re(l_2)\}>0, \omega\geq 0, \Re(\sigma_3)>\Re(\sigma_2)>0, \Re(d)>0,$ *then*

$$\int_0^\infty e^{-st}t^{\frac{d}{k}-1}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;vt)dt = \sum_{m=0}^{\infty}\frac{(\sigma_1)_{m,k}\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+mk,\sigma_3-\sigma_2)\Gamma_k(d+mk)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)k^{\frac{d}{k}+m-1}s^{\frac{d}{k}+m}} \times\frac{v^m}{m!}. \tag{44}$$

Proof. Consider left hand side of equation (44), we have

$$\int_0^\infty e^{-st}t^{\frac{d}{k}-1}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;vt)dt = \int_0^\infty e^{-st}t^{\frac{d}{k}-1}\sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+mk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{v^m t^m}{m!}dt.$$

By changing the order of integration and summation, we have

$$\begin{aligned}\int_0^\infty e^{-st}t^{\frac{d}{k}-1}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;vt)dt &= \sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+mk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{v^m}{m!}\int_0^\infty e^{-st}t^{\frac{d}{k}+m-1}dt\\ &= \sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+mk,\sigma_3-\sigma_2)\Gamma_k(\frac{d}{k}+m)k}{\beta_k(\sigma_2,\sigma_3-\sigma_2)k^{\frac{d}{k}+m-1}s^{\frac{d}{k}+m}}\frac{v^m}{m!}.\end{aligned}$$

6. Derivative of Generalized Extended Confluent Hypergeometric k-Function

Theorem 6. *If* $k > 0, \min\{\Re(\delta_1), \Re(\delta_2), \Re(l_1), \Re(l_2)\} > 0, \omega \geq 0, \Re(\sigma_3) > \Re(\sigma_2) > 0, l_1, l_2 \geq 1$, *then*

$$\frac{d^l}{dv^l}[\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;v)] = \frac{(\sigma_2)_{l,k}}{(\sigma_3)_{l,k}}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2+lk,\sigma_3+lk;v). \tag{45}$$

Proof. From equation (30), we have

$$\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;v) = \sum_{l=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+lk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{v^l}{l!}. \tag{46}$$

After taking derivative of (46) w.r.t v and simlification, we obtain

$$\begin{aligned}\frac{d}{dv}[\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;v)] &= \sum_{l=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+lk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{lv^{l-1}}{l!}\\ &= \sum_{l=1}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+lk,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{v^{l-1}}{(l-1)!}.\end{aligned}$$

By replacing l by *l+1*, we get

$$\frac{d}{dv}[\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;v)] = \sum_{l=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+(l+1)k,\sigma_3-\sigma_2)}{\beta_k(\sigma_2,\sigma_3-\sigma_2)}\frac{v^l}{l!} \tag{47}$$

By using following property

$$\beta_k(\sigma_2,\sigma_3-\sigma_2) = \frac{(\sigma_3)_k}{(\sigma_2)_k}\beta_k(\sigma_2+k,\sigma_3-\sigma_2),$$

then equation (47) can be written as

$$\frac{d}{dv}[\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;v)] = \frac{(\sigma_2)_k}{(\sigma_3)_k}\sum_{l=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\sigma_2+(l+1)k,\sigma_3-\sigma_2)}{\beta_k(\sigma_2+k,\sigma_3-\sigma_2)}\frac{v^l}{l!}.$$

Suppose result is true for *l-1*, then

$$\frac{d^{l-1}}{dv^{l-1}}[\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;v)] = \frac{(\sigma_2,)_{l-1,k}}{(\sigma_3)_{l-1,k}} \tag{48}$$

$$\times\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2+(l-1)k,\sigma_3+(l-1)k;v).$$

By taking derivative of equation (48) w.r.t. v, we get

$$\begin{aligned}\frac{d^l}{dv^l}[\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2,\sigma_3;v)] &= \frac{(\sigma_2)_{l-1,k}(\sigma_2+(l-1)k)}{(\sigma_3)_{l-1,k}(\sigma_3+(l-1)k)}\\ &\times\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_1+lk,\sigma_2+lk,\sigma_3+lk;v)\\ &= \frac{(\sigma_2)_{l,k}}{(\sigma_3)_{l,k}}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\sigma_2+lk,\sigma_3+lk;v).\end{aligned}$$

Remark 8. *If we take $k=1$ into equation* (45)*, we get derivative of generalized extended confluent hypergeometric function which investigated by Khan* et al. *[1].*

7. Generalized Extended Whittaker k-Function

In this section, we introduce generalized extended whittaker k-function with the help of generalized extended confluent hypergeometric k-function. Further, we investigate Mellin transforms, Hankel transformation, Laplace transformation, fractional integral and derivative of these new generalized extended Whittaker k-function.

Definition 3. *If $\xi\geq 0, l_1,l_2\geq 1, \Re(\delta_1),\Re(\delta_2)>0, k>0, \Re(\mu)>-\frac{1}{2}, \Re(\mu+p)>-\frac{1}{2}, \Re(\mu-p)>-\frac{1}{2}$,then generalized extended Whittaker k-function we define as follows*

$$M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z) = z^{\mu+\frac{1}{2}}\exp\left(-\frac{z}{2}\right)\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\mu-p+\frac{1}{2};2\mu+1;z). \tag{49}$$

where $\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(u,v;z)$ is generalized extended confluent hypergeometric k-function which is defined in (30)*.*

Remark 9. *If we take $k=1$ into* (49)*, we get generalized extended Whittaker function defined by Khan* et al. *[1]. If we take $l_2=1$, then we get extended Whittaker function defined in [32],also see[33]. Further by putting $\delta_1=\delta_2$, and $l_1=l_2$, we get extended Whittaker function which investigated by Khan and Ghayasuddin in [31]. Further for $l_2=1$, we get extended Whittaker due to Nagar* et al. *[30]. For $\xi=0$ gives classical Whittaker function defined in [29]*

8. Integral Representation of Generalized Extended Whittaker k-Function

Theorem 7. *If $\xi\geq 0, \Re(\mu+p)>-\frac{1}{2}, \Re(\mu-p)>-\frac{1}{2}), l_1,l_2\geq 1, \Re(\delta_1),\Re(\delta_2)>0, k>0$, then following integral representations hold true*

$$M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z) = \frac{z^{\mu+\frac{1}{2}}\exp(\frac{-z}{2})}{k\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})}\int_0^1 t^{\frac{\mu-p+\frac{1}{2}}{k}-1}(1-t)^{\frac{\mu+p+\frac{1}{2}}{k}-1}\exp(zt)$$

$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{kt^{l_1}(1-t)^{l_2}})dt. \tag{50}$$

$$M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z) = \frac{(q-s)^{1-\frac{2\mu+1}{k}}z^{\mu+\frac{1}{2}}\exp(\frac{-z}{2})}{k\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})}\int_s^q (u-s)^{\frac{\mu-p+\frac{1}{2}}{k}-1}(q-u)^{\frac{\mu+p+\frac{1}{2}}{k}-1}$$
$$\times \exp\left[\frac{z(u-s)}{q-s}\right] {}_1F_{1,k}(\delta_1;\delta_2;\frac{-(q-s)^{l_1+l_2}\xi^k}{k(u-s)^{l_1}(q-u)^{l_2}})du. \tag{51}$$

$$M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z) = \frac{z^{\mu+\frac{1}{2}}\exp(\frac{-z}{2})}{k\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})}\int_0^\infty u^{\frac{\mu-p+\frac{1}{2}}{k}-1}(u+1)^{\frac{-(2\mu+1)}{k}}$$
$$\times \exp(\frac{zu}{1+u}){}_1F_{1,k}(p_1,q_1;\frac{-\xi^k(u+1)^{l_1+l_2}}{ku^{l_1}})du. \tag{52}$$

$$M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z) = \frac{(2)^{1-\frac{2\mu+1}{k}}z^{\mu+\frac{1}{2}}}{k\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})}\int_{-1}^1 (u+1)^{\frac{\mu-p+\frac{1}{2}}{k}-1}(1-u)^{\frac{\mu+p+\frac{1}{2}}{k}-1}$$
$$\times \exp(\frac{zu}{2}){}_1F_{1,k}(\delta_1;\delta_2;\frac{-(2)^{l_1+l_2}\xi^k}{k(u+1)^{l_1}(1-u)^{l_2}})du. \tag{53}$$

$$M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z) = \frac{z^{\mu+\frac{1}{2}}\exp(\frac{z}{2})}{k\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})}\int_0^1 (1-u)^{\frac{\mu-p+\frac{1}{2}}{k}-1}(u)^{\frac{\mu+p+\frac{1}{2}}{k}-1}\exp(-zu)$$
$$\times {}_1F_{1,k}(\delta_1;\delta_2;\frac{-\xi^k}{k(1-u)^{l_1}(u)^{l_2}})du. \tag{54}$$

Proof. By using equation (31) in (49), we get (50) and further by putting $t=\frac{u-s}{q-s}$, $t=\frac{u}{1+u}$, $t=1-u$, in (50), we get (51), (52), (54). If we take $s=-1$ and $q=1$ in (51), we get (53)

Remark 10. *If we take* $k=1$ *in* (50), (51), (52), (53), *and* (54), *we get integral representation of generalized extended Whittaker function investigated by Khan* et al. *[1]. Further by putting* $\delta_1=\delta_2$, *and* $l_1=l_2$ *we get integral representation of extended Whittaker function which investigated by Khan and Ghayasuddin in [31], further for* $l_2=1$, *we get integral representation of extended whittaker due to Nagar* et al. *[30].*

9. Integral Transform of Generalized Extended Whittaker k-Function

Theorem 8. *If $k > 0$, $\Re(r) > 0$, $\Re(\delta_1 + r) > 0$, $\Re(\delta_2 + r) > 0$, $\Re(\xi) \geq 0$, $\Re(\delta_1) > 0$, $\Re(\delta_2) > 0$, $l_1, l_2 \geq 1$, $\Re(\mu + l_2 r + k) > \frac{-1}{2}$, $R(\mu + l_1 r - k) > \frac{-1}{2}$, then following Mellin transforms holds true*

$$\int_0^\infty \xi^{r-1} M^{(\delta_1,\delta_2,l_1,l_2)}_{\xi,k,p,\mu}(z)d\xi = \frac{\Gamma^{(\delta_1,\delta_2)}_k(r) z^{\mu+\frac{1}{2}} \exp(\frac{-z}{2}) \beta_k(\mu - p + \frac{1}{2} + l_1 r, \mu + p + \frac{1}{2} + l_2 r)}{\beta_k(\mu - p + \frac{1}{2}, \mu + p + \frac{1}{2})}$$
$$\times \psi_k(\mu - p + \frac{1}{2} + l_1 r, 2\mu + (l_1 + l_2)r + 1). \tag{55}$$

Proof. Consider left hand side of equation (55) then using (50) and changing the order of integration, we get

$$\int_0^\infty \xi^{r-1} M^{(\delta_1,\delta_2,l_1:l_2)}_{\xi,k,p,\mu}(z)d\xi = \frac{z^{\mu+\frac{1}{2}} \exp(\frac{-z}{2})}{k\beta_k(\mu - p + \frac{1}{2}, \mu + p + \frac{1}{2})} \int_0^1 s^{\frac{\mu-p+\frac{1}{2}}{k}-1} (1-s)^{\frac{\mu+p+\frac{1}{2}}{k}-1} \exp(zs)$$
$$\times \int_0^\infty \xi^{r-1} {}_1F_{1,k}(\delta_1; \delta_2; \frac{-\xi^k}{k s^{l_1} (1-s)^{l_2}}) d\xi.$$

By substituting $\lambda = \frac{\xi}{s^{\frac{m_1}{k}} (1-s)^{\frac{n_1}{k}}}$, we get

$$\int_0^\infty \xi^{r-1} M^{(\delta_1,\delta_2,l_1:l_2)}_{\xi,k,p,\mu}(z)d\xi = \frac{\Gamma^{(p_1,q_1)}_k(r) z^{\mu+\frac{1}{2}} \exp(\frac{-z}{2})}{k\beta_k(\mu - p + \frac{1}{2}, \mu + p + \frac{1}{2})} \int_0^1 s^{\frac{\mu-p+\frac{1}{2}+m_1 r}{k}-1} (1-s)^{\frac{\mu+p+\frac{1}{2}+n_1 r}{k}-1} \exp(zs) ds$$
$$= \frac{\Gamma^{(p_1,q_1)}_k(r) z^{\mu+\frac{1}{2}} \exp(\frac{-z}{2})}{k\beta_k(\mu - p + \frac{1}{2}, \mu + p + \frac{1}{2})} \frac{\beta_k(\mu - p + \frac{1}{2} + m_1 r, \mu + p + \frac{1}{2} + n_1 r)}{\beta_k(\mu - p + \frac{1}{2} + m_1 r, \mu + p + \frac{1}{2} + n_1 r)}$$
$$\times \int_0^1 s^{\frac{\mu-p+\frac{1}{2}+m_1 r}{k}-1} (1-s)^{\frac{\mu+p+\frac{1}{2}+n_1 r}{k}-1} \exp(zs) ds.$$

By using the integral representation of generalized extended confluent hypergeometric k-function we get the above result.

Remark 11. *If we take $k = 1$ in (55), then we get Mellin transfrmation of generalized extended confluent hypergeometric function which was investigated by Khan* et al. *[1].*

Theorem 9. *If $\xi \geq 0$, $2p > b > 0$, $\Re(a + \mu) > -\frac{1}{2}$, $k > 0$, then we have following integral transfom holds true*

$$\int_0^\infty \exp(-pz) z^{a-1} M^{(\delta_1,\delta_2,l_1:l_2)}_{\xi,k,p,\mu}(bz)dz = \frac{(b)^{\mu+\frac{1}{2}} \Gamma(a + \mu + \frac{1}{2})}{(p + \frac{b}{2})^{a+\mu+\frac{1}{2}}}$$

$$\times F_{\xi,k}^{\delta_1,\delta_2,l_1,l_2}(a+\mu+\frac{1}{2},\mu-p+\frac{1}{2};2\mu+1;\frac{2b}{2p+b}). \quad (56)$$

Proof. Consider the left hand side of equation (56), then using integral representation of generalized extended Whittaker k-function and by changing the order of interation and summation, we get

$$\int_0^\infty \exp(-pz)z^{a-1}M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(bz)dz = \sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+mk,\mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})}$$
$$\times\frac{(b)^{m+\mu+\frac{1}{2}}}{m!}\int_0^\infty \exp(-(p+\frac{b}{2})z)z^{(a+\mu+m+\frac{1}{2})-1}dz.$$

By using the following formula

$$\frac{\Gamma(u)}{s^u} = \int_0^\infty \exp(-st)t^{u-1}dt,$$

we get the above result.

Corollary 2. *If we put a=b=1 in* (56)*, then we get following integral transform*

$$\int_0^\infty \exp(-pz)M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)dz = \frac{(2)^{\mu+\frac{3}{2}}\Gamma(\mu+\frac{3}{2})}{(2p+1)^{\mu+\frac{3}{2}}}$$
$$\times F_{\xi,k}^{\delta_1,\delta_2,l_1,l_2}(\mu+\frac{3}{2},\mu-p+\frac{1}{2};2\mu+1;\frac{2}{2p+1}).$$

Theorem 10. *If $k>0$, $\Re(\mu+p)>-\frac{1}{2}$, $\Re(\mu-p)>-\frac{1}{2}$, $\Re(\mu+v)>0$, $l_1,l_2\geq 1$, then following Hankel transformation holds true*

$$\int_0^\infty zM_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)J_v(az)dz = \frac{\Gamma(\mu+v+\frac{5}{2})}{(a^2+\frac{1}{4})^{\frac{\mu}{2}+\frac{5}{4}}}\sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+mk,\mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})}$$
$$\times\frac{\Gamma(\mu+v+\frac{5}{2})_m}{(a^2+\frac{1}{4})^{\frac{m}{2}}m!}P_{\mu+m+\frac{3}{2}}^{-v}(\frac{1}{\sqrt{4a^2+1}}). \quad (57)$$

Proof. By using (26) and (46), then by changing the order of integration and summation, we get

$$\int_0^\infty zM_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)J_v(az)dz = \sum_{m=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+mk,\mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})m!}$$

$$\times \int_0^\infty z^{(\mu+m+\frac{3}{2})} \exp(\frac{-z}{2}) J_v(az)dz.$$

By using the following formula

$$\int_0^\infty \exp(-pz) z^\mu J_v(az)dz \qquad = \Gamma(\mu+v+1) r^{-\mu-1} P_\mu^{-v}(\frac{p}{r}).$$

$R(\mu+v) > -1$, $r = \sqrt{p^2+a^2}$, $P_\mu^{-v}(z)$ is Legendre function.
By taking $p = \frac{1}{2}$, $\mu = \mu + m + \frac{3}{2}$ and after simlification, we get requir result.

Theorem 11. *For generalzed extended Whittaker k-function the following relation holds true*

$$M_{\xi,p,\mu,k}^{(\delta_1,\delta_2,l_1:l_2)}(-z) \qquad = (-1)^{\mu+\frac{1}{2}} M_{\xi,k,-p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z), \tag{58}$$

where $\xi \geq 0, \Re(\mu) > -\frac{1}{2}, \Re(\mu+p) > -\frac{1}{2}, \Re(\mu-p) > -\frac{1}{2}, l_1, l_2 \geq 1.$

Proof. By replacing z by $-z$ in equation (49), we get

$$M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(-z) \qquad = (-z)^{\mu+\frac{1}{2}} \exp\left(\frac{z}{2}\right) \psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\mu-p+\frac{1}{2}, 2\mu+1; -z), \tag{59}$$

Now using (30) in (59) and after simplication, we get desired result.

10. Laplace Transformation of Generalized Extended Whittaker k-Function

Theorem 12. *If* $k > 0$, $\Re(\xi+\mu) > \frac{1}{2}$, $\Re(\xi-\mu) > \frac{1}{2}$, $l_1, l_2 \geq 1$, *then*

$$\begin{aligned} L[\exp(\frac{z}{2}) M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)] \qquad &= \sum_{m=0}^\infty \frac{\Gamma(\mu+m+\frac{3}{2})}{(s)^{\mu+m+\frac{3}{2}}} \\ &\times \frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+mk, \mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2}, \mu+p+\frac{1}{2}) \quad m!}. \end{aligned} \tag{60}$$

Proof. By using equation (49) and definition of Laplace transform, we get

$$\exp(\frac{z}{2}) M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z) \qquad = \int_0^\infty \exp(-sz) z^{\mu+\frac{1}{2}} \psi_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}; 2\mu+1; z)dz.$$

By using (26) and by changing the order of integration and summation, we have

$$= \sum_{m=0}^\infty \frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+mk, \mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2}, \mu+p+\frac{1}{2}) m!}$$

$$\times \int_0^\infty \exp(-sz) z^{(\mu+m+\frac{3}{2})-1} dz,$$

By using the definition of classical gamma function, we get desired result.

11. Riemann-Liouville Fractional Integral of Generalized Extended Whittaker k-Function

Theorem 13. *If $\Re(\lambda) < 0$, $\xi \in R_0^+$, $\Re(\mu+p) > -\frac{1}{2}$, $\Re(\mu-p) > -\frac{1}{2}$, $k > 0$, $l_1, l_2 \geq 1$, then*

$$\begin{aligned}\beth_z^\lambda[\exp(\frac{z}{2}) M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)] &= \frac{z^{\mu-\lambda+\frac{1}{2}}}{\Gamma(-\lambda)} \sum_{l=0}^\infty \frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+lk, \mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2}, \mu+p+\frac{1}{2})} \frac{z^l}{l!} \\ &\times \beta(\mu+l+\frac{3}{2}; -\lambda). \end{aligned} \tag{61}$$

Proof. By using definition of Riemann-Liouville fractional integral, we get

$$\beth_z^\lambda[\exp(\frac{z}{2}) M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)] = \frac{1}{\Gamma(-\lambda)} \int_0^z t^{\frac{1}{2}+\mu} \psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\mu-p+\frac{1}{2}, 2\mu+1; t)(z-t)^{-\lambda-1} dt.$$

Now using equation (49), then use (31) and by changing the order of integration and summation, we have

$$\begin{aligned}\beth_z^\lambda[\exp(\frac{z}{2}) M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)] &= \sum_{l=0}^\infty \frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+lk, \mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2}, \mu+p+\frac{1}{2}) l!} \\ &\times \frac{1}{\Gamma(-\lambda)} \int_0^z t^{\mu+l+\frac{1}{2}} (z-t)^{-\lambda-1} dt.\end{aligned}$$

By substituting $t = vz$, we get

$$\begin{aligned}\beth_z^\lambda[\exp(\frac{z}{2}) M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)] &= \frac{z^{\mu-\lambda+\frac{1}{2}}}{\Gamma(-\lambda)} \int_0^1 v^{\mu+l+\frac{1}{2}} (1-v)^{-\lambda-1} dv \\ &\times \sum_{l=0}^\infty \frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+lk, \mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2}, \mu+p+\frac{1}{2})} \frac{z^l}{l!},\end{aligned}$$

By using definition of beta function, we get desired result.

12. Riemann-Liouville k-fractional Integral of Generalized Extended Whittaker k-Function

Theorem 14. *If $k > 0$, $\Re(\lambda) < 0$, $\xi \in R_0^+$, $\Re(\mu + p) > -\frac{1}{2}$, $\Re(\mu - p) > -\frac{1}{2}$, $l_1, l_2 \geq 1$, then*

$$\begin{aligned} {}_k\beth_z^{\lambda}[\exp(\frac{z}{2})M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)] &= \frac{z^{\mu-\frac{\lambda}{k}+\frac{1}{2}}}{k\Gamma_k(-\lambda)}\sum_{l=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+lk,\mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})}\frac{z^l}{l!} \\ &\times\beta(\mu+l+\frac{3}{2};\frac{-\lambda}{k}). \end{aligned} \tag{62}$$

Proof. By using definition of Riemann-Liouville k-fractional integral, we get

$$ {}_k\beth_z^{\lambda}[\exp(\frac{z}{2})M_{\xi,p,\mu,k}^{(\delta_1,\delta_2,l_1:l_2)}(z)] = \frac{1}{k\Gamma_k(-\lambda)}\int_0^z t^{\frac{1}{2}+\mu}\psi_{\xi,k}^{(\delta_1,\delta_2,l_1:l_2)}(\mu-p+\frac{1}{2},2\mu+1;t)(z-t)^{\frac{-\lambda}{k}-1}dt.$$

First use equation (49), then using (31) and by changing the order of integration and summation, we have

$$\begin{aligned} {}_k\beth_z^{\lambda}[\exp(\frac{z}{2})M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)] &= \sum_{l=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+lk,\mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})l!}. \\ &\times\frac{1}{k\Gamma_k(-\lambda)}\int_0^z t^{\mu+l+\frac{1}{2}}(z-t)^{\frac{-\lambda}{k}-1}dt. \end{aligned}$$

By substituting $t = vz$, we get

$$\begin{aligned} {}_k\beth_z^{\lambda}[\exp(\frac{z}{2})M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)] &= \frac{z^{\mu-\frac{\lambda}{k}+\frac{1}{2}}}{k\Gamma_k(-\lambda)}\int_0^1 v^{\mu+l+\frac{1}{2}}(1-v)^{\frac{-\lambda}{k}-1}dv \\ &\times\sum_{l=0}^{\infty}\frac{\beta_{\xi,k}^{(\delta_1,\delta_2,l_1,l_2)}(\mu-p+\frac{1}{2}+lk,\mu+p+\frac{1}{2})}{\beta_k(\mu-p+\frac{1}{2},\mu+p+\frac{1}{2})}\frac{z^l}{l!}. \end{aligned}$$

By using definition of beta k-function, we get desired result.

13. Derivative of Generalized Extended Whittaker k-Function

Theorem 15. *If $\xi \geq 0, \Re(\mu+p) > -\frac{1}{2}, \Re(\mu-p) > -\frac{1}{2}), l_1, l_2 \geq 1, \Re(\delta_1), \Re(\delta_2) > 0, k > 0$, then*

$$\begin{aligned} \frac{d^l}{dz^l}[\exp(\frac{z}{2})z^{-\mu-\frac{1}{2}}M_{\xi,k,p,\mu}^{(\delta_1,\delta_2,l_1:l_2)}(z)] &= \frac{(\mu-p+\frac{1}{2})_{l,k}}{(2\mu+1)_{l,k}} \\ &\times\exp(\frac{z}{2})z^{-\mu-\frac{lk}{2}-\frac{1}{2}}M_{\xi,p-\frac{lk}{2},\mu+\frac{lk}{2},k}(z). \end{aligned} \tag{63}$$

Proof. From equation (45), we have

$$\frac{d^l}{dv^l}[\psi^{(\delta_1,\delta_2,l_1:l_2)}_{\xi,k}(\sigma_2,\sigma_3;v)] \qquad = \frac{(\sigma_2)_{l,k}}{(\sigma_3)_{l,k}}\psi^{(\delta_1,\delta_2,l_1:l_2)}_{\xi,k}(\sigma_2+lk,\sigma_3+lk;v). \tag{64}$$

$$\begin{aligned} M^{(\delta_1,\delta_2,l_1:l_2)}_{\xi,p,\mu,k}(z) &= (z)^{\mu+\frac{1}{2}}\exp\left(-\frac{z}{2}\right) \\ &\times\psi^{(\delta_1,\delta_2,l_1:l_2)}_{\xi,k}(\mu-p+\frac{1}{2},2\mu+1:z). \end{aligned} \tag{65}$$

Now consider left hand side of (63) and using definition of generalized extended Whittaker k-function, we have

$$\frac{d^l}{dz^l}[\exp(\frac{z}{2})z^{-\mu-\frac{1}{2}}M^{(\delta_1,\delta_2,l_1:l_2)}_{\xi,p,\mu,k}(z)] \qquad = \frac{d^l}{dz^l}[\psi^{(\delta_1,\delta_2,l_1:l_2)}_{\xi,k}(\mu-p+\frac{1}{2},2\mu+1+:z)].$$

Now applying (64) in above equation and after simplification, we get desired result.

14. Conclusion

In this paper, we have introduced the generalizations of extended hypergeometric, beta, and Whittaker functions in terms of new parameter $k>0$. We have also proved some integral representations, Mellin transforms, Laplace transformation, Hankel transformation, Riemann-Liouville fractional and Riemann-Liouville k-fractional integral for these generalized extended k-functions. We have also investigated the derivative of generalized extended hypergeometric and Whittaker k-functions. In future, further we generalized the Confluent and Whittaker functions by introducing another parameters. We can define a fractional operator by using above generalization.

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