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A New Trend of Fractional Inequalities for Differentiable Monotone Convexities Through Generalized Operators with Applications

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Abstract. This paper's main goal is to describe the new fractional operators for monotone differentiable function equipped with generalized Mittag-Leffler functions as its kernel, and develop the fractional inequalities for a new family of continuous differentiable convex functions by implementation of newly described fractional operators. Due to the generalized fractional operators to obtain the new version of the Hermite Hadamard type inequalities, and their refinements for continuous differentiable monotone convexities, all the results have a significant behavior in the field of analysis, and open new horizon for the modification of inequalities through a new class of convexities. We also spoke about a few unique situations involving the acquired outcome in the framework of the corollaries.

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1. Introduction

Fractional calculus (FC) is the generalization of classical calculus which has made a great contribution in many areas of mathematics, due to its numerous applications. Fractional operators (integral and differential of arbitrary order) play a vital role in the advancement of modern FC. In recent years, the multi-index special function has been used in extension of fractional operators by means of its kernel. Many researchers, including Riemann-Liouville (RL), Abel, Laurent, Hardy, and Littlewood [1–4], and others were interested in this field. The history of fractional calculus was covered in detail in [5, 6]; here, we wanted to concentrate on a few significant developments in the many fields of mathematics.

Although Leibniz's "paradoxes" were overcome by other writers, there are still some unanswered questions in the domain of fractional calculus. The availability of several contradictory definitions has been a persistent problem throughout the ages. One version was created by Liouville via differentiating of the exponential functions, and another one was introduced by Lacroix using the integral formula for inverse power functions. Some critics have concluded that one definition by Liouville and one by Lacroix is "correct" while the other is "wrong" since they cannot be used interchangeably. But according to the De Morgan's writings [7], *"Both these systems, then, may very possibly be parts of a more general systems."*

Similarly to Leibniz's views from hundreds of years prior, his statements were predictive. In actuality, the Riemann-Liouville formulation of fractional calculus was a particular case of both Liouville's formula and Lacroix's. For this, an arbitrary integration constant c was needed. Setting it to zero resulted in the Liouville's formula, whereas setting it to $-\infty$ resulted in the Lacroix's formula.

Through the use of complex analysis, this generic Riemann-Liouville definition for the fractional derivative and fractional integral of any function was developed in the late 1800s. The Riemann-Liouville formula is primarily utilized in real-analysis contexts; however, it was originally inspired by a generalization of the Cauchy integral formula for repeated derivatives of a complex analytic function. Currently, the most commonly used definition of fractional calculus is attributed to Riemann-Liouville's definition. The left and right side of ξ-RL fractional integrals of a function f with respect to the function $\xi(x)$ on $[\alpha, \rho]$ are respectively defined as:

$$I_{\alpha^+}^{\wp,\xi} f(x) = \frac{1}{\Gamma(\wp)} \int_{\alpha}^{x} \xi'(\beta)(\xi(x) - \xi(\beta))^{\wp-1} f(\beta) d\beta,$$

$$I_{\rho^-}^{\wp,\xi} f(x) = \frac{1}{\Gamma(\wp)} \int_{x}^{\rho} \xi'(\beta)(\xi(\beta) - \xi(x))^{\wp-1} f(\beta) d\beta, \quad \wp > 0.$$

Nevertheless, there are still other proposed definitions for fractional calculus. Several contradictory formulae are still in use today, which confuses many beginners in the field who assume that there is only one definition for fractional derivatives, just as there is only one definition for the first-order derivative. Although there are other methods to extend meaning, fractional calculus is sometimes referred to as a "extension of meaning". Because of the power-function kernel in the integral transform description, the Riemann-Liouville model can be used to explain processes with power-law behavior. However, there are many other kinds of behavior found in nature that are not amenable to simple power functions.

The Hermite Hadamard (H-H) dual inequality is the foundational discovery for convex functions on a real-valued interval, with clear mathematical significance and limited potential for specific inequalities. This research delves into the basics of the Hermite Hadamard inequality for convex functions and presents specific results for particular means. It also introduces the Hermite Hadamard type inequalities for various forms of convexity and emphasizes the characteristics of functions, functionals, and sequences that can be used to modify the Hermite Hadamard result. Many researchers have worked to modify the Hermite-Hadamard fractional inequalities and their refinements, which have made a contribution in the literature [8–10].

Fractional calculus saw a substantial rise in research output and popularity in the late twentieth century. Since then, there have been several specialized journals dedicated to fractional calculus, making it a very active area of study. Numerous scientific domains have found applications, as enumerated in [11–13] and the associated references. Specifically, the modeling of some intermediate physical processes, such as viscoelasticity, depends on the intermediate feature of fractional-calculus operators [14]. In several universities, fractional calculus is now taught as a required subject in the graduate mathematics program. Several textbooks [15–17] serve as introductory resources for students and aspiring researchers in the topic.

The study underscored the characteristics of several functions and sequences that can be utilized to adapt the H-H theorem. Research-wise, there are currently a number of alternative viewpoints and lines of inquiry that may be at odds with one another in some situations.

The gradual development of new technologies has increased the demand for fractional operators and special functions. In the last few decades, many researchers have worked to develop fractional operators having generalized special functions as its kernel, which have many applications in the field of operator theory, fractional inequalities. Such type of fractional operators have resolved many issues which are facing the research communities. The formation of fractional and differential operators can be possible by series functions in Riemann-Liouville system.

Convexity has greatly benefited mathematics ever since Jensen's first convex inequality was introduced. Convexity was used to derive many inequalities; see books [18, 19]. Applications of inequality include probability theory, optimization, and analysis difficulties. We direct readers to the papers [20–26] for applications. The Hermite Hadamard inequality is among the highly elegant conclusions in the study of convex inequalities. Many mathematicians have been interested in the well-known Hermite Hadamard inequality, which was independently established by Jacques Hadamard and Charles Hermite. They have employed different kinds of convex functions to produce numerous generalizations of this inequality in the literature.

The extensive range of applications of convexity has captured the interest of many researchers, leading to the development of several new interpretations of traditional convexity in various studies.Convexity has piqued the interest of many researchers due to its wide range of applications. As a result, the literature has seen the emergence of several new adaptations of classical convexity.

Numerous prominent integral inequality for the convex functions exist in the literature; these include the following: Ostrowski integral inequality [27], Simpson's integral inequality [28], Hardy integral inequality [29], Olsen integral inequality [30], Gagliardo Nirenberg integral inequality [31], Fejr-Hermite Hadamard inequality [32], and q-Hermite Hadamard integral inequality [33]. Researchers have been described many classical and fractional integral inequalities after introducing the Hermite Hadamard type inequalities. Wu et al., recently presented a new family of convex sets and convex functions called $\check{\Upsilon}$-convex sets and $\check{\Upsilon}$-convex functions in [34]. We have many more fractional inequalities as well, but Hermite Hadamard type inequality is the most renowned. Hermite Hadamard inequalities of many kinds have recently been investigated and generalized for numerous kinds of convex functions under various circumstances and parameters. The classical and fractional inequalities [35, 36] are respectively defined as follows

$$f\left(\frac{\alpha+\rho}{2}\right) \leq \frac{1}{\rho-\alpha}\int_{\alpha}^{\rho} f(x)dx \leq \frac{f(\alpha)+f(\rho)}{2}, \tag{1}$$

and

$$f\left(\frac{\alpha+\rho}{2}\right) \leq \frac{\Gamma(\tau+1)}{2(\rho-\alpha)^{\tau}}\left[\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\alpha^{+}} f(\alpha) + \Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\rho^{-}} f(\rho)\right] \leq \frac{f(\alpha)+f(\rho)}{2}, \tag{2}$$

where $f :\subseteq \mathbb{R}$ is considered to be a convex function on , $f \in L^1(\alpha,\rho)$ with $\alpha < \rho$.

The Mittag-Leffler function is near to the exponential function in solving fractional integro-differential equations of arbitrary order. These functions need more recognition because of their extensive applications across various fields. They are instrumental in defining new fractional integral operators, which in turn are used to extend mathematical inequalities. In this paper, we will explore and examine an integral operator with a kernel that is a generalized Mittag-Leffler function, and we will also identify its familiar special cases.

2. Preliminaries

In this section, we discuss some definitions that help us to understand our main results. Throughout this section, Q denotes a subset of $\mathbb{R}$.

Definition 1. *([34]) Let $\check{\Upsilon}$ be continuous, differentiable and strictly monotone function; then the $\check{\Upsilon}$-convex set is denoted as $\mathcal{N}[\check{\Upsilon},\eta](\alpha,\rho) := \check{\Upsilon}^{-1}(\eta\check{\Upsilon}(\alpha)+(1-\eta)\check{\Upsilon}(\rho))$, and is defined, for each $\alpha,\rho \in Q, \eta \in [0,1]$, as follows*

$$\mathcal{N}[\check{\Upsilon},\eta](\alpha,\rho) \in Q. \tag{3}$$

Definition 2. *([34]) $f : Q \to \mathbb{R}$ is $\check{\Upsilon}$-convex function w.r.t. $\check{\Upsilon}$ if*

$$f(\mathcal{N}[\check{\Upsilon},\eta](\alpha,\rho)) \leq \eta f(\alpha) + (1-\eta)f(\rho), \tag{4}$$

for each $\alpha, \rho \in Q, \eta \in [0,1]$.

Remark 1.

- *If the inequality(4) is to be held as a strict inequality for all $\eta \in (0,1)$ and $\alpha, \rho \in Q$, $\alpha \neq \rho$, then f is a strictly monotone $\check{\Upsilon}$-convex function on Q.*
- *If $-f$ is $\check{\Upsilon}$-convex on Q in (4), then f is $\check{\Upsilon}$-concave function on Q.*
- *If $-f$ is strictly monotone $\check{\Upsilon}$-convex on Q, then f is a strictly monotone $\check{\Upsilon}$-concave function on Q.*

Definition 3. *([37]) Let $\mu \in \mathbb{R}$, then for positive real numbers $\nu, \tau, j, \vartheta, z, \kappa$, the generalized Mittag-Leffler function is defined as follows*

$$\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}\big(\xi(\mu)^v\big) = \sum_{n=0}^{\infty} \frac{(\vartheta)_{\kappa n}\big(\xi(\mu)^v\big)}{\Gamma(vn+\tau)(z)_{\delta n}}. \tag{5}$$

3. Modification of Hermite Hadamard type inequalities for $\check{\Upsilon}$-convex function

In this section, we modify the Hermite Hadamard type inequalities and the related refinements for $\check{\Upsilon}$-convex function by implementation of generalized fractional operators for monotone differentiable function having extended Mittag-Leffler function as a kernel.

Definition 4. *Let $(\alpha, \rho) \subseteq \mathbb{R}, \Phi(x)$ be differentiable monotone-positive function on $(\alpha, \rho]$, and $\Phi'(x)$ be continuous on (α, ρ). Then, the left and right-side of generalized fractional integral operators of a function f with respect to a function $\Phi(x)$ on $[\alpha, \rho]$, for positive real numbers $\nu, \tau, j, \vartheta, z, \kappa$, and $\omega \in \mathbb{R}$, are respectively defined as follows*

$$\begin{aligned}\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\alpha^+} f(x) &= \int_{\alpha}^{x} \big(\Phi(x)-\Phi(\mu)\big)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}\big(\xi(\mu)^n\big)\Phi'(\mu)f(\mu)d\mu,\\ \Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\rho^-} f(x) &= \int_{x}^{\rho} \big(\Phi(\mu)-\Phi(x)\big)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}\big(\xi(\mu)^n\big)\Phi'(\mu)f(\mu)d\mu, \tau > 0.\end{aligned} \tag{6}$$

Theorem 1. *Let $f : [\alpha, \rho] \subseteq \mathbb{R} \to \mathbb{R}$ be integrable $\check{\Upsilon}$-convex and $f \in L^1(\alpha, \rho)$ with $0 \leq \alpha < \rho$, and the function $\check{\Upsilon}$ be positive and monotonically increasing on $(\alpha, \rho]$ and $\check{\Upsilon}'(x)$ is continuous on (α, ρ). Then, we have, for $\tau > 0$ and $\beta \in \mathbb{R}$,*

$$\begin{aligned}&f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^v) \leq \frac{1}{2\big(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)\big)^{\tau}}\left[\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\alpha^+} f(\alpha) + \Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\rho^-} f(\rho)\right]\\ &\leq \frac{f(\alpha)+f(\rho)}{2}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^v),\end{aligned} \tag{7}$$

where $(\xi(\mu)^v)^n = \left(\left(\frac{\check{\Upsilon}(\mu)-\check{\Upsilon}(\alpha)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^v\right)^n$ and $(\xi(\omega)^v)^n = \left(\left(\frac{\check{\Upsilon}(\rho)-\check{\Upsilon}(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^v\right)^n$.

Proof. Consider the f as a $\check{\Upsilon}$-convex function, i.e.,

$$f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(x)+\check{\Upsilon}(y)}{2}\right)\right) \leq \frac{f(x)+f(y)}{2}. \tag{8}$$

Putting the values $x = \check{\Upsilon}^{-1}\big(\beta\check{\Upsilon}(\alpha)+(1-\beta)\check{\Upsilon}(\rho)\big)$ and $y = \check{\Upsilon}^{-1}\big((1-\beta)\check{\Upsilon}(\alpha)+\beta\check{\Upsilon}(\rho)\big)$, in equation (8), we get

$$\begin{aligned} 2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right) &\leq f\big(\check{\Upsilon}^{-1}\big(\beta\check{\Upsilon}(\alpha)+(1-\beta)\check{\Upsilon}(\rho)\big)\big) \\ &+ f\big(\check{\Upsilon}^{-1}\big((1-\beta)\check{\Upsilon}(\alpha)+\beta\check{\Upsilon}(\rho)\big)\big). \end{aligned} \tag{9}$$

Multiplying both sides by $(1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^{v})$ of equation (9) and then integrating the resulting inequality with respect to β over $[0,1]$, we get

$$\begin{aligned} &\int_0^1 (1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^{v})\ f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)d\beta \\ &\leq \int_0^1 (1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^{v})\ f\big(\check{\Upsilon}^{-1}\big(\beta\check{\Upsilon}(\alpha)+(1-\beta)\check{\Upsilon}(\rho)\big)\big)d\beta \\ &+ \int_0^1 (1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^{v})\ f\big(\check{\Upsilon}^{-1}\big((1-\beta)\check{\Upsilon}(\alpha)+\beta\check{\Upsilon}(\rho)\big)\big)d\beta. \end{aligned}$$

Therefore,

$$\begin{aligned} &2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\int_0^1 (1-\beta)^{\tau-1}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}((1-\beta)^{v})^{n}}{\Gamma(vn+\tau)(z)_{\delta n}}d\beta \\ &\leq \int_0^1 (1-\beta)^{\tau-1}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}((1-\beta)^{v})^{n}}{\Gamma(vn+\tau)(z)_{\delta n}}f\big(\check{\Upsilon}^{-1}\big(\beta\check{\Upsilon}(\alpha)+(1-\beta)\check{\Upsilon}(\rho)\big)\big)d\beta \\ &\leq \int_0^1 (1-\beta)^{\tau-1}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}((1-\beta)^{v})^{n}}{\Gamma(vn+\tau)(z)_{\delta n}}f\big(\check{\Upsilon}^{-1}\big((1-\beta)\check{\Upsilon}(\alpha)+\beta\check{\Upsilon}(\rho)\big)\big)d\beta. \end{aligned} \tag{10}$$

Putting the values $\mu = \check{\Upsilon}^{-1}\big(\beta\check{\Upsilon}(\alpha)+(1-\beta)\check{\Upsilon}(\rho)\big)$ and $\omega = \check{\Upsilon}^{-1}\big((1-\beta)\check{\Upsilon}(\alpha)+\beta\check{\Upsilon}(\rho)\big)$ in equation (10), then we obtain

$$\begin{aligned} &2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}}{\Gamma(vn+\tau)(z)_{\delta n}}\int_0^1 (1-\beta)^{\tau-1}((1-\beta)^{v})^{n}d\beta \\ &\leq \sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}}{\Gamma(vn+\tau)(z)_{\delta n}}\int_\rho^\alpha \left(1-\big(\frac{\check{\Upsilon}(\mu)-\check{\Upsilon}(\rho)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\big)\right)^{\tau-1}\left(\left(1-\big(\frac{\check{\Upsilon}(\mu)-\check{\Upsilon}(\rho)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\big)\right)^{v}\right)^{n} f(\mu).\frac{\check{\Upsilon}'(\mu)d\mu}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)} \end{aligned}$$

$$+\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}}{\Gamma(vn+\tau)(z)_{\delta n}}\int_{\alpha}^{\rho}\left(1-(\frac{\check{\Upsilon}(\omega)-\check{\Upsilon}(\alpha)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)})\right)^{\tau-1}\left(\left(1-(\frac{\check{\Upsilon}(\omega)-\check{\Upsilon}(\alpha)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)})\right)^{v}\right)^{n}f(\omega).\frac{\check{\Upsilon}'(\omega)d\omega}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}.$$

Hence,

$$2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}}{\Gamma(vn+\tau)(z)_{\delta n}}\left(\frac{1}{\tau+vn}\right)$$

$$\le\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}}{\Gamma(vn+\tau)(z)_{\delta n}}\int_{\rho}^{\alpha}\left(\frac{\check{\Upsilon}(\mu)-\check{\Upsilon}(\alpha)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{\tau-1}\left(\left(\frac{\check{\Upsilon}(\mu)-\check{\Upsilon}(\alpha)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{v}\right)^{n}\frac{\check{\Upsilon}'(\mu)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}f(\mu)d\mu$$

$$+\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}}{\Gamma(vn+\tau)(z)_{\delta n}}\int_{\alpha}^{\rho}\left(\frac{\check{\Upsilon}(\rho)-\check{\Upsilon}(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{\tau-1}\left(\left(\frac{\check{\Upsilon}(\rho)-\check{\Upsilon}(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{v}\right)^{n}\frac{\check{\Upsilon}'(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}f(\omega)d\omega. \tag{11}$$

So,

$$2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}((1-\beta)^{v})^{n}}{\Gamma(vn+\tau+1)(z)_{\delta n}}$$

$$\le\frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}}{\Gamma(vn+\tau)(z)_{\delta n}}\left(\left(\frac{\check{\Upsilon}(\mu)-\check{\Upsilon}(\alpha)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{v}\right)^{n}\int_{\rho}^{\alpha}(\check{\Upsilon}(\mu)-\check{\Upsilon}(\alpha))^{\tau-1}\check{\Upsilon}'(\mu)f(\mu)d\mu$$

$$+\frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}}{\Gamma(vn+\tau)(z)_{\delta n}}\left(\left(\frac{\check{\Upsilon}(\rho)-\check{\Upsilon}(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{v}\right)^{n}\int_{\alpha}^{\rho}(\check{\Upsilon}(\rho)-\check{\Upsilon}(\omega))^{\tau-1}\check{\Upsilon}'(\omega)f(\omega)d\omega.$$

It becomes

$$2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}((1-\beta)^{v})$$

$$\le\frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n})(\xi(\mu)^{v})}{\Gamma(vn+\tau)(z)_{\delta n}}\int_{\rho}^{\alpha}(\check{\Upsilon}(\mu)-\check{\Upsilon}(\alpha))^{\tau-1}\check{\Upsilon}'(\mu)f(\mu)d\mu$$

$$+\frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n})(\xi(\omega)^{v})}{\Gamma(vn+\tau)(z)_{\delta n}}\int_{\alpha}^{\rho}(\check{\Upsilon}(\rho)-\check{\Upsilon}(\omega))^{\tau-1}\check{\Upsilon}'(\omega)f(\omega)d\omega,$$

and finally,

$$2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}((1-\beta)^{v})$$

$$\le\frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\Bigg[\mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}(\xi(\mu)^{v})\int_{\rho}^{\alpha}(\check{\Upsilon}(\mu)-\check{\Upsilon}(\alpha))^{\tau-1}\check{\Upsilon}'(\mu)f(\mu)d\mu$$

$$+ \mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}(\xi(\omega)^{v}) \int_{\alpha}^{\rho} (\check{\Upsilon}(\rho) - \check{\Upsilon}(\omega))^{\tau-1}\check{\Upsilon}'(\omega)f(\omega)d\omega\Bigg],$$

and thus,

$$f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}((1-\beta)^{v})$$

$$\leq \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\left[\Im_{\nu,\tau,j,\omega,\alpha^{+}}^{\vartheta,z,\kappa} f(\alpha) + \Im_{\nu,\tau,j,\omega,\rho^{-}}^{\vartheta,z,\kappa} f(\rho)\right].$$

Now, for the second inequality, we consider the $\check{\Upsilon}$-convexity

$$f(\check{\Upsilon}^{-1}(\beta\check{\Upsilon}(\alpha) + (1-\beta)\check{\Upsilon}(\rho))) \leq \beta f(\alpha) + (1-\beta)f(\rho), \tag{12}$$

and

$$f(\check{\Upsilon}^{-1}((1-\beta)\check{\Upsilon}(\alpha) + \beta\check{\Upsilon}(\rho))) \leq (1-\beta)f(\alpha) + \beta f(\rho). \tag{13}$$

By adding the two inequalities given in equations (12) and (13), we get

$$f(\check{\Upsilon}^{-1}(\beta\check{\Upsilon}(\alpha) + (1-\beta)\check{\Upsilon}(\rho))) + f(\check{\Upsilon}^{-1}((1-\beta)\check{\Upsilon}(\alpha) + \beta\check{\Upsilon}(\rho))) \leq f(\alpha) + f(\rho). \tag{14}$$

Multiplying both sides by $(1-\beta)^{\tau-1}\mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}(1-\beta)^{v}$ of equation (14) and then integrating with respect to β over $[0,1]$, we can obtain

$$\int_{0}^{1}(1-\beta)^{\tau-1}\mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}(1-\beta)^{v} f(\check{\Upsilon}^{-1}(\beta\check{\Upsilon}(\alpha) + (1-\beta)\check{\Upsilon}(\rho)))d\beta$$

$$+ \int_{0}^{1}(1-\beta)^{\tau-1}\mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}(1-\beta)^{v} f(\check{\Upsilon}^{-1}((1-\beta)\check{\Upsilon}(\alpha) + \beta\check{\Upsilon}(\rho)))d\beta$$

$$\leq (f(\alpha)+f(\rho))\int_{0}^{1}(1-\beta)^{\tau-1}\mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}(1-\beta)^{v}d\beta.$$

Hence, the required result is

$$\frac{1}{2(\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho))^{\tau}}\left[\Im_{\nu,\tau,j,\omega,\alpha^{+}}^{\vartheta,z,\kappa} f(\alpha) + \Im_{\nu,\tau,j,\omega,\rho^{-}}^{\vartheta,z,\kappa} f(\rho)\right] \leq \frac{f(\alpha)+f(\rho)}{2}\mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}((1-\beta)^{v}),$$

and this completes the proof.

Corollary 1. *By the assumption of Theorem 1, replace* $\check{\Upsilon}(x) = x$*; then the inequality* (7) *reduces to inequality* (2)*.*

Corollary 2. *By the assumption of Theorem 1, replace* $\tau = 1$*; then we have*

$$f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right) \leq \frac{1}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\int_{\alpha}^{\rho} f(x)\check{\Upsilon}'(x)dx \leq \frac{f(\alpha)+f(\rho)}{2}, \tag{15}$$

which was already established in [25].

Corollary 3. *By the assumption of Theorem 1, replace* $\check{\Upsilon}(x) = x,\ \tau = 1$*; then we have the inequality* (1).

Theorem 2. *Let* $f : [\alpha, \rho] \subseteq \mathbb{R} \to \mathbb{R}$ *be integrable* $\check{\Upsilon}$*-convex, and* $f \in L^1(\alpha, \rho)$ *with* $0 \le \alpha < \rho$*. Moreover, the function* $\check{\Upsilon}$ *is also monotone and positive on* $(\alpha, \rho]$*, and* $\check{\Upsilon}'(x)$ *be continuous on* (α, ρ)*. Then for* $\tau > 0$*, we have the following inequality*

$$\begin{aligned} & f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^v) \\ & \le \frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^\tau}\left[\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^+} f(\alpha) + \Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^-} f(\rho)\right] \\ & \le \frac{f(\alpha)+f(\rho)}{2}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^v), \end{aligned} \tag{16}$$

where $(\xi(\mu)^v)^n = \left(\left(\frac{\check{\Upsilon}(\mu)-\check{\Upsilon}(\alpha)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^v\right)^n$ *and* $(\xi(\omega)^v)^n = \left(\left(\frac{\check{\Upsilon}(\rho)-\check{\Upsilon}(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^v\right)^n$.

Proof. Consider the $\check{\Upsilon}$-convex function f. We have

$$f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(x)+\check{\Upsilon}(y)}{2}\right)\right) \le \frac{\check{\Upsilon}(x)+\check{\Upsilon}(y)}{2}. \tag{17}$$

Put the values $x = \check{\Upsilon}^{-1}(\frac{\beta}{2}\check{\Upsilon}(\alpha) + \frac{2-\beta}{2}\check{\Upsilon}(\rho))$ and $y = \check{\Upsilon}^{-1}(\frac{2-\beta}{2}\check{\Upsilon}(\alpha) + \frac{\beta}{2}\check{\Upsilon}(\rho))$ in equation (17) to get

$$\begin{aligned} 2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right) & \le f\big(\check{\Upsilon}^{-1}(\frac{\beta}{2}\check{\Upsilon}(\alpha) + (\frac{2-\beta}{2})\check{\Upsilon}(\rho))\big) \\ & \quad + f\big(\check{\Upsilon}^{-1}(\frac{2-\beta}{2})\check{\Upsilon}(\alpha) + \frac{\beta}{2}\check{\Upsilon}(\rho))\big). \end{aligned} \tag{18}$$

Multiplying both sides by $(1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(1-\beta)^v$ of equation (18), and then integrating the resulting inequality with respect to β over $[0, 1]$, give

$$\begin{aligned} & 2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\int_0^1 (1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^v)d\beta \\ & \le \int_0^1 (1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^v) f(\check{\Upsilon}^{-1}(\frac{\beta}{2}\check{\Upsilon}(\alpha) + (\frac{2-\beta}{2})\check{\Upsilon}(\rho)))d\beta \\ & + \int_0^1 (1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^v) f(\check{\Upsilon}^{-1}(\frac{2-\beta}{2})\check{\Upsilon}(\alpha) + \frac{\beta}{2}\check{\Upsilon}(\rho))d\beta. \end{aligned} \tag{19}$$

By changing variables $\mu = \check{\Upsilon}^{-1}(\frac{\beta}{2}\check{\Upsilon}(\alpha) + (\frac{2-\beta}{2})\check{\Upsilon}(\rho))$ and $\omega = \check{\Upsilon}^{-1}(\frac{2-\beta}{2})\check{\Upsilon}(\alpha) + \frac{\beta}{2}\check{\Upsilon}(\rho))$ in (19), we obtain

$$\begin{aligned}
&2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\int_0^1 (1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^v)d\beta \\
&\leq \sum_{n=0}^{\infty}\frac{\vartheta_{\kappa n}}{\Gamma(\tau+vn)(\mu)_{\delta n}}\int_{\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2})}^{\rho}\left(1-2\left(\frac{\check{\Upsilon}(\mu)-\check{\Upsilon}(\rho)}{\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho)}\right)\right)^{\tau-1} \\
&\times\left(\left(1-2\left(\frac{\check{\Upsilon}(\mu)-\check{\Upsilon}(\rho)}{\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho)}\right)\right)^{v}\right)^{n} f(\mu)\frac{2\check{\Upsilon}'(\mu)}{\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho)} \\
&+\sum_{n=0}^{\infty}\frac{\vartheta_{\kappa n}}{\Gamma(\tau+vn)(z)_{\delta n}}\int_{\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2})}^{\alpha}\left(1-2\left(\frac{\check{\Upsilon}(\omega)-\check{\Upsilon}(\alpha)}{\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho)}\right)\right)^{\tau-1} \\
&\times\left(\left(1-2\left(\frac{\check{\Upsilon}(\omega)-\check{\Upsilon}(\alpha)}{\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho)}\right)\right)^{v}\right)^{n} f(\omega)\frac{2\check{\Upsilon}'(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}d\omega.
\end{aligned}$$

It gives that

$$\begin{aligned}
&2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\sum_{n=0}^{\infty}\frac{\vartheta_{\kappa n}}{\Gamma(\tau+vn)(z)_{\delta n}}\int_0^1 (1-\beta)^{\tau+vn-1}d\beta \\
&\leq 2\sum_{n=0}^{\infty}\frac{\vartheta_{\kappa n}}{\Gamma(\tau+vn)(z)_{\delta n}}\int_{\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2})}^{\rho}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{\tau-1} \\
&\times\left(\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{v}\right)^{n}\frac{\check{\Upsilon}'(\mu)}{(\check{\Upsilon}(v)-\check{\Upsilon}(u))}f(\mu)d\mu \\
&+\int_{\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2})}^{\alpha}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{\tau-1} \\
&\times\left(\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\right)^{v}\right)^{n}\frac{\check{\Upsilon}'(\omega)}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}f(\omega)d\omega.
\end{aligned}$$

Hence,

$$\begin{aligned}
&2f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^v) \leq \frac{2}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\Bigg[\sum_{n=0}^{\infty}\frac{\vartheta_{\kappa n}(\xi(\mu)^v)^n}{\Gamma(\tau+vn)(z)_{\delta n}} \\
&\times\int_{\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2})}^{\rho}\left(\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu)\right)^{\tau-1}\check{\Upsilon}'(\mu)f(\mu)d\mu \\
&+\sum_{n=0}^{\infty}\frac{\vartheta_{\kappa n}(\xi(\omega)^v)^n}{\Gamma(\tau+vn)(z)_{\delta n}}\int_{\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2})}^{\alpha}\left(\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\omega)\right)^{\tau-1}\check{\Upsilon}'(\omega)f(\omega)d\omega\Bigg],
\end{aligned}$$

and finally,

$$f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^{v}) \tag{20}$$

$$\leq \frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\left[\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^{+}}f(\alpha)+\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^{-}}f(\rho)\right].$$

Now, for the second inequality, we consider the $\check{\Upsilon}$-convexity

$$f\left(\check{\Upsilon}^{-1}\left(\frac{\beta}{2}\check{\Upsilon}(\alpha)+(\frac{2-\beta}{2})\check{\Upsilon}(\rho)\right)\right)\leq \frac{\beta}{2}f(\alpha)+(\frac{2-\beta}{2})f(\rho), \tag{21}$$

and

$$f\left(\check{\Upsilon}^{-1}\left((\frac{2-\beta}{2})\check{\Upsilon}(\alpha)+\frac{\beta}{2}\check{\Upsilon}(\rho)\right)\right)\leq (\frac{2-\beta}{2})f(\alpha)+\frac{\beta}{2}f(\rho). \tag{22}$$

By adding the two inequalities given in (21) and (22), we get

$$f\left(\check{\Upsilon}^{-1}\left(\frac{\beta}{2}\check{\Upsilon}(\alpha)+(\frac{2-\beta}{2})\check{\Upsilon}(\rho)\right)\right)+f\left(\check{\Upsilon}^{-1}\left((\frac{2-\beta}{2})\check{\Upsilon}(\alpha)+\frac{\beta}{2}\check{\Upsilon}(\rho)\right)\right)\leq f(\alpha)+f(\rho). \tag{23}$$

Multiplying both sides by $(1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(1-\beta)^{v}$ of equation (23), then integrating the resulting inequality with respect to β over $[0,1]$, we get

$$\int_{1}^{0}(1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(1-\beta)^{v}f\left(\check{\Upsilon}^{-1}\left(\frac{\beta}{2}\check{\Upsilon}(\alpha)+(\frac{2-\beta}{2})\check{\Upsilon}(\rho)\right)\right)d\beta$$

$$+\int_{1}^{0}(1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(1-\beta)^{v}f\left(\check{\Upsilon}^{-1}\left((\frac{2-\beta}{2})\check{\Upsilon}(\alpha)+\frac{\beta}{2}\check{\Upsilon}(\rho)\right)\right)d\beta$$

$$\leq f(\alpha)+f(\rho)(1-\beta)^{\tau-1}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(1-\beta)^{v}d\beta\frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}$$

$$\times\left[\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^{+}}f(\alpha)+\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^{-}}f(\rho)\right]$$

$$\leq \frac{f(\alpha)+f(\rho)}{2}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}((1-\beta)^{v}). \tag{24}$$

From the inequalities (20) and (24), we have the required result.

Corollary 4. *Specifically, in Theorem 2, if we take $\check{\Upsilon}(x)=x$, then the inequality* (16) *is simplified to the following inequality:*

$$f(\frac{\alpha+\rho}{2})\leq\frac{1}{(\rho-\alpha)^{\tau}}\left[\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\frac{\alpha+\rho}{2})^{+}}f(\alpha)+\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\frac{\alpha+\rho}{2})^{-}}f(\rho)\right]\leq\frac{f(\alpha)+f(\rho)}{2}, \tag{25}$$

which was already established in [29].

Corollary 5. *If we take $\tau = 1$ in Theorem 2, then*

$$f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right) \leq \frac{1}{\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)}\int_{\alpha}^{\rho} f(x)\check{\Upsilon}'(x)dx \leq \frac{f(\alpha)+f(\rho)}{2}, \tag{26}$$

holds which was already established in [25].

Corollary 6. *If we take $\check{\Upsilon}(x) = x$ and $\tau = 1$ in Theorem 2, then inequality* (16) *reduces to the inequality* (1).

4. Further Consequences

In this section, we are going to discuss some properties of Hermite Hadamard's type inequalities via different convexities and check their behavior for generalized Mittag-Leffler function as a kernel. As the consequences for the theorem 1 and Theorem 2, we get the following results.

Theorem 3. *Let $f : [\alpha, \rho] \subseteq \mathbb{R} \to \mathbb{R}$ be an L^1 integrable $\check{\Upsilon}$-convex function and $f' \in L^1(\alpha, \rho)$ for $0 \leq \alpha < \rho$. Moreover, the function $\check{\Upsilon}$ is also monotone and positive on $(\alpha, \rho]$ and $\check{\Upsilon}'(x)$ be continuous on (α, ρ). Then, for $\tau > 0$,*

$$\frac{f(\alpha)+f(\rho)}{2} - \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\left[\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\alpha^+} f(\alpha) + \Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\rho^-} f(\rho)\right]$$

$$= \frac{1}{2[\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha)]^{\tau}}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(\xi(\mu)^{v})\int_{\alpha}^{\rho}((\check{\Upsilon}(\alpha)-\check{\Upsilon}(\mu))^{\tau} - (\check{\Upsilon}(\rho)-\check{\Upsilon}(\mu))^{\tau})f'(\mu)d\mu.$$

Proof. Consider the integrals

$$I_1 := \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\alpha^+} f(\alpha),$$

and

$$I_2 := \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,v^-} f(\rho).$$

In this case, by Definition 4, and integrating by parts, we may write

$$\begin{aligned} I_1 :&= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\alpha^+} f(\alpha) \\ &= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\left[\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(\tau+vn)(z)_{\delta n}}\int_{\alpha}^{\rho}(\check{\Upsilon}(\mu)-\check{\Upsilon}(\rho))^{\tau-1}\check{\Upsilon}'(\mu)f(\mu)d\mu\right] \\ &= \frac{-1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(\tau+vn)(z)_{\delta n}}\int_{\alpha}^{\rho} f(\mu)d(\check{\Upsilon}(\mu)-\check{\Upsilon}(\rho))^{\tau} \end{aligned}$$

$$= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(\xi(\mu)^{v}) \left[(\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho))^{\tau} f(\alpha) + \int_{\alpha}^{\rho} (\check{\Upsilon}(\alpha)-\check{\Upsilon}(\mu))^{\tau} f'(\mu) d\mu \right]. \tag{27}$$

Similarly, we have

$$\begin{aligned} I_2 :&= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\rho^-} f(\rho) \\ &= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \left[\sum_{n=0}^{\infty} \frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(vn+\tau)(z)_{\delta n}} \int_{\rho}^{\alpha} (\check{\Upsilon}(\rho)-\check{\Upsilon}(\mu))^{\tau-1} \check{\Upsilon}'(\mu) f(\mu) d\mu \right] \\ &= \frac{-1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \sum_{n=0}^{\infty} \frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(vn+\tau)(z)_{\delta n}} \int_{\rho}^{\alpha} f(\mu) d(\check{\Upsilon}(\rho)-\check{\Upsilon}(\mu))^{\tau} \\ &= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(\xi(\mu)^{v}) \left[(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau} f(\rho) + \int_{\rho}^{\alpha} (\check{\Upsilon}(\rho)-\check{\Upsilon}(\mu))^{\tau} f'(\mu) d\mu \right]. \end{aligned} \tag{28}$$

From the identities in equations (27) and (28), we have

$$\begin{aligned} \frac{f(\alpha)+f(\rho)}{2} - (I_1+I_2) &= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(\xi(\mu)^{v}) \\ &\quad \times \int_{\alpha}^{\rho} [(\check{\Upsilon}(\alpha)-\check{\Upsilon}(\mu))^{\tau} - (\check{\Upsilon}(\rho)-\check{\Upsilon}(\mu))^{\tau}] f'(\mu) d\mu, \end{aligned}$$

which is the required result.

Corollary 7. *If we take $\check{\Upsilon}(x) = x$ and $\tau = 1$ in Theorem 3, then we have the following identity:*

$$\begin{aligned} &\frac{f(\alpha)+f(\rho)}{2} - \frac{1}{2(\rho-\alpha)} \left[\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\alpha^+} f(\alpha) + \Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,\rho^-} f(\rho) \right] \\ &= \frac{1}{2(\rho-\alpha)} \mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(\xi(\mu)^{v}) \int_{\alpha}^{\rho} (\alpha-\rho) f'(\mu) d\mu. \end{aligned}$$

Theorem 4. *Let $f : [\alpha, \rho] \subseteq \mathbb{R} \to \mathbb{R}$ be an L^1 integrable $\check{\Upsilon}$-convex function and $f' \in L^1(\alpha, \rho)$ for $0 \le \alpha < \rho$. Moreover, the function $\check{\Upsilon}$ be monotone and positive on $(\alpha, \rho]$ and $\check{\Upsilon}'(x)$ is continuous on (α, ρ). Then, for $\tau > 0$,*

$$\begin{aligned} &\frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \left[\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^+} f(\alpha) + \Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^-} f(\rho) \right] - f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2} \right) \right) \\ &= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(\xi(\mu)^{v}) \left(\int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^+}^{\rho} (\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu))^{\tau} f'(\mu) d\mu \right. \end{aligned}$$

$$-\int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^{+}}^{\alpha}(\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu))^{\tau}f'(\mu)d\mu\Bigg), \tag{29}$$

for $\tau > 0$.

Proof. Consider the integrals

$$J_1 := \frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^{+}}f(\alpha),$$

and

$$J_2 := \frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^{-}}f(\rho).$$

Definition 4 and integration by parts give

$$\begin{aligned}
J_1 :&= \frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^{+}}f(\alpha)\\
&= \frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(\tau+vn)(z)_{\delta n}}\int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\rho}(\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu))^{\tau}\check{\Upsilon}'(\mu)f(\mu)d\mu\\
&= \frac{-1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(\tau+vn)(z)_{\delta n}}\int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\rho}f(\mu)d(\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu))^{\tau}\\
&= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(\tau+vn)(z)_{\delta n}}\Bigg[(\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho))^{\tau}f\Bigg(\check{\Upsilon}^{-1}\Bigg(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\Bigg)\Bigg)\\
&+\int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\rho}(\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon})^{\tau}f'(\mu)d\mu\Bigg]\\
&= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\mathbb{E}^{\vartheta,z,\kappa}_{\nu,\tau,j}(\xi(\mu)^{v})\Bigg[(\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho))^{\tau}f\Bigg(\check{\Upsilon}^{-1}\Bigg(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\Bigg)\Bigg)\\
&+\int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\rho}(\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon})^{\tau}f'(\mu)d\mu\Bigg].
\end{aligned}\tag{30}$$

Similarly, we have

$$\begin{aligned}
J_2 :&= \frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\Im^{\vartheta,z,\kappa}_{\nu,\tau,j,\omega,(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))^{-}}f(\rho)\\
&= \frac{1}{(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(\tau+vn)(z)_{\delta n}}\int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\alpha}(\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu))^{\tau}\check{\Upsilon}'(\mu)f(\mu)d\mu\\
&= \frac{-1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}}\sum_{n=0}^{\infty}\frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(\tau+vn)(z)_{\delta n}}\int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\alpha}f(\mu)d(\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu))^{\tau}
\end{aligned}$$

$$= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \sum_{n=0}^{\infty} \frac{(\vartheta)_{\kappa n}(\xi(\mu)^{v})^{n}}{\Gamma(\tau+vn)(z)_{\delta n}} \Bigg[(\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho))^{\tau} f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)$$

$$+ \int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\alpha} (\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon})^{\tau} f'(\mu) d\mu\Bigg]$$

$$= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}(\xi(\mu)^{v}) \Bigg[(\check{\Upsilon}(\alpha)-\check{\Upsilon}(\rho))^{\tau} f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)$$

$$+ \int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\rho} (\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon})^{\tau} f'(\mu) d\mu\Bigg]. \tag{31}$$

From the identities in (30) and (31), we have

$$J_1 + J_2 - f\left(\check{\Upsilon}^{-1}\left(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}\right)\right)$$

$$= \frac{1}{2(\check{\Upsilon}(\rho)-\check{\Upsilon}(\alpha))^{\tau}} \mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}(\xi(\mu)^{v}) \Bigg(\int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\rho} (\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu))^{\tau} f'(\mu) d\mu$$

$$- \int_{(\check{\Upsilon}^{-1}(\frac{\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)}{2}))}^{\alpha} (\check{\Upsilon}(\alpha)+\check{\Upsilon}(\rho)-2\check{\Upsilon}(\mu))^{\tau} f'(\mu) d\mu\Bigg),$$

which is the required result.

Corollary 8. *If we take $\check{\Upsilon}(x) = x$ and $\tau = 1$ in Theorem 4, then we have the following identity:*

$$\frac{1}{\rho-\alpha}\left[\Im_{\nu,\tau,j,\omega,(\frac{\alpha+\rho}{2})^{+}}^{\vartheta,z,\kappa} f(\alpha) + \Im_{\nu,\tau,j,\omega,(\frac{\alpha+\rho}{2})^{-}}^{\vartheta,z,\kappa} f(\rho)\right]$$

$$- f\left(\frac{\alpha+\rho}{2}\right)$$

$$= \frac{1}{2(\rho-\alpha)} \mathbb{E}_{\nu,\tau,j}^{\vartheta,z,\kappa}(\xi(\mu)^{v}) \Bigg(\int_{\frac{\alpha+\rho}{2}^{+}}^{\rho} (\alpha+\rho-2\mu) f'(\mu) d\mu$$

$$- \int_{\frac{\alpha+\rho}{2}^{+}}^{\alpha} (\alpha+\rho-2\mu) f'(\mu) d\mu\Bigg).$$

5. Conclusion

In this research work, we discussed the generalized fractional operators for the differentiable monotone function with generalized Mittag-Leffler functions as its kernel and established some relations, by implementation of newly defined fractional operators to modify some well-known inequalities for the $\check{\Upsilon}$-convex function. We discussed the behavior of Hermite Hadamard inequalities and their consequences by utilizing newly defined operators for continuous differentiable strictly monotone function. In future, many researchers can work to develop such a type of fractional operators and make alterations to some inequalities for differentiable monotone (p, q)-convexities.

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Conflict of Interest

The authors declare that there are no conflicts of interest.

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