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Strong β-$\mathcal{I}$-Submaximality and β-$\mathcal{I}$-Paracompactness in Ideal Topological Spaces

Chawalit Boonpok[1] , Palin Raktaow[2], Areeyuth Sama-Ae[3,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Applied Mathematics and Innovation of Mathematics Teaching, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*
[3] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

Abstract. This work introduces and examines the concepts of strong β-$\mathcal{I}$-submaximality and strong β-$\mathcal{I}$-paracompactness in ideal topological spaces, presenting them as natural extensions of the classical notions of submaximality and paracompactness. The study emphasizes the analysis of submaximal spaces through the lens of strong β-$\mathcal{I}$-open sets. Additionally, it offers several characterizations of strong β-$\mathcal{I}$-paracompact spaces and investigates the preservation of this property under mappings.

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1. Introduction and Preliminaries

General topology has demonstrated its efficacy in both theoretical and practical domains. Topology's significance has emerged in various domains, including computational topology, geometric design, computer-aided design, and engineering. Khalimsky et al. [1] and Kong and Kopperman [2] advanced digital topology and computer graphics by the application of connected topologies on finite ordered sets. Moore and Peters [3] examined computational topology in geometric and molecular design, whereas Rosen and Peters [4] employed topological approaches in engineering design research.

The concept of submaximality in general topological spaces was first introduced by Hewitt [5], who defined a space as submaximal if every dense subset is open. This property plays a significant role in topology, often serving as a crucial condition in the study of

*Corresponding author.
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Email addresses: chawalit.b@msu.ac.th (C. Boonpok), palin.rt@gmail.com (P. Raktaow), areeyuth.s@psu.ac.th (A. Sama-Ae)

maximal topologies related to various topological invariants. The original idea of submaximal spaces can be traced back to Bourbaki [6]. Hewitt's foundational work laid the basis for further developments by Arhangel'skii and Collins [7], who provided necessary and sufficient conditions for submaximality and showed that all such spaces are left-separated. Their contributions also sparked interest in whether every submaximal space is σ-discrete. Additionally, they established that any connected Hausdorff space that does not admit a strictly finer connected topology is necessarily submaximal [7]. Dontchev [8] characterized submaximal spaces using various topological notions studied the connections between submaximal and related spaces. Tokgoz and Yalvac [9] developed the concept of submaximality and offer some new results.

Paracompact spaces are considered one of the key classes of topological spaces, as they generalize both metrizable and compact spaces. These spaces were quickly recognized by topologists and analysts. A paracompact space is defined as a topological space in which every open cover has an open refinement that is locally finite. This concept was first introduced by Dieudonné [10] in 1944. A Hausdorff space is paracompact if and only if it admits partitions of unity that are subordinate to any open cover. Additionally, all paracompact Hausdorff spaces are normal, as shown in [11]. Various generalized forms of paracompactness, such as S-paracompactness [12], P_3-paracompactness [13], and β-paracompactness [14], have been explored in the literature. In 2006, Al-Zoubi [12] used semi-open sets to define S-paracompact spaces, a generalization of paracompact spaces, and studied their relationships. Li and Song [15] constructed a Hausdorff S-paracompact space that is not paracompact and further investigated the characterizations of S-paracompact spaces.

The concept of ideal topological spaces was first introduced by Kuratowski [16] and Vaidyanathaswamy [17]. The integration of ideals into topological structures has led to the generalization of certain classical topological ideas. The topology τ of a space can be augmented to a topology $\tau^\star$ through an ideal $\mathcal{I}$, resulting in the creation of an ideal topological space. This framework has been utilized in areas such as ideal resolvability [18], paracompactness concerning ideals [19], and continuity decomposition [20]. Additionally, novel topological constructs founded on ideals were presented in [21]. Jankovic and Hamlett [21] examined and elucidated the basic characteristics of these spaces, proposing the notion of $\mathcal{I}$-open sets and performing comprehensive analyses of topologies employing ideals. Abd. El-Monsef et al. [22] conducted a comprehensive analysis of $\mathcal{I}$-open sets. The notion of I_g-closed sets was established by Dontchev et al. in 1999 [18]. Furthermore, Abd El-Monsef et al. [23] introduced the concept of the s-local function, which was subsequently examined by Khan and Noiri [24].

Ekici and Noiri [25] introduced the notion of $\mathcal{I}$-submaximal ideal topological spaces and studied several characterizations and further properties of $\mathcal{I}$-submaximal. Recently, Boonpok [26] presented the notion of semi-$\mathcal{I}$-submaximal ideal topological spaces and examined their characterizations. Inspired by these advancements, this work aims to offer the concept of strong β-$\mathcal{I}$-submaximal ideal topological spaces, which serves as a natural extension of the previously examined semi-$\mathcal{I}$-submaximal spaces.

The notion of paracompactness with respect to an ideal was first introduced by Zahid [27] and later examined by Hamlett et al. [19]. Sathiyasundari and Renukadevi [28] investigated $\mathcal{I}$-paracompactness and its properties, extending certain results from paracompact spaces to $\mathcal{I}$-paracompact spaces. The concept of S-paracompactness in ideal topological spaces was studied by Sanabria et al. [29], who introduced $\mathcal{I}$-S-paracompact spaces, a new type of space that includes both S-paracompact and $\mathcal{I}$-paracompact spaces. In 2013, Demir and Ozbakir [14] proposed a modified version of expandable and paracompact spaces, termed β-expandable and β-paracompact spaces, respectively. They demonstrated that every β-paracompact space is essentially a β-expandable space. Yildirim et al. [30] introduced the concept of β-paracompactness within ideal topological spaces and compared it with existing forms of paracompactness. Recently, Boonpok et al. [31] proposed the concept of δ-$\beta_{\mathcal{I}}$-paracompactness in the context of ideal topological spaces as a weaker variant of β-paracompactness. Additionally, Boonpok and Sama-Ae [32] provided characterizations of δ_1-$\beta_{\mathcal{I}}$-paracompactness with respect to an ideal. This paper constructs strong β-$\mathcal{I}$-paracompact spaces using strong β-$\mathcal{I}$-open sets, and the spaces under study are examined in detail with respect to β-paracompactness, as described in reference [30]. Multiple characterizations of strong β-$\mathcal{I}$-paracompactness are presented to enhance the theory of ideal topology and give a wider framework for future research.

Throughout this paper, unless otherwise specified, the symbols (X, τ), or simply X, refer to a general topological space without assuming any separation axioms. For a subset A of a topological space (X, τ), the closure and interior of A are denoted by $\mathrm{cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathcal{I}$ on a set X is a nonempty family of subsets of X such that if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$, and if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. A topological space (X, τ) together with an ideal $\mathcal{I}$ is called an ideal topological space and denoted by $(X, \tau, \mathcal{I})$. The set of all subsets of X is denoted as $P(X)$. A set operator $(\cdot)^* : P(X) \to P(X)$, known as a *local function* [16], is defined with respect to a topology τ and an ideal $\mathcal{I}$. For any subset $A \subseteq X$, it is given by

$$A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau : x \in U\}$ represents the set of open neighborhoods of x in the topology τ. This operator induces a finer topology on X, called the $*$-*topology*, denoted by $\tau^*(\mathcal{I})$. The $*$-topology is generated by the Kuratowski closure operator, which is defined as $\mathrm{cl}^*(A) = A \cup A^*$ [21]. For any ideal topological space, a finer topology $\tau^*(\mathcal{I})$, or simply τ^*, always exists. It is generated by the subbasis

$$\beta(\mathcal{I}, \tau) = \{U - I : U \in \tau, I \in \mathcal{I}\}.$$

However, $\beta(\mathcal{I}, \tau)$ does not necessarily form a topology in general [21]. It is clear that $A^* \subseteq B^*$ and $\mathrm{cl}^*(A) \subseteq \mathrm{cl}^*(B)$ if $A \subseteq B$.

Definition 1. [33] *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called strong β-$\mathcal{I}$-open if $A \subseteq \mathrm{cl}^*(\mathrm{Int}(\mathrm{cl}^*(A)))$. The complement of a strong β-$\mathcal{I}$-open set is called a strong β-$\mathcal{I}$-closed set.*

According to Definition 1, we have the following lemma.

Lemma 1. *In an ideal topological space, the following properties are satisfied:*

(1) *The arbitrary union of strong β-$\mathcal{I}$-open sets is itself a strong β-$\mathcal{I}$-open set; and*

(2) *The arbitrary intersection of strong β-$\mathcal{I}$-closed sets remains a strong β-$\mathcal{I}$-closed set.*

Definition 2. [33] *The strong β-$\mathcal{I}$-closure of a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $s\beta\operatorname{cl}_{\mathcal{I}}(A)$, is defined as the intersection of all strong β-$\mathcal{I}$-closed sets containing A, while the strong β-$\mathcal{I}$-interior of A, denoted by $s\beta\operatorname{Int}_{\mathcal{I}}(A)$, is defined as the union of all strong β-$\mathcal{I}$-open sets contained in A.*

Utilizing Definition 2, we have the following lemma.

Lemma 2. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. The following statements hold:*

(1) $A \subseteq s\beta\operatorname{cl}_{\mathcal{I}}(A)$;

(2) $s\beta\operatorname{Int}_{\mathcal{I}}(A) \subseteq A$;

(3) *If* $A \subseteq B$, *then* $s\beta\operatorname{Int}_{\mathcal{I}}(A) \subseteq s\beta\operatorname{Int}_{\mathcal{I}}(B)$ *and* $s\beta\operatorname{cl}_{\mathcal{I}}(A) \subseteq s\beta\operatorname{cl}_{\mathcal{I}}(B)$;

(4) $X - s\beta\operatorname{cl}_{\mathcal{I}}(A) = s\beta\operatorname{Int}_{\mathcal{I}}(X - A)$; *and*

(5) $X - s\beta\operatorname{Int}_{\mathcal{I}}(A) = s\beta\operatorname{cl}_{\mathcal{I}}(X - A)$.

Proof. Statements (1), (2), and (3) follow directly from the definitions of $s\beta\operatorname{cl}_{\mathcal{I}}(A)$ and $s\beta\operatorname{Int}_{\mathcal{I}}(A)$. To establish statement (4), observe that

$$\begin{aligned} X - s\beta\operatorname{cl}_{\mathcal{I}}(A) &= X - \cap\{F \mid A \subseteq F,\ F \text{ is strong } \beta\text{-}\mathcal{I}\text{-closed}\} \\ &= \cup\{X - F \mid X - F \subseteq X - A,\ X - F \text{ is strong } \beta\text{-}\mathcal{I}\text{-open}\} \\ &= \cup\{G \mid G \subseteq X - A,\ G \text{ is strong } \beta\text{-}\mathcal{I}\text{-open}\} \\ &= s\beta\operatorname{Int}_{\mathcal{I}}(X - A), \end{aligned}$$

which confirms statement (4). Similarly, for statement (5), note that

$$\begin{aligned} X - s\beta\operatorname{Int}_{\mathcal{I}}(A) &= X - \cup\{G \mid G \subseteq A,\ G \text{ is strong } \beta\text{-}\mathcal{I}\text{-open}\} \\ &= \cap\{X - G \mid X - A \subseteq X - G,\ X - G \text{ is strong } \beta\text{-}\mathcal{I}\text{-closed}\} \\ &= \cap\{F \mid X - A \subseteq F,\ F \text{ is strong } \beta\text{-}\mathcal{I}\text{-closed}\} \\ &= s\beta\operatorname{cl}_{\mathcal{I}}(X - A), \end{aligned}$$

thereby verifying statement (5).

The following lemma outlines fundamental characteristics and conditions related to strong β-$\mathcal{I}$-open and strong β-$\mathcal{I}$-closed sets within the context of ideal topological spaces.

Lemma 3. *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. The following properties hold:*

(1) *If A is open, then A is strong β-$\mathcal{I}$-open;*

(2) *If A is closed, then A is strong β-$\mathcal{I}$-closed;*

(3) $s\beta\operatorname{Int}_{\mathcal{I}}(A)$ *is strong β-$\mathcal{I}$-open, and* $\operatorname{Int}(A) \subseteq s\beta\operatorname{Int}_{\mathcal{I}}(A)$*;*

(4) $s\beta\operatorname{cl}_{\mathcal{I}}(A)$ *is strong β-$\mathcal{I}$-closed, and* $s\beta\operatorname{cl}_{\mathcal{I}}(A) \subseteq \operatorname{cl}(A)$*;*

(5) *A is strong β-$\mathcal{I}$-open if and only if* $A = s\beta\operatorname{Int}_{\mathcal{I}}(A)$*;*

(6) *A is strong β-$\mathcal{I}$-closed if and only if* $A = s\beta\operatorname{cl}_{\mathcal{I}}(A)$*; and*

(7) $x \in s\beta\operatorname{cl}_{\mathcal{I}}(A)$ *if and only if $U \cap A \neq \emptyset$ for every strong β-$\mathcal{I}$-open set U containing x.*

Proof. (1): Let A be an open set. Then $A = \operatorname{Int}(A)$, and since $A \subseteq \operatorname{cl}^*(A)$, we have $A = \operatorname{Int}(A) \subseteq \operatorname{Int}(\operatorname{cl}^*(A))$. It follows that $A \subseteq \operatorname{cl}^*(A) \subseteq \operatorname{cl}^*(\operatorname{Int}(\operatorname{cl}^*(A)))$, which implies that A is strong β-$\mathcal{I}$-open.

(2): The result follows directly from (1).

(3): Let U_α be any strong β-$\mathcal{I}$-open set with $U_\alpha \subseteq A$. By the definition of a strong β-$\mathcal{I}$-open set, we have that $U_\alpha \subseteq s\beta\operatorname{Int}_{\mathcal{I}}(A)$. Since U_α is strong β-$\mathcal{I}$-open, we have $U_\alpha \subseteq \operatorname{cl}^*(\operatorname{Int}(\operatorname{cl}^*(U_\alpha)))$. Then,

$$s\beta\operatorname{Int}_{\mathcal{I}}(A) = \cup_\alpha U_\alpha \subseteq \cup_\alpha \operatorname{cl}^*(\operatorname{Int}(\operatorname{cl}^*(U_\alpha))) \subseteq \operatorname{cl}^*(\operatorname{Int}(\operatorname{cl}^*(s\beta\operatorname{Int}_{\mathcal{I}}(A)))),$$

and therefore $s\beta\operatorname{Int}_{\mathcal{I}}(A)$ is a strong β-$\mathcal{I}$-open set. From the definitions of $\operatorname{Int}(A)$ and $s\beta\operatorname{Int}_{\mathcal{I}}(A)$, we deduce that $\operatorname{Int}(A) \subseteq s\beta\operatorname{Int}_{\mathcal{I}}(A)$.

(4): By part (4) of Lemma 2, it follows that $s\beta\operatorname{cl}_{\mathcal{I}}(A)$ is strong β-$\mathcal{I}$-closed. Moreover, from the definitions of $s\beta\operatorname{cl}_{\mathcal{I}}(A)$ and $\operatorname{cl}(A)$, we have that $s\beta\operatorname{cl}_{\mathcal{I}}(A) \subseteq \operatorname{cl}(A)$.

(5): Since $s\beta\operatorname{Int}_{\mathcal{I}}(A)$ is the union of all strong β-$\mathcal{I}$-open sets contained in A, it follows that $s\beta\operatorname{Int}_{\mathcal{I}}(A) \subseteq A$. Therefore, A is strong β-$\mathcal{I}$-open if and only if $A = s\beta\operatorname{Int}_{\mathcal{I}}(A)$.

(6): As $s\beta\operatorname{cl}_{\mathcal{I}}(A)$ is the intersection of all strong β-$\mathcal{I}$-closed sets containing A, it implies that $A \subseteq s\beta\operatorname{cl}_{\mathcal{I}}(A)$. Hence, A is strong β-$\mathcal{I}$-closed if and only if $A = s\beta\operatorname{cl}_{\mathcal{I}}(A)$.

(7): Assume that $x \in s\beta\operatorname{cl}_{\mathcal{I}}(A)$. Suppose there exists a strong β-$\mathcal{I}$-open set U containing x such that $U \cap A = \emptyset$. Then it follows that $A \subseteq X - U$. Since $X - U$ is strong β-$\mathcal{I}$-closed, this would imply $x \notin s\beta\operatorname{cl}_{\mathcal{I}}(A)$, which contradicts the assumption.

Conversely, suppose that for every strong β-$\mathcal{I}$-open set U containing x, we have $U \cap A \neq \emptyset$. Assume, for the sake of contradiction, that $x \notin s\beta\operatorname{cl}_{\mathcal{I}}(A)$. Then there exists a strong β-$\mathcal{I}$-closed set F such that $A \subseteq F$ and $x \notin F$. Consequently, $x \in X - F$, which is a strong β-$\mathcal{I}$-open set disjoint from A, leading to a contradiction. Therefore, it follows that $x \in s\beta\operatorname{cl}_{\mathcal{I}}(A)$.

2. Strong β-$\mathcal{I}$-Submaximality

In this section, we explore a collection of equivalent conditions that provide a thorough characterization of when an ideal topological space $(X, \tau, \mathcal{I})$ can be regarded as strong β-$\mathcal{I}$-submaximal, thereby establishing a foundational understanding of the underlying properties that define this class of spaces.

Definition 3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset $A \subseteq X$ is said to be:*

(1) *Strong β-$\mathcal{I}$-dense if $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) = X$; and*

(2) *Strong β-$\mathcal{I}$-codense if $X - A$ is strong β-$\mathcal{I}$-dense.*

Definition 4. *An ideal topological space $(X, \tau, \mathcal{I})$ is called strong β-$\mathcal{I}$-submaximal if each strong β-$\mathcal{I}$-dense subset of X is strong β-$\mathcal{I}$-open.*

Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$, and $\mathcal{I} = \{\emptyset, \{c\}\}$. It is clear that the only strong β-$\mathcal{I}$-dense sets are $\{a\}$, $\{a, b\}$, and $\{a, b, c\}$, all of which are also strong β-$\mathcal{I}$-open. Hence, the space $(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal.

Definition 5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset $A \subseteq X$ is said to be:*

(1) *Locally strong β-$\mathcal{I}$-closed if A is the intersection of a strong β-$\mathcal{I}$-open set and a strong β-$\mathcal{I}$-closed set; and*

(2) *Co-locally strong β-$\mathcal{I}$-closed if A is the union of a strong β-$\mathcal{I}$-open set and a strong β-$\mathcal{I}$-closed set.*

Theorem 1 presents five equivalent conditions that characterize when a subset A of an ideal topological space is locally strong β-$\mathcal{I}$-closed.

Theorem 1. *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *A is locally strong β-$\mathcal{I}$-closed;*

(2) *$A = U \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(A)$ for some strong β-$\mathcal{I}$-open set U;*

(3) *$s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed;*

(4) *$A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A))$ is strong β-$\mathcal{I}$-open; and*

(5) *$A \subseteq s\beta\,\mathrm{Int}_{\mathcal{I}}(A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)))$.*

Proof. (1) ⇒ (2): Suppose A is locally strong β-$\mathcal{I}$-closed. Then there exists a strong β-$\mathcal{I}$-open set U and a strong β-$\mathcal{I}$-closed set F such that $A = U \cap F$. Given that F is strong β-$\mathcal{I}$-closed, it follows that $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \subseteq s\beta\,\mathrm{cl}_{\mathcal{I}}(F) = F$, so $A \subseteq U \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \subseteq U \cap F = A$. Consequently, $A = U \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(A)$.

(2) ⇒ (3): Suppose $A = U \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(A)$ for some strong β-$\mathcal{I}$-open set U. Since

$$\begin{aligned} s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A &= s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \cap (X - A) \\ &= (X - (U \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(A))) \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \\ &= (X - U) \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(A), \end{aligned}$$

$s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed.

(3) $\Rightarrow$ (4): Assume that the set $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed. Given that $X - (s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A) = (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \cup A$, it follows that $A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A))$ is strong β-$\mathcal{I}$-open.

(4) $\Rightarrow$ (5): The evidence is clear.

(5) $\Rightarrow$ (1): Suppose that $A \subseteq s\beta\,\mathrm{Int}_{\mathcal{I}}(A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)))$. Since $X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)$ is strong β-$\mathcal{I}$-open, we have

$$X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A) = s\beta\,\mathrm{Int}_{\mathcal{I}}(X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)).$$

It follows that

$$X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A) = s\beta\,\mathrm{Int}_{\mathcal{I}}(X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \subseteq s\beta\,\mathrm{Int}_{\mathcal{I}}(A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A))).$$

Therefore,

$$A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \subseteq s\beta\,\mathrm{Int}_{\mathcal{I}}(A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A))),$$

which shows that $A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A))$ is strong β-$\mathcal{I}$-open. Since

$$A = (A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A))) \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(A),$$

it follows that A is locally strong β-$\mathcal{I}$-closed.

The subsequent theorem presents five mutually equivalent conditions, each involving strong β-$\mathcal{I}$-closed sets, strong β-$\mathcal{I}$-dense sets, and B-$s\mathcal{I}$- sets, which together provide a comprehensive characterization of when an ideal topological space qualifies as strong β-$\mathcal{I}$-submaximal. To proceed, we begin by introducing the concept of a B-$s\mathcal{I}$ set within the context of an ideal topological space.

Definition 6. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called a B-$s\mathcal{I}$ set if it can be written as $A = U \cap V$, where U is a strong β-$\mathcal{I}$-open set, and V is a set that satisfies the condition $s\beta\,\mathrm{Int}_{\mathcal{I}}(V) = s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(V))$.*

Theorem 2. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal;*

(2) *For every subset A of X, $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed;*

(3) *Every subset of X is locally strong β-$\mathcal{I}$-closed;*

(4) *Every subset of X is a B-$s\mathcal{I}$ set; and*

(5) *Every strong β-$\mathcal{I}$-dense subset of X is a B-$s\mathcal{I}$ set.*

Proof. (1) $\Rightarrow$ (2): Let $(X, \tau, \mathcal{I})$ be strong β-$\mathcal{I}$-submaximal and let $A \subseteq X$. As

$$\begin{aligned} X &= s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \\ &\subseteq s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \cup (X - s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \end{aligned}$$

$$
\begin{aligned}
&= s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \cup s\beta\,\mathrm{cl}_{\mathcal{I}}(X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \\
&\subseteq s\beta\,\mathrm{cl}_{\mathcal{I}}(A \cup (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A))) \\
&= s\beta\,\mathrm{cl}_{\mathcal{I}}(X - (s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A)),
\end{aligned}
$$

we have $s\beta\,\mathrm{cl}_{\mathcal{I}}(X - (s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A)) = X$ and hence $X - (s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A)$ is strong β-$\mathcal{I}$-dense. According to the hypothesis, $X - (s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A)$ is strong β-$\mathcal{I}$-open. Consequently, $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed.

(2) $\Rightarrow$ (3): By Theorem 1, a subset A is locally strong β-$\mathcal{I}$-closed if and only if $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed.

(3) $\Rightarrow$ (4): If every subset of X is locally strong β-$\mathcal{I}$-closed, then for any $A \subseteq X$, we can write $A = U \cap V$ where U is strong β-$\mathcal{I}$-open and V is strong β-$\mathcal{I}$-closed. Since $V = s\beta\,\mathrm{cl}_{\mathcal{I}}(V)$, we get $s\beta\,\mathrm{Int}_{\mathcal{I}}(V) = s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(V))$. Therefore A is a B-$s\mathcal{I}$ set.

(4) $\Rightarrow$ (5): It is obvious.

(5) $\Rightarrow$ (1): Let A be a strong β-$\mathcal{I}$-dense subset, and assume that every strong β-$\mathcal{I}$-dense subset of X is a B-$s\mathcal{I}$ set. Then A can be expressed as $A = U \cap V$, where U is a strong β-$\mathcal{I}$-open set and V is a set satisfying $s\beta\,\mathrm{Int}_{\mathcal{I}}(V) = s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(V))$. Given that $A \subseteq V$, it implies that $X = s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \subseteq s\beta\,\mathrm{cl}_{\mathcal{I}}(V)$, and therefore, $X = s\beta\,\mathrm{cl}_{\mathcal{I}}(V)$. Thus $X = s\beta\,\mathrm{Int}_{\mathcal{I}}(X) = s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(V)) = s\beta\,\mathrm{Int}_{\mathcal{I}}(V)$. This indicates that $V = X$, hence $A = U \cap V = U \cap X = U$. Therefore, A is classified as strong β-$\mathcal{I}$-open. Accordingly, $(X, \tau, \mathcal{I})$ is defined as strong β-$\mathcal{I}$-submaximal.

Theorem 3 presents three equivalent conditions involving co-locally strong β-$\mathcal{I}$-closed sets and strong β-$\mathcal{I}$-closed sets that characterize when a topological space is strong β-$\mathcal{I}$-submaximal.

Theorem 3. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal;*

(2) *Every subset of X is co-locally strong β-$\mathcal{I}$-closed;*

(3) *For every subset A of X such that $s\beta\,\mathrm{Int}_{\mathcal{I}}(A) = \emptyset$ is strong β-$\mathcal{I}$-closed;*

Proof. (1) $\Rightarrow$ (2): Assume that $(X, \tau, \mathcal{I})$ is a strong β-$\mathcal{I}$-submaximal space. For any subset $A \subseteq X$, Theorem 2 guarantees the existence of a strong β-$\mathcal{I}$-open set U and a strong β-$\mathcal{I}$-closed set V such that $X - A = U \cap V$. It follows that

$$A = (X - U) \cup (X - V),$$

where $X - U$ is strong β-$\mathcal{I}$-closed and $X - V$ is strong β-$\mathcal{I}$-open. Therefore, A can be expressed as the union of a strong β-$\mathcal{I}$-closed set and a strong β-$\mathcal{I}$-open set, which means A is co-locally strong β-$\mathcal{I}$-closed.

(2) $\Rightarrow$ (3): Suppose that every subset of X is co-locally strong β-$\mathcal{I}$-closed. Let $A \subseteq X$ be such that $s\beta\,\mathrm{Int}_{\mathcal{I}}(A) = \emptyset$. Then, by assumption, A can be written as $A = U \cup V$, where U is strong β-$\mathcal{I}$-open and V is strong β-$\mathcal{I}$-closed. Since $U \subseteq A$, it follows that

$$U = s\beta\,\mathrm{Int}_{\mathcal{I}}(U) \subseteq s\beta\,\mathrm{Int}_{\mathcal{I}}(A) = \emptyset,$$

which implies $U = \emptyset$. Thus, $A = V$, and since V is strong β-$\mathcal{I}$-closed, it follows that A is strong β-$\mathcal{I}$-closed.

(3) $\Rightarrow$ (1): Let $A \subseteq X$ be a strong-$\mathcal{I}$-dense subset. We aim to show that A is strong β-$\mathcal{I}$-open. Since

$$X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A) = s\beta\,\mathrm{Int}_{\mathcal{I}}(X - A) = \emptyset,$$

it follows that the complement $X - A$ has empty strong β-$\mathcal{I}$-interior. By assumption, such a subset must be strong β-$\mathcal{I}$-closed. Hence, $X - A$ is strong β-$\mathcal{I}$-closed, and therefore A is strong β-$\mathcal{I}$-open.

Definition 7. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is referred to as strong β-$\mathcal{I}$-discrete if for every point $x \in A$, the singleton set $\{x\}$ is strong β-$\mathcal{I}$-closed.*

Theorem 4 establishes five equivalent conditions involving strong β-$\mathcal{I}$-discreteness and strong β-$\mathcal{I}$-closed sets, which collectively characterize when an ideal topological space $(X, \tau, \mathcal{I})$ can be regarded as strong β-$\mathcal{I}$-submaximal. The proof of Theorem 4 relies on the following lemma.

Lemma 4. *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then,*

$$s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A) = \emptyset.$$

Proof. Let A be a subset of X. Since $s\beta\,\mathrm{Int}_{\mathcal{I}}(X - A) = X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)$, we have

$$\begin{aligned} s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A) &= s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \cap (X - A)) \\ &\subseteq s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \cap s\beta\,\mathrm{Int}_{\mathcal{I}}(X - A) \\ &= s\beta\,\mathrm{Int}_{\mathcal{I}}(s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \cap (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \\ &\subseteq s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \cap (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(A)) \\ &= \emptyset. \end{aligned}$$

Theorem 4. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal;*

(2) *For every subset A of X with $s\beta\,\mathrm{Int}_{\mathcal{I}}(A) = \emptyset$ is strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete;*

(3) *For every subset A of X, the set $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete;*

(4) *Every strong β-$\mathcal{I}$-codense subset of X is strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete; and*

(5) *Every strong β-$\mathcal{I}$-codense subset of X is strong β-$\mathcal{I}$-closed.*

Proof. (1) $\Rightarrow$ (2): Assume that $(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal. Let A be a subset of X such that $s\beta \operatorname{Int}_{\mathcal{I}}(A) = \emptyset$. Then,

$$s\beta \operatorname{cl}_{\mathcal{I}}(X - A) = X - s\beta \operatorname{Int}_{\mathcal{I}}(A) = X,$$

so $X - A$ is strong β-$\mathcal{I}$-dense. By assumption, $X - A$ is strong β-$\mathcal{I}$-open. Thus, A is strong β-$\mathcal{I}$-closed.

Next, we verify that A is strong β-$\mathcal{I}$-discrete. Let $x \in A$. Since $\{x\} \subseteq A$, we have

$$s\beta \operatorname{Int}_{\mathcal{I}}(\{x\}) \subseteq s\beta \operatorname{Int}_{\mathcal{I}}(A) = \emptyset,$$

which implies $s\beta \operatorname{Int}_{\mathcal{I}}(\{x\}) = \emptyset$. Then,

$$s\beta \operatorname{cl}_{\mathcal{I}}(X - \{x\}) = X - s\beta \operatorname{Int}_{\mathcal{I}}(\{x\}) = X,$$

so $X - \{x\}$ is strong β-$\mathcal{I}$-open, which means $\{x\}$ is strong β-$\mathcal{I}$-closed. Since this holds for each $x \in A$, it follows that A is strong β-$\mathcal{I}$-discrete.

(2) $\Rightarrow$ (3): Suppose every subset A of X with $s\beta \operatorname{Int}_{\mathcal{I}}(A) = \emptyset$ is strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete. By Lemma 4, we have

$$s\beta \operatorname{Int}_{\mathcal{I}}(s\beta \operatorname{cl}_{\mathcal{I}}(A) - A) = \emptyset.$$

By the hypothesis, this implies that the set $s\beta \operatorname{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete.

(3) $\Rightarrow$ (4): Suppose every subset A of X satisfies that $s\beta \operatorname{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete.

Let A be a strong β-$\mathcal{I}$-codense subset of X. Then $X - A$ is strong β-$\mathcal{I}$-dense, i.e.,

$$X = s\beta \operatorname{cl}_{\mathcal{I}}(X - A).$$

Hence,

$$A = X - (X - A) = s\beta \operatorname{cl}_{\mathcal{I}}(X - A) - (X - A).$$

By the assumption, A is strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete.

(4) $\Rightarrow$ (5): This is obvious.

(5) $\Rightarrow$ (1): Suppose that every strong β-$\mathcal{I}$-codense subset of X is strong β-$\mathcal{I}$-closed. Let A be a strong β-$\mathcal{I}$-dense subset of X. Then its complement $X - A$ is strong β-$\mathcal{I}$-codense. By the hypothesis, $X - A$ is strong β-$\mathcal{I}$-closed. Thus, A is strong β-$\mathcal{I}$-open. Therefore, the space $(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal.

The final theorem in this section outlines three equivalent conditions that determine when an ideal topological space $(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal.

Theorem 5. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal;*

(2) *Every subset of X is locally strong β-$\mathcal{I}$-closed; and*

(3) *Every strong β-$\mathcal{I}$-dense subset of X is locally strong β-$\mathcal{I}$-closed.*

Proof. (1) $\Rightarrow$ (2): Assume that $(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal. Let $A \subseteq X$. As the same proof in Theorem 2, we have $s\beta\,\mathrm{cl}_{\mathcal{I}}(X - (s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A)) = X$ and hence $X - (s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A)$ is strong β-$\mathcal{I}$-dense. By the hypothesis, $X - (s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A)$ is strong β-$\mathcal{I}$-open. Therefore, $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed. By Theorem 1, A is locally strong β-$\mathcal{I}$-closed.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Assume that every strong β-$\mathcal{I}$-dense subset of X is locally strong β-$\mathcal{I}$-closed. Let A be a strong β-$\mathcal{I}$-dense subset of X. This implies $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) = X$. By assumption, A is locally strong β-$\mathcal{I}$-closed, meaning there exist a strong β-$\mathcal{I}$-open set U and a strong β-$\mathcal{I}$-closed set V such that $A = U \cap V$. Since $A \subseteq V$, we have that $X = s\beta\,\mathrm{cl}_{\mathcal{I}}(A) \subseteq s\beta\,\mathrm{cl}_{\mathcal{I}}(V) = V$, which gives $V = X$. Thus, $A = U \cap V = U \cap X = U$ and so A is strong β-$\mathcal{I}$-open. Consequently, $(X, \tau, \mathcal{I})$ is strong β-$\mathcal{I}$-submaximal.

3. Stong β-$\mathcal{I}$-Paracompactness and Characterizations

This section explores the notion of $s\beta$-$\mathcal{I}$-paracompactness, which is a variant of the $\mathcal{I}$-β-paracompactness concept introduced by Yildirim et al. [30]. We aim to present a formal description of this concept.

Let $\mathcal{A} = \{U_\alpha : \alpha \in \Lambda_1\}$ and $\mathcal{B} = \{V_\mu : \mu \in \Lambda_2\}$ be two families of subsets in a topological space X. We say that $\mathcal{A}$ is a *refinement* of $\mathcal{B}$ if for every $\alpha \in \Lambda_1$, there exists $\mu \in \Lambda_2$ such that $U_\alpha \subseteq V_\mu$.

A family $\mathcal{A}$ of subsets of a topological space (X, τ) is called *β-locally finite* [14] if, for each point $x \in X$, there exists a β-open neighborhood U of x that intersects only finitely many sets from $\mathcal{A}$. Yildirim et al. [30] defined an ideal topological space $(X, \tau, \mathcal{I})$ to be *$\mathcal{I}$-β-paracompact* if every open cover of X has a β-locally finite refinement $\mathcal{V}$ consisting of β-open sets, and the set $X - \cup\{V : V \in \mathcal{V}\}$ is an element of $\mathcal{I}$.

Definition 8. *A collection $\mathcal{A}$ of subsets of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $s\beta$-$\mathcal{I}$-locally finite if for each $x \in X$, there exists a strong β-$\mathcal{I}$-open set U containing x and U intersects at most finitely many members of $\mathcal{A}$.*

Lemma 5. *Let $\mathcal{A}$ be a collection of subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If $\mathcal{A}$ is $s\beta$-$\mathcal{I}$-locally finite, then it is β-locally finite.*

Proof. Assume that $\mathcal{A}$ is $s\beta$-$\mathcal{I}$-locally finite. We aim to show that $\mathcal{A}$ is β-locally finite. Let $x \in X$. Since $\mathcal{A}$ is $s\beta$-$\mathcal{I}$-locally finite, there exists a strong β-$\mathcal{I}$-open set G_x containing x that intersects only finitely many members of $\mathcal{A}$. Because every strong β-$\mathcal{I}$-open set is also β-open, G_x is a β-open neighborhood of x with the same finiteness property. Thus, $\mathcal{A}$ is β-locally finite.

Definition 9. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $s\beta$-$\mathcal{I}$-paracompact if every open cover of X has an $s\beta$-$\mathcal{I}$-locally finite refinement $\mathcal{A}$ consisting of strong β-$\mathcal{I}$-open sets (not necessarily a cover) such that $X - \cup\{V : V \in \mathcal{A}\} \in \mathcal{I}$. The collection $\mathcal{A}$ of subsets of X such that $X - \cup\{V : V \in \mathcal{A}\} \in \mathcal{I}$ is called an $\mathcal{I}$-cover. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $s\beta$-$\mathcal{I}$-paracompact if for any open cover of A has an $s\beta$-$\mathcal{I}$-locally finite refinement $\mathcal{A}$ consisting of strong β-$\mathcal{I}$-open sets such that $A - \cup\{V : V \in \mathcal{A}\} \in \mathcal{I}$.*

The two theorems that follow arise from the fact that every open set is strong β-$\mathcal{I}$-open, every strong β-$\mathcal{I}$-open set is β-open and $\emptyset$ is in any ideal.

Theorem 6. *If a topological space (X, τ) is paracompact, then $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-paracompact.*

Proof. It is evident, as $\emptyset \in \mathcal{I}$.

Theorem 7. *If $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-paracompact then it is $\mathcal{I}$-β-paracompact.*

Proof. Every $s\beta$-$\mathcal{I}$-locally finite collection of subsets of X is β-locally finite, as demonstrated by Lemma 5. Furthermore, each strong β-$\mathcal{I}$-open set is a β-open set. We can continue with the proof by following to the definitions of β-$\mathcal{I}$-paracompactness and $s\beta$-$\mathcal{I}$-paracompactness.

Consider the set $\mathbb{N}$ of all positive integers. Define a topology τ on $\mathbb{N}$ by

$$\tau = \{\emptyset\} \cup \{\mathbb{N}\} \cup \{\{1, 2, \dots, n\} : n \in \mathbb{N}\}.$$

Let $\mathcal{I} = \{A \subseteq \mathbb{N} : 1 \notin A\}$. It is easy to see that $\mathcal{I}$ is an ideal on $\mathbb{N}$. Consider the open cover $\mathcal{U} = \{\{1, 2, 3, \dots, n\} : n \in \mathbb{N}\}$ of $\mathbb{N}$. It is evident that there does not exist a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$ that still covers $\mathbb{N}$. Therefore, the space $(\mathbb{N}, \tau)$ is not paracompact. Nevertheless, $\mathbb{N}$ is an $s\beta$-$\mathcal{I}$-paracompact space. For any open cover $\mathcal{U}$ of $\mathbb{N}$, there exists refinement $\mathcal{V} = \{\{1\}\}$ of a strong β-$\mathcal{I}$-open set $\{1\}$ that $\mathcal{V}$ is $s\beta$-$\mathcal{I}$-locally finite. This holds because the complement $\mathbb{N} - \{1\} = \{2, 3, \dots\} \in \mathcal{I}$, satisfying the required condition.

Theorem 8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then $G \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(A) = \emptyset$ if and only if $G \cap A = \emptyset$, for all strong β-$\mathcal{I}$-open subset G of X.*

Proof. It follows from part (7) of Lemma 3, together with the fact that $A \subseteq s\beta\,\mathrm{cl}_{\mathcal{I}}(A)$.

Theorem 9. *Let $\mathcal{A} = \{V_\lambda : \lambda \in \Lambda\}$ be a collection of subsets of an ideal topological space $(X, \tau, \mathcal{I})$. The following statements are true.*

(1) *If $\mathcal{A}$ is $s\beta$-$\mathcal{I}$-locally finite and $H_\lambda \subseteq V_\lambda$ for all $\lambda \in \Lambda$, then $\mathcal{B} = \{H_\lambda : \lambda \in \Lambda\}$ is $s\beta$-$\mathcal{I}$-locally finite.*

(2) *$\mathcal{A}$ is $s\beta$-$\mathcal{I}$-locally finite if and only if $\{s\beta\,\mathrm{cl}_{\mathcal{I}}(V_\lambda) : \lambda \in \Lambda\}$ is $s\beta$-$\mathcal{I}$-locally finite.*

Proof. (1): Let $x \in X$. Since $\mathcal{A}$ is $s\beta$-$\mathcal{I}$-locally finite, there exists a strong β-$\mathcal{I}$-open set U containing x, which intersects at most finitely many elements of $\mathcal{A}$. As $H_\lambda \subseteq V_\lambda$ for all $\lambda \in \Lambda$, it follows that U intersects at most finitely many of the sets in $\mathcal{B} = \{H_\lambda : \lambda \in \Lambda\}$. Hence, $\mathcal{B} = \{H_\lambda : \lambda \in \Lambda\}$ is $s\beta$-$\mathcal{I}$-locally finite.

(2): Let $\mathcal{A}$ be $s\beta$-$\mathcal{I}$-locally finite and let $x \in X$. Then, there exists a strong β-$\mathcal{I}$-open set G containing x that satisfies $G \cap V_\lambda = \emptyset$ for every $\lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n$. By Theorem 8, we obtain that $G \cap s\beta\,\mathrm{cl}_{\mathcal{I}}(V_\lambda) = \emptyset$ for every $\lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n$. Therefore, $\{s\beta\,\mathrm{cl}_{\mathcal{I}}(V_\lambda) : \lambda \in \Lambda\}$ is $s\beta$-$\mathcal{I}$-locally finite.

The converse follows from (1).

Theorem 10. *If $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-paracompact and $\mathcal{J}$ is an ideal on X with $\mathcal{I} \subseteq \mathcal{J}$, then $(X, \tau, \mathcal{J})$ is $s\beta$-$\mathcal{J}$-paracompact.*

Proof. Let $(X, \tau, \mathcal{I})$ be $s\beta$-$\mathcal{I}$-paracompact, and suppose $\mathcal{I} \subseteq \mathcal{J}$. Let $\mathcal{A} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Since $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-paracompact, by definition, there exists an $s\beta$-$\mathcal{I}$-locally finite refinement $\mathcal{A}'$ of $\mathcal{A}$ consisting of strong β-$\mathcal{I}$-open sets such that:

$$X - \cup\{V : V \in \mathcal{A}'\} \in \mathcal{I}.$$

Because $\mathcal{I} \subseteq \mathcal{J}$, it follows that:

$$X - \cup\{V : V \in \mathcal{A}'\} \in \mathcal{J}.$$

Thus, $(X, \tau, \mathcal{J})$ is $s\beta$-$\mathcal{J}$-paracompact.

Lemma 6. *If an open cover $\mathcal{A} = \{U_\lambda : \lambda \in \Lambda\}$ of an ideal topological space $(X, \tau, \mathcal{I})$ has an $s\beta$-$\mathcal{I}$-locally finite strong β-$\mathcal{I}$-open refinement $\mathcal{B}$ such that $X - \cup\{V : V \in \mathcal{B}\} \in \mathcal{I}$, then there exists a precise $s\beta$-$\mathcal{I}$-locally finite strong β-$\mathcal{I}$-open refinement $\mathcal{C} = \{H_\lambda : \lambda \in \Lambda\}$ of $\mathcal{A}$ such that $X - \cup\{H_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$.*

Proof. A similar technique is employed as in the proof of Lemma 1.3 in [29].

Definition 10. *An ideal topological space $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-regular if for any closed subset F of X and $x \notin F$, there exist disjoint strong β-$\mathcal{I}$-open sets U and V such that $x \in U$ and $F - V \in \mathcal{I}$.*

Theorem 11. *If $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-paracompact and Hausdorff, then $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-regular.*

Proof. Let F be a closed subset of X, and let $x \notin F$. For each $y \in F$, since X is Hausdorff, there exist disjoint open sets V_x and O_{xy} such that $x \in V_x$ and $y \in O_{xy}$. This implies that $y \notin \mathrm{cl}(V_x)$. Now, consider the family $\mathcal{A} = \{O_{xy} : y \in F\} \cup \{X - F\}$, which is an open cover of X. By assumption, there exists an $s\beta$-$\mathcal{I}$-locally finite strong β-$\mathcal{I}$-open refinement $\mathcal{B} = \{H_{xy} : y \in F\} \cup \{W\}$ such that: $H_{xy} \subseteq O_{xy}$ for each $y \in F$, $W \subseteq X - F$, and $X - (\cup\{H_{xy} : y \in F\} \cup \{W\}) \in \mathcal{I}$.

Next, let us define the sets $V = \cup\{H_{xy} : y \in F\}$ and $U = X - \{s\beta\,\mathrm{cl}_{\mathcal{I}}(\cup(H_{xy})) : y \in F\}$. We assert that U and V are disjoint strong β-$\mathcal{I}$-open subsets of X. Given that $H_{xy} \subseteq O_{xy}$ and $s\beta\,\mathrm{cl}_{\mathcal{I}}(H_{xy}) \subseteq \mathrm{cl}(H_{xy})$, it follows that $s\beta\,\mathrm{cl}_{\mathcal{I}}(H_{xy}) \subseteq \mathrm{cl}(O_{xy})$. As $x \notin \mathrm{cl}(O_{xy})$, it consequently follows that $x \notin s\beta\,\mathrm{cl}_{\mathcal{I}}(H_{xy})$, and thus $x \in U$. Furthermore, we observe that

$$F - V = F - \cup\{H_{xy} : y \in F\} \subseteq X - (\cup\{H_{xy} : y \in F\} \cup W) \in \mathcal{I}.$$

Thus, U and V are indeed disjoint $s\beta$-$\mathcal{I}$-open sets, satisfying $x \in U$ and $F - V \in \mathcal{I}$. This confirms that the space $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-regular.

Theorem 12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following statements are equivalent:*

(1) *For every closed subset F of X and every $x \notin F$, there exist disjoint strong β-$\mathcal{I}$-open sets U and V such that $x \in U$ and $F - V \in \mathcal{I}$.*

(2) *For every open subset G of X and every $x \in G$, there exists a strong β-$\mathcal{I}$-open set U such that $x \in U$ and $s\beta\,\mathrm{cl}_{\mathcal{I}}(U) - G \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): Let G be an open set and $x \in G$. Then $X - G$ is closed, and since $x \notin X - G$, by assumption, there exist disjoint strong β-$\mathcal{I}$-open sets U and V such that $x \in U$ and $(X - G) - V \in \mathcal{I}$. Since U and V are disjoint, by Theorem 8, we have $s\beta\,\mathrm{cl}_{\mathcal{I}}(U) \subseteq X - V$. Therefore, $s\beta\,\mathrm{cl}_{\mathcal{I}}(U) \cap (X - G) \subseteq (X - G) - V$. Hence, we conclude that $s\beta\,\mathrm{cl}_{\mathcal{I}}(U) \cap (X - G) = s\beta\,\mathrm{cl}_{\mathcal{I}}(U) - G \in \mathcal{I}$.

(2) $\Rightarrow$ (1): Let F be a closed set and $x \notin F$. This implies that $X - F$ is open, and $x \in X - F$. By assumption, there exists a strong β-$\mathcal{I}$-open set U such that $x \in U$ and $s\beta\,\mathrm{cl}_{\mathcal{I}}(U) - (X - F) \in \mathcal{I}$. Thus, we define $V = X - s\beta\,\mathrm{cl}_{\mathcal{I}}(U)$, which is a strong β-$\mathcal{I}$-open set. Since U and V are disjoint, we have $F - V = F - (X - s\beta\,\mathrm{cl}_{\mathcal{I}}(U)) = s\beta\,\mathrm{cl}_{\mathcal{I}}(U) - (X - F) \in \mathcal{I}$.

By Theorem 11 and Theorem 12, we have the following colollary.

Corollary 1. *An ideal topological space $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-paracompact and Hausdorff if and only if for any open set $G \subseteq X$ and for every point $x \in G$, there exists a strong β-$\mathcal{I}$-open set U such that $x \in U$ and $s\beta\,\mathrm{cl}_{\mathcal{I}}(U) - G \in \mathcal{I}$.*

Theorem 13. *If an ideal topological space $(X, \tau, \mathcal{I})$ is $s\beta$-$\mathcal{I}$-paracompact and regular, then every open cover of X has an $s\beta$-$\mathcal{I}$-locally finite $\mathcal{I}$-cover refinement of strong β-$\mathcal{I}$-closed sets.*

Proof. Let $\mathcal{A}$ be an open cover of X. By the regularity of X, for each $x \in X$ and $U_x \in \mathcal{A}$ containing x, there exists an open set G_x such that $x \in G_x$ and $\mathrm{cl}(G_x) \subseteq U_x$. Thus, the family $\mathcal{A}_1 = \{G_x : x \in X\}$ is an open cover of X. As X is $s\beta$-$\mathcal{I}$-paracompact, the cover $\mathcal{A}_1$ has an $s\beta$-$\mathcal{I}$-locally finite refinement $\mathcal{B}_1 = \{V_\lambda : \lambda \in \Lambda\}$ consisting of strong β-$\mathcal{I}$-open sets such that $X - \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Since $V_\lambda \subseteq s\beta\,\mathrm{cl}_{\mathcal{I}}(V_\lambda)$ for each λ, and $\mathcal{I}$ is an ideal, we conclude that $X - \cup\{s\beta\,\mathrm{cl}_{\mathcal{I}}(V_\lambda) : \lambda \in \Lambda\} \in \mathcal{I}$. By Theorem 9, the collection

$\mathcal{B} = \{s\beta \operatorname{cl}_{\mathcal{I}}(V_\lambda) : V_\lambda \in \mathcal{B}_1\}$ is $s\beta$-$\mathcal{I}$-locally finite. Since $\mathcal{B}_1$ refines $\mathcal{A}_1$, for each $\lambda \in \Lambda$, there exists some $G_x \in \mathcal{A}_1$ such that $V_\lambda \subseteq G_x$. Therefore, we have:

$$s\beta \operatorname{cl}_{\mathcal{I}}(V_\lambda) \subseteq \operatorname{cl}(V_\lambda) \subseteq \operatorname{cl}(G_x).$$

Consequently, $s\beta \operatorname{cl}_{\mathcal{I}}(V_\lambda) \subseteq U_x$. This shows that $\mathcal{B}$ refines $\mathcal{A}$. Thus, the collection $\mathcal{B} = \{s\beta \operatorname{cl}_{\mathcal{I}}(V_\lambda) : V_\lambda \in \mathcal{B}_1\}$ is an $s\beta$-$\mathcal{I}$-locally finite $\mathcal{I}$-cover refinement of strong β-$\mathcal{I}$-closed sets.

Theorem 14. *If an ideal topological space $(X, \tau, \mathcal{I})$ is Hausdorff and A is an $s\beta$-$\mathcal{I}$-paracompact subset of X, then A is a closed set in (X, τ^*).*

Proof. Let $x \in X - A$. Since X is Hausdorff, for each $y \in A$, there exists an open set $G_y \in \tau$ such that $y \in G_y$ and $x \notin G_y$. Thus, the family $\mathcal{U} = \{G_y : y \in A\}$ forms an open cover of A. Since A is an $s\beta$-$\mathcal{I}$-paracompact subset of X, the cover $\mathcal{U}$ has an $s\beta$-$\mathcal{I}$-locally finite strong β-$\mathcal{I}$-open refinement $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ such that $A - \cup\{V_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$.

Since $x \notin \operatorname{cl}(V_\alpha)$ for all $\alpha \in \Lambda$, it follows that $x \notin \cup\{\operatorname{cl}(V_\alpha) : \alpha \in \Lambda\}$. Moreover, since the locally finite family is closure-preserving, we have:

$$\cup\{\operatorname{cl}(V_\alpha) : \alpha \in \Lambda\} = \operatorname{cl}\left(\cup\{V_\alpha : \alpha \in \Lambda\}\right),$$

so that $x \notin \operatorname{cl}\left(\cup\{V_\alpha : \alpha \in \Lambda\}\right)$.

Let $G = X - \operatorname{cl}\left(\cup\{V_\alpha : \alpha \in \Lambda\}\right)$ and $J = A - \operatorname{cl}\left(\cup\{V_\alpha : \alpha \in \Lambda\}\right)$. We know that $G \in \tau$ and $J \subseteq A - \cup\{V_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$, and additionally, $(G - J) \cap A = \emptyset$. Thus, $x \notin A^*$, confirming that $A^* \subseteq A$.

Theorem 15. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If A and B are $s\beta$-$\mathcal{I}$-paracompact subsets of X, then $A \cup B$ is also an $s\beta$-$\mathcal{I}$-paracompact subset of X*

Proof. Let $\mathcal{A} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $A \cup B$. This cover $\mathcal{A}$ also serves as an open cover for both A and B. By assumption, there exist $s\beta$-$\mathcal{I}$-locally finite strong β-$\mathcal{I}$-open families $\mathcal{B} = \{V_\alpha : \alpha \in \Lambda_1\}$ for A and $\mathcal{C} = \{W_\mu : \mu \in \Lambda_2\}$ for B, which refine $\mathcal{A}$ such that:

$$A - \cup\{V_\alpha : \alpha \in \Lambda_1\} \in \mathcal{I}, \quad B - \cup\{W_\mu : \mu \in \Lambda_2\} \in \mathcal{I}.$$

This implies that:

$$A \subseteq \cup\{V_\alpha : \alpha \in \Lambda_1\} \cup I_1, \quad B \subseteq \cup\{W_\mu : \mu \in \Lambda_2\} \cup I_2,$$

where $I_1, I_2 \in \mathcal{I}$. Therefore, we have:

$$A \cup B \subseteq \cup(\{V_\alpha : \alpha \in \Lambda_1\} \cup \{W_\mu : \mu \in \Lambda_2\}) \cup (I_1 \cup I_2).$$

It follows that:

$$A \cup B - \cup\{V_\alpha \cup W_\mu : \alpha \in \Lambda_1, \mu \in \Lambda_2\} \subseteq I_1 \cup I_2 \in \mathcal{I}.$$

We see that the collection $\mathcal{D} = \{V_\alpha \cup W_\mu : \alpha \in \Lambda_1, \mu \in \Lambda_2\}$ of strong β-$\mathcal{I}$-open sets is $s\beta$-$\mathcal{I}$-locally finite and refines $\mathcal{A}$. Consequently, $A \cup B$ is an $s\beta$-$\mathcal{I}$-paracompact subset of X.

Theorem 16. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is an $s\beta$-$\mathcal{I}$-paracompact subset of X and B is a closed subset of X, then $A \cap B$ is also an $s\beta$-$\mathcal{I}$-paracompact subset of X.*

Proof. Let $\mathcal{A} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $A \cap B$. Since $X - B$ is open in X, the collection $\mathcal{A}_1 = \{U_\lambda : \lambda \in \Lambda\} \cup \{X - B\}$ forms an open cover of A. By assumption and Lemma 6, $\mathcal{A}_1$ has a precise $s\beta$-$\mathcal{I}$-locally finite strong β-$\mathcal{I}$-open refinement $\mathcal{B} = \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}$ such that: $V_\lambda \subseteq U_\lambda$ for all $\lambda \in \Lambda$, $V \subseteq X - B$, $A - (\cup\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}) \in \mathcal{I}$.

Now, observe that:

$$A \cap B - \cup\{V_\lambda : \lambda \in \Lambda\} = A \cap B - (\cup\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}) \subseteq A - (\cup\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}) \in \mathcal{I}.$$

Thus, we have $A \cap B - \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. It follows that the collection $\mathcal{B}_1 = \{V_\lambda : \lambda \in \Lambda\}$, consisting of strong β-$\mathcal{I}$-open sets, is $s\beta$-$\mathcal{I}$-locally finite and refines $\mathcal{A}$. Therefore, $A \cap B$ is an $s\beta$-$\mathcal{I}$-paracompact subset of X.

As a consequence of Theorem 16, we obtain the following corollaries.

Corollary 2. *If A is a closed subset of an $s\beta$-$\mathcal{I}$-paracompact space $(X, \tau, \mathcal{I})$, then A is also an $s\beta$-$\mathcal{I}$-paracompact subset of X.*

Corollary 3. *Let A and B be closed subsets of an $s\beta$-$\mathcal{I}$-paracompact space of $(X, \tau, \mathcal{I})$, then $A \cup B$ is also an $s\beta$-$\mathcal{I}$-paracompact subset of X*

Corollary 4. *If A is an $s\beta$-$\mathcal{I}$-paracompact subset of X and B is an open set contained A, then $A - B$ is an $s\beta$-$\mathcal{I}$-paracompact subset of X.*

Lemma 7. [19] *Let $\mathcal{I}$ be an ideal on a topological space X. If Y is a subset of X, then $\mathcal{I}_Y = \{I \cap Y : I \in \mathcal{I}\}$ is an ideal on Y.*

Theorem 17. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If A is an $s\beta$-$\mathcal{I}_{\mathcal{B}}$-paracompact subset of B and B is a subset of X whose intersection with any strong β-$\mathcal{I}$-open is again strong β-$\mathcal{I}$-open, then A is an $s\beta$-$\mathcal{I}$-paracompact subset of X.*

Proof. Let $\mathcal{A} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of A in X. Then, $\mathcal{U}_{\mathcal{A}} = \{U_\alpha \cap B : \alpha \in \Lambda\}$ is an open cover of A in B. Since A is an $s\beta$-$\mathcal{I}_{\mathcal{B}}$-paracompact subset of B, the collection $\mathcal{U}_{\mathcal{A}}$ has a precise $s\beta$-$\mathcal{I}_{\mathcal{B}}$-locally finite strong β-$\mathcal{I}$-open refinement $\mathcal{V}_{\mathcal{A}} = \{V_\alpha \cap B : \alpha \in \Lambda\}$ such that $V_\alpha \subseteq U_\alpha$ for all $\alpha \in \Lambda$, and $A - \cup\{V_\alpha \cap B : \alpha \in \Lambda\} \in \mathcal{I}_{\mathcal{B}}$. Since V_α is a strong β-$\mathcal{I}$-open subset of X for all $\alpha \in \Lambda$, the collection $\mathcal{B} = \{V_\alpha : \alpha \in \Lambda\}$ of strong β-$\mathcal{I}$-open sets of X is $s\beta$-$\mathcal{I}$-locally finite and refines $\mathcal{A}$. Now, we have

$$A - \cup\{V_\alpha : \alpha \in \Lambda\} \subseteq A - \cup\{V_\alpha \cap B : \alpha \in \Lambda\} \in \mathcal{I}_{\mathcal{B}} \subseteq \mathcal{I}.$$

Therefore, A is an $s\beta$-$\mathcal{I}$-paracompact subset of X.

4. Preservation of Strong β-$\mathcal{I}$-Paracompactness

In this part, we will show that the property of $s\beta$-$\mathcal{I}$-paracompactness is maintained under specific circumstances. We begin by introducing the following definition.

Definition 11. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau', \mathcal{J})$ be ideal topological spaces, and let $f : X \to Y$ be a function.*

(1) *The function f is said to be* $s\beta$-$\mathcal{I}$-open *if the image of every strong β-$\mathcal{I}$-open set in X is a strong β-$\mathcal{J}$-open set in Y.*

(2) *The function f is said to be* $s\beta$-$\mathcal{I}$-closed *if the image of every strong β-$\mathcal{I}$-closed set in X is a strong β-$\mathcal{J}$-closed set in Y.*

(3) *The function f is said to be* $s\beta$-$\mathcal{I}$-irresolute *if the preimage of every strong β-$\mathcal{J}$-open set in Y is a strong β-$\mathcal{I}$-open set in X.*

Observe that $f^{-1}(\mathcal{J})$ forms an ideal on X if $f : X \to Y$ is a function and Y is a topological space endowed with an ideal $\mathcal{J}$. Furthermore, if f is surjective and X has an ideal $\mathcal{I}$, it follows that $f(\mathcal{I})$ forms an ideal in Y.

We now present the characteristics of a function mapped between two ideal topological spaces, where one space reflects the same properties as the other. To begin, we introduce the notion of $s\beta$-$\mathcal{I}$-compactness and state a lemma that will be employed in proving Theorem 18.

Definition 12. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $s\beta$-$\mathcal{I}$-compact if every cover $\mathcal{A}$ of strong β-$\mathcal{I}$-open subsets of X has a finite subcover, i.e., there exist sets $A_1, A_2, \ldots, A_n \in \mathcal{A}$ such that*

$$X \subseteq A_1 \cup A_2 \cup \cdots \cup A_n.$$

Lemma 8. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau', \mathcal{J})$ be ideal topological spaces, and $f : X \to Y$ be surjective. Then f is strong β-$\mathcal{I}$-closed if and only if for every $y \in Y$ and for every strong β-$\mathcal{I}$-open set U in X containing $\{f^{-1}(y)\}$, there exists a strong β-$\mathcal{J}$-open set V containing y such that $f^{-1}(V) \subseteq U$.*

Proof. Let $y \in Y$ and let U be a strong β-$\mathcal{I}$-open subset of X such that $\{f^{-1}(y)\} \subseteq U$. Define the set $V = Y - f(X - U)$, which is strong β-$\mathcal{J}$-open. Clearly, $y \in V$ and $f^{-1}(V) \subseteq U$. Therefore, the necessity is established.

Now, let F be a strong β-$\mathcal{I}$-closed subset of X, and let $y \in Y - f(F)$. This implies that $\{f^{-1}(y)\} \subseteq X - F$. By assumption, there exists a strong β-$\mathcal{J}$-open set V_y containing y such that $f^{-1}(V_y) \subseteq X - F$, and consequently, $y \in V_y \subseteq Y - f(F)$. Therefore, the set $Y - f(F) = \cup\{V_y : y \in Y\}$ is a strong β-$\mathcal{J}$-open set. Hence, $f(F)$ is a strong β-$\mathcal{J}$-closed set in Y.

Theorem 18. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau', \mathcal{J})$ be ideal topological spaces. Suppose that $f : X \to Y$ satisfies the following statements:*

(1) *f is continuous;*

(2) *f is $s\beta$-$\mathcal{I}$-open;*

(3) *f is $s\beta$-$\mathcal{I}$-closed;*

(4) *f is surjective with $\{f^{-1}(y)\}$ being $s\beta$-$\mathcal{I}$-compact for every $y \in Y$; and*

(5) *$f(\mathcal{I}) \subseteq \mathcal{J}$.*

If X is $s\beta$-$\mathcal{I}$-paracompact, then Y is $s\beta$-$\mathcal{J}$-paracompact.

Proof. Let $\mathcal{A} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of Y. This implies that $\mathcal{B} = \{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is an open cover of X. Since X is $s\beta$-$\mathcal{I}$-paracompact, the collection $\mathcal{B}$ has a precise $s\beta$-$\mathcal{I}$-locally finite refinement $\mathcal{C} = \{V_\lambda : \lambda \in \Lambda\}$, where each V_λ is strong β-$\mathcal{I}$-open and

$$X - \cup_{\lambda\in\Lambda} V_\lambda \in \mathcal{I}.$$

Since f is $s\beta$-$\mathcal{I}$-open, the family $f(\mathcal{C}) = \{f(V_\lambda) : \lambda \in \Lambda\}$ consists of strong β-$\mathcal{J}$-open sets and is a refinement of $\mathcal{A}$. Moreover, we have

$$Y - \cup_{\lambda\in\Lambda} f(V_\lambda) \in \mathcal{J}.$$

Next, we check that $f(\mathcal{C})$ is $s\beta$-$\mathcal{J}$-locally finite. Let $y \in Y$. Since $\mathcal{C}$ is $s\beta$-$\mathcal{I}$-locally finite, for each $x \in \{f^{-1}(y)\}$, there exists a strong β-$\mathcal{I}$-open set G_x containing x such that G_x intersects at most finitely many elements of $\mathcal{C}$. Because $\{f^{-1}(y)\}$ is $s\beta$-$\mathcal{I}$-compact, and the family $\{G_x : f(x) = y\}$ forms a strong β-$\mathcal{I}$-open cover of $\{f^{-1}(y)\}$, there exists a finite subcover $\{H_y\}$ such that

$$\{f^{-1}(y)\} \subseteq \cup H_y,$$

and $\cup H_y$ intersects at most finitely many elements of $\mathcal{C}$. As f is $s\beta$-$\mathcal{I}$-closed, applying Lemma 8, there exists a strong β-$\mathcal{J}$-open set W_y containing y such that

$$f^{-1}(W_y) \subseteq \cup H_y.$$

Thus, $f^{-1}(W_y)$ intersects at most finitely many elements of $\mathcal{C}$, which implies that W_y intersects at most finitely many elements of $f(\mathcal{C})$. Therefore, $f(\mathcal{C})$ is $s\beta$-$\mathcal{J}$-locally finite in Y. Consequently, $(Y, \tau', \mathcal{J})$ is $s\beta$-$\mathcal{J}$-paracompact.

The next theorem characterizes a function from an $s\beta$-$\mathcal{I}$-paracompact ideal topological space $(X, \tau, \mathcal{I})$ to a topological space (Y, τ'), ensuring that Y retains the same structural properties as X.

Theorem 19. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and (Y, τ') a topological space. Suppose that $f : X \to Y$ satisfies the following statements:*

(1) *f is $s\beta$-$\mathcal{I}$-irresolute;*

(2) *f is continuous;*

(3) *f is $s\beta$-$\mathcal{I}$-open;*

(4) *f is surjective; and*

(5) *$f(\mathcal{V})$ is $s\beta$-$f(\mathcal{I})$-locally finite in Y for every $s\beta$-$\mathcal{I}$-locally finite $\mathcal{V}$ in X.*

If $(X,\tau,\mathcal{I})$ is $s\beta$-$\mathcal{I}$-paracompact, then $(Y,\tau',f(\mathcal{I}))$ is $s\beta$-$f(\mathcal{I})$-paracompact.

Proof. Let $\mathcal{A}=\{U_\lambda:\lambda\in\Lambda\}$ be an open cover of Y. This implies that $\mathcal{B}=\{f^{-1}(U_\lambda):\lambda\in\Lambda\}$ forms an open cover of X. Since X is $s\beta$-$\mathcal{I}$-paracompact, the collection $\mathcal{B}$ has a precise $s\beta$-$\mathcal{I}$-locally finite strong β-$\mathcal{I}$-open refinement $\mathcal{C}=\{V_\lambda:\lambda\in\Lambda\}$ such that

$$X-\cup_{\lambda\in\Lambda}V_\lambda\in\mathcal{I}.$$

Since

$$Y-\cup_{\lambda\in\Lambda}f(V_\lambda)\subseteq f\left(X-\cup_{\lambda\in\Lambda}V_\lambda\right),$$

and $f\left(X-\cup_{\lambda\in\Lambda}V_\lambda\right)\in f(\mathcal{I})$, it follows that

$$Y-\cup_{\lambda\in\Lambda}f(V_\lambda)\in f(\mathcal{I}).$$

Because f is surjective, $f(\mathcal{I})$ is an ideal in Y. By assumption, $f(\mathcal{C})=\{f(V_\lambda):\lambda\in\Lambda\}$ is a precise $s\beta$-$f(\mathcal{I})$-open refinement of strong β-$f(\mathcal{I})$-open sets in Y.

Now, we proceed to verify that $f(\mathcal{C})$ refines $\mathcal{A}$. For each $f(V_\lambda)\in f(\mathcal{C})$, we have $V_\lambda\in\mathcal{C}$, and there exists $U_\lambda\in\mathcal{A}$ such that $V_\lambda\subseteq f^{-1}(U_\lambda)$, as $\mathcal{C}$ refines $\mathcal{B}$. This implies that

$$f(V_\lambda)\subseteq f(f^{-1}(U_\lambda))\subseteq U_\lambda.$$

Consequently, $(Y,\tau',f(\mathcal{I}))$ is $s\beta$-$f(\mathcal{I})$-paracompact.

The following theorem outlines conditions under which a function from a topological space X to an $s\beta$-$\mathcal{J}$-paracompact ideal topological space Y ensures that X shares the same properties as Y.

Theorem 20. *Let (X,τ) be a topological space and $(Y,\tau',\mathcal{J})$ an ideal topological space. Suppose that $f:X\to Y$ satisfies the following statements:*

(1) *f is open;*

(2) *f is $s\beta$-$f^{-1}(\mathcal{J})$-irresolute; and*

(3) *f is bijective.*

If $(Y,\tau',\mathcal{J})$ is $s\beta$-$\mathcal{J}$-paracompact, then $(X,\tau,f^{-1}(\mathcal{J}))$ is $s\beta$-$f^{-1}(\mathcal{J})$-paracompact.

Proof. Let $\mathcal{A}=\{U_\lambda:\lambda\in\Lambda\}$ be an open cover of X. Since f is open, the collection $f(\mathcal{A})=\{f(U_\lambda):\lambda\in\Lambda\}$ is an open cover of Y. By hypothesis, $f(\mathcal{A})$ has a precise $s\beta$-$\mathcal{J}$-locally finite strong β-$\mathcal{J}$-open refinement $\mathcal{B}=\{V_\lambda:\lambda\in\Lambda\}$ such that

$$Y-\cup_{\lambda\in\Lambda}V_\lambda\in\mathcal{J}.$$

This implies that $Y - \cup_{\lambda\in\Lambda} V_\lambda = J$ for some $J \in \mathcal{J}$, which means

$$f^{-1}(Y) - \cup_{\lambda\in\Lambda} f^{-1}(V_\lambda) = f^{-1}(Y) - f^{-1}(\cup_{\lambda\in\Lambda} V_\lambda) = f^{-1}(J).$$

Hence,

$$X - \cup_{\lambda\in\Lambda} f^{-1}(V_\lambda) \in f^{-1}(\mathcal{J}).$$

Let $\mathcal{I} = f^{-1}(\mathcal{J})$. Since f is $s\beta$-$\mathcal{I}$-irresolute, the collection $\mathcal{C} = \{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ forms an $s\beta$-$\mathcal{I}$-locally finite collection of strong β-$\mathcal{I}$-open sets. For each $f^{-1}(V_\lambda) \in \mathcal{C}$, since $V_\lambda \in \mathcal{B}$, there exists $U_\lambda \in \mathcal{A}$ such that $V_\lambda \subseteq f(U_\lambda)$ as $\mathcal{B}$ refines $f(\mathcal{A})$. Thus,

$$f^{-1}(V_\lambda) \subseteq f^{-1}(f(U_\lambda)) = U_\lambda.$$

The refinement of $\mathcal{A}$ by $\mathcal{C}$ is then asserted. Therefore, $(X, \tau, \mathcal{I})$ is shown to be $s\beta$-$\mathcal{I}$-paracompact.

5. Conclusions

Conclusions: This paper introduces and investigates the concept of strong β-$\mathcal{I}$-submaximal ideal topological spaces, which generalizes submaximality in the context of ideal topology. We have demonstrated that the following statements are equivalent:

- $(X, \tau, \mathcal{I})$ possesses the property of strong β-$\mathcal{I}$-submaximality.
- For every subset A of X, the set $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is strong β-$\mathcal{I}$-closed.
- Each subset of X is locally strong β-$\mathcal{I}$-closed.
- Every subset of X qualifies as a B-$s\mathcal{I}$ set.
- Any strong β-$\mathcal{I}$-dense subset of X is a B-$s\mathcal{I}$ set.
- Every subset of X is co-locally strong β-$\mathcal{I}$-closed.
- If a subset A of X satisfies $s\beta\,\mathrm{Int}_{\mathcal{I}}(A) = \emptyset$, then it is strong β-$\mathcal{I}$-closed.
- All strong β-$\mathcal{I}$-codense subsets of X are strong β-$\mathcal{I}$-closed.
- Every subset A of X for which $s\beta\,\mathrm{Int}_{\mathcal{I}}(A) = \emptyset$ holds is both strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete.
- For any subset A of X, the set $s\beta\,\mathrm{cl}_{\mathcal{I}}(A) - A$ is both strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete.
- Each strong β-$\mathcal{I}$-codense subset of X is strong β-$\mathcal{I}$-closed and strong β-$\mathcal{I}$-discrete.

This study explores various characterizations of $s\beta$-$\mathcal{I}$-paracompactness in ideal topological spaces, highlighting it as a stronger variant of β-paracompactness. It is established that every $s\beta$-$\mathcal{I}$-paracompact space is necessarily $\mathcal{I}$-β-paracompact, and in the case of Hausdorff spaces, $s\beta$-$\mathcal{I}$-paracompactness implies $s\beta$-$\mathcal{I}$-regularity.

Moreover, it is shown that the union of two $s\beta$-$\mathcal{I}$-paracompact subsets remains $s\beta$-$\mathcal{I}$-paracompact, and the intersection of an $s\beta$-$\mathcal{I}$-paracompact subset with a closed set also retains the property.

The work further demonstrates the preservation of $s\beta$-$\mathcal{I}$-paracompactness under certain mappings. Specifically, if a map $f : X \to Y$ is continuous, $s\beta$-$\mathcal{I}$-open, $s\beta$-$\mathcal{I}$-closed, and surjective, with each fiber $\{f^{-1}(y)\}$ being $s\beta$-$\mathcal{I}$-compact, and if $f(\mathcal{I}) \subseteq \mathcal{J}$, then $s\beta$-$\mathcal{I}$-paracompactness of X implies the same for Y.

Similarly, if f is $s\beta$-$\mathcal{I}$-irresolute, continuous, $s\beta$-$\mathcal{I}$-open, and surjective, and for every $s\beta$-$\mathcal{I}$-locally finite family $\mathcal{V}$, the image $f(\mathcal{V})$ is $s\beta$-$\mathcal{J}$-locally finite, then the $s\beta$-$\mathcal{I}$-paracompactness of X ensures that Y is $s\beta$-$\mathcal{J}$-paracompact.

Finally, if f is open, $s\beta$-$\mathcal{I}$-irresolute, and bijective, and Y is $s\beta$-$\mathcal{J}$-paracompact, then it follows that X is also $s\beta$-$\mathcal{I}$-paracompact.

Applications: Strong β-$\mathcal{I}$-submaximality and paracompactness contribute significantly to fields such as digital topology, computational geometry, and data analysis by utilizing generalized open sets and locally finite covers. These properties broaden the scope of continuity results within ideal spaces and find relevance in areas like fuzzy and soft topology, effectively linking abstract topological theory with computational approaches to support better approximation techniques and structural interpretations.

Future Works: Potential research avenues include extending the concepts to fuzzy and neutrosophic ideals, exploring their impact on separation axioms and product topologies, investigating connections with measure theory, analyzing relationships among various generalized open sets, and creating algorithms for detecting these properties in finite spaces—thereby enriching both the theoretical framework and real-world applications.

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