EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6282
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Fixed Point Results on Interpolative Metric Spaces via Simulation Functions

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Abstract. The aim of this paper is to bring generalized fixed point theorems via the interesting simulation functions in the context of interpolative metric spaces. The notion of interpolative metric space is introduced by Karapinar[6], very recently. This new metric structure is a concrete generalization of the usual metric space which is indicated by examples.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Interpolative metric space, metric space, simulation functions

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1. Introduction and Preliminaries

The last century has witnessed so many developments in mathematics, of such great beauty and quality, that it seems impossible for one person to grasp, collect and explain them. Very serious and important contributions have been made from every field of mathematics, especially in analysis, number theory and applied mathematics. In this article, we will focus on the fixed point theorem, which can be considered a drop in this mathematical ocean.

Fixed point theorem is actually a very important field of study that is the common denominator of both topology, analysis and applied mathematics. In fact, the concept of fixed point is a very natural concept that we encounter in every area of life; therefore, it has a very wide potential for study and application. In this study, we will limit ourselves

DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6282

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to metric fixed points. The first theorem historically presented is the Brouwer fixed-point theorem Brouwer. It guarantees that every continuous function has a fixed point on a closed interval. On the other hand, it does not discuss how this point can be found or whether it is unique. This theorem's topology structure is emphasized, not its metric structure; that is why it is not seen as the first metric fixed point.

In the literature, the first metric fixed point theory is attributed to the famous and genius Polish mathematician S. Banach [1]. In addition, some researchers claim that the theorem put forward by Banach was actually implicitly expressed by Picard. Indeed, the "successive approximation" method used by Picard [11] is the backbone of the proof of the Banach fixed-point theorem. That is why we come across sources where the expression Picard-Banach fixed point theorem is used. On the other hand, the theorem put forward by Banach is constructed within a complete normed space. The famous Italian mathematician Caccioppoli [5] contributed to its current form. That's why it is expressed as the Banach-Caccioppoli fixed point in some sources.

It is very useful to underline that the Banach-Caccioppoli theorem is actually a generalization study: The expression "every contraction has a single fixed point" was first expressed in a complete norm space, but now it has been generalized as a complete metric space. Due to the nature of mathematics and human beings, every concept, every structure is tried to be expanded and developed. The subject of this article is the attempt to add a final link to such generalizations, see e.g. [6–9].

We underline that the Picard's successive approximation method should be accepted as an initial application for the usage of fixed point hypothesis. In other words, Picard result is exceptionally critical and basic to indicate the connection of the theory with the applications in mathematics. Indeed, the metric fixed point has a key role not only in solving mathematics problems, but in nearly all quantitative sciences. In other words, the metric fixed-point techniques are very crucial in numerous research areas. Subsequently, this theory has been alluring to several mathematicians. [3, 8, 9, 12], see also [2, 10, 14, 16, 19]. It turned out that numerous publications in the related literature overlap the published results, whereas others were identical to the results in the existing publications. For more details and concrete examples about this discussion, see e.g. [8] and related references therein.

On the other hand, as a nature of science and in particular mathematics, researchers have been continuing to improve, generalize and extend the existing results. Indeed, the advancement of the hypothesis has two primary cornerstone: The first one is to debilitate, generalize, and progress the definition of the existing contraction. The other is to expand the construction of the abstract space where the presence of the fixed point is examined and questioned. The examples of the second trend, one can list quasi-metric spaces, modular metric spaces, Banach-Valued metric spaces, partial metric spaces, metric-like (dislocated) spaces, symmetric spaces, b-metric spaces, ultra-metric spaces, G-metric space, A-metric space, Quaternion-Valued metric spaces, TVS-valued metric spaces, S-metric spaces, D-metric space, Intuitionistic Fuzzy metric space, multiplicative metric spaces, interpolative metric spaces, and so on. As it was discussed above, some of these abstract structures are novel and interesting generalizations of the usual metric space, whereas some of them are original, see e.g. [8].

This paper focuses on the second trend, and we consider one of the recently presented improvements of the standard metric spaces, namely, interpolative metric spaces [6, 7, 25]. Shortly and simply expressing, an interpolative metric structure is the natural extension of the standard metric concept by changing the triangle inequality condition with another new inequality, so-called, interpolative inequality. This modern condition can be exceptionally valuable in estimations in possible applications.

Next, we review the idea of the interpolative metric, that can be expressed as an inevitable generalization of a usual metric structure.

Definition 1. *[6, 7]. For a nonempty set S, a distance function $d : S \times S \to [0, +\infty)$ is called (ξ, c)-interpolative metric if*

$(im0)$ *$d(t, y) > 0$, for all distinct $t, y \in S$;*

$(im1a)$ *$d(t, y) = 0$, implies that $t = y$*

$(im1b)$ *$t = y$ implies that $d(t, y) = 0$;*

$(im2)$ *$d(y, t) = d(t, y)$, for all $t, y \in S$;*

$(im3)$ *there exist $k \geq 0$ and $\xi \in (0, 1)$ in a way that the following inequality is satisfied for any choice of $t, y, w \in S$.*

$$d(t, y) \leq d(t, w) + d(w, y) + k\left[(d(t, w))^{\xi}\,(d(w, y))^{1-\xi}.\right]$$

Here, the pair (S, d) is said to be an (ξ, c)-interpolative metric space (in short, IMS).

Remark 1. *[6, 7]. As it is clear, for $c = 0$, an (ξ, c)-interpolative metric space (S, d) turns into the usual metric space.*

Inspired from the examples in [6, 7], we shall construct the following example to indicate that the converse is false.

Example 1. *Set $S = [0, 1]$ and consider a mapping $d : S \times S \to [0, \infty)$ which is defined by*

$$d(t, y) := |t - y|^6, \text{ for all } t, y \in S.$$

It is obvious to see that axioms $(im1)$ and $(im2)$ are easily derived. For $(im3)$, we have

$$
\begin{aligned}
d(t,y) = |t-y|^6 &= |t-w+w-y|^6 \\
&= |t-w|^6 + 6|t-w|^5|w-y| + 15|t-w|^4|w-y|^2 \\
&+20|t-w|^3|w-y|^3 + 15|t-w|^2|w-y|^4 \\
&+6|t-w||w-y|^5 + |w-y|^6 \\
&\leq |t-w|^6 + |w-y|^6 + 6|t-w|^5|w-y| + 6|t-w||w-y|^5 \\
&+15|t-w|^4|w-y|^2 + 15|t-w|^2|w-y|^4 \\
&+20|t-w|^3|w-y|^3 \qquad (1.1) \\
&\leq d(t,w) + d(w,y) + 6\left[(d(t,w))^{\frac{5}{6}}(d(w,y))^{\frac{1}{6}}\right] \\
&+6\left[(d(t,w))^{\frac{1}{6}}(d(w,y))^{\frac{5}{6}}\right] + 15\left[(d(t,w))^{\frac{4}{6}}(d(w,y))^{\frac{2}{6}}\right] \\
&+15\left[(d(t,w))^{\frac{2}{6}}(d(w,y))^{\frac{4}{6}}\right] + 20\left[(d(t,w))^{\frac{3}{6}}(d(w,y))^{\frac{3}{6}}\right] \\
&\text{without loss of generality, we assume } d(t,w) \geq d(w,y) \\
&\leq d(t,w) + d(w,y) + 30\left[(d(t,w))^{\frac{1}{6}}(d(w,y))^{\frac{5}{6}\cdot}\right]
\end{aligned}
$$

So, we find that $(im3)$ is satisfied for $c \geq 14$ for all $\xi = \frac{1}{6} \in (0,1)$. As a result, (S,d) forms $(\frac{1}{6}, 62)$-interpolative metric space.

In conclusion, (S,d) is $(\frac{1}{6}, 62)$-interpolative metric space.

Inspired by the example in [6, 7], we consider the following sample:

Example 2. *The (S,ρ) denotes a usual metric space. We consider the following distance function $d : S \times S \to [0,\infty)$ defined by*

$$d(t,y) := [\rho(t,y)]^2 + \Delta\rho(t,y), \text{ for all } t,y \in S,$$

where $\Delta \geq 1$. Due to fact that ρ is a standard metric on S, we have $(im1)$ and $(im2)$. Furthermore, to indicate that $(im3)$ is fulfilled, it is enough to let $c \geq 2$ for any $\xi \in (0,1)$. Therefore, (S,d) forms a $(\frac{1}{2}, 2)$-interpolative metric space.

In fact, it is easy to find that

$$
\begin{aligned}
d(t,y) &= [\rho(t,y)]^2 + \Delta\rho(t,y) \\
&\leq (\rho(t,w)+\rho(w,y))(\rho(t,w)+\rho(w,y)+\Delta) \\
&\leq (\rho(t,w)+\rho(w,y))(\rho(t,w)+\rho(w,y)+\Delta) \\
&\leq [\rho(t,w)(\rho(t,w)+\Delta)+\rho(t,w)\rho(w,y)] \\
&+[\rho(w,y)(\rho(w,y)+\Delta)+\rho(w,y)\rho(t,w)] \\
&\leq [\rho(t,w)(\rho(t,w)+\Delta)] + [\rho(w,y)(\rho(w,y)+\Delta)] + 2\rho(t,w)\rho(w,y) \\
&\leq d(t,w)+d(w,y)+2(\rho(t,w))^{\frac{1}{2}}(\rho(t,w))^{\frac{1}{2}}(\rho(w,y))^{\frac{1}{2}}(\rho(w,y))^{\frac{1}{2}} \\
&\leq d(t,w)+d(w,y)+2(\rho(t,w))^{\frac{1}{2}}[\rho(t,w)+\Delta]^{\frac{1}{2}}(\rho(w,y))^{\frac{1}{2}}[\rho(w,y)+\Delta]^{\frac{1}{2}} \\
&\leq d(t,w)+d(w,y)+2(d(t,w))^{\frac{1}{2}}(d(w,y))^{\frac{1}{2}}.
\end{aligned}
$$

As a result, the distance function $d(t,y)$ can not fulfills the metric conditions.

For $r > 0$ and $t \in S$, in the setting of (ξ, c)-interpolative metric space (S, d) we set an open ball

$$\mathfrak{B}(t, r) = \{w \in S : d(t, w) < r\}.$$

Definition 2. *(See, e.g. [6]) Suppose a self-mapping T defined on a (ξ, c)-interpolative metric (S, d), construct an iterative sequence $\{T^{n-1}m\}$ in S. The sequence $\{T^{n-1}m\}$ is convergent to m^* in S when $d(T^{n-1}m, x) \to 0$, as $n \to \infty$. In addition, it is called a Cauchy sequence in S, when $\lim_{n\to\infty} \sup\{d(T^{n-1}m, T^{s-1}m) : m > n\} = 0$. Furthermore, a (ξ, c)-interpolative metric space (S, d) is called complete when every Cauchy sequence converges in S.*

The thought of the simulation function can be displayed in [14]:

Definition 3. *[14] If the mapping $\alpha : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a simulation function when*

(α1) *$\alpha(0, 0) = 0$;*

(α2) *$\alpha(m, n) < n - m$ for all $m, n > 0$;*

(α3) *if $\{m_n\}$, $\{k_n\}$ are sequences in $(0, \infty)$ such a way that $\lim_{n\to\infty} m_n = \lim_{n\to\infty} k_n > 0$ then*

$$\limsup_{n\to\infty} \alpha(m_n, k_n) < 0.$$

Let $\mathcal{T}$ represent the collection of all simulation functions. We infer from the above definition that $\alpha(m, m) < 0$ for all $m > 0$.

A. Roldan *et al.* [23] reconsider Definition 3 in order to widen the class of all simulation functions by replacing the assumption $(\alpha 3)$ by the following one:

$(\alpha 3)'$ if $\{m_n\}$ and $\{k_n\}$ are two positive sequences in a way that

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty} k_n > 0 \text{ and } m_n < k_n,$$

then

$$\limsup_{n\to\infty} \alpha(m_n, k_n) < 0.$$

A number of outlines of the diversion work are given in [14]:

Example 3. *Let the functions $h, g : [0, \infty) \to [0, \infty)$ be continuous in a way that $g(t) = h(t) = 0$ if and only if $t = 0$ and $g(t) < t \leq h(t)$ for all $t > 0$. For $i = 1, 2, 3$, we define $\alpha_i : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, as*

(*i*) *Then, for every $t, s \in [0, \infty)$, $\alpha_1(t, s) = g(s) - h(t) \in \mathcal{T}$.*

(*ii*) *For every $t, s \in [0, \infty)$, $\alpha_2(t, s) = s - \frac{H(t,s)}{F(t,s)} t$, where the functions $H, T : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous in each variable such that $H(t, s) > F(t, s)$ for all $t, s > 0$.*

(iii) *For any $t, s \in [0, \infty)$, $\alpha_3(t, s) = s - p(s) - t \in \mathcal{T}$, where $p : [0, \infty) \to [0, \infty)$ is a continuous function such that $p(t) = 0$ if and only if $t = 0$.*

Next, we recall the definition of $\mathcal{Z}_\alpha$-contraction, Khojasteh e*t al*. [14]. For more examples and details, see e.g. [13, 15–22, 24].

Definition 4. *[14] Let (S, d) form an usual metric space. Let T be a self-mapping on S, and $\alpha \in \mathcal{T}$. For any $m, n \in S$, T is referred to be a $\mathcal{Z}_\alpha$-contraction if*

$$\alpha(d(Tm, Tn), d(m, n)) \geq 0. \tag{1.2}$$

Remark 2. *[14]*

(i) *The famous Banach fixed point theorem is a straightforward consequence of S_α-contraction, which can be derived by taking $\lambda \in [0, 1)$ and $\alpha(n, m) = \lambda m - n$ for all $m, n \in [0, \infty)$ in the formulation above.*

(ii) *It is crystal clear from the simulation function formulation that for every $b \geq a > 0$, $\alpha(b, a) < 0$. Thus, for every unique $m, n \in S$, we get*

$$d(Tm, Tn) < d(m, n).$$

This indicates that each $\mathcal{Z}_\alpha$-contraction mapping is continuous since it is contractive.

We now present the lemmas and results demonstrated in [14]:

Lemma 1. *[14] Let $T : S \to S$ be a $\mathcal{Z}_\alpha$-contraction where (S, d) is a metric space. Then,*

1. *the mapping T possesses a unique fixed point in S (provided it exists);*
2. *T is asymptotically regular at every $m \in S$;*
3. *Let $m_n = Tm_{n-1}$ for each $n \in \mathbb{N}$ with an initial value $m_0 \in M$ (arbitrarily chosen point m renamed as $m_0 := m$). Then, the constructed sequence $\{m_n\}$, generated by Picard operator T, is bounded.*

Theorem 1. *If a self-mapping T, on a metric space (S, d) , forms an $\mathcal{Z}_\alpha$-contraction, then T has a unique fixed point m^* in M. Furthermore, for arbitrary initial point $m_0 \in M$ the constructed sequence $\{m_n\}$, converges to the fixed point of T, where $m_n = Tm_{n-1}$ for all $n \in \mathbb{N}$.*

2. Main Results

We start this section by stating the very important auxiliary function in the metric fixed point theory. A function $\psi : [0, \infty) \to [0, \infty)$ is called c-comparison function when

(Ψ_1) ψ is increasing;

(Ψ_2) $\sum_{n=1}^{+\infty} \psi^n(x) < \infty$ for any $x > 0$, where ψ^n is observed by recursively compose the function ψ with itself n times.

The letter Ψ to hold forth the family of all c-comparison functions.

Lemma 2. *([3], [12],[24]) If $\psi : [0,\infty) \to [0,\infty)$ is a comparison function, then:*

(1) each iterate ψ^k of ψ, $k \geq 1$, is also a comparison function;

(2) ψ is continuous at 0;

(3) $\psi(t) < t$, for any $t > 0$.

Here, we shall introduce the notion of $\mathcal{Z}_\alpha$-contraction, a new contraction condition defined within interpolative metric spaces:

Definition 5. *Let $T : S \to S$ be a mapping, where (S,d) is a (ξ,c)-interpolative metric space, and $\psi \in \Sigma$. The mapping T is called an interpolative $\mathcal{Z}_\alpha$-contraction in case:*

$$\alpha(d(Tm,Tl),\psi(d(m,l))) \geq 0, \tag{2.1}$$

for each $m,l \in M$.

Proposition 1. *Let $T : S \to S$ be an interpolative $\mathcal{Z}_\alpha$-contraction, where (S,d) is an interpolative metric space. Then the fixed point of T in S is unique, provided that there exists a fixed point.*

Proof.
Suppose that our assertion fails. It means that there are two fixed points $m,l \in S$ that are distinct $T(m) = m \neq l = T(l)$. Naturally, $d(m,l) > 0$. On account of the inequality, (2.1) one can derive

$$0 \leq \alpha(d(Tm,Tl),\psi(d(m,l)) < \psi(d(m,l)) - d(m,l),$$

yields from the fact, $\psi(t) < t$ for any $t > 0$, that

$$d(m,l) \leq \psi(d(m,l)) < d(m,l),$$

a contradiction.

Theorem 2. *Let $T : S \to S$ be an interpolative $\mathcal{Z}_\alpha$-contraction, where (S,d) is a complete (ξ,c)-interpolative metric space.*

Then T possesses a unique fixed point m^* in M and for every $m_0 \in S$ the Picard sequence $\{m_n\}$, where $m_n = Tm_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T.

Proof. Due to Proposition 1, it guarantees that if there is a fixed point, the mapping T, it is unique. To finalize the proof, it is enough to guarantee the existence of a fixed point of the mapping T.

For this goal, it is sufficient to construct a recursive sequence starting with an arbitrary choice $m \in S$ that sequence has a limit. We start with renaming this arbitrarily chosen initial point as $m_0 := m$ that produces the sequence with the general term $m_n = Tm_{n-1}$ for all $n \in \mathbb{N}$.

Without loss of generality, one can suppose that the successive terms in this hand made sequence are pairwise distinct. Suppose, on the contrary, that there exists a $k \in \mathbb{N}$ such that consecutive k^{th} and $k+1^{th}$ terms equal, then the proof is fulfilled. In fact, we have $T^k m = T^{k-1}m$, that is, $Tl = l$ is claimed fixed point inequality, with $l = T^{k-1}m$.

Throughout the proof, without loss of generality, we suppose that for all $n \in \mathbb{N}$, that $T^n m \neq T^{n-1}m$. On account of the inequality (2.1), we find that

$$\begin{aligned} 0 &\leq \alpha(d(T^{n+1}m, T^n m), \psi(d(T^n m, T^{n-1}m))) \\ &= \alpha(d(TT^n m, TT^{n-1}m), \psi(d(T^n m, T^{n-1}m))) \\ &< \psi(d(T^n m, T^{n-1}m)) - d(T^{n+1}m, T^n m). \end{aligned}$$

Regarding the successive terms being distinct, the inequality above yields

$$d(T^{n+1}m, T^n m) \leq \psi(d(T^n m, T^{n-1}m)) < d(T^n m, T^{n-1}m)). \tag{2.2}$$

Recursively, one can derive from (2.2)

$$d(T^{n+1}m, T^n m) \leq \psi^{n-1}(d(T^2 m, Tm)). \tag{2.3}$$

Thus, we conclude, from the inequality (2.2), that the sequence of nonnegative reals $\{d(T^n m, T^{n-1}m)\}$ is monotonically decreasing. Further, it is necessarily a convergent sequence. Let us denote $\lim_{n\to\infty} d(T^n m, T^{n+1}m) = \kappa \geq 0$. We claim that $\kappa = 0$. In this step, we use the method of Reductio Ad Absurdum. For this purpose, we suppose, on the contrary, that $\kappa > 0$. By keeping the fact that T is (ξ, c)-interpolative $\mathcal{Z}_\alpha$-contraction in mind, and considering the assumption (α3), we find

$$0 \leq \lim_{n\to\infty} \sup \alpha(d(T^{n+1}m, T^n m), d(T^n m, T^{n-1}m)) < 0,$$

a contradiction. Consequently, the limit $\kappa = 0$, that is,

$$\lim_{n\to\infty} d(T^n m, T^{n+1}m) = 0. \tag{2.4}$$

Hence, we conclude that the self-mapping T is an asymptotically regular mapping in S.

For this purpose, we claim that

$$\lim_{n\to\infty} d(T^{n-1}m, T^{n+r}m) = 0. \tag{2.5}$$

The method of induction shall be used to prove the claim above. First, we consider the inequality below,

$$\begin{aligned} d(T^{n-1}m, T^{n+1}m) &\leq d(x_n, T^n m) + d(T^n m, T^{n+1}m) \\ &+c\left[(d(x_n, T^n m))^{\xi}\left(d(T^n m, T^{n+1}m)\right)^{1-\xi}\right] \end{aligned} \tag{2.6}$$

By letting $n \to \infty$ in the inequality above, together with expression (2.4), we find that

$$\lim_{n\to\infty} d(T^{n-1}m, T^{n+1}m) = 0. \tag{2.7}$$

In addition, we have

$$\begin{aligned} d(T^{n-1}m, T^{n+2}m) &\leq d(x_n, T^{n+1}m) + d(T^{n+1}m, T^{n+2}m) \\ &+c\left[\left(d(x_n, T^{n+1}m)\right)^{\xi}\left(d(T^n m, T^{n+2}m)\right)^{1-\xi}\right] \end{aligned} \tag{2.8}$$

Taking (2.4) and (2.8) into account together by taking $n \to \infty$ in the above inequality we find that

$$\lim_{n\to\infty} d(T^{n-1}m, T^{n+2}m) = 0. \tag{2.9}$$

In this step, we presume our claims hold for some natural number $r > 1$; that is,

$$\lim_{n\to\infty} d(T^{n-1}m, T^{n+r-1}m) = 0, \text{ for some } r \in \mathbb{N}. \tag{2.10}$$

Accordingly, employing of the Theorem, we find

$$\begin{aligned} d(T^{n-1}m, T^{n+r}m) &\leq d(T^{n-1}m, T^{n+r-1}m) + d(T^{n+r-1}m, T^{n+r}m) \\ &+c\left[\left(d(T^{n-1}m, T^{n+r-1}m)\right)^{\xi}\left(d(T^{n+r-1}m, T^{n+r}m)\right)^{1-\xi}\right] \end{aligned} \tag{2.11}$$

Employing the limits (2.4) and (2.10), together with letting $n \to \infty$ in the inequality above, we obtain

$$\lim_{n\to\infty} d(T^{n-1}m, T^{n+r}m) = 0. \tag{2.12}$$

As a result, our claim (2.5) holds. Keeping this observation in mind, it is concluded that

$$d(T^n m, T^{s-1}m) < 1, \tag{2.13}$$

for $m > n > M_1$ for some $M_1 \in \mathbb{N}$.

As a next step, we shall discuss whether the sequence $\{T^{n-1}m\}$ is Cauchy.

For for $m > n > k = \{M, M_1\}$, we have

$$\begin{aligned} d(T^{n-1}m, T^{s-1}m) &\leq d(T^{n-1}m, T^n m) + d(T^n m, T^{s-1}m) \\ &+c[(d(T^{n-1}m, T^n m))^{\xi}(d(T^n m, T^{s-1}m))^{1-\xi}] \\ &\leq \psi^{n-k}(d(T^{k-1}m, T^k m)) + d(T^n m, T^{s-1}m) \\ &\quad +c[(\psi^{n-k}(d(T^{k-1}m, T^k m)))^{\xi}(d(T^n m, T^{s-1}m))^{1-\xi}]. \end{aligned} \tag{2.14}$$

Due to (2.13), we get $d(T^n m, T^{s-1}m) < 1$. Consequently, we find

$$(d(T^n m, T^{s-1}m))^{1-\xi} < 1. \tag{2.15}$$

Keeping this observation in mind, we shall estimate the right-hand side of (2.14):

$$\begin{aligned}
&\leq \psi^{n-k}(d(T^{k-1}m, T^k m)) + d(T^n m, T^{s-1}m) \\
&+[c(\psi^{n-k}(d(T^{k-1}m, T^k m)))^{\xi}(d(T^n m, T^{s-1}m))^{\xi}]d(T^n m, T^{s-1}m) \\
&\leq \psi^{n-k}(d(T^{k-1}m, T^k m)) \\
&+[1 + c(\psi^{n-k}(d(T^{k-1}m, T^k m)))^{\xi}(d(T^n m, T^{s-1}m))^{\xi}]d(T^n m, T^{s-1}m)(\text{ due to (2.15)}) \\
&\leq \psi^{n-k}(d(T^{k-1}m, T^k m)) \\
&+[1 + c(\psi^{n-k}(d(T^{k-1}m, T^k m)))^{\xi}]d(T^n m, T^{s-1}m).
\end{aligned} \tag{2.16}$$

In conclusion, we have

$$d(T^{n-1}m, T^{s-1}m) \leq \psi^{n-k}(d(T^{k-1}m, T^k m)) + [1 + c(\psi^{n-k}(d(T^{k-1}m, T^k m)))^{\xi}]d(T^n m, T^{s-1}m). \tag{2.17}$$

Notice also that

$$\begin{aligned}
d(T^n m, T^{s-1}m) &\leq d(T^n m, T^{n+1}m) + d(T^{n+1}m, T^{s-1}m) \\
&+c[(d(T^{k-1}m, T^k m))^{\xi}(d(T^{n+1}m, T^{s-1}m))^{1-\xi}] \\
&\leq \psi^{n-k+1}(d(T^{k-1}m, T^k m)) + d(T^{n+1}m, T^{s-1}m) \\
&+[c(\psi^{n-k+1}(d(T^{k-1}m, T^k m)))^{\xi}(d(T^{n+1}m, T^{s-1}m))^{\xi}]d(T^{n+1}m, T^{s-1}m) \\
&\leq \psi^{n-k+1}(d(T^{k-1}m, T^k m)) + [1 + c(\psi^{n-k+1}(d(T^{k-1}m, T^k m)))^{\xi}]d(T^{n+1}m, T^{s-1}m).
\end{aligned} \tag{2.18}$$

Gathering (2.17) and (2.18) yields that

$$\begin{aligned}
d(T^{n-1}m, T^{s-1}m) &\leq \psi^{n-k}(d(T^{k-1}m, T^k m)) + \psi^{n-k+1}(d(T^{k-1}m, T^k m))[1 + c(\psi^{n-k})^{\xi}] \\
&+[1 + c(\psi^{n-k}(d(T^{k-1}m, T^k m)))^{\xi}][1 + c(\psi^{n-k+1}(d(T^{k-1}m, T^k m)))^{\xi}]d(T^{n+1}m, T^{s-1}m)
\end{aligned} \tag{2.19}$$

By collecting all observations above, we find

$$\begin{aligned}
d(T^{n-1}m, T^{s-1}m) &\leq \psi^{n-k}(d(T^{k-1}m, T^k m)) \textstyle\sum_{i=0}^{m-n-1} \psi^i \prod_{j=0}^{i-1}(1 + c\psi^{n-k+j}(d(T^{k-1}m, T^k m)))^{\xi} \\
&\leq \psi^{n-k}(d(T^{k-1}m, T^k m)) \textstyle\sum_{i=0}^{m-n-1} \psi^i(d(T^{k-1}m, T^k m)) \prod_{j=0}^{i-1}(1 + c\psi^j(d(T^{k-1}m, T^k m)))^{\xi}.
\end{aligned} \tag{2.20}$$

Notice that the sequence $\sum_{i=0}^{\infty} S_i$ dominates the right-hand side of (2.20). Hence, by letting $n, m \to \infty$, we conclude that it converges, where,

$$S_i = \psi^i(d(T^{k-1}m, T^k m)) \prod_{j=0}^{i-1}(1 + c\psi^j(d(T^{k-1}m, T^k m)))^{\xi} < \psi^i(d(T^{k-1}m, T^k m)) \prod_{j=0}^{i-1}(1 + cd(T^{k-1}m, T^k m))^{\xi}.$$

In conclusion, $\{T^{n-1}m\}$ is Cauchy.

We recall that (S,d) is a complete (ξ,c)-interpolative metric space. It implies that the sequence $\{T^{n-1}m\}$ converges to $m^* \in S$. In this step, we need to show that m^* is the fixed point of T. To prove it, assume not, that is, $d(m^*,Tm^*) > 0$. Note that

$$d(T^n m, Tm^*) = d(TT^{n-1}m, Tm^*) \quad \leq \psi\left(\max\{d(T^{n-1}m, m^*), d(T^{n-1}m, T^n m), d(Tm^*, m^*)\}\right) \tag{2.21}$$

Applying lim sup to (2.21), we find

$$\begin{aligned} d(Tm^*, m^*) &\leq \psi(d(Tm^*, m^*)) \\ &< d(Tm^*, m^*) \text{ due to assumption} d(m^*, Tm^*) > 0 \end{aligned} \tag{2.22}$$

a contradiction. Therefore, $Tm^* = m^*$ forms a fixed point of T in S.

3. Consequences

In this part, we shall consider the immediate consequences of this paper's main result.

Corollary 1. *Let (S,d) be a complete (ξ,c)-IMS, and let $T : S \to S$ be a mapping. If there is an auxiliary mapping $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \psi(d(x,y)), \tag{3.1}$$

for all $x,y \in S$, then, T guarantees to possess a unique fixed point in S.

Sketch of the Proof 1. *Note that the full proof mimics the proof of the main theorem, thus we prefer the avoid the repetition. Instead, we give the sketch of the proof: It is sufficient to consider the proper function $\alpha(t,s) = \psi(s) - t$ in the main theorem.*

Note that the famous Banach's Contraction Principle can be transformed into an interpolative metric space as follows.

Corollary 2. *Let (S,d) be a complete (ξ,c) IMS, and let $T : S \to S$ be a mapping. If there is an real number $q \in (0,1)$ such that*

$$d(Tx,Ty) \leq q d(x,y), \tag{3.2}$$

for all $x,y \in S$, then, T guarantees to possess a unique fixed point in S.

Sketch of the Proof 2. *For this corollary, the full proof is the mimics of the proof of the main theorem, thus we skip it. Indeed, it is sufficient to consider $\psi(t) = qt$ where $q \in (0,1)$.*

Note that regarding Example 3, we may list several new fixed-point theorems as consequences of Theorem 2. In other words, by considering different simulation functions, we get different fixed-point theorems that are generalizations of the existing results in the setting of interpolative metric structure. Further, employing different types of c-comparison functions, we shall get more results. Note that all these results improve and generalize the published results. Additionally, it is important to note that the findings achieved here are also applicable in the normal metric spaces.

4. Conclusion

This study aims to explore the existence of fixed points for certain contractions defined with the help of simulation functions in interpolative metric spaces, which is a newly defined abstract structure. These simulation functions are quite special and interesting, and they enable us to make the numerous contractions in the literature the consequences of our presented result. This is our first result in this newly defined structure, and it is a very basic result. Consequently, this result can be enriched in several ways that can be potential future works. As a next step, it is also quite reasonable to investigate the possible applications, especially in the economy and industrial engineering.

In such generalizations, one of the remarkable studies is finding an answer to the following question: In which abstract structure does, Picard sequence converge fastest? In other words, how can one compare the rates of convergence in these distinct structures?

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