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A Combined Study of Continuities and Boundedness in Neutrosophic Pseudo-Normed Linear Spaces

Pandiselvi. M[1], Jeyaraman. M[1], Mohammad Akram[2,*]

[1] *PG and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamil Nadu, India*
[2] *Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia*

Abstract. In this work, we investigate various notions of neutrosophic boundedness and continuity within the framework of neutrosophic pseudo normed linear spaces. Specifically, we introduce and analyze different types of neutrosophic continuity such as pointwise, uniform, and sequential neutrosophic continuity and examine their interrelationships. We also explore several forms of neutrosophic boundedness and establish connections between boundedness and continuity under neutrosophic settings. Illustrative examples are provided to demonstrate the applicability of the introduced concepts.

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1. Introduction

In 1992, Felbin [1] introduced the concept of fuzzy norms on linear spaces. Subsequently, Xiao and Zhu [2] extended this framework by investigating the topological properties of fuzzy normed linear spaces. Bag and Samanta proposed another form of fuzzy norm [3] and further advanced the theory by developing notions such as weak and strong fuzzy boundedness, weak and sequential fuzzy continuity, fuzzy continuity, and the fuzzy norm of linear operators with respect to an associated fuzzy norm [4]. The concept of fuzzy pseudo norms was introduced by S. Nadaban [5]. Dinda et al. [6] explored intuitionistic fuzzy pseudo normed linear spaces and demonstrated that these possess a more general structure than intuitionistic fuzzy normed spaces. The foundation for intuitionistic fuzzy sets, as a generalization of fuzzy sets, was laid by Atanassov [7], while J. H. Park [8] introduced the notion of intuitionistic fuzzy metric spaces and examined several

*Corresponding author.
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Email addresses: mpandiselvi2612@gmail.com (Pandiselvi. M), jeya.math@gmail.com (Jeyaraman. M), akramkhan_20@rediffmail.com, akram@iu.edu.sa (M. Akram)

of their fundamental properties. In 1998, Smarandache [9] proposed neutrosophic logic and neutrosophic sets. Building on this, Kirisci and Simsek [10] introduced the concept of neutrosophic metric spaces, which account for degrees of membership, non-membership, and indeterminacy (neutrality). In this context, Jeyaraman et al. [11] investigated multivalued mappings in Hausdorff neutrosophic metric spaces and established several fixed point results. Their work emphasizes the structural richness of neutrosophic metric spaces and provides a foundation for various applications in fixed point theory.

The present work focuses on the study of neutrosophic boundedness and various forms of neutrosophic continuity for linear operators within neutrosophic pseudo normed spaces—a generalization of neutrosophic normed spaces. Section 3 highlights the concept of neutrosophic continuities and the intra-relationships among their various types. Section 4 delves into different notions of neutrosophic boundedness. Finally, the interconnections between different forms of continuity and boundedness are discussed, following an initial investigation of intra-relations among the various types of neutrosophic boundedness.

2. Preliminaries

This section recalls essential definitions and concepts. These preliminaries form the foundation for the results developed in subsequent sections

Definition 1. *[12] Let $\mathfrak{F}$ be a linear space over a field $\mathbb{R}$. A mapping $\|\cdot\| : \mathfrak{F} \to \mathbb{R}$ is named to be a pseudo norm on $\mathfrak{F}$ if it holds the following assertions:*
1. $\|\varpi\| \geq 0$, *for all* $\varpi \in \mathfrak{F}$,
2. $\|\varpi\| = 0 \Leftrightarrow \varpi = 0$,
3. $\|\mathfrak{k}\varpi\| \leq \|\varpi\|$, *for all* $\varpi \in \mathfrak{F}$, *for all* $\mathfrak{k} \in \mathfrak{K}$ *with* $|\mathfrak{k}| \leq 1$,
4. The strong triangle inequality $\|\varpi + \mathfrak{w}\| \leq \|\varpi\| + \|\mathfrak{w}\|$, *for all* $\varpi, \mathfrak{w} \in \mathfrak{F}$.

Definition 2. *Let $\mathfrak{F}$ is a vector space over a field $\mathbb{R}$, and η, ρ, ς are neutrosophic sets on $\mathfrak{F} \times \mathbb{R} \times \mathbb{R}$, if it meets the following conditions for every $\varpi, \mathfrak{w} \in \mathfrak{F}$ and $\mathfrak{o}, \varphi \in \mathbb{R}$*
(n1) $0 \leq \eta(\varpi, \varphi) \leq 1; 0 \leq \rho(\varpi, \varphi) \leq 1; 0 \leq \varsigma(\varpi, \varphi) \leq 1$;
(n2) $\eta(\varpi, \varphi) + \nu(\varpi, \varphi) + \rho(\varpi, \varphi) \leq 3$;
(n3) for all $\varphi \in \mathbb{R}$ *with* $\varphi \leq 0, \eta(\varpi, \varphi) = 0$;
(n4) for all $\varphi \in \mathbb{R}^+, \eta(\varpi, \varphi) = 1 \Leftrightarrow \varpi = \varphi$;
(n5) for all $\varphi \in \mathbb{R}^+, \eta(\sigma\varpi, \varphi) \geq \eta(\varpi, \varphi)$ *if* $|\sigma| \leq 1$ *for all* $\sigma \in \mathfrak{F}$;
(n6) $\eta(\varpi + \mathfrak{w}, \mathfrak{o} + \varphi) \geq \min(\eta(\varpi, \mathfrak{o}), \eta(\mathfrak{w}, \varphi))$ *for all* $\mathfrak{o}, \varphi \in \mathbb{R}^+$;
(n7) $\lim_{\varphi \to \infty} \eta(\varpi, \varphi) = 1$;
(n8) if there exists $0 < \delta < 1$ *such that* $\eta(\varpi, \varphi) > \delta, \forall \varphi \in \mathbb{R}^+$ *then* $\varpi = 0$;
(n9) $\eta(\varpi, \cdot)$ *is left continuous on* $\mathbb{R}$, *for all* $\varpi \in \mathfrak{F}$;
(n10) for all $\varphi \in \mathbb{R}$ *with* $\varphi \leq 0, \rho(\varpi, \varphi) = 1$;
(n11) for all $\varphi \in \mathbb{R}^+, \rho(\varpi, \varphi) = 0 \Leftrightarrow \varpi = \varphi$;
(n12) for all $\varphi \in \mathbb{R}^+, \rho(\sigma\varpi, \varphi) \leq \rho(\varpi, \varphi)$ *if* $|\sigma| \leq 1$ *for all* $\sigma \in \mathfrak{F}$;
(n13) $\rho(\varpi + \mathfrak{w}, \mathfrak{o} + \varphi) \leq \max(\rho(\varpi, \mathfrak{o}), \rho(\mathfrak{w}, \varphi))$ *for all* $\mathfrak{o}, \varphi \in \mathbb{R}^+$;
(n14) $\lim_{\varphi \to \infty} \rho(\varpi, \varphi) = 0$;

(n15) If there exists $0 < \delta < 1$ *such that* $\rho(\varpi, \varphi) < \delta, \forall \varphi \in \mathbb{R}^+$ *then* $\varpi = 0$*;*
(n16) $\rho(\varpi, \cdot)$ *is left continuous on* $\mathbb{R}$*, for all* $\varpi \in \mathfrak{F}$*;*
(n17) for all $\varphi \in \mathbb{R}$ *with* $\varphi \leq 0, \varsigma(\varpi, \varphi) = 1$*;*
(n18) for all $\varphi \in \mathbb{R}^+, \varsigma(\varpi, \varphi) = 0 \Leftrightarrow \varpi = 0$*;*
(n19) for all $\varphi \in \mathbb{R}^+, \varsigma(\sigma\varpi, \varphi) \leq \varsigma(\varpi, \varphi)$ *if* $|\sigma| \leq 1$ *for all* $\sigma \in \mathfrak{F}$*;*
(n20) $\varsigma(\varpi + \mathfrak{w}, \mathfrak{o} + \varphi) \leq \max(\varsigma(\varpi, \mathfrak{o}), \varsigma(w, \varphi))$ *for all* $\mathfrak{o}, \varphi \in \mathbb{R}^+$*;*
(n21) $\lim_{\varphi \to \infty} \varsigma(\varpi, \varphi) = 0$*;*
(n22) If there exists $0 < \delta < 1$ *such that* $\varsigma(\varpi, \varphi) < \delta, \forall \varphi \in \mathbb{R}^+$ *then* $\varpi = 0$*;*
(n23) $\varsigma(\varpi, \cdot)$ *is left continuous on* $\mathbb{R}$*, for all* $\varpi \in \mathfrak{F}$*;*
Then the 4-tuple $(\mathfrak{F}, \eta, \rho, \varsigma)$ *is named to be a Neutrosophic Pseudo Normed Linear Space [NPNLS].*

Note 1. $\mathfrak{r} * \mathfrak{s} = \mathfrak{r}$ *and* $\mathfrak{r} \Diamond \mathfrak{s} = \mathfrak{r}, \forall \mathfrak{r} \in [0, 1]$ *is satisfied only when* $\mathfrak{r} * \mathfrak{s} = \max\{\mathfrak{r}, \mathfrak{s}\}$ *and* $\mathfrak{r} \Diamond \mathfrak{s} = \max\{\mathfrak{r}, \mathfrak{s}\}$.

Definition 3. *Let* $(\mathfrak{F}, \eta, \rho, \varsigma)$ *be NPNLS. A sequence* $\{\mathfrak{r}_n\}$ *converges to* $\mathfrak{r} \in \mathfrak{F}$ *if and only if* $\lim_{\varphi \to \infty} \eta(\mathfrak{r}_n - \mathfrak{r}, \varphi) = 1$, $\lim_{\varphi \to \infty} \rho(\mathfrak{r}_n - \mathfrak{r}, \varphi) = 0$ *and* $\lim_{\varphi \to \infty} \varsigma(\mathfrak{r}_n - \mathfrak{r}, \varphi) = 0$.

Theorem 2. *Let* $(\mathfrak{F}, \eta, \rho, \varsigma)$ *be NPNLS. Then for any* $0 < \delta < 1$ *the functions* $\|\varpi\|_\delta, \|\varpi\|_\delta^* : \mathfrak{F} \to [0, \infty)$ *defined as* $\|\varpi\|_\delta = \wedge\{\varphi > 0 : \eta(\varpi, \varphi) \geq \delta\}$ *is ascending family of pseudo norm on* $\mathfrak{F}$. $\|\varpi\|_\delta^* = \wedge\{\varphi > 0 : \rho(\varpi, \varphi) \leq \delta$ *and* $\varsigma(\varpi, \varphi) \leq \delta\}$ *is a descending family pseudo norm on* $\mathfrak{F}$.

Theorem 3. *Let* $(\mathfrak{F}, \eta, \rho, \varsigma)$ *be NPNLS and let*

$$\eta'(\varpi, \varphi) = \begin{cases} \vee\{0 < \delta < 1 : \|\varpi\|_\delta \leq \varphi\} & if \varphi > 0 \\ 0 & if \varphi \leq 0 \end{cases},$$

$$\rho'(\varpi, \varphi) = \begin{cases} \wedge\{0 < \delta < 1 : \|\varpi\|_\delta^* \leq \varphi\} & if \varphi > 0 \\ 1 & if \varphi \leq 0 \end{cases} \quad and$$

$$\varsigma'(\varpi, \varphi) = \begin{cases} \wedge\{0 < \delta < 1 : \|\varpi\|_\delta^* \leq \varphi\} & if \varphi > 0 \\ 1 & if \varphi \leq 0 \end{cases} \quad then$$

(1). $(\eta', \rho', \varsigma')$ *is a neutrosophic pseudo norm on* $\mathfrak{F}$.
(2). $\eta = \eta', \rho = \rho'$ *and* $\varsigma = \varsigma'$*, where* $\|\varpi\|_\delta$ *is an ascending family of pseudo norms and* $\|\varpi\|_\delta^*$ *is descending family of pseudo norms defined in Theorem (2).*

3. Neutrosophic continuities of an operators on Neutrosophic Pseudo Normed Linear Spaces

This section investigates various forms of continuity for operators on NPNLS, which are essential for understanding operator behavior in neutrosophic settings.

Definition 4. *Let* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *be NPNLS.*
A mapping $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is named to be neutrosophic continuous at* ϖ_0 *if for any given* $\epsilon > 0$ *and* $0 < \delta < 1$ *there exist* $\gamma = \gamma(\delta, \epsilon)$ *such that for all* $\varpi \in \mathfrak{F}$,
$\eta_1(\varpi - \varpi_0, \gamma) > 1 - \alpha \Rightarrow \eta_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \gamma) > 1 - \delta$

$\rho_1(\varpi - \varpi_0, \gamma) < \alpha \Rightarrow \rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \gamma) < \delta$ *and*
$\varsigma_1(\varpi - \varpi_0, \gamma) < \alpha \Rightarrow \varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \gamma) < \delta.$

Definition 5. *Let* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *be NPNLS.*
A mapping $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is referred as sequentially neutrosophic continuous at* ϖ_0 *if for any sequence* $\{\varpi_n\}, \varpi_n \in \mathfrak{F}$ *and* $\varphi > 0$,

$$\lim_{\varphi\to\infty} \eta_1(\varpi_n - \varpi_0, \varphi) = 1 \Rightarrow \lim_{\varphi\to\infty} \eta_2(\Upsilon(\varpi_n) - \Upsilon(\varpi_0), \varphi) = 1,$$
$$\lim_{\varphi\to\infty} \rho_1(\varpi_n - \varpi_0, \varphi) = 0 \Rightarrow \lim_{\varphi\to\infty} \rho_2(\Upsilon(\varpi_n) - \Upsilon(\varpi_0), \varphi) = 0 \text{ and}$$
$$\lim_{\varphi\to\infty} \varsigma_1(\varpi_n - \varpi_0, \varphi) = 0 \Rightarrow \lim_{\varphi\to\infty} \varsigma_2(\Upsilon(\varpi_n) - \Upsilon(\varpi_0), \varphi) = 0.$$

Theorem 4. *If a linear operator* $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is sequentially neutrosophic continuous at* $\varpi_0 \in \mathfrak{F}$ *then it is sequentially neutrosophic continuous on* $\mathfrak{F}$*, where* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *are NPNLS.*

Proof. Let $\{\varpi_n\}$ be a sequence in $\mathfrak{F}$ and $\varpi_n \to \varpi$. Then for all $\varphi > 0$,
$\lim_{\varphi\to\infty} \eta_1(\varpi_n - \varpi, \varphi) = 1$, $\lim_{\varphi\to\infty} \rho_1(\varpi_n - \varpi, \varphi) = 0$ and $\lim_{\varphi\to\infty} \varsigma_1(\varpi_n - \varpi, \varphi) = 0$.
Therefore, $\lim_{\varphi\to\infty} \eta_1((\varpi_n - \varpi + \varpi_0) - \varpi_0, \varphi) = 1$, $\lim_{\varphi\to\infty} \rho_1((\varpi_n - \varpi + \varpi_0) - \varpi_0, \varphi) = 0$ and $\lim_{\varphi\to\infty} \varsigma_1((\varpi_n - \varpi + \varpi_0) - \varpi_0, \varphi) = 0$.
Since Υ is sequentially neutrosophic continuous at ϖ_0, for all $\varphi > 0$ we have

$$\lim_{\varphi\to\infty} \eta_1(\Upsilon(\varpi_n - \varpi + \varpi_0) - \Upsilon(\varpi_0), \varphi) = 1,$$
$$\lim_{\varphi\to\infty} \rho_1(\Upsilon(\varpi_n - \varpi + \varpi_0) - \Upsilon(\varpi_0), \varphi) = 0 \text{ and}$$
$$\lim_{\varphi\to\infty} \varsigma_1(\Upsilon(\varpi_n - \varpi + \varpi_0) - \Upsilon(\varpi_0), \varphi) = 0$$
$$\Rightarrow \lim_{\varphi\to\infty} \eta_1(\Upsilon(\varpi_n) - \Upsilon(\varpi) + \Upsilon(\varpi_0) - \Upsilon(\varpi_0), \varphi) = 1,$$
$$\lim_{\varphi\to\infty} \rho_1(\Upsilon(\varpi_n) - \Upsilon(\varpi) + \Upsilon(\varpi_0) - \Upsilon(\varpi_0), \varphi) = 0 \text{ and}$$
$$\lim_{\varphi\to\infty} \varsigma_1(\Upsilon(\varpi_n) - \Upsilon(\varpi) + \Upsilon(\varpi_0) - \Upsilon(\varpi_0), \varphi) = 0,$$

since Υ is linear.

$$\lim_{\varphi\to\infty} \eta_1(\Upsilon(\varpi_n) - \Upsilon(\varpi), \varphi) = 1,$$
$$\lim_{\varphi\to\infty} \rho_1(\Upsilon(\varpi_n) - \Upsilon(\varpi), \varphi) = 0 \text{ and}$$
$$\lim_{\varphi\to\infty} \varsigma_1(\Upsilon(\varpi_n) - \Upsilon(\varpi), \varphi) = 0.$$

Since $\varpi \in \mathfrak{F}$ was chosen arbitrarily, it follows that Υ is sequentially neutrosophic continuous on $\mathfrak{F}$.

Theorem 5. *If a linear operator* $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is sequentially neutrosophic continuous at* $\varpi_0 \in \mathfrak{F}$ *then it is sequentially neutrosophic continuous if and only if it is neutrosophic continuous, where* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *are NPNLS.*

Proof. Suppose Υ be neutrosophic continuous at $\varpi_0 \in \mathfrak{F}, \{\varpi_n\}$ be a sequence in $\mathfrak{F}$ and $\varpi_n \to \varpi_0$. Then for any given $\epsilon > 0, 0 < \delta < 1$ there exist $\gamma = \gamma(\delta, \epsilon)$ and $\alpha = \alpha(\delta, \epsilon)$ such that for all $\varpi \in \mathfrak{F}$,

$$\begin{aligned}
&\eta_1(\varpi - \varpi_0, \gamma) > 1 - \alpha \Rightarrow \eta_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \gamma) > 1 - \delta \\
&\rho_1(\varpi - \varpi_0, \gamma) < \alpha \Rightarrow \rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \gamma) < \delta \,\text{and} \\
&\varsigma_1(\varpi - \varpi_0, \gamma) < \alpha \Rightarrow \varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \gamma) < \delta.
\end{aligned}$$

Since $\{\varpi_n\}$ converges to ϖ_0 there exists $n_0 \in \mathbb{N}$ such that for all $n_0 \geq n$,
$\eta_1(\varpi - \varpi_0, \gamma) > 1 - \alpha, \rho_1(\varpi - \varpi_0, \gamma) < \alpha \,\text{and}\, \varsigma_1(\varpi - \varpi_0, \gamma) < \alpha$
and since Υ is neutrosophic continuous at $\varpi_0 \in \mathfrak{F}$, we have
$\eta_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) > 1 - \delta, \rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) < \delta \,\text{and}\, \varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) < \delta$.
Hence, $\Upsilon(\varpi_n) \to \Upsilon(\varpi_0)$.
Conversely, suppose Υ be not neutrosophic continuous at $\varpi_0 \in \mathfrak{F}$. Then there exist $\mathfrak{w} \in \mathfrak{F}$ such that for any given $\epsilon > 0, 0 < \delta < 1$ there exist $\gamma > 0$ and $\alpha \in (0, 1)$,

$$\begin{aligned}
&\eta_1(\mathfrak{w} - \varpi_0, \gamma) > 1 - \alpha \Rightarrow \eta_2(\Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0), \gamma) > 1 - \delta, \\
&\rho_1(\mathfrak{w} - \varpi_0, \gamma) < \alpha \Rightarrow \rho_2(\Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0), \gamma) < \delta \,\text{and} \\
&\varsigma_1(\mathfrak{w} - \varpi_0, \gamma) < \alpha \Rightarrow \varsigma_2(\Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0), \gamma) < \delta.
\end{aligned}$$

Hence for $\gamma = \alpha = \frac{1}{n+1}$ there exist $\mathfrak{w}_n$ for $n = 1, 2, \ldots,$ such that

$$\begin{aligned}
&\eta_1(\mathfrak{w}_n - \varpi_0, \gamma) = \eta_1\left(\mathfrak{w}_n - \varpi_0, \frac{1}{n+1}\right) > 1 - \frac{1}{n+1} \\
&\Rightarrow \eta_2(\Upsilon(\mathfrak{w}_n) - \Upsilon(\varpi_0), \epsilon) \leq 1 - \delta, \\
&\rho_1(\mathfrak{w}_n - \varpi_0, \gamma) = \rho_1\left(\mathfrak{w}_n - \varpi_0, \frac{1}{n+1}\right) < \frac{1}{n+1} \\
&\Rightarrow \rho_2(\Upsilon(\mathfrak{w}_n) - \Upsilon(\varpi_0), \epsilon) \geq \delta \,\text{and} \\
&\varsigma_1(\mathfrak{w}_n - \varpi_0, \gamma) = \varsigma_1\left(\mathfrak{w}_n - \varpi_0, \frac{1}{n+1}\right) < \frac{1}{n+1} \\
&\Rightarrow \varsigma_2(\Upsilon(\mathfrak{w}_n) - \Upsilon(\varpi_0), \epsilon) \geq \delta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \eta_1(\mathfrak{w}_n - \varpi_0, \gamma) = 1 \Rightarrow \lim_{n\to\infty} \eta_2(\Upsilon(\mathfrak{w})_n - \Upsilon(\varpi_0), \epsilon) \neq 1, \\
&\lim_{n\to\infty} \rho_1(\mathfrak{w}_n - \varpi_0, \gamma) = 0 \Rightarrow \lim_{n\to\infty} \rho_2(\Upsilon(\mathfrak{w})_n - \Upsilon(\varpi_0), \epsilon) \neq 0 \,\text{and} \\
&\lim_{n\to\infty} \varsigma_1(\mathfrak{w}_n - \varpi_0, \gamma) = 0 \Rightarrow \lim_{n\to\infty} \varsigma_2(\Upsilon(\mathfrak{w})_n - \Upsilon(\varpi_0), \epsilon) \neq 0.
\end{aligned}$$

Hence Υ is not sequentially neutrosophic continuous at ϖ_0.

Definition 6. *Let* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *be NPNLS. We say that the mapping* $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is strongly neutrosophic continuous at* ϖ_0 *if, for each positive real number* ϵ*, one can find a* $\delta \in (0, 1)$ *such that a certain set of conditions is satisfied for all elements* ϖ *in* $\mathfrak{F}$.

$$\eta_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \geq \eta_1(\varpi - \varpi_0, \gamma),$$
$$\rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \rho_1(\varpi - \varpi_0, \gamma) \text{ and}$$
$$\varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \varsigma_1(\varpi - \varpi_0, \gamma).$$

Theorem 6. *If a linear operator* $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is strongly neutrosophic continuous at* $\varpi_0 \in \mathfrak{F}$ *then it is strongly neutrosophic continuous, where* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *are NPNLS.*

Proof. Given that Υ possesses strong neutrosophic continuity at the point ϖ_0, then corresponding to each $\epsilon > 0$, there exists a positive number γ such that the condition $\eta_2\big(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon\big) \geq \eta_1\big(\varpi - \varpi_0, \gamma\big)$ is satisfied for all $\varpi \in \mathfrak{F}$,
$\rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \rho_1(\varpi - \varpi_0, \gamma)$ and $\varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \varsigma_1(\varpi - \varpi_0, \gamma)$.
Taking $\mathfrak{w} \in \mathfrak{F}$ we have $\varpi + \varpi_0 - \mathfrak{w} \in \mathfrak{F}$. Therefore replacing ϖ by $\varpi + \varpi_0 - \mathfrak{w}$.
We have,

$$\eta_2(\Upsilon(\varpi + \varpi_0 - \mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \geq \eta_1(\varpi + \varpi_0 - \mathfrak{w} - \varpi_0, \gamma),$$
$$\rho_2(\Upsilon(\varpi + \varpi_0 - \mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \leq \rho_2(\varpi + \varpi_0 - \mathfrak{w} - \varpi_0, \gamma) \text{ and}$$
$$\varsigma_2(\Upsilon(\varpi + \varpi_0 - \mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \leq \varsigma_1(\varpi + \varpi_0 - \mathfrak{w} - \varpi_0, \gamma).$$

Therefore,

$$\eta_2(\Upsilon(\varpi) + \Upsilon(\varpi_0) - \Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0)), \epsilon) \geq \eta_1(\varpi - \mathfrak{w}, \gamma),$$
$$\rho_2(\Upsilon(\varpi) + \Upsilon(\varpi_0) - \Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0)), \epsilon) \leq \rho_1(\varpi - \mathfrak{w}, \gamma) \text{ and}$$
$$\varsigma_2(\Upsilon(\varpi) + \Upsilon(\varpi_0) - \Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0)), \epsilon) \leq \varsigma_1(\varpi - \mathfrak{w}, \gamma).$$

Hence,

$$\eta_2(\Upsilon(\varpi) - \Upsilon(\mathfrak{w}), \epsilon) \geq \eta_1(\varpi - \mathfrak{w}, \gamma),$$
$$\rho_2(\Upsilon(\varpi) - \Upsilon(\mathfrak{w})), \epsilon) \leq \rho_1(\varpi - \mathfrak{w}, \gamma) \text{ and}$$
$$\varsigma_2(\Upsilon(\varpi) - \Upsilon(\mathfrak{w})), \epsilon) \leq \varsigma_1(\varpi - \mathfrak{w}, \gamma).$$

Hence Υ is strongly neutrosophic continuous at $\mathfrak{w}$. Since $\mathfrak{w} \in \mathfrak{F}$ is arbitrary, Υ is strongly neutrosophic continuous on $\mathfrak{F}$.

Definition 7. *Let* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *be NPNLS.*
A mapping $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is referred as weakly neutrosophic continuous at* ϖ_0 *if for any given* $\epsilon > 0$ *there exists* $0 < \gamma$ *such that for all* $\varpi \in \mathfrak{F}$,

$$\eta_1(\varpi - \varpi_0, \gamma) \geq \delta \Rightarrow \eta_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \geq \delta,$$
$$\rho_1(\varpi - \varpi_0, \gamma) \leq \delta \Rightarrow \rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \delta \text{ and}$$
$$\varsigma_1(\varpi - \varpi_0, \gamma) \leq \delta \Rightarrow \varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \delta.$$

Theorem 7. *If a linear operator* $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is strongly neutrosophic continuous at* $\varpi_0 \in \mathfrak{F}$ *then it is weakly neutrosophic continuous on* $\mathfrak{F}$*, where* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *are NPNLS.*

Proof. Since Υ is weakly neutrosophic continuous at ϖ_0, for given $\epsilon > 0$ there exists $0 < \delta$ and $0 < \gamma$ such that for all $\varpi \in \mathfrak{F}$,

$$\begin{aligned} &\eta_1(\varpi - \varpi_0, \gamma) \geq \delta \Rightarrow \eta_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \geq \delta, \\ &\rho_1(\varpi - \varpi_0, \gamma) \leq \delta \Rightarrow \rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \delta \text{ and} \\ &\varsigma_1(\varpi - \varpi_0, \gamma) \leq \delta \Rightarrow \varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \delta. \end{aligned}$$

Taking $\mathfrak{w} \in \mathfrak{F}$ we have $\varpi + \varpi_0 - \mathfrak{w} \in \mathfrak{F}$. Therefore replacing ϖ by $\varpi + \varpi_0 - \mathfrak{w}$. We have,

$$\begin{aligned} \eta_1(\varpi - \varpi_0, \gamma) \geq \delta &\Rightarrow \eta_2(\Upsilon(\varpi + \varpi_0 - \mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \geq \delta \\ &\Rightarrow \eta_2(\Upsilon(\varpi) + \Upsilon(\varpi_0) - \Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \geq \delta \\ &\Rightarrow \eta_2(\Upsilon(\varpi) - \Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \geq \delta, \\ \rho_1(\varpi - \varpi_0, \gamma) \geq \delta &\Rightarrow \rho_2(\Upsilon(\varpi + \varpi_0 - \mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \leq \delta \\ &\Rightarrow \rho_2(\Upsilon(\varpi) + \Upsilon(\varpi_0) - \Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \leq \delta \\ &\Rightarrow \rho_2(\Upsilon(\varpi) - \Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \leq \delta \text{ and} \\ \varsigma_1(\varpi - \varpi_0, \gamma) \geq \delta &\Rightarrow \varsigma_2(\Upsilon(\varpi + \varpi_0 - \mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \leq \delta \\ &\Rightarrow \varsigma_2(\Upsilon(\varpi) + \Upsilon(\varpi_0) - \Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \leq \delta \\ &\Rightarrow \varsigma_2(\Upsilon(\varpi) - \Upsilon(\mathfrak{w}) - \Upsilon(\varpi_0), \epsilon) \leq \delta. \end{aligned}$$

Since $\mathfrak{w} \in \mathfrak{F}$ is arbitrary, Υ is weakly neutrosophic continuous on $\mathfrak{F}$.

Example 1. *Let* $(\mathfrak{F}, \|\cdot\|)$ *be a pseudo normed linear space and* $\eta, \rho, \varsigma : \mathfrak{F} \times \mathfrak{R} \to [0, 1]$ *be defined by*

$$\eta(\varpi, \varphi) = \begin{cases} 1 & \text{if } \varphi > 0, \|\varpi\| < \varphi \\ \frac{\varphi}{\varphi + \|\varpi\|} & \text{if } \varphi > 0, \|\varpi\| \geq \varphi \\ 0 & \text{if } \varphi \leq 0 \end{cases},$$

$$\rho(\varpi, \varphi) = \begin{cases} 0 & \text{if } \varphi > 0, \|\varpi\| < \varphi \\ \frac{\|\varpi\|}{\varphi + \|\varpi\|} & \text{if } \varphi > 0, \|\varpi\| \geq \varphi \\ 1 & \text{if } \varphi \geq 0 \end{cases} \quad \text{and}$$

$$\varsigma(\varpi, \varphi) = \begin{cases} 0 & \text{if } \varphi > 0, \|\varpi\| < \varphi \\ \frac{\|\varpi\|}{\varphi} & \text{if } \varphi > 0, \|\varpi\| \geq \varphi \\ 1 & \text{if } \varphi \geq 0 \end{cases}$$

then $(\mathfrak{F}, \eta, \rho, \varsigma)$ *is a NPNLS. Let* $\Upsilon : (\mathfrak{F}, \eta, \rho, \varsigma) \to (\mathfrak{F}, \eta, \rho, \varsigma)$ *be a linear operator defined by* $\Upsilon(\varpi) = \frac{\varpi^3}{1+\varpi}$*. Let* $\varpi_0 \in \mathfrak{F}$ *then for each* $\varpi \in \mathfrak{F}, \epsilon > 0$ *there exists* $0 < \delta < 1$ *and* $0 < \gamma$,

$$\eta(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \geq \delta$$

$$\Rightarrow \frac{\epsilon}{\epsilon + \|\Upsilon(\varpi) - \Upsilon(\varpi_0)\|} \geq \delta \Rightarrow \frac{\epsilon}{\epsilon + \left\|\frac{\varpi^3}{1+\varpi} - \frac{\varpi_0^3}{1+\varpi_0}\right\|} \geq \delta$$

$$\frac{\epsilon \|(1+\varpi)(1+\varpi_0)\|}{\epsilon \|(1+\varpi)(1+\varpi_0)\| + \left\|\varpi^3 + \varpi_0\varpi^3 - \varpi_0^3 - \varpi\xi_0^3\right\|} \geq \delta$$

$$\frac{\epsilon \|1+\varpi+\varpi_0+\varpi\xi_0\|}{\|1+\varpi+\varpi_0+\varpi\xi_0\| + \left\|(\varpi-\varpi_0)(\varpi^2+\varpi\xi_0+\varpi_0^2) + \varpi\xi_0(\varpi+\varpi_0)(\varpi-\varpi_0)\right\|} \geq \delta$$

$$\frac{\epsilon \|1+\varpi+\varpi_0+\varpi\xi_0\|}{\|1+\varpi+\varpi_0+\varpi\xi_0\| + \left\|(\varpi-\varpi_0)(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|} \geq \delta$$

$$\frac{\epsilon \frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|}}{\epsilon \frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|} + \|(\varpi-\varpi_0)\|} \geq \delta$$

$$\frac{\epsilon \|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2\right\|} \geq \delta \frac{\epsilon \|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2\right\|} + \delta \|(\varpi-\varpi_0)\|,$$

$$\epsilon \geq \delta.\epsilon + \delta \|(\varpi-\varpi_0)\| \frac{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|}{\|1+\varpi+\varpi_0+\varpi\xi_0\|} \geq \delta.\epsilon + \delta \|(\varpi-\varpi_0)\|,$$

since $\frac{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|}{\|1+\varpi+\varpi_0+\varpi\xi_0\|} \geq 1.$
$\gamma \geq \delta.\gamma + \delta \|(\varpi-\varpi_0)\|$, *by taking* $\epsilon = \gamma. \Rightarrow \frac{\gamma}{\gamma + \|(\varpi-\varpi_0)\|} \geq \delta \Rightarrow \eta(\varpi-\varpi_0, \gamma) \geq \delta.$

$$\rho(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \delta$$

$$\frac{\|\Upsilon(\varpi) - \Upsilon(\varpi_0)\|}{\epsilon + \|\Upsilon(\varpi) - \Upsilon(\varpi_0)\|} \leq \delta \Rightarrow \frac{\left\|\frac{\varpi^3}{1+\varpi} - \frac{\varpi_0^3}{1+\varpi_0}\right\|}{\epsilon + \left\|\frac{\varpi^3}{1+\varpi} - \frac{\varpi_0^3}{1+\varpi_0}\right\|} \leq \delta$$

$$\frac{\left\|\varpi^3 + \varpi_0\varpi^3 - \varpi_0^3 - \varpi\xi_0^3\right\|}{\epsilon \|(1+\varpi)(1+\varpi_0)| + \left\|\varpi^3 + \varpi_0\varpi^3 - \varpi_0^3 - \varpi\xi_0^3\right\|} \leq \delta$$

$$\frac{\left\|(\varpi-\varpi_0)(\varpi^2+\varpi\xi_0+\varpi_0^2) + \varpi\xi_0(\varpi+\varpi_0)(\varpi-\varpi_0)\right\|}{\epsilon \|1+\varpi+\varpi_0+\varpi\xi_0\| + \left\|(\varpi-\varpi_0)(\varpi^2+\varpi\xi_0+\varpi_0^2) + \varpi\xi_0(\varpi+\varpi_0)(\varpi-\varpi_0)\right\|} \leq \delta$$

$$\frac{\left\|(\varpi-\varpi_0)(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2\right\|}{\|1+\varpi+\varpi_0+\varpi\xi_0\| + \left\|(\varpi-\varpi_0)(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|} \leq \delta$$

$$\frac{\|\varpi-\varpi_0\| \left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|}{\|1+\varpi+\varpi_0+\varpi\xi_0\| + \|\varpi-\varpi_0\| \left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|} \leq \delta$$

$$\frac{\|(\varpi-\varpi_0)\|}{\epsilon \frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|} + \|(\varpi-\varpi_0)\|} \leq \delta$$

$$\delta \frac{\epsilon \|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2\right\|} + \delta \|(\varpi-\varpi_0)\| \leq \|(\varpi-\varpi_0)\|$$

$$\|(\varpi-\varpi_0)\| (1-\delta) \leq \delta\epsilon \frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|} \leq \delta.\epsilon,$$

since $\frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|} \leq 1.$
$\|(\varpi-\varpi_0)\| - \delta\|(\varpi-\varpi_0)\| \leq \delta\gamma,$ *by taking* $\epsilon = \gamma \Rightarrow \|(\varpi-\varpi_0)\| \leq \delta(\gamma + \|(\varpi-\varpi_0)\|)$
$\frac{\|(\varpi-\varpi_0)\|}{\gamma+\|(\varpi-\varpi_0)\|} \leq \delta \Rightarrow \rho(\varpi-\varpi_0,\gamma) \leq \delta.$

$$\begin{aligned}
&\varsigma(\Upsilon(\varpi)-\Upsilon(\varpi_0),\epsilon) \leq \delta \\
&\Rightarrow \frac{\|\Upsilon(\varpi)-\Upsilon(\varpi_0)\|}{\epsilon} \leq \delta \Rightarrow \frac{\left\|\frac{\varpi^3}{1+\varpi} - \frac{\varpi_0^3}{1+\varpi_0)}\right\|}{\epsilon} \leq \delta \\
&\frac{\left\|\varpi^3+\varpi_0\varpi^3-\varpi_0^3-\varpi\xi_0^3\right\|}{\epsilon\|(1+\varpi)(1+\varpi_0)\|} \leq \delta \\
&\frac{\left\|(\varpi-\varpi_0)(\varpi^2+\varpi\xi_0+\varpi_0^2)+\varpi\xi_0(\varpi+\varpi_0)(\varpi-\varpi_0)\right\|}{\epsilon\|1+\varpi+\varpi_0+\varpi\xi_0\|} \leq \delta \\
&\frac{\left\|(\varpi-\varpi_0)(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2\right\|}{\epsilon\|1+\varpi+\varpi_0+\varpi\xi_0\|} \leq \delta \\
&\frac{\|\varpi-\varpi_0\|\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|}{\epsilon\|1+\varpi+\varpi_0+\varpi\xi_0\|} \leq \delta \\
&\frac{\|(\varpi-\varpi_0)\|}{\epsilon\frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|}} \leq \delta \\
&\delta\frac{\epsilon\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2\right\|} + \delta\|(\varpi-\varpi_0)\| \leq \|(\varpi-\varpi_0)\| \\
&\|(\varpi-\varpi_0)\|\,\delta \leq \delta\epsilon\frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|} \leq \delta.\epsilon,
\end{aligned}$$

since $\frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right|} \leq 1.$
$\Rightarrow \delta\|(\varpi-\varpi_0)\| \leq \delta\gamma,$ *by taking* $\epsilon = \gamma \Rightarrow \|(\varpi-\varpi_0)\| \leq \delta\gamma$
$\frac{\|(\varpi-\varpi_0)\|}{\gamma} \leq \delta \Rightarrow \varsigma(\varpi-\varpi_0,\gamma) \leq \delta.$ *Thus for every*

$$\begin{aligned}
&\eta(\Upsilon(\varpi)-\Upsilon(\varpi_0),\epsilon) \geq \eta(\varpi-\varpi_0,\gamma) \\
&\frac{\epsilon}{\epsilon+\left\|\frac{\varpi^3}{1+\varpi}-\frac{\varpi_0^3}{1+\varpi_0}\right\|} \geq \frac{\delta}{\delta+\|(\varpi-\varpi_0)\|} \\
&\frac{\epsilon}{\epsilon+\frac{\|(\varpi^3+\varpi^3\varpi_0-\varpi_0^3-\varpi\xi_0^3)\|}{\|(1+\varpi)(1+\varpi_0)\|}} \geq \frac{\gamma}{\gamma+\|(\varpi-\varpi_0)\|} \\
&\epsilon\|(\varpi-\varpi_0)\|\,\|1+\varpi+\varpi_0+\varpi\xi_0\| \geq \gamma\|(\varpi-\varpi_0)\|\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\| \\
&\gamma \leq \epsilon\frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2\right\|}.
\end{aligned}$$

Now $\inf\left\{\frac{\|1+\varpi+\varpi_0+\varpi\xi_0\|}{\left\|(\varpi^2+\varpi\xi_0+\varpi^2\varpi_0+\varpi\xi_0^2)\right\|}\right\} = 0,$ *for all* $\varpi \in \mathfrak{F}$. *Therefore,* $\gamma = 0$, *which is not possible. This shows that* Υ *is not strongly neutrosophic continuous.*

Theorem 8. *If a linear operator $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ is strongly neutrosophic continuous then it is sequentially neutrosophic continuous, where $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ and $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ are NPNLS.*

Proof. Let $\{\varpi_n\}$ be a sequence in $\mathfrak{F}$ and $\varpi_n \to \varpi_0$.
$\lim_{n\to\infty} \eta_1(\varpi_n - \varpi_0, \varphi) = 1$, $\lim_{n\to\infty} \rho_1(\varpi_n - \varpi_0, \varphi) = 0$ and $\lim_{n\to\infty} \varsigma_1(\varpi_n - \varpi_0, \varphi) = 0$ for all $\varphi > 0$.
Now since Υ is strongly neutrosophic continuous at $\varpi_0 \in \mathfrak{F}$.
Then for any given $\epsilon > 0$, there exist $\gamma = \gamma(\epsilon) > 0$ such that for all $\varpi \in \mathfrak{F}$,
$\eta_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \geq \eta_1(\varpi - \varpi_0, \gamma)$,
$\rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \rho_1(\varpi - \varpi_0, \gamma)$ and
$\varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \varsigma_1(\varpi - \varpi_0, \gamma)$.
$\lim_{n\to\infty} \eta_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \geq \lim_{n\to\infty} \eta_1(\varpi_n - \varpi_0, \varphi) = 1 \Rightarrow \lim_{n\to\infty} \eta_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) = 1$
$\lim_{n\to\infty} \rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \lim_{n\to\infty} \rho_1(\varpi_n - \varpi_0, \varphi) = 0 \Rightarrow \lim_{n\to\infty} \rho_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) = 0$
$\lim_{n\to\infty} \varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) \leq \lim_{n\to\infty} \varsigma_1(\varpi_n - \varpi_0, \varphi) = 0 \Rightarrow \lim_{n\to\infty} \varsigma_2(\Upsilon(\varpi) - \Upsilon(\varpi_0), \epsilon) = 0.$
Since ϵ is arbitrary small positive number, Υ is sequentially neutrosophic continuous.

Example 2. *Consider the NPNLS $(\mathfrak{F}, \eta, \rho, \varsigma)$ as in Example (1) and the linear operator Υ is defined by $\Upsilon(\varpi) = \frac{\varpi^3}{1+\varpi}$. Let $\{\varpi_n\}$ be a sequence in $\mathfrak{F}$ and $\varpi_n \to \varpi_0$.*
$\lim_{n\to\infty} \eta_1(\varpi_n - \varpi_0, \varphi) = 1$, $\lim_{n\to\infty} \rho_1(\varpi_n - \varpi_0, \varphi) = 0$ *and* $\lim_{n\to\infty} \varsigma_1(\varpi_n - \varpi_0, \varphi) = 0$

$$\Rightarrow \lim_{n\to\infty} \frac{\varphi}{\varphi + \|(\varpi_n - \varpi_0)\|} = 1,\ \lim_{n\to\infty} \frac{\|(\varpi_n - \varpi_0)\|}{\varphi + \|(\varpi_n - \varpi_0)\|} = 0 \text{ and } \lim_{n\to\infty} \frac{\|(\varpi_n - \varpi_0)\|}{\varphi} = 0.$$

$$\Rightarrow \lim_{n\to\infty} \|(\varpi_n - \varpi_0)\| = 0 \tag{3.1}$$

$$\eta(\Upsilon(\varpi_n) - \Upsilon(\varpi_0), \varphi) = \frac{\varphi}{\varphi + \left\| \frac{\varpi_n^3}{1+\varpi_n} - \frac{\varpi_0^3}{1+\varpi_0} \right\|} = \frac{\varphi}{\varphi + \frac{\left\|(\varpi_n^3 + \varpi_n^3\varpi_0 - \varpi_0^3 - \varpi_n\varpi_0^3)\right\|}{\|(1+\varpi_n)(1+\varpi_0)\|}}$$

$$\Rightarrow \frac{\varphi\|(1+\varpi_n)(1+\varpi_0)\|}{\varphi\|(1+\varpi_n)(1+\varpi_0)\| + \|(\varpi_n - \varpi_0)\| \left\|(\varpi_n^2 + \varpi_n\varpi_0 + \varpi_n^2\varpi_0 + \varpi_n\varpi_0^2)\right\|} = 1 \text{ as } n \to \infty \text{ by Equation (3.1).}$$

And,

$$\rho(\Upsilon(\varpi_n) - \Upsilon(\varpi_0), \varphi) = \frac{\left\| \frac{\varpi_n^3}{1+\varpi_n} - \frac{\varpi_0^3}{1+\varpi_0} \right\|}{\varphi + \left\| \frac{\varpi_n^3}{1+\varpi_n} - \frac{\varpi_0^3}{1+\varpi_0} \right\|} = \frac{\frac{\left\|(\varpi_n^3 + \varpi_n^3\varpi_0 - \varpi_0^3 - \varpi_n\varpi_0^3)\right\|}{\|(1+\varpi_n)(1+\varpi_0)\|}}{\varphi + \frac{\left\|(\varpi_n^3 + \varpi_n^3\varpi_0 - \varpi_0^3 - \varpi_n\varpi_0^3)\right\|}{\|(1+\varpi_n)(1+\varpi_0)\|}}$$

$$= \frac{\|(\varpi_n - \varpi_0)\| \left\|(\varpi_n^2 + \varpi_n\varpi_0 + \varpi_n^2\varpi_0 + \varpi_n\varpi_0^2)\right\|}{\varphi\|(1+\varpi_n)(1+\varpi_0)\| + \|(\varpi_n - \varpi_0)\| \left\|(\varpi_n^2 + \varpi_n\varpi_0 + \varpi_n^2\varpi_0 + \varpi_n\varpi_0^2)\right\|} = 0 \text{ as } n \to \infty \text{ by Equation (3.1).}$$

Also,

$$\rho(\Upsilon(\varpi_n) - \Upsilon(\varpi_0), \varphi) = \frac{\left\| \frac{\varpi_n^3}{1+\varpi_n} - \frac{\varpi_0^3}{1+\varpi_0} \right\|}{\varphi} = \frac{\frac{\left\|(\varpi_n^3 + \varpi_n^3\varpi_0 - \varpi_0^3 - \varpi_n\varpi_0^3)\right\|}{\|(1+\varpi_n)(1+\varpi_0)\|}}{\varphi}$$

$$= \frac{\|(\varpi_n - \varpi_0)\| \left\|(\varpi_n^2 + \varpi_n\varpi_0 + \varpi_n^2\varpi_0 + \varpi_n\varpi_0^2)\right\|}{\varphi\|(1+\varpi_n)(1+\varpi_0)\|} = 0 \text{ as } n \to \infty \text{ by Equation (3.1).}$$

It follows that Υ exhibits sequential neutrosophic continuity at $\varpi_0 \in \mathfrak{F}$, and thus this property extends over the entire space $\mathfrak{F}$. Nonetheless, Example (1) clearly illustrates that Υ does not satisfy the criteria for strong neutrosophic continuity.

Corollary 1. *If a linear operator $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ is strongly neutrosophic continuous then it is neutrosophic continuous, where $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ and $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ are NPNLS.*

Proof. The corollary is a direct consequence of Theorem (5) and Theorem (8).

4. Neutrosophic Boundedness of Operators on Neutrosophic Pseudo Normed Linear Space

This section aims to generalize classical notions of boundedness for operators within neutrosophic pseudo normed linear spaces, offering new perspectives in the neutrosophic framework.

Definition 8. *Let* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *NPNLS.*
A mapping $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is said to be strongly neutrosophic bounded if for all* $\varpi \in \mathfrak{F}$ *and* $\varphi \in \mathbb{R}^+$, $\eta_2(\Upsilon(\varpi), \varphi) \geq \eta_1(\varpi, \varphi)$, $\rho_2(\Upsilon(\varpi), \varphi) \leq \rho_1(\varpi, \varphi)$ *and* $\varsigma_2(\Upsilon(\varpi), \varphi) \leq \varsigma_1(\varpi, \varphi)$.

Definition 9. *Let* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *NPNLS.*
A mapping $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is said to be weakly neutrosophic bounded if for any* $0 < \delta < 1$, *for all* $\varpi \in \mathfrak{F}$ *and* $\varphi \in \mathbb{R}^+$, $\eta_1(\varpi, \varphi) \geq \delta \Rightarrow \eta_2(\Upsilon(\varpi), \varphi) \geq \delta$,
$\rho_1(\varpi, \varphi) \leq 1 - \delta \Rightarrow \rho_2(\Upsilon(\varpi), \varphi) \leq 1 - \delta$ *and* $\varsigma_1(\varpi, \varphi) \leq 1 - \delta \Rightarrow \varsigma_2(\Upsilon(\varpi), \varphi) \leq 1 - \delta$.

Theorem 9. *If a linear operator* $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is strongly neutrosophic bounded then it is weakly neutrosophic bounded, where* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *are NPNLS.*

Proof. The result can be readily derived from the definitions of strong and weak neutrosophic boundedness for linear operators.

Definition 10. *Let* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1), (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *be NPNLS.*
A mapping $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is named to be uniformly neutrosophic bounded if there exists* $\epsilon > 0, 0 < \gamma < 1$ *such that* $\|\Upsilon\tilde{\varpi}\|_\delta^2 \leq \|\varpi\|_\delta^1$, *that* $\|\Upsilon\tilde{\varpi}\|_\delta^{2^*} \leq \|\varpi\|_\delta^{1^*}$, *where* $\|\cdot\|_\delta^1$ *and* $\|\cdot\|_\delta^2$ *are ascending family of pseudo norms and* $\|\cdot\|_\delta^{1^*}$ *and* $\|\cdot\|_\delta^{2^*}$ *are descending family of pseudo norm defined by*
$\|\varpi\|_\delta^1 = \wedge\{\varphi > 0 : \eta_1(\varpi, \varphi) \geq \delta\}, \|\Upsilon(\varpi)\|_\delta^2 = \wedge\{\varphi > 0 : \eta_2(\varpi, \varphi) \geq \delta\}$,
$\|\varpi\|_\delta^{1^*} = \wedge\{\varphi > 0 : \rho_1(\varpi, \varphi) \leq 1 - \delta \, and \, \varsigma_1(\varpi, \varphi) \leq 1 - \delta\}$,
$\|\Upsilon\tilde{\varpi}\|_\delta^{2^*} = \wedge\{\varphi > 0 : \rho_2(\varpi, \varphi) \leq 1 - \delta \, and \, \varsigma_2(\varpi, \varphi) \leq 1 - \delta\}$.

Theorem 10. *Consider the neutrosophic pseudo normed linear spaces* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$. *A linear mapping* $\Upsilon : \mathfrak{F} \to \mathfrak{G}$ *is strongly neutrosophic bounded if and only if it satisfies the condition of uniform neutrosophic boundedness relative to the corresponding* δ*-norms, where* δ *is any fixed real number in the interval* $(0, 1)$.

Proof. Suppose Υ is strongly neutrosophic bounded. Then for all $\varpi \in \mathfrak{F}$ and $\varphi \in \mathbb{R}^+$,

$$\Rightarrow \eta_2(\Upsilon(\varpi), \varphi) \geq \eta_1(\varpi, \varphi), \rho_2(\Upsilon(\varpi), \varphi) \leq \rho_1(\varpi, \varphi) \, \text{and} \, \varsigma_2(\Upsilon(\varpi), \varphi) \leq \varsigma_1(\varpi, \varphi). \quad (4.1)$$

$\|\varpi\|_\delta^1 = \wedge\{\mathfrak{o} > 0 : \eta_1(\varpi, \varphi) \geq \mathfrak{o}\}$.
Hene, there exist $\mathfrak{o}_0 > \varphi$ such that $\eta_1(\varpi, \mathfrak{o}_0) \geq \mathfrak{o}$.

There exist $\mathfrak{o}_0 > \varphi$ such that $\eta_2(\Upsilon(\varpi), \mathfrak{o}_0) \geq \mathfrak{o}$. (by Equation (4.1)
$\|\Upsilon(\varpi)\|_\delta^2 \leq \mathfrak{o}_0 < \varphi$. Thus, $\|\Upsilon(\varpi)\|_\delta^2 \leq \|\varpi\|_\delta^1$.
Also, let $\|\varpi\|_\delta^{1^*} > \varphi \Rightarrow \wedge\{\varphi > 0 : \rho_1(\varpi, \varphi) \leq \delta$ and $\varsigma_1(\varpi, \varphi) \leq \delta\} > \varphi$
Hence there exist $\mathfrak{o}_0 > \varphi$ such that $\rho_1(\varpi, \mathfrak{o}_0) \leq \mathfrak{o}$ and $\varsigma_1(\varpi, \mathfrak{o}_0) \leq \mathfrak{o}$
There exist $\mathfrak{o}_0 > \varphi$ such that $\rho_2(\Upsilon(\varpi), \mathfrak{o}_0) \leq \mathfrak{o}$ and $\varsigma_2(\Upsilon(\varpi), \mathfrak{o}_0) \leq \mathfrak{o}$ (by Equation (4.1)
$\|\Upsilon(\varpi)\|_\delta^2 \geq \mathfrak{o}_0 > \varphi$. Thus, $\|\Upsilon(\varpi)\|_\delta^{2^*} \geq \|\varpi\|_\delta^{1^*}$.
Hence Υ is uniformly neutrosophic bounded. Conversely, suppose Υ is uniformly neutrosophic bounded with respect to to corresponding δ-norms. Then $0 < \delta < 1$,

$$\|\Upsilon(\varpi)\|_\delta^2 \leq \|\varpi\|_\delta^1, \|\Upsilon(\varpi)\|_\delta^{2^*} \geq \|\varpi\|_\delta^{1^*} \tag{4.2}$$

Let $\eta_1(\varpi, \varphi) > \mathfrak{r} \Rightarrow \vee\left\{0 < \delta < 1 : \|\varpi\|_\delta^1 \leq \varphi\right\} > \mathfrak{r}$.
Hence there exists $0 < \delta_0 < 1$ such that $\delta_0 > \mathfrak{r}$ and $\|\varpi\|_{\delta_0}^1 \leq \varphi$
There exists $0 < \delta_0 < 1$ such that $\delta_0 > \mathfrak{r}$ and $\|\Upsilon\tilde{\varpi}\|_{\delta_0}^2 \leq \varphi$ (by equation (4.2)
$\eta_2(\Upsilon(\varpi), \varphi) \geq \delta_0 > \mathfrak{r}$. Therefore, $\eta_2(\Upsilon(\varpi), \varphi) \geq \eta_1(\varpi, \varphi)$.
Let $\rho_1(\varpi, \varphi) < \mathfrak{s} \Rightarrow \wedge\{0 < \delta < 1 : \|\varpi\|_\delta^{1^*} \geq \varphi\} < \mathfrak{s}$.
Hence there exists $0 < \delta_0 < 1$ such that $\delta_0 < \mathfrak{s}$ and $\|\varpi\|_{\delta_0}^{1^*} \leq \varphi$
Therefore there exists $0 < \delta_0 < 1$ such that $\delta_0 < \mathfrak{s}$ and $\|\Upsilon(\varpi)\|_\delta^{2^*} \leq \varphi$ (by equation (4.2)
$\rho_2(\Upsilon(\varpi), \varphi) \leq \delta_0 < \mathfrak{s}$. Therefore, $\rho_2(\Upsilon(\varpi), \varphi) \leq \rho_1(\varpi, \varphi)$.
Let $\varsigma_1(\varpi, \varphi) < \mathfrak{s} \Rightarrow \wedge\{0 < \delta < 1 : \|\varpi\|_\delta^{1^*} \geq \varphi\} < \mathfrak{s}$.
Therefore there exists $0 < \delta_0 < 1$ such that $\delta_0 < \mathfrak{s}$ and $\|\varpi\|_{\delta_0}^{1^*} \leq \varphi$
Hence there exists $0 < \delta_0 < 1$ such that $\delta_0 < \mathfrak{s}$ and $\|\Upsilon(\varpi)\|_\delta^{2^*} \leq \varphi$ (by equation (4.2))
$\varsigma_2(\Upsilon(\varpi), \varphi) \leq \delta_0 < \mathfrak{s}$. Therefore, $\varsigma_2(\Upsilon(\varpi), \epsilon) \leq \varsigma_1(\varpi, \varphi)$.
Hence Υ is strongly neutrosophic bounded.

Theorem 11. *If a linear operator* $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *is strongly neutrosophic bounded if and if it is strongly neutrosophic continuous, where* $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ *and* $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ *are NPNLS.*

Proof. Suppose Υ is strongly neutrosophic bounded then for all $\varpi \in \mathfrak{F}$ and $\epsilon \in \mathbb{R}^+$, we have
$\Rightarrow \eta_2(\Upsilon(\varpi), \epsilon) \geq \eta_1(\varpi, \epsilon)$, $\rho_2(\Upsilon(\varpi), \epsilon) \leq \rho_1(\varpi, \epsilon)$ and $\varsigma_2(\Upsilon(\varpi), \epsilon) \leq \varsigma_1(\varpi, \epsilon)$.
$\eta_2(\Upsilon(\varpi - \vartheta), \epsilon) \geq \eta_1(\varpi - \vartheta, \epsilon)$, $\rho_2(\Upsilon(\varpi - \vartheta), \epsilon) \leq \rho_1(\varpi - \vartheta, \epsilon)$ and
$\varsigma_2(\Upsilon(\varpi - \vartheta), \epsilon) \leq \varsigma_1(\varpi - \vartheta, \epsilon)$.
$\eta_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \epsilon) \geq \eta_1(\varpi - \vartheta, \gamma)$, $\rho_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \epsilon) \leq \rho_1(\varpi - \vartheta, \gamma)$ and
$\varsigma_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \epsilon) \leq \varsigma_1(\varpi - \vartheta, \gamma)$.
Therefore Υ is strongly neutrosophic continuous at ϑ and hence by Theorem (6) Υ is strongly neutrosophic continuous on $\mathfrak{F}$. Conversely, suppose Υ is strongly neutrosophic continuous on $\mathfrak{F}$. Then Υ is strongly neutrosophic continuous at any point of $\mathfrak{F}$, say ϑ, for all $\varpi \in \mathfrak{F}$ take $\epsilon = \varphi = \delta$, then
$\eta_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \varphi) \geq \eta_1(\varpi - \vartheta, \varphi)$, $\rho_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \varphi) \leq \rho_1(\varpi - \vartheta, \varphi)$ and
$\varsigma_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \varphi) \leq \varsigma_1(\varpi - \vartheta, \varphi)$.
Hence, $\eta_2(\Upsilon(\varpi), \varphi) \geq \eta_1(\varpi, \varphi)$, $\rho_2(\Upsilon(\varpi), \varphi) \leq \rho_1(\varpi, \varphi)$ and $\varsigma_2(\Upsilon(\varpi), \varphi) \leq \varsigma_1(\varpi, \varphi)$.

If $\varpi = \vartheta, \varphi > 0$ then
$\eta_2(\Upsilon(\vartheta), \varphi) = \eta_2(\vartheta, \varphi) = 1 = \eta_1(\vartheta, \varphi)$, $\rho_2(\Upsilon(\vartheta), \varphi) = \rho_2(\vartheta, \varphi) = 0 = \rho_1(\vartheta, \varphi)$ and
$\varsigma_2(\Upsilon(\vartheta), \varphi) = \varsigma_2(\vartheta, \varphi) = 0 = \varsigma_1(\vartheta, \varphi)$.
For any $\varpi, \varphi \leq 0, \Rightarrow \eta_2(\Upsilon(\varpi), \varphi) = 0 = \eta_1(\varpi, \varphi)$,
$\rho_2(\Upsilon(\varpi), \varphi) = 1 = \rho_1(\varpi, \varphi)$ and $\varsigma_2(\Upsilon(\varpi), \varphi) = 0 = \varsigma_1(\varpi, \varphi)$.
Hence for all $\varpi \in \mathfrak{F}$ and $\varphi \in \mathbb{R}$, Υ is strongly neutrosophic bounded.

Corollary 2. *If a linear operator $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ is strongly neutrosophic bounded then it is sequentially neutrosophic bounded, where $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ and $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ are NPNLS.*

Proof. The corollary arises as a consequence of Theorem (8) and Theorem (11).

Corollary 3. *A linear operator $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ is strongly neutrosophic bounded then it is neutrosophic continuous, where $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ and $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ are NPNLS.*

Theorem 12. *A linear operator $\Upsilon : (\mathfrak{F}, \eta_1, \rho_1, \varsigma_1) \to (\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ is weakly neutrosophic continuous if and if it is weakly neutrosophic bounded, where $(\mathfrak{F}, \eta_1, \rho_1, \varsigma_1)$ and $(\mathfrak{G}, \eta_2, \rho_2, \varsigma_2)$ are NPNLS.*

Proof. Suppose Υ is weakly neutrosophic bounded. Then for any $0 < \delta < 1, \varpi \in \mathfrak{F}$ and $\varphi \in \mathbb{R}^+$,
$\eta_1(\varpi, \varphi) \geq \delta \Rightarrow \eta_2(\Upsilon(\varpi), \varphi) \geq \delta$, $\rho_1(\varpi, \varphi) \leq \delta \Rightarrow \rho_2(\Upsilon(\varpi), \varphi) \leq \delta$ and
$\varsigma_1(\varpi, \varphi) \leq \delta \Rightarrow \varsigma_2(\Upsilon(\varpi), \varphi) \leq \delta$.
$\eta_1(\varpi - \vartheta, \varphi) \geq \delta \Rightarrow \eta_2(\Upsilon(\varpi - \vartheta), \varphi) \geq \delta$, $\rho_1(\varpi - \vartheta, \varphi) \leq \delta \Rightarrow \rho_2(\Upsilon(\varpi - \vartheta), \varphi) \leq \delta$ and
$\varsigma_1(\varpi - \vartheta, \epsilon) \leq \delta \Rightarrow \varsigma_2(\Upsilon(\varpi - \vartheta), \epsilon) \leq \delta$.
$\eta_1(\varpi - \vartheta, \varphi) \geq \delta \Rightarrow \eta_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \varphi) \geq \delta$, $\rho_1(\varpi - \vartheta, \varphi) \leq \delta \Rightarrow \rho_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \varphi) \leq \delta$ and
$\varsigma_1(\varpi - \vartheta, \epsilon) \leq \delta \Rightarrow \varsigma_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \epsilon) \leq \delta$,
where $\epsilon = \varphi = \delta$. Therefore, Υ is weakly neutrosophic continuous at ϑ and hence by Theorem (7), Υ is weakly neutrosophic continuous.
Conversely, suppose Υ is weakly neutrosophic continuous on $\mathfrak{F}$. Then Υ is weakly neutrosophic continuous at any point of $\mathfrak{F}$, say ϑ, for all $\varpi \in \mathfrak{F}$ take $\epsilon = \varphi = \delta$, then

$$\begin{aligned}
&\eta_1(\varpi - \vartheta, \varphi) \geq \delta \Rightarrow \eta_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \varphi) \geq \delta, \\
&\rho_1(\varpi - \vartheta, \varphi) \leq \delta \Rightarrow \rho_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \varphi) \leq \delta \\
&\varsigma_1(\varpi - \vartheta, \varphi) \leq \delta \Rightarrow \varsigma_2(\Upsilon(\varpi) - \Upsilon(\vartheta), \varphi) \leq \delta. \\
&\eta_1(\varpi, \varphi) \geq \delta \Rightarrow \eta_2(\Upsilon(\varpi), \varphi) \geq \delta, \\
&\rho_1(\varpi, \varphi) \leq \delta \Rightarrow \rho_2(\Upsilon(\varpi), \varphi) \leq \delta \text{ and} \\
&\varsigma_1(\varpi, \varphi) \leq \delta \Rightarrow \varsigma_2(\Upsilon(\varpi), \varphi) \leq \delta.
\end{aligned}$$

If $\varpi = \vartheta, \varphi > 0$ then $\Rightarrow \eta_2(\Upsilon(\vartheta), \varphi) = 1 = \eta_1(\vartheta, \varphi)$, $\rho_2(\Upsilon(\vartheta), \varphi) = 0 = \rho_1(\vartheta, \varphi)$ and $\varsigma_2(\Upsilon(\vartheta), \varphi) = 0 = \varsigma_1(\vartheta, \varphi)$.

For any $\varpi, \varphi \leq 0, \Rightarrow \eta_2(\Upsilon(\varpi), \varphi) = 0 = \eta_1(\varpi, \varphi)$, $\rho_2(\Upsilon(\varpi), \varphi) = 1 = \rho_1(\varpi, \varphi)$ and $\varsigma_2(\Upsilon(\varpi), \varphi) = 0 = \varsigma_1(\varpi, \varphi)$.
Hence for any $0 < \delta < 1$, for all $\varpi \in \mathfrak{F}$ and $\varphi \in \mathbb{R}$, Υ is weakly neutrosophic bounded.

5. Conclusions

In this paper, we explored various forms of neutrosophic continuity and boundedness in NPNLS, establishing significant relationships between these properties. We introduced new characterizations and provided illustrative examples to validate the theoretical developments. These contributions not only enhance the foundational understanding of operator behavior in NPNLS but also extend the existing framework of neutrosophic neutrosophic normed linear spaces. Future work may involve studying these notions in more generalized settings, such as neutrosophic b-normed or intuitionistic fuzzy normed spaces. Additionally, exploring applications of these results in fields like decision theory, control systems, or differential equations could provide valuable real-world insights.

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