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Bayesian Analysis of the Lomax-Power Rayleigh (T-X) Distribution

Muhammad Ijaz[1], Naeem Khan[1], M. I. Khan[2,*], Ghulam Mustafa[3], Hana N. Alqifari[4]

[1] *Department of Mathematics and Statistics, The University of Haripur, Haripur, Pakistan*
[2] *Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, 42351, Saudi Arabia*
[3] *Department of Statistics, Islamia College Peshawar, Peshawar, Pakistan*
[4] *Department of Statistics and Operations Research, College of Science, Qassim University, Saudi Arabia*

Abstract. This study presents a Bayesian analysis of the Lomax-Power Rayleigh (T-X) distribution using both uniform and Jeffery priors. Bayes estimators are derived under four different loss functions: squared error loss, quadratic loss, weighted loss, and precautionary loss. The performance of the proposed estimators is demonstrated through simulation studies and a real-world dataset. Results indicate that the quadratic loss function yields the most reliable parameter estimates.

2020 Mathematics Subject Classifications: 47N30, 62C10, 62C12, 62XX

Key Words and Phrases: Uniform and Jeffery priors, Loss functions, application, Bayes theorem, simulation

1. Introduction

Probability distributions are central to modeling uncertainty in a variety of fields, including reliability analysis, life testing problems, engineering, agriculture sciences, and econometrics etc. Recently, many advance probability distributions have been derived by researchers to model some more complex and large data structures better than the existing probability distributions. For example, recently a lot of new versions of the Lomax distribution have been derived, e.g [1] derived X-Gamma Lomax distribution, [2] presented Inverse Power Lomax distribution, [3] discuss Lomax Power Rayleigh distribution. For more study, we refer to see [4–8].

*Corresponding author.
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Email addresses: m.ijaz@uoh.edu.pk (M. Ijaz), prof.naeemkhan1@gmail.com (N. Khan), izhar.stats@gmail.com, khanizhar@iu.edu.sa (M. I. Khan), ghulam29@gmail.com (G. Mustafa), hn.alqifari@qu.edu.sa (H. N. Alqifari)

Bayesian analysis is always preferable when we are given some prior information. Bayesian analyses are found to be better than classical statistics when there are extreme values in data. A large number of problems have been tackle using Bayesian analysis under different probability distributions, for example, [9] discussed Bayesian analysis of for skew symmetric distribution, [10] explore Bayesian and non-Bayesian inference for Truncated Cauchy power Weibull-G class of distributions, [11] considered Lomax distribution with four loss functions, [12] discussed the Bayesian analysis of the Extended Lomax distribution.
Although the Lomax-Power Rayleigh (LPR) distribution is highly flexible for modeling real-world phenomena, it has received little attention within a Bayesian framework. This study aims to bridge this gap by providing a comprehensive Bayesian analysis of the LPR distribution. By deriving Bayes estimators under different priors and loss functions, we offer researchers and practitioners a novel and practical approach for modeling and analyzing data that follows this distribution. This research is motivated by the growing need for accurate and reliable statistical methods in fields such as reliability analysis, survival analysis, and actuarial science.

1.1. Motivation

Bayesian estimation combines prior information with new data to inform statistical inference. [3] introduced the Lomax-Power Rayleigh Distribution (LPRD), a four-parameter distribution with various applications in real-world datasets. The LPRD performance in fitting real-life data sets, particularly in bioscience, motivated us to explore its Bayesian estimation. This paper aims to estimate the parameters of the LPRD using a Bayesian approach and compare the results with the classical approach proposed by [3]. The LPRD (T-X) flexibility and performance make it suitable for applications in fields like engineering, survival analysis, hydrology, economics, and bioscience.
In this study, we considered uniform and Jeffery priors which are commonly used because of the non-informative nature. These priors contribute little subjective information, making them suitable when there is no strong information about the parameter values. Secondly, prior information is not available and hence these priors can be used as benchmarks or as reference priors. Thirdly, Jeffery priors are dominant over others because of their invariance property. The paper is structured as follows: Section 2 introduces the LPRD (T-X)and explores its likelihood-based parameter estimation. Section 3 derives Bayesian estimation methods for the LPRD parameters. Section 4 discusses the simulation setup used in the study. Section 5 presents the analysis and results, while Section 6 concludes the study, outlining future research directions and contributions.

2. Methods and Material

2.1. Lomax-Power Rayleigh (T-X) distribution

Lomax-Power Rayleigh (T-X) distribution introduced by [3] is a statistical distribution combining Lomax and Power Rayleigh distributions using Transformed-Transformer (T-X) family method. It is a flexible distribution for modeling various data types, particularly those with heavy tails and positive skewness. Let a random variable X follow a Lomax-Power Rayleigh (T-X) distribution with the following density function:

$$f(x) = \frac{\alpha\theta\lambda}{\beta^2} x^{2\alpha-1}\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)^{-(\theta+1)}, \;\; \alpha,\beta,\lambda,\theta \geq 0,\;\; x \geq 0 \tag{1}$$

where, α, λ are shape parameters and β and θ are scale parameters.
The corresponding CDF is given by

$$F(x) = \left(\frac{2\beta^2}{2\beta^2+\lambda x^{2\alpha}}\right)^{\theta}, \;\; \alpha,\beta,\lambda,\theta \geq 0,\;\; x \geq 0$$

The hazard rate function of LPRD (T-X) is defined as

$$h(x) = 2\alpha\theta\lambda\left(\frac{2\beta^2 x^{2\alpha}}{x^{2\alpha}}+\lambda\right)^{-1}, \;\; \alpha,\beta,\lambda,\theta \geq 0,\;\; x \geq 0$$

1 defines the plot of PDF and CDF while 2 explains different shapes of the hazard rate functions.

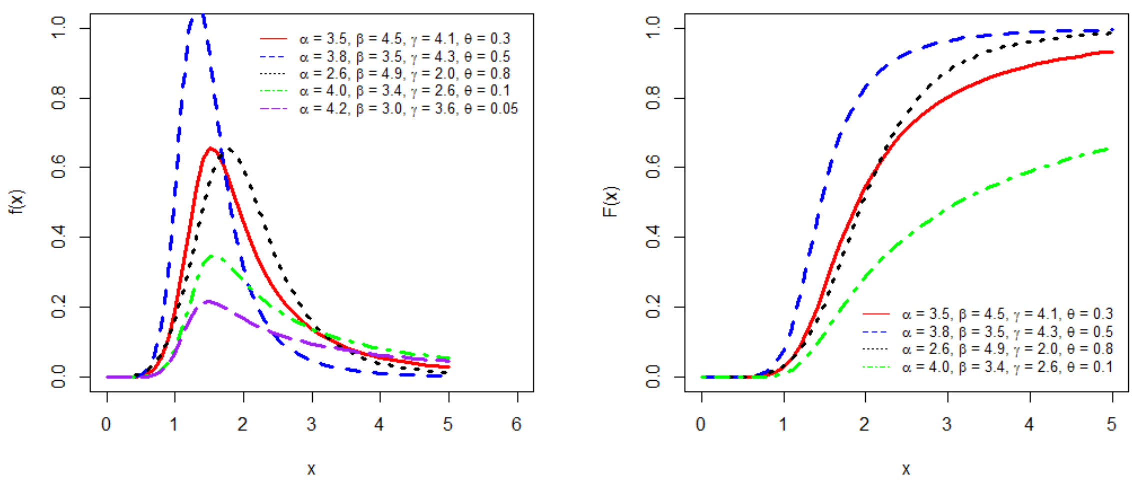

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Figure 1: Pdf and CDF Plots of LPRD (T-X)

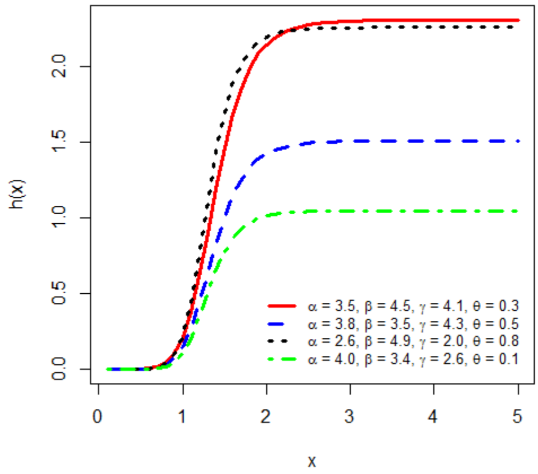

Figure 2: Hazard rate function of LPRD (T-X)

2.2. The classical estimation

The likelihood function of (1) which introduced by [3] becomes

$$\prod_{i=0}^{n} f(x) = \left(\frac{\alpha\theta\lambda}{\beta^2}\right)^n \prod_{i=0}^{n} x^{2\alpha-1} e^{\sum_{i=0}^{n}\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)^{-(\theta+1)}} \tag{2}$$

$$L = \prod_{i=1}^{n} f(x) = \left(\frac{\alpha\theta\lambda}{\beta^2}\right)^n \prod_{i=0}^{n} x^{2\alpha-1}\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)^{-(\theta+1)}$$

The log- likelihood function

$$\log L = n\log\alpha + n\log\theta + n\log\lambda \ - 2n\log\beta + (2\alpha-1)\sum_{i=1}^{n}\log(x_i)$$

$$-(\theta+1)\sum_{i=1}^{n}\log\left(2\beta^2+\lambda x^{2\alpha}\right) + (\theta+1)\sum_{i=1}^{n}\log\left(2\beta^2\right)$$

By partially differentiating the above equation with respect to α, θ and λ respectively and setting it equal to zero, we obtain the following estimators for the parameters:

$$\hat{\alpha} = \frac{n}{2\sum_{i=1}^{n} x_i\left[(\theta+1)\lambda-1\right]}, \hat{\theta} = \frac{n}{2\alpha\lambda\sum_{i=1}^{n} x_i}, \hat{\lambda} = \frac{n}{2\alpha(\theta+1)\sum_{i=1}^{n} x_i}.$$

These estimators are also presented by [3].

3. Bayesian estimation

3.1. Posterior distribution under Uniform prior

The posterior distribution of LPR under a uniform prior is defined by

$$f(\theta\ /x_i) = \frac{\prod_{i=0}^{n} f(x_i\ /\theta)g(\theta)}{\int_0^{\infty}\prod_{i=0}^{n} f(x_i\ /\theta)g(\theta)d\theta} \tag{3}$$

where $g(\theta) \propto 1$ is the prior probability function and $\prod_{i=0}^{n} f(x_i/\theta)$ is the MLE of the Lomax-Power Rayleigh (T-X) distribution. Using Eq (2), we get

$$f(\theta\ /x_i) = \frac{C^{(n+1)}}{\Gamma(n+1)}\theta^n e^{-\theta C}.$$

This indicates that $f(\theta/x_i)$, $\Gamma(n{+}1, C)$ and $C{=}\sum_{i=0}^{n} log\left(1{+}\frac{\lambda x^{2\alpha}}{2\beta^2}\right)$.

3.2. Posterior distribution under Jeffery prior

The posterior distribution of LPR under Jeffery prior is defined by

$$f(\theta\ /x_i) = \frac{\prod_{i=0}^{n} f(x_i\ /\theta)g(\theta)}{\int_0^{\infty}\prod_{i=0}^{n} f(x_i\ /\theta)g(\theta)d\theta} \tag{4}$$

Where $g(\theta) = E\left[\frac{\partial^2 \log L(x_i,\theta)}{\partial\lambda^2}\right] = \frac{1}{\theta}$ is the Jeffery prior and $\prod_{i=0}^{n} f(x_i/\theta)$ is the MLE of the Lomax-Power Rayleigh (T-X) distribution. Using Eq (2) in (4), we get

$$f(x_i/\theta) = \frac{C^{(n)}}{\Gamma n}\theta^{n-1}e^{-\theta C}$$

This indicates that $f(\theta/x_i)$, $\Gamma(n+1, C)$ and $C = \sum_{i=0}^{n} log\left(1 + \frac{\lambda x^{2\alpha}}{2\beta^2}\right)$.

3.3. Bayesian estimates under Uniform prior

In this section, Bayesian estimates are derived under four loss functions by using a unifor prior

3.3.1. Bayesian Estimates under Squared error loss function (SELF)

The SELF is defined by

$$\hat{\theta}_{SELF} = (\widehat{\theta}{-}\theta)^2$$

The above expression yields

$$\hat{\theta}\int \frac{C^{(n+1)}}{\Gamma(n+1)}\theta^n e^{-\theta C}d\theta - \int \theta\frac{C^{(n+1)}}{\Gamma(n+1)}\theta^n e^{-\theta C}d\lambda{=}\ 0.$$

The result for θ is given below

$$\hat{\theta}_{SELF} = \frac{(n+1)}{C} \tag{5}$$

3.3.2. Bayesian Estimates under Quadratic error loss function (QELF)

The QELF is defined by

$$\hat{\theta}_{QELF} = \left(\frac{\hat{\theta}-\theta}{\theta}\right)^2$$

Simplifying the above equation for θ, we get

$$\hat{\theta}\int\frac{1}{\theta^2}\frac{C^{(n+1)}}{\Gamma\left(n+1\right)}\theta^n e^{-\theta C}d\theta - \int\frac{1}{\theta}\frac{C^{(n+1)}}{\Gamma\left(n+1\right)}\theta^n e^{-\theta C}d\theta = 0$$

Simplifying the above expression, we get

$$\hat{\theta}_{QELF} = \frac{(n-1)}{C} \tag{6}$$

3.3.3. Bayesian Estimates under Weighted error loss function (WELF)

The WELF is given by

$$\hat{\theta}_{WELF} = \frac{\left(\hat{\theta}-\theta\right)^2}{\theta}$$

Simplifying the above equation for θ, we get

$$\hat{\theta}\int\frac{1}{\theta}\frac{C^{(n+1)}}{\Gamma\left(n+1\right)}\theta^n e^{-\theta C}d\lambda - \int\frac{C^{(n+1)}}{\Gamma\left(n+1\right)}\theta^n e^{-\theta C}d\lambda = 0$$

Simplifying the above expression, we get

$$\hat{\theta}_{WELF} = \frac{n}{C}. \tag{7}$$

3.3.4. Bayesian Estimates under Precautionary error loss function (PELF)

The PELF [13] can be defined as

$$\hat{\theta}_{PELF} = \frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}}$$

Simplifying further the above equation for λ

$$\hat{\theta}\int\frac{C^{(n+1)}}{\Gamma\left(n+1\right)}\theta^n e^{-\theta C}d\theta - \frac{1}{\hat{\theta}^2}\int\theta^2\frac{C^{(n+1)}}{\Gamma\left(n+1\right)}\theta^n e^{-\theta C}d\theta = 0$$

Hence, we obtained the following expression

$$\hat{\theta}_{PELF} = \frac{\sqrt{\{(n+1)\,(n+2)\}}}{C}. \tag{8}$$

3.4. Bayesian estimates under Jeffery prior

This section elloborates Bayesian estimates under Jeffery priors by using four loss functions.

3.4.1. Bayesian estimates under Squared error loss function (SELF)

The SELF is defined by

$$\hat{\theta}_{SELF} = \left(\hat{\theta} - \theta\right)^2$$

The above expression produced

$$\hat{\theta} \int \frac{C^{(n)}}{\Gamma(n)} \theta^{(n-1)} e^{-\theta C} d\theta - \int \theta \frac{C^{(n)}}{\Gamma(n)} \theta^{(n-1)} e^{-\theta C} d\theta = 0$$

We obtained the following result

$$\hat{\theta}_{SELF} = \frac{n}{C} \tag{9}$$

3.4.2. Bayesian estimates under Quadratic error loss function (QELF)

The QELF is given by

$$\hat{\theta}_{QELF} = \left(\frac{\hat{\theta} - \theta}{\theta}\right)^2$$

After simplification, the following expression is obtained

$$\hat{\theta} \int \frac{1}{\theta^2} \frac{C^{(n)}}{\Gamma(n)} \theta^{(n-1)} e^{-\theta C} d\theta - \int \frac{1}{\theta} \frac{C^{(n)}}{\Gamma(n)} \theta^{(n-1)} e^{-\theta C} d\theta = 0$$

Finally, we obtainedthe result

$$\hat{\theta}_{QELF} = \frac{(n-2)}{C}. \tag{10}$$

3.4.3. Bayesian estimates under Weighted error loss function (WELF)

The WELF can be explained by

$$\hat{\theta}_{WELF} = \frac{\left(\hat{\theta} - \theta\right)^2}{\theta}$$

Simplifying the above equation, we get

$$\widehat{\theta} \int \frac{1}{\theta} \frac{C^{(n)}}{\Gamma(n)} \theta^{(n-1)} e^{-\theta C} d\theta - \int \frac{C^{(n)}}{\Gamma(n)} \theta^{(n-1)} e^{-\theta C} d\lambda = 0$$

Lastly, we obtained the following result

$$\hat{\theta}_{WELF} = \frac{n}{C}. \tag{11}$$

3.4.4. Bayesian estimates under Precautionary error loss function (PELF)

The PELF [13] is defined by

$$\hat{\theta}_{PELF} = \frac{\left(\hat{\theta} - \theta\right)^2}{\hat{\theta}}$$

The Bayes estimator $\hat{\theta}$ can be defined as

$$\hat{\theta} \int \frac{C^{(n)}}{\Gamma(n)} \theta^{(n-1)} e^{-\theta C} d\lambda - \frac{1}{\hat{\theta}^2} \int \theta^2 \frac{C^{(n)}}{\Gamma(n)} \theta^{(n-1)} e^{-\theta C} d\theta = 0$$

Hence, we determined the result

$$\hat{\theta}_{PELF} = \frac{\sqrt{\{n(n+2)\}}}{C} \tag{12}$$

4. Practical Applications

Bayesian Analysis Application

This section illustrates the practical application of the Bayesian analysis of the Lomax-Power Rayleigh (T-X) distribution developed in this study. By employing both uniform and Jeffery priors, we evaluate the performance of the Bayesian estimators using real and simulated datasets. A real dataset, sourced from existing literature [14], is analyzed to demonstrate the effectiveness of the proposed approach.

4.1. Real Data Analysis

The real data set defines the failure time of 84 windshields for a particular model of aircraft (the unit for measurement is 1000 hours) with the following values:
0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557,1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981,2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890,4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255,1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615,2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300,3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Table 1 demonstrates that with the fixed values of all other parameters rather than θ, the result of QELF perform better than others when the uniform and Jeffery priors are considered. Similarly, Table 2 declared that as the sample size grows, the MSE decreases. Figure 3 clearly demonstrates that with the increase in sample size, the MSE of Bayesian estimates are decreased in both Uniform and Jeffery priors.

Table 1: Estimated value and MSE under uniform and Jeffery Priors

Uniform Prior, when $n = 50, \alpha = 3.5, \beta = 0.05, \lambda = 1.5$					**Jeffery's Prior, when** $n = 50, \alpha = 1.5, \beta = 1.0, \lambda = 0.5$				
θ	SELF	QELF	WELF	PELF	θ	SELF	QELF	WELF	PELF
1.5	0.0681	1.4998	1.5145	1.5385	1.5	0.0001	0.5969	0.6096	0.6284
	2.0503	0.0227	0.0238	0.0258		0.2500	0.0104	0.0130	0.0176
1	2.0377	1.9935	2.0186	2.0434	2.5	0.6216	0.5975	0.6091	0.6281
	0.3305	0.2830	0.3109	0.3378		0.0159	0.0105	0.0129	0.0175
2.5	2.5481	2.5007	2.5263	2.5604	3.5	0.6213	0.5968	0.6095	0.6282
	1.1682	1.0641	1.1191	1.1907		0.0158	0.0103	0.0130	0.0175
3	3.0657	3.0060	3.0342	3.0753	4.5	0.6224	0.5977	0.6092	0.6279
	2.5454	2.3598	2.4488	2.5758		0.0161	0.0106	0.0129	0.0175
3.5	3.5619	3.5101	0.0685	3.5855	5.5	0.6227	0.5970	0.6103	0.6282
	4.4103	4.1661	2.0494	4.4808		0.0161	0.0104	0.0132	0.0175

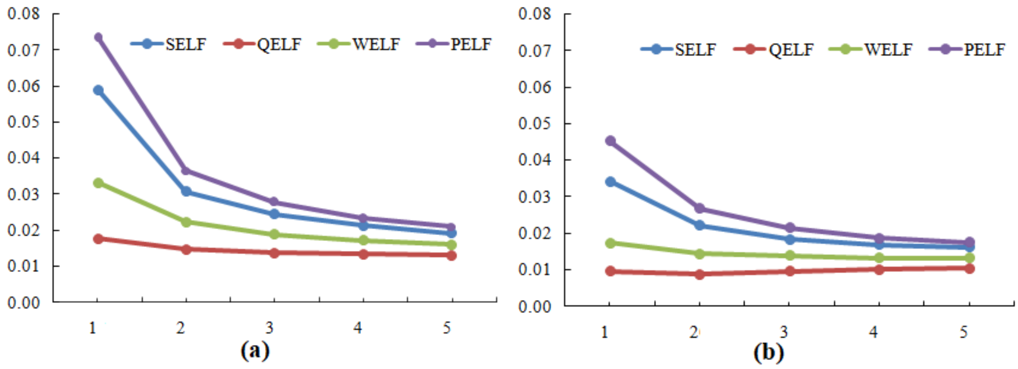

Figure 3: (a) MSE under a Uniform Prior; (b) MSE under Jeffery Prior

5. Simulation Analysis

In this section, a simulation study has been carried by choosing different parameter values with replication of 5000 times. The following quantile function was used to simulate data

$$x = \left[\frac{2\beta^2}{\lambda}\left\{(1-U)^{\frac{-1}{\theta}} - 1\right\}\right]^{\frac{1}{2\alpha}}$$

Table 3, and Table 4 conclude that with different parameter values of θ and n, the MSE of QELF is lower than the MSE of all other error loss functions.

Figure 4 declares that the smaller MSE are attained in both under a uniform and Jeffery priors when the QELF is considered. As we increase the sample of size n, the results of WELF becomes closure to QELF.

Table 2: Estimated value and MSE under uniform and Jeffery Priors

Uniform Prior, when $\alpha = 1.5, \beta = 1.0, \lambda = 0.5, \theta = 0.8$					**Jeffery's Prior, when** $\alpha = 1.5, \beta = 1.0, \lambda = 0.5, \theta = 0.5$				
n	SELF	QELF	WELF	PELF	n	SELF	QELF	WELF	PELF
10	0.7028	0.5753	0.6379	0.7336	10	0.6402	0.5124	0.5755	0.6714
	0.0588	0.0176	0.0331	0.0733		0.0341	0.0095	0.0173	0.0451
20	0.6576	0.5982	0.6290	0.6744	20	0.6288	0.5653	0.5966	0.6447
	0.0306	0.0147	0.0222	0.0365		0.0220	0.0086	0.0142	0.0266
30	0.6455	0.6040	0.6252	0.6558	30	0.6239	0.5836	0.6049	0.6355
	0.0243	0.0136	0.0187	0.0276		0.0182	0.0096	0.0137	0.0214
40	0.6390	0.6074	0.6237	0.6457	40	0.6225	0.5911	0.6066	0.6298
	0.0212	0.0132	0.0171	0.0231		0.0167	0.0099	0.0131	0.0186
50	0.6343	0.6095	0.6220	0.6406	50	0.6221	0.5965	0.6097	0.6280
	0.0191	0.0130	0.0160	0.0209		0.0160	0.0103	0.0131	0.0175

Table 3: Estimated value and MSE under uniform and Jeffery Priors

Uniform Prior, when $n = 50, \alpha = 1.5, \beta = 1.0, \lambda = 0.5$					**Jeffery's Prior, when** $n = 50, \alpha = 1.5, \beta = 1.0, \lambda = 0.5$				
θ	SELF	QELF	WELF	PELF	θ	SELF	QELF	WELF	PELF
0.5	0.5092	0.5006	0.5060	0.5124	1.5	1.5143	1.4853	1.4985	1.5247
	0.0027	0.0025	0.0026	0.0029		1.0515	0.9929	1.0196	1.0732
1.5	1.5303	1.4993	1.5149	1.5360	2.5	2.5214	2.4706	2.5057	2.5343
	1.0852	1.0212	1.0541	1.0968		4.1497	3.9441	4.0864	4.2042
2.5	2.5473	2.5050	2.5202	2.5605	3.5	3.5350	3.4562	3.5074	3.5508
	4.2569	4.0851	4.1461	4.3124		9.3372	8.8572	9.1747	9.4351
3.5	3.5702	3.4924	3.5343	3.5895	4.5	4.5520	4.4474	4.5057	4.5817
	9.5584	9.0796	9.3410	9.6821		16.6341	15.7817	16.2528	16.8798
4.5	4.5915	4.5126	0.5050	4.6059	5.5	5.5583	5.4544	5.5171	5.5846
	16.9504	16.3146	0.0026	17.0719		25.8990	24.8465	25.4738	26.1739

6. Conclusion

In this paper, we developed a Bayesian framework for estimating LPRD(T-X) distribution parameters under various loss functions. Simulation and real data set results confirm dominancy of the quadratic loss function over others. Moreover, when the sample size increases, the MSE of the WELF tends to the MSE of QELF. Hence, it can be claimed that when we choose such priors for the LPRD(T-X) distribution, the estimates of QELF are preferable as compared to others. Future research could explore Bayesian predictive modeling and real-time inference using this distribution in engineering and medical survival datasets.

Table 4: Estimated value and MSE under uniform and Jeffery Priors

Uniform Prior, when $\alpha = 1.5, \beta = 1.0, \lambda = 0.5, \theta = 0.8$					**Jeffery's Prior, when** $\alpha = 1.5, \beta = 1.0, \lambda = 0.5, \theta = 0.5$				
n	SELF	QELF	WELF	PELF	n	SELF	QELF	WELF	PELF
10	0.9814	0.7967	0.8827	1.0219	10	0.5553	0.4460	0.4979	0.5888
	0.3482	0.1689	0.2427	0.4050		0.0400	0.0277	0.0320	0.0529
20	0.8835	0.8008	0.8417	0.9091	20	0.5257	0.4748	0.5012	0.5385
	0.1911	0.1244	0.1568	0.2144		0.0160	0.0131	0.0137	0.0174
30	0.8540	0.7981	0.8297	0.8644	30	0.5184	0.4827	0.4990	0.5270
	0.1516	0.1122	0.1338	0.1587		0.0101	0.0086	0.0088	0.0106
40	0.8406	0.8034	0.8221	0.8550	40	0.5129	0.4859	0.4988	0.5184
	0.1342	0.1098	0.1214	0.1465		0.0071	0.0064	0.0064	0.0074
50	0.8320	0.7993	0.8156	0.8394	50	0.5111	0.4904	0.5008	0.5146
	0.1244	0.1025	0.1133	0.1297		0.0057	0.0052	0.0053	0.0058

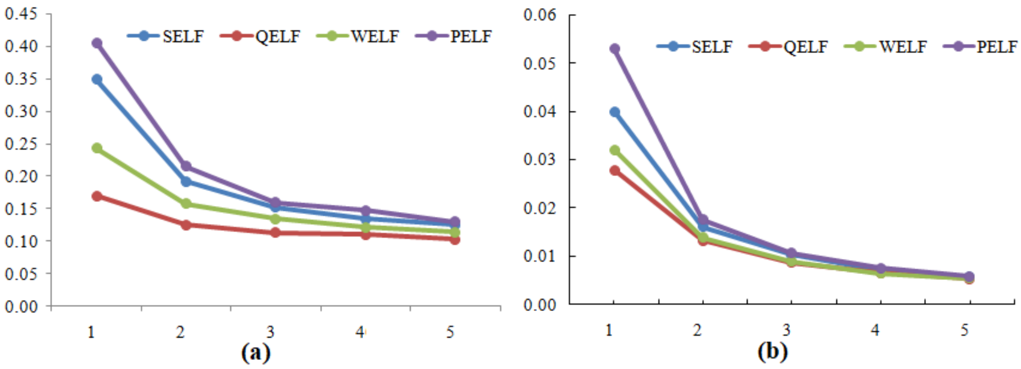

Figure 4: (a) MSE under a Uniform Prior; (b) MSE under Jeffery Prior

Author Contributions

Conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing-original draft preparation, writing-review and editing, visualization, supervision, project administration, funding acquisition, have done by M.I. and N.K., M.I.K., G.M., and H.N.A. All authors have read and agreed to the published version of the manuscript.

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