EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6294
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Representation of Bi-Univalent Functions to Lucas Balancing Polynomials with Geometric Properties and Coefficient Bounds

Stalin Thangamani[1], Pshtiwan Othman Mohammed[2,3,*], Jeno Francis Devadoss[4], Majeed A. Yousif[5], Meraa Arab[6,*], Dumitru Baleanu[7]

[1] *Department of Mathematics, Vel Tech Rangarajan Dr.Sagunthala R & D Institute of Science and Technology, Avadi, Chennai 600062, India*
[2] *Research and Development Center, University of Sulaimani, Sulaymaniyah 46001, Iraq*
[3] *Research Center, University of Halabja, Halabja 46018, Iraq*
[4] *Department of Mathematics, Prathyusha Engineering College, Anna University, Chennai 602025, India*
[5] *Department of Mathematics, College of Education, University of Zakho, Zakho 42002, Iraq*
[6] *Department of Mathematics and Statistics, College of Science, King Faisal University, Hofuf 31982, Al Ahsa, Saudi Arabia*
[7] *Department of Computer Science and Mathematics, Lebanese American University, Beirut 11022801, Lebanon*

Abstract. In this paper, we introduce and analyze a new subclass of bi-univalent functions defined in the open unit disk, associated with Lucas and Lucas balancing polynomials. By employing the Taylor-Maclaurin series expansion, precise bounds for the second and third coefficients, $|a_2|$ and $|a_3|$, are obtained. These estimates lead to important geometric interpretations related to the distortion, growth, and structural behavior of the functions near the origin. Furthermore, the mapping characteristics of these functions are examined in connection with existing bi-univalent subclasses. Special emphasis is placed on deriving a Fekete-Szegö type inequality for the proposed class, thereby extending earlier contributions in the field. The findings presented here are valuable for applications in geometric function theory and areas like fluid mechanics, conformal mappings, and engineering models where the analytic structure of bi-univalent functions plays a significant role.

2020 Mathematics Subject Classifications: 30C45, 30C50

Key Words and Phrases: Bi-univalent functions, Lucas polynomial, Fekete-Szegö inequality

---

*Corresponding author.
*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6294

Email addresses: drstalint@veltech.edu.in (S. Thangamani), pshtiwansangawi@gmail.com (P.O. Mohammed), jenofrancis29@gmail.com (J. F. Devadoss), majeed.yousif@uoz.edu.krd (M.A. Yousif), marab@kfu.edu.sa (M. Arab), dumitru.baleanu@lau.edu.lb (D. Baleanu)

1. Introduction

In the open unit disk $\mathbb{D} = \{\Theta \in \mathbb{C} : |\Theta| < 1\}$, $\mathbb{A}$ often denotes the class of normalized and analytic functions $f(\Theta)$ of the form

$$f(\Theta) = \Theta + \sum_{k=2}^{\infty} a_k \Theta^k \tag{1}$$

and $\mathbb{S}$ act as a subclass of all functions in $\mathbb{A}$. As stated by Koebe one-quarter theorem [1], if $f \in \mathbb{S}$ is a univalent function, then the image of unit disk under f will contain a disk of radius $\frac{1}{4}$. As a results, for all univalent functions $f \in \mathbb{S}$, there exists an inverse function f^{-1} that satisfies $f^{-1}[f(\Theta)] = \Theta$ and $f[f^{-1}(\Phi)] = g(\Phi)$ for $|\Phi| < r_0(f) : r_0(f) \geq \frac{1}{4}$. A power series in $f(\Theta)$ given by Equation (1) can be used to define the inverse function, where $a_2, a_3, \ldots$ are complex coefficients. If both a function f and a function f^{-1} are univalent in $\mathbb{D}$, the function is said to be bi-univalent in the unit disc. Let Σ signify the class of bi-univalent functions in $\mathbb{D}$ written by (1). Some familiar function for the class Σ are specified below: $\frac{\Theta}{1-\Theta}$, $\frac{1}{2} log\left(\frac{1+\Theta}{1-\Theta}\right)$, $-log(1-\Theta)$.

Lewin [2] initially derived the bi-univalent functions class Σ, and found that $|a_2| < 1.51$. For the generalized subclass of class Σ Brannan and Clunie [3] later enhanced this outcome as $|a_2| \leq \sqrt{2}$. Brannan and Taha [4] focused on a specific subclass of class Σ, also estimate the initial coefficients $|a_2|$ and $|a_3|$. For convex and starlike functions, Ma Minda [5] employed a number of subclasses that meet the requirement that either $1 + \frac{\Theta f''(\Theta)}{f'(\Theta)}$ or $\frac{\Theta f'(\Theta)}{f(\Theta)}$ is subordinate to a more comprehensive subordinate function. In order to do this, Ma Minda looked at an analytic function ϕ, it uses the unit disk to map $\mathbb{D}$ onto an area that is starlike with regard to 1 and also meets the requirements given that it is symmetrical relating to the real axis, also full files the conditions $\phi(0) = 1$ and $\phi'(0) > 0$. The function f in the Ma Minda starlike class satisfies subordination $\frac{\Theta f'(\Theta)}{f(\Theta)} \prec \phi(\Theta)$. The Ma Minda convex class , in a similar vein, consists of functions f that fulfill the subordination $1 + \frac{\Theta f''(\Theta)}{f'(\Theta)} \prec \phi(\Theta)$.

A large number of integer number sequences have surfaced due to the elegance of their recurrence relations, including Fibonacci, Jacobsthal, Lucas, Fermat, Pell, Chebyshev, Telephone and others. The Lucas numbers are a brand-new integer series that Behera and Panda [6] revealed.In recent years,[7–11] several authors have explored subclasses of biunivalent (or univalent) functions Over the past few decades, the features of this unique number sequence have been intensively investigated, and certain generalizations have been developed. A definition and a number of interesting properties of the polynomials are given by Lee et al. [12] and Ray [13]. The polynomials are a innate consequence of the Lucas numbers and Balancing Lucas numbers. Due to their applications in fluid dynamics, geometric function theory, and conformal mapping, bi-univalent functions have attracted a lot of attention in complex analysis. These functions have intriguing geometric properties that can provide important information on growth behavior, distortion, and mapping properties. A greater comprehension of the structural characteristics of bi-univalent functions and how they interact with classical inequalities can be gained by studying their subclasses. Lucas polynomials are one such class of functions, and their special characteristics can be used to study how these functions behave. In recent years, a multitude of writers have made great strides in implementing sharp bounds for a variety of bi-univalent function subclasses that commonly interact with specific polynomial families, including Lucas and balancing Lucas polynomials [14–24] and [25] .

The primary objective of this work is to determine the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and their influence on the geometric and analytic features of the bi-univalent subclass of Lucas and Lucas balancing polynomial functions. By presenting new results and building on previous work, this study contributes to our understanding of bi-univalent functions.

The main objective of this study is to provide upper bounds and initial coefficients for the Taylor-Maclaurin and Fekete-Szegö functionals of the function of the stated subclass. This research is divided into three parts. In the first section, some basic concepts of bi-univalent function theory are discussed, including information on Lucas and balancing Lucas polynomials. Parts two and three introduce new subclasses of bi-univalent functions using the Lucas and balancing Lucas polynomials, respectively.

The motivation for this study stems from the need to better understand the intricate geometric and analytic properties of bi-univalent functions. These functions play a crucial role in complex analysis, particularly in areas such as conformal mappings and fluid dynamics, where the behavior of functions near boundaries and their growth rates are of significant importance. By focusing on the Taylor-Maclaurin coefficients and their implications, this work aims to provide deeper insights into how these functions behave in the unit disk and their connections to other classes of functions.

In addition to their theoretical significance, the initial coefficients have practical implications. These include calculating the degree to which bi-univalent functions can resemble other kinds of functions, creating numerical algorithms to find bi-univalent functions using the initial coefficients, and using these functions in complex analysis, particularly when examining boundary behavior of mappings from the unit disk and conformal mappings. Furthermore, a function's growth rate is frequently estimated using the initial coefficients. In particular, they help define limits on the function's maximum modulus and behavior near the unit disk's edge. By setting upper constraints for these coefficients, we can regulate how much the function "spreads" or "compresses" inside the unit disk.
The structure of the paper is organized systematically to facilitate a coherent flow of ideas and results. Section 2 presents the initial coefficient estimates for the subclass $\mathcal{G}_{u,v}^{\Sigma,G}(\tau)$, laying the foundation for the subsequent analysis. Building on this, Section 3 derives the Fekete-Szegö inequality for the same subclass, highlighting important functional bounds. In Section 4, attention shifts to a related subclass $\mathcal{G}^{\Sigma,G}(\tau)$, where initial coefficient estimates are again established. Following this, Section 5 investigates the Fekete-Szegö

inequality for $\mathcal{G}^{\Sigma,G}(\tau)$, offering a comparative view of the functional behavior. Section 6 is dedicated to a discussion on the influence of the parameter s, supported by surface plot interpretations that provide visual insights into the parameter's effect. Finally, Section 7 concludes the paper by summarizing the key findings and highlighting potential directions for future research.

2. The initial Coefficient estimates for the subclass $\mathcal{G}_{u,v}^{\Sigma,G}(\tau)$

The Lucas polynomials $L_n(u(s), v(s), s)$ are defined by the recurrence relation that follows:

$$L_n(s) = u(s)L_{n-1}(s) + v(s)L_{n-2}(s)$$

with initial conditions:

$$L_0(s) = 2, \quad L_1(s) = u(s),$$

The generating function for $L_n(s)$ is given by:

$$G(s, \Theta) = \frac{2 - u(s)\Theta}{1 - u(s)\Theta - v(s)\Theta^2}.$$

The generating function $G(s, \Theta)$ is related to the sequence $L_n(u(s), v(s), s)$ by the expansion:

$$G(s, \Theta) = \sum_{n=0}^{\infty} L_n(u(s), v(s), s)\Theta^n$$

Thus, we can express the generating function as:

$$G(s, \Theta) = \frac{2 - u(s)\Theta}{1 - u(s)\Theta - v(s)\Theta^2} = 2+u(s)\Theta+(2v(s)+u^2(s))\Theta^2+(3u(s)v(s)+u^3(s))\Theta^3+..., \tag{2}$$

$$\therefore G(s, \Theta) = \sum_{n=0}^{\infty} L_n(u(s), v(s), s)\Theta^n.$$

where

$$L_0(s) = 2, \quad L_1(s) = u(s), \quad L_2(s) = 2v(s) + u^2(s) \qquad L_3(s) = u^3(s) + 3u(s)v(s), ...$$

Remark 1.
It should be noted that the (u, v)-polynomial $L_n(s)$ leads to different polynomials for certain values of u and v. Here, we highlight a few examples from those:

(i) For $u(s) = 2s$ and $v(s) = 1$, then obtain the polynomials of Pell-Lucas $Q_n(s)$.

(ii) For $u(s) = 1$ and $v(s) = 2s$, then acquire polynomials of Jacobsthal-Lucas $j_n(s)$.

(iii) For $u(s) = 3s$ and $v(s) = -2$, then obtain the polynomials of Fermat-Lucas $f_n(s)$.

(iv) For $u(s) = 2s$ and $v(s) = -1$, then acquire first kind Chebyshev polynomials $T_n(s)$.

The following lemma is to be kept forefront in order to get at our primary conclusions.

Lemma 1. *[26] If $p(\Theta) \in P$, then $|p_n| \leq 2$ for each n, where P is the family of all functions p analytic in $\mathbb{D}$ for which*

$$Re\,(p(\Theta)) > 0, \quad p(\Theta) = 1 + p_1\,\Theta + p_2\,\Theta^2 + ...; \quad \Theta \in \mathbb{D}.$$

In the following, it is assumed that $G(s, \Theta)$ is form of Taylor series nature

$$G(s,\Theta)-1 = 1+u(s)\Theta+2v(s)+u^2(s))\Theta^2+(3u(s)v(s)+u^3(s))\Theta^3+(u^4(s)+4u^2(s)+2v(s))\Theta^4+..., \tag{3}$$

$$G(t,\Phi)-1 = 1+u(t)\Phi+2v(t)+u^2(t))\Phi^2+(3u(t)v(t)+u^3(t))\Phi^3+(u^4(t)+4u^2(t)+2v(t))\Phi^4+..., \tag{4}$$

Definition 1. *If the function $f \in \Sigma$ is considered to be in the class $\mathcal{G}_{u,v}^{\Sigma,G}(\tau)$, then the resulting subordination holds,*

$$(1-\tau)\left(\frac{\Theta f'(\Theta)}{f(\Theta)}\right) + \tau\left(1 + \frac{\Theta f''(\Theta)}{f'(\Theta)}\right) \prec G(s,\Theta) - 1 \tag{5}$$

and

$$(1-\tau)\left(\frac{\Phi f'(\Phi)}{f(\Phi)}\right) + \tau\left(1 + \frac{\Phi f''(\Phi))}{f'(\Phi)}\right) \prec G(t,\Phi) - 1 \tag{6}$$

Example 1. *When replacing $\tau = 0$ in definition (1) then the particular subordination is hold for the class $\mathcal{G}_{u,v}^{\Sigma,G}(0)$.*

$$\left(\frac{\Theta f'(\Theta)}{f(\Theta)}\right) \prec G(s,\Theta) - 1$$

$$\left(\frac{\Phi f'(\Phi)}{f(\Phi)}\right) \prec G(t,\Phi) - 1$$

Example 2. *When replacing $\tau = 1$ in definition (1) then the particular subordination is hold for the class $\mathcal{G}_{u,v}^{\Sigma,G}(1)$.*

$$\left(1 + \frac{\Theta f''(\Theta)}{f'(\Theta)}\right) \prec G(s,\Theta) - 1$$

$$\left(1 + \frac{\Phi f''(\Phi))}{f'(\Phi)}\right) \prec G(t,\Phi) - 1$$

Theorem 1. *If $f(\Theta)$ defined by the equation (1) is belongs to the class $\mathcal{G}_{u,v}^{\Sigma,G}(\tau)$. Then*

$$|a_2| \leq \sqrt{\frac{u^2(s)+u^2(t)+2(v(s)+v(t))}{(2+2\tau)}}$$

and

$$|a_3| \leq \frac{u^2(s)+u^2(t)}{2(1+\tau)^2} + \frac{u^2(s)-u^2(t)+2(v(s)-v(t))}{4(1+2\tau)} \tag{7}$$

Proof. Let $f \in \mathcal{G}_{u,v}^{\Sigma,G}(\tau)$. Then there are two analytic functions $f, g : \mathbb{D} \to \mathbb{D}$ given by (5) *and* (6) *such that*

$$(1-\tau)\left(\frac{\Theta f'(\Theta)}{f(\Theta)}\right) + \tau\left(1+\frac{\Theta f''(\Theta)}{f'(\Theta)}\right) = G(s,\Theta) - 1$$

and

$$(1-\tau)\left(\frac{\Phi f'(\Phi)}{f(\Phi)}\right) + \tau\left(1+\frac{\Phi f''(\Phi))}{f'(\Phi)}\right) = G(t,\Phi) - 1$$

Since

$$(1-\tau)\left(\frac{\Theta f'(\Theta)}{f(\Theta)}\right) + \tau\left(1+\frac{\Theta f''(\Theta)}{f'(\Theta)}\right) = 1+(1+\tau)a_2\Theta+(2(1+2\tau)a_3-(1+3\tau)a_2^2)\Theta^2+... \tag{8}$$

and

$$(1-\tau)\left(\frac{\Phi f'(\Phi)}{f(\Phi)}\right) + \tau\left(1+\frac{\Phi f''(\Phi))}{f'(\Phi)}\right) = 1-(1+\tau)a_2\Phi+((3+5\tau)a_2^2-2(1+2\tau)a_3)\Phi^2+... \tag{9}$$

From the equations (3) *and* (8)*, By comparing coefficients of Θ and Θ^2 respectively, we get coefficient of Θ :*

$$(1+\tau)a_2 = u(s) \tag{10}$$

coefficient of Θ^2 :

$$2(1+2\tau)a_3 - (1+3\tau)a_2^2 = u^2(s) + 2v(s) \tag{11}$$

From the equations (4) *and* (9)*, equating the coefficients of Φ and Φ^2 respectively, we get coefficient of Φ :*

$$-(1+\tau)a_2 = u(t) \tag{12}$$

coefficient of Φ^2 :

$$(3+5\tau)a_2^2 - 2(1+2\tau)a_3 = u^2(t) + 2v(t) \tag{13}$$

Now, Adding the equations (10) *and* (12)*, we have*

$$u(s) = -u(t) \tag{14}$$

Squaring and adding the equations (10) *and* (12)*, we get*

$$2(1+\tau)^2a_2^2 = u^2(s) + u^2(t)$$

$$a_2^2 = \frac{u^2(s) + u^2(t)}{2(1+\tau)^2} \tag{15}$$

subtracting the equations (11) *and* (13), *we get*

$$2(1+2\tau)a_3 - 4(1+2\tau)a_2^2 + 2(1+2\tau)a_3 = u^2(s) - u^2(t) + 2(v(s) - v(t))$$

$$4(1+2\tau)a_3 = 4(1+2\tau)a_2^2 - u(t)^2 + u^2(s) + 2(v(s) - v(t))$$

$$a_3 = a_2^2 + \frac{u^2(s) - u^2(t) + 2(v(s) - v(t))}{4(1+2\tau)} \tag{16}$$

substituting the equations (11) *in equation* (13), *we get*

$$(2+2\tau)a_2^2 = u^2(s) + 2v(s) + u^2(t) + 2v(t)$$

$$a_2^2 = \frac{u^2(s) + u^2(t) + 2(v(s) + v(t))}{(2+2\tau)} \tag{17}$$

$$|a_2| \leq \sqrt{\frac{u^2(s) + u^2(t) + 2(v(s) + v(t))}{(2+2\tau)}} \tag{18}$$

substituting the equations (15) *in equation* (16), *we get*

$$|a_3| \leq \frac{u^2(s) + u^2(t)}{2(1+\tau)^2} + \frac{u^2(s) - u^2(t) + 2(v(s) - v(t))}{4(1+2\tau)} \tag{19}$$

substituting the equation (17) *in* (16), *we attain*

$$a_3 = \frac{u^2(s) + u^2(t) + 2(v(s) + v(t))}{(2+2\tau)} + \frac{u^2(s) - u^2(t) + 2(v(s) - v(t))}{4(1+2\tau)}$$

$$|a_3| \leq \frac{u^2(s) + 2v(s)}{(2+2\tau)} \tag{20}$$

Hence,

$$|a_2| \leq \sqrt{\frac{u^2(s) + u^2(t) + 2(v(s) + v(t))}{(2+2\tau)}}$$

and

$$|a_3| \leq \frac{u^2(s) + u^2(t)}{2(1+\tau)^2} + \frac{u^2(s) - u^2(t) + 2(v(s) - v(t))}{4(1+2\tau)}$$

Specializing the parameter values of $\tau = 0$ and $\tau = 1$ in the above theorem (1), then the subsequent corollaries are obtained, in that order.

Corollary 1. *If $f(\Theta)$ is derived by (1) which is belongs to class $\mathcal{G}^{\Sigma,G}_{u,v}(\tau)$. Then*

$$|a_2| \leq \sqrt{\frac{u^2(s)+u^2(t)+2(v(s)+v(t))}{2}}$$

and

$$|a_3| \leq \frac{u^2(s)+u^2(t)}{2} + \frac{u^2(s)-u^2(t)+2(v(s)-v(t))}{4}$$

Corollary 2. *If the function $f(\Theta) \in \mathcal{G}^{\Sigma,G}_{u,v}(\tau)$. Then*

$$|a_2| \leq \sqrt{\frac{u^2(s)+u^2(t)+2(v(s)+v(t))}{4}}$$

and

$$|a_3| \leq \frac{u^2(s)+u^2(t)}{8} + \frac{u^2(s)-u^2(t)+2(v(s)-v(t))}{12}$$

3. Fekete-Szegö inequality for the subclass $\mathcal{G}^{\Sigma,G}_{u,v}(\tau)$

Theorem 2. *Assume $f(\Theta)$ is provided by (1) and is a member of the class $\mathcal{G}^{\Sigma,G}_{u,v}(\tau)$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{v(s)}{(2+2\tau)} & \text{if } 0 \leq |h_1(\eta)| \leq \frac{1}{(2+2\tau)}, \\ v(s)|\frac{2(1-\eta)}{(2+2\tau)-4(1+\tau)^2}| & \text{if } |h_1(\eta)| \geq \frac{1}{(2+2\tau)}, \end{cases} \tag{21}$$

where

$$h_1(\eta) = \frac{(1-\eta)}{(2+2\tau)-4(1+\tau)^2}.$$

Proof. From the equation (16) and (17), we get

$$a_3 - \eta a_2^2 = a_2^2 + \frac{u^2(s)-u^2(t)+2(v(s)-v(t))}{4(1+\tau)^2} - \eta a_2^2$$

$$a_3 - \eta a_2^2 = (1-\eta)a_2^2 + \frac{u^2(s)-u^2(t)+2(v(s)-v(t))}{4(1+\tau)^2}$$

$$a_3 - \eta a_2^2 = (1-\eta)\frac{2(v(s)+v(t)}{(2+2\tau))-4(1+\tau)^2} + \frac{u^2(s)-u^2(t)+2(v(s)-v(t))}{4(1+\tau)^2}$$

$$a_3 - \eta a_2^2 = (1-\eta)\frac{2v(s)}{(2+2\tau)-4(1+\tau)^2} + \frac{u^2(s)-u^2(t)}{4(1+\tau)^2} + \frac{2(v(s)-v(t))}{4(1+\tau)^2}$$

$$\begin{aligned} a_3 - \eta a_2^2 = v(s)\left(\frac{2(1-\eta)}{(2+2\tau)-4(1+\tau)^2} + \frac{1}{2(1+\tau)^2}\right) \\ + v(t)\left(\frac{2(1-\eta)}{(2+2\tau)-4(1+\tau)^2} - \frac{1}{2(1+\tau)^2}\right) \end{aligned}$$

$$+\frac{u^2(s)-u^2(t)}{4(1+\tau)^2}$$

$$a_3-\eta a_2^2 = v(s)\left(h_1(\eta)+\frac{1}{2(1+\tau)^2}\right) + v(t)\left(h_1(\eta)-\frac{1}{2(1+\tau)^2}\right)$$

where

$$h_1(\eta)=\frac{2(1-\eta)}{(2+2\tau)-4(1+\tau)^2} \tag{22}$$

Hence,

$$|a_3-\eta a_2^2| \leq \begin{cases} \frac{v(s)}{(2+2\tau)} & \text{if } 0\leq |h_1(\eta)| \leq \frac{1}{(2+2\tau)}, \\ v(s)|\frac{2(1-\eta)}{(2+2\tau)-4(1+\tau)^2}| & \text{if } |h_1(\eta)| \geq \frac{1}{(2+2\tau)}. \end{cases}$$

Specializing the parameter values of $\tau = 0$ and $\tau = 1$ in the theorem (2), we obtain the following corollaries respectively.

Corollary 3. *Suppose that $f(\Theta)$ belongs to the class $\mathcal{G}_{u,v}^{\Sigma,G}(\tau)$ and is provided by* (1)*. Then*

$$|a_3-\eta a_2^2| \leq \begin{cases} v(s) & \text{if } 0\leq |h_1(\eta)| \leq \frac{1}{2}, \\ v(s)|(1-\eta)| & \text{if } |h_1(\eta)| \geq \frac{1}{2}, \end{cases}$$

where

$$|h_1(\eta)| = \frac{(1-\eta)}{2}.$$

Corollary 4. *Suppose that $f(\Theta)$ is belonging to the class $\mathcal{G}_{u,v}^{\Sigma,G}(\tau)$ and is provided by* (1)*. Then*

$$|a_3-\eta a_2^2| \leq \begin{cases} \frac{v(s)}{4} & \text{if } 0\leq |h_1(\eta)| \leq \frac{1}{4}, \\ v(s)|\frac{(1-\eta)}{6}| & \text{if } |h_1(\eta)| \geq \frac{1}{4}, \end{cases}$$

where

$$|h_1(\eta)| = \frac{(1-\eta)}{6}.$$

4. The initial Coefficient estimates for the subclass $\mathcal{G}^{\Sigma,G}(\tau)$

For $u(s) = s$ and $v(s) = 1$ in (2), the following Lucas polynomials $L_n(s)$ were produced. In the following, it is assumed that $G(s,\Theta)$ is a Taylor series of the form

$$G(s,\Theta)-1 = 1 + s\Theta + (2+s^2)\Theta^2 + (3s+s^3)\Theta^3 + (s^4+4s^2+2)\Theta^4 + ..., \tag{23}$$

$$G(t,\Phi)-1 = 1 + t\Phi + (2+t^2)\Phi^2 + (3t+t^3)\Phi^3 + (t^4+4t^2+2)\Phi^4 + ..., \tag{24}$$

Definition 2. *If a function $f \in \Sigma$ is regarded as a member of the class $\mathcal{G}^{\Sigma,G}(\tau)$, then the resulting subordination is valid.*

$$(1-\tau)\left(\frac{\Theta f'(\Theta)}{f(\Theta)}\right)+\tau\left(1+\frac{\Theta f''(\Theta)}{f'(\Theta)}\right)\prec G(s,\Theta)-1 \tag{25}$$

and

$$(1-\tau)\left(\frac{\Phi f'(\Phi)}{f(\Phi)}\right)+\tau\left(1+\frac{\Phi f''(\Phi))}{f'(\Phi)}\right)\prec G(t,\Phi)-1 \tag{26}$$

Example 3. *For the function f to be classified as belonging to the class $\mathcal{G}^{\Sigma,G}(\tau)$ when $\tau = 0$, it must meet a subordination condition. This constrains how f behaves in relation to other functions, frequently in terms of growth, distortion, or mapping properties in the unit disc.*

$$\left(\frac{\Theta f'(\Theta)}{f(\Theta)}\right))\prec G(s,\Theta)-1$$

and

$$\left(\frac{\Phi f'(\Phi)}{f(\Phi)}\right)\prec G(t,\Phi)-1$$

Example 4. *A function $f \in \Sigma$ is regarded as belonging to the class $\mathcal{G}^{\Sigma,G}(\tau)$ when $\tau = 1$ in definition (2) if it meets the subsequent subordination condition:*

$$\left(1+\frac{\Theta f''(\Theta)}{f'(\Theta)}\right)\prec G(s,\Theta)-1$$

and

$$\left(1+\frac{\Phi f''(\Phi))}{f'(\Phi)}\right)\prec G(t,\Phi)-1$$

Theorem 3. *If a function $f \in \Sigma$ is regarded as belonging to the class $\mathcal{G}^{\Sigma,G}(\tau)$, then the resulting subordination condition is valid:*

$$|a_2| \leq \sqrt{\frac{s^2+t^2+4}{4(1+2\tau)}} \tag{27}$$

and

$$|a_3| \leq \frac{s^2+t^2}{8(1+\tau)^2}+\frac{s^2-t^2}{4(1+2\tau)} \tag{28}$$

Proof: *Let $f \in \mathcal{G}^{\Sigma,G}(\tau)$. Then $f, g : \mathbb{D} \to \mathbb{D}$ given by (25) and (26) such that*

$$(1-\tau)\left(\frac{\Theta f'(\Theta)}{f(\Theta)}\right)+\tau\left(1+\frac{\Theta f''(\Theta)}{f'(\Theta)}\right)=G(s,\Theta)-1 \tag{29}$$

and

$$(1-\tau)\left(\frac{\Phi f'(\Phi)}{f(\Phi)}\right)+\tau\left(1+\frac{\Phi f''(\Phi))}{f'(\Phi)}\right)=L(t,\Phi)-1 \tag{30}$$

Since

$$(1-\tau)\left(\frac{\Theta f'(\Theta)}{f(\Theta)}\right)+\tau\left(1+\frac{\Theta f''(\Theta)}{f'(\Theta)}\right)=1+(1+\tau)a_2\Theta+(2(1+2\tau)a_3-(1+3\tau)a_2^2)\Theta^2+... \tag{31}$$

and

$$(1-\tau)\left(\frac{\Phi f'(\Phi)}{f(\Phi)}\right)+\tau\left(1+\frac{\Phi f''(\Phi))}{f'(\Phi)}\right)=1-(1+\tau)a_2\Phi+((3+5\tau)a_2^2-2(1+2\tau)a_3)\Phi^2+... \tag{32}$$

From the equations (23) *and* (31)*, By comparing coefficients of* Θ *and* Θ^2 *respectively, we get coefficient of* Θ :

$$(1+\tau)a_2=s \tag{33}$$

coefficient of Θ^2 :

$$2(1+2\tau)a_3-(1+3\tau)a_2^2=2+s^2 \tag{34}$$

From the equations (24) *and* (32)*, equating the coefficients of* Φ *and* Φ^2 *respectively, we get coefficient of* Φ :

$$-(1+\tau)a_2=t \tag{35}$$

coefficient of Φ^2 :

$$(3+5\tau)a_2^2-2(1+2\tau)a_3=2+t^2 \tag{36}$$

Now, adding the equations (33) *and* (35)*, we have*

$$s=-t \tag{37}$$

Squaring and adding the equations (33) *and* (35)*, we get*

$$2(1+\tau)^2a_2^2=s^2+t^2$$

$$a_2^2=\frac{s^2+t^2}{2(1+\tau)^2} \tag{38}$$

subtracting the equations (34) *and* (36)*, we get*

$$2(1+2\tau)a_3-(1+3\tau)a_2^2-(3+5\tau)a_2^2+2(1+2\tau)a_3=s^2-t^2$$

$$4(1+2\tau)a_3=s^2-t^2+(4+8\tau)a_2^2$$

$$a_3=a_2^2+\frac{s^2-t^2}{4(1+2\tau)} \tag{39}$$

substituting the equations (38) *in equation* (39)*, we get*

$$a_3=\frac{s^2-t^2}{4(1+2\tau)}+\frac{s^2+t^2}{2(1+\tau)^2} \tag{40}$$

substituting the equations (34) *in equation* (36), *we get*

$$2(1+2\tau)a_3-(1+3\tau)a_2^2+(3+5\tau)a_2^2+(2+2\tau)a_3=2+t^2$$

$$(2+2\tau)a_2^2=2+t^2+2+s^2$$

$$a_2^2=\frac{s^2+t^2+4}{2(1+\tau)} \tag{41}$$

substituting the equation (41) *in* (39), *we get*

$$a_3=\frac{s^2+t^2+4}{(2+2\tau)}+\frac{s^2-t^2}{4(1+2\tau)}$$

$$|a_3|\leq\frac{s^2+2}{2(1+\tau)} \tag{42}$$

Hence,

$$|a_2|\leq\sqrt{\frac{s^2+t^2+4}{(2+2\tau)}}$$

$$|a_3|\leq\frac{s^2+t^2}{2(1+\tau)^2}+\frac{s^2-t^2}{4(1+2\tau)}$$

The next couple of consequences are obtained by specializing the parameter values of $\tau=0$ and $\tau=1$ in the theorem (3), respectively.

Corollary 5. *The function $f(\Theta)$ satisfies a couple of conditions to be included in this subclass of bi-univalent functions if it is provided by* (1) *and is a member of the class $\mathcal{G}^{\Sigma,G}(\tau)$. Then*

$$|a_2|\leq\sqrt{\frac{s^2+t^2+4}{2}}$$

and

$$|a_3|\leq\frac{s^2+t^2}{2}+\frac{s^2-t^2}{4}$$

Corollary 6. *The function $f(\Theta)$ satisfies a couple of conditions to be included in this subclass of bi-univalent functions if it is provided by* (1) *and is a member of the class $\mathcal{G}^{\Sigma,G}(\tau)$. Then*

$$|a_2|\leq\sqrt{\frac{s^2+t^2+4}{4}}$$

and

$$|a_3|\leq\frac{s^2+t^2}{8}+\frac{s^2-t^2}{12}$$

5. Fekete-Szegö inequality for the subclass $\mathcal{G}^{\Sigma,G}(\tau)$

Theorem 4. *If $f(\Theta)$ is given by (1) be in the class $\mathcal{G}^{\Sigma,G}(\tau)$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{s^2}{(1+2\tau)} & \text{if } 0 \leq |h_1(\eta)| \leq \frac{1}{4(1+2\tau)}, \\ 2s^2|\frac{(1-\eta)}{2(1+\tau)^2}| & \text{if } |h_1(\eta)| \geq \frac{1}{4(1+2\tau)}, \end{cases} \tag{43}$$

where

$$h_1(\eta) = \frac{(1-\eta)}{2(1+\tau)^2}.$$

Proof. From the equations (38) *and* (39)*, we get*

$$a_3 - \eta a_2^2 = a_2^2 + \frac{s^2 - t^2}{4(1+2\tau)} - \eta a_2^2$$

$$a_3 - \eta a_2^2 = (1-\eta)a_2^2 + \frac{s^2 - t^2}{4(1+2\tau)}$$

$$a_3 - \eta a_2^2 = (1-\eta)\frac{s^2 + t^2}{2(1+\tau^2)} + \frac{s^2 - t^2}{4(1+2\tau)}$$

$$a_3 - \eta a_2^2 = s^2\left(\frac{(1-\eta)}{2(1+\tau)^2} + \frac{1}{4(1+2\tau)}\right) + t^2\left(\frac{(1-\eta)}{2(1+\tau)^2} - \frac{1}{4(1+2\tau)}\right)$$

$$a_3 - \eta a_2^2 = s^2\left(h_1(\eta) + \frac{1}{4(1+2\tau)}\right) + t^2\left(h_1(\eta) - \frac{1}{4(1+2\tau)}\right)$$

where

$$h_1(\eta) = \frac{(1-\eta)}{2(1+\tau)^2} \tag{44}$$

Hence,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{s^2}{(1+2\tau)} & \text{if } 0 \leq |h_1(\eta)| \leq \frac{1}{4(1+2\tau)}, \\ 2s^2|\frac{(1-\eta)}{2(1+\tau)^2}| & \text{if } |h_1(\eta)| \geq \frac{1}{4(1+2\tau)}. \end{cases}$$

The two subsequent consequences are obtained by changing the parameter values in the theorem (4) to $\tau = 0$ and $\tau = 1$, respectively.

Corollary 7. *If the function $f(\Theta)$ is defined by* (1) *and belongs to the class $\mathcal{G}^{\Sigma,G}(\tau)$, then it exhibits specific geometric and analytic property $|a_3 - \eta a_2^2|$ that characterize this subclass of functions. i.e.,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} s^2 & \text{if } 0 \leq |h_1(\eta)| \leq \frac{1}{4}, \\ 2s^2|\frac{(1-\eta)}{2}| & \text{if } |h_1(\eta)| \geq \frac{1}{4}, \end{cases}$$

This implies

$$|a_3 - \eta a_2^2| \leq \begin{cases} s^2 & \text{if } 0 \leq |h_1(\eta)| \leq \frac{3}{4}, \\ s^2|(1-\eta)| & \text{if } |h_1(\eta)| \geq \frac{3}{4}, \end{cases}$$

where

$$h_1(\eta) = \frac{(1-\eta)}{2}.$$

Corollary 8. *The function $f(\Theta)$ possesses particular geometric feature $|a_3 - \eta a_2^2|$ of function $f(\Theta)$, it is defined by* (1) *and it is a member of the class $\mathcal{G}^{\Sigma,G}(\tau)$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{s^2}{3} & \text{if } 0 \leq |h_1(\eta)| \leq \frac{1}{12}, \\ 2s^2|\frac{(1-\eta)}{8}| & \text{if } |h_1(\eta)| \geq \frac{1}{12}, \end{cases}$$

This implies

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{s^2}{3} & \text{if } 0 \leq |h_1(\eta)| \leq \frac{7}{8}, \\ s^2|\frac{(1-\eta)}{4}| & \text{if } |h_1(\eta)| \geq \frac{7}{8}, \end{cases}$$

where

$$h_1(\eta) = \frac{(1-\eta)}{8}.$$

6. Discussion: Influence of Parameter s and Surface Plot Interpretation

For special values of s, the expansion may reduce to well-known families of polynomials:

- When $s = 0$: only even powers of Θ survive with relatively simple coefficients — suggesting a resemblance to Chebyshev-type behavior.
- When $s = 1$: the sequence does not exactly match Fibonacci or Lucas, but the structure mimics a nonlinear recurrence.
- When $s = 2$: the pattern aligns closely with growth seen in Bell or Motzkin-like structures, potentially capturing partition-related or moment-generating behavior.

From a generating function perspective, the parameter s may encode a deformation, weight, or even a flow parameter in a functional or combinatorial system. However, because the expansion is polynomial in s, the Fibonacci sequence is not recovered directly for any constant value of s. Moreover, the appearance of higher powers of s than allowed in classical recursions indicates that the structure diverges from a standard Lucas recurrence.

The parameter s plays a crucial role in shaping the behavior of the Fekete–Szegö inequality, particularly in its effect on the sharpness and structure of the bound for the third coefficient. From the theorem (4) bound is given by:

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{s^2}{1+2\tau}, & \text{if } |h_1(\eta)| \leq \dfrac{1}{4(1+2\tau)}, \\ 2s^2 \left| \dfrac{1-\eta}{2(1+\tau)^2} \right|, & \text{if } |h_1(\eta)| \geq \dfrac{1}{4(1+2\tau)}, \end{cases}$$

where $h_1(\eta) = \dfrac{1-\eta}{2(1+\tau)^2}$.

The corresponding surface plot of the bound $|a_3 - \eta a_2^2|$ over the complex η-plane reveals a piecewise smooth, radially structured surface centered at $\eta = 1$. The height of the surface at each point corresponds to the maximum allowable deviation of the coefficient a_3 in terms of a_2, modulated by the geometry of η and the analytic parameter τ. The region defined by:

$$|1 - \eta| \leq \frac{1}{2}$$

marks a flat plateau where the bound remains constant. Beyond this threshold, the bound increases linearly with $|1 - \eta|$, creating a rising surface outward from $\eta = 1$.

For a fixed value of $\tau = 1$, the shape and scale of the surface are strongly influenced by the parameter s. This dependency is clearly shown in Figures 1-3, where the behavior of the surface is illustrated for different values of s. Variations in s significantly alter the surface characteristics, showcasing how the bound's structure evolves as the parameter changes.

The surface plot of the bound $|a_3 - \eta a_2^2|$ provides a clear visual interpretation of this analytic bound. When $s = 0$, the bound becomes zero, reflecting a highly constrained structure similar to Chebyshev-like behavior in the generating function. As s increases to 1, the surface shows a mild elevation and nonlinear coefficient growth, but it does not align with Fibonacci or Lucas forms. For $s = 2$, the surface rises more sharply, exhibiting coefficient growth that resembles Bell or Motzkin-like structures.

The interaction between s and η demonstrates how geometric deviations and deformations control the flexibility and complexity of the function class $\mathcal{G}^{\Sigma,G}(\tau)$. This interplay is crucial in determining the structure of the function, providing a deeper understanding of the role of second-order coefficient bounds in geometric function theory.

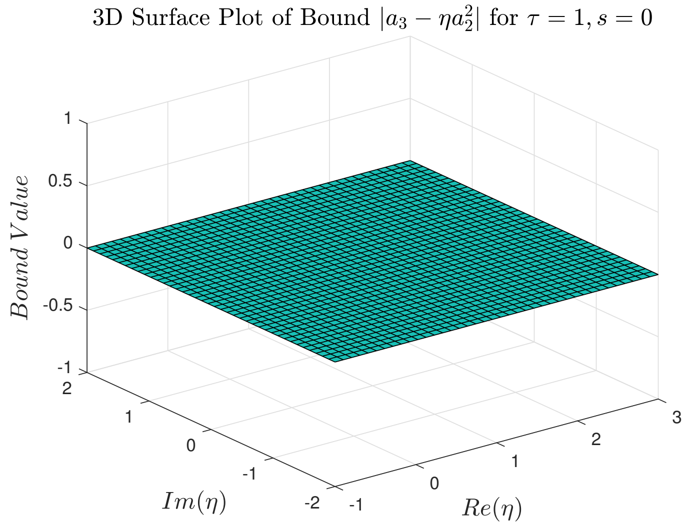

Figure 1: For $s = 0$, the bound is zero, indicating constrained geometry with a Chebyshev-like generating function.

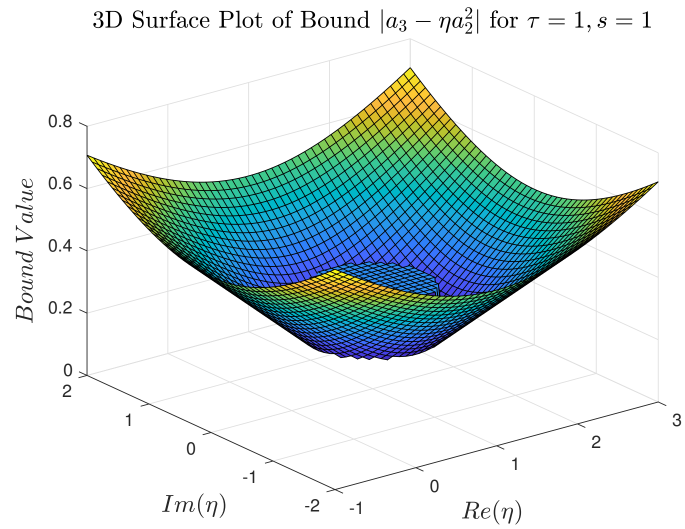

Figure 2: For $s = 1$, the surface shows mild elevation and nonlinear coefficient growth, without matching Fibonacci or Lucas forms.

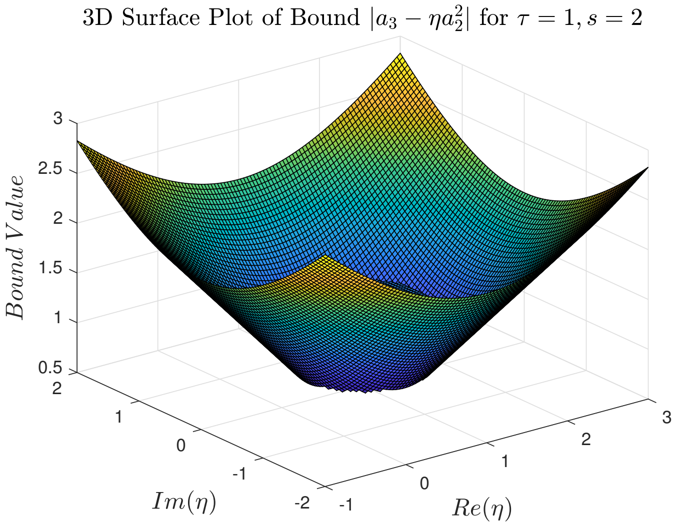

Figure 3: For $s = 2$, the surface rises steeply, resembling coefficient behavior in Bell or Motzkin-like structures.

7. Conclusion

In conclusion, the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for two new subclasses of bi-univalent functions related to Lucas polynomial functions are precisely determined in this study. The obtained coefficients and Fekete-Szegö estimates reveal that our subclass of bi-univalent functions demonstrates improved control over its first few coefficients compared to the previously studied subclasses. This has implications for understanding the geometric properties of these functions, particularly in their growth and distortion behavior. The classical theory is interestingly improved by the novel subclass of bi-univalent functions, especially in relation to initial coefficients and the Fekete-Szegö inequality. Future studies might concentrate on applying these discoveries to particular geometric issues in univalent function theory or expanding them to higher-order coefficients. Analyzing these initial coefficients helps us better understand these functions' geometric, growth, and distortion characteristics, as well as how they behave close to the origin and relate to other bi-univalent subclasses. This work builds upon and improves upon earlier studies by using the Fekete-Szegö inequality, providing a better understanding of the functional and mapping properties. These results have wide-ranging implications in fields where the exact behavior of bi-univalent functions is crucial, such as engineering, fluid dynamics, and complex analysis.

Acknowledgements

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant

No. KFU252106].

Credit authorship contribution statement

Stalin Thangamani: Investigation, Writing – review & editing. **Dumitru Baleanu:** Conceptualization, Project administration, Software. **Jeno Francis:** Conceptualization, Project administration. **Majeed Ahmad Yousif:** Software, Writing – review & editing. **Meraa Arab:** Funding acquisition, Methodology, Writing – review & editing. **Pshtiwan Othman Mohammed:** Conceptualization, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and material

No data was used for the research described in the article.

References

[1] P. Koebe. Ueber die uniformisierung beliebiger analytischer kurven ii. *Nachr. k. Ges. Wissenschaft. Gottinger Math. Phys.*, pages 177–198, 1907.

[2] M. Lewin. On a coefficient problem for bi-univalent functions. *Proc. Amer. Math. Soc.*, 18:63–68, 1967.

[3] D. A. Brannan and J. G. Clunie. *Aspects of Contemporary Complex Analysis.* Academic Press, New York and London, 1980.

[4] D. A. Brannan and T. S. Taha. On some classes of bi-valent functions. *Studia Univ. Babes-Bolyai Math.*, 31:70–77, 1986.

[5] W. C. Ma and D. Minda. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis, Tianjin*, Conf. Proc. Lecture Notes Anal., pages 157–169, Cambridge, MA, 1992.

[6] A. Behera and G. K. Panda. On the square roots of triangular numbers. *Fibonacci Quart.*, 37:98–105, 1999.

[7] B. A. Frasin and M. K. Aouf. New subclasses of bi-univalent functions. *Appl. Math. Lett.*, 24:1569–1573, 2011.

[8] D. Bansal and J. Sokol. Coefficient bound for a new class of analytic and bi-univalent functions. *J. Fract. Calc. Appl.*, 5(1):122–128, 2014.

[9] A. Y. Lashin. On certain subclasses of analytic and bi-univalent functions. *J. Egyptian Math. Soc.*, 24:220–225, 2016.

[10] A. O. Pall-Szabo and G. I. Oros. Coefficient related studies for new classes of bi-univalent functions. *Mathematics*, 8:1110, 2020.

[11] K. A. Jassim, R. O. Rasheed, and R. H. Jassim. Generalized subclass of analytic bi-univalent functions defined by differential operator. *J. Interdiscip. Math.*, 24:961–970, 2021.

[12] G. Lee and M. Asci. Some properties of the (p, q)-fibonacci and (p, q)-lucas polynomials. *Journal of Applied Mathematics*, 2012:264842, 2012.

[13] P. K. Ray. Certain matrices associated with balancing and lucas-balancing numbers. *Matematika*, 28:15–22, 2012.

[14] S. Altinkaya and S. Yacin. On the (p, q)-lucas polynomial coefficient bounds of bi-univalent function class. *Boletín de la Sociedad Matemática Mexicana*, 25:567–575, 2019.

[15] Y. Almalki, A. K. Wanas, T. G. Shaba, A. Alb Lupaş, and M. Abdalla. Coefficient bounds and fekete–szegö inequalities for a two families of bi-univalent functions related to gegenbauer polynomials. *Axioms*, 12:1018, 2023.

[16] Arzu Akgül. (p, q)-lucas polynomial coefficient inequalities of the bi-univalent function class. *Turkish Journal of Mathematics*, 43(5):2170–2176, 2019.

[17] S. R. Swamy, A. K. Wanas, and Y. Sailaja. Some special families of holomorphic and salagean type bi-univalent functions associated with (m,n)-lucas polynomials. *Mathematics*, 11(4):563–574, 2020.

[18] N. Magesh, C. Abirami, and Ş. Altınkaya. Initial bounds for certain classes of bi-univalent functions defined by the (p, q)-lucas polynomials. *TWMS Journal of Applied and Engineering Mathematics*, 11(1):282–288, 2021.

[19] A. K. Wanas and L. I. Cotîrlă. Applications of (m, n)-lucas polynomials on a certain family of bi-univalent functions. *Mathematics*, 10(4):595, 2022.

[20] A. Hussen and M. Illafe. Coefficient bounds for a certain subclass of bi-univalent functions associated with lucas-balancing polynomials. *Mathematics*, 11:4941, 2023.

[21] İ. Aktaş and İ. Karaman. On some new subclasses of bi-univalent functions defined by balancing polynomials. *Karamanoğlu Mehmetbey Univ. Journal of Engineering and Natural Sciences*, 5:25–32, 2023.

[22] A. K. Wanas, G. Ş. Sălăgean, and Á. P. S. Orsolya. Coefficient bounds and fekete-szegő inequality for a certain family of holomorphic and bi-univalent functions defined by (m, n)-lucas polynomials. *Filomat*, 37(4):1037–1044, 2023.

[23] A. K. Wanas, E. K. Wanas, A. Cătaş, and M. Abdalla. Applications of (m,n)-lucas polynomials for a certain family of bi-univalent functions associating λ-pseudo-starlike functions with sakaguchi type functions. *Earthline Journal of Mathematical Sciences*, 15(1):1–10, 2024.

[24] A. Hussen, M. S. A. Madi, and A. M. M. Abominjil. Bounding coefficients for certain subclasses of bi-univalent functions related to lucas-balancing polynomials. *AIMS Mathematics*, 9(7):18034–18047, 2024.

[25] M. Buyankara and M. Çağlar. Coefficient inequalities for two new subclasses of bi-univalent functions involving lucas-balancing polynomials. *Eastern Anatolian Journal of Science*, 10(2):5–11, 2024.

[26] P. L. Duren. *Univalent Functions*, volume 259 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, New York, 1983.