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Fuzzy Geodetic and Detour Spectra: Geodetic-Laplacian Energy in Fuzzy Graphs

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Abstract. This research study delves into the spectral characteristics and energy measures associated with fuzzy graphs. We present the fuzzy geodetic spectrum and fuzzy detour spectrum, which capture the spectral properties of fuzzy graphs under geodetic and detour constraints. In addition, we propose and derive novel expressions for the fuzzy geodetic-Laplacian and detour-Laplacian energies, incorporating both upper and lower bounds. Towards the end, a real-world application of fuzzy geodetic-Laplacian energy pertaining to illegal immigration and human trafficking is described. It emphasizes its potential to improve decision-making measures, allowing more effective and coordinated crime prevention and resolution initiatives.

2020 Mathematics Subject Classifications: 05C72, 05C50, 05C20

Key Words and Phrases: Fuzzy graphs, geodetic spectrum, detour spectrum, geodetic-Laplacian energy

1. Introduction

Lotfi A. Zadeh set forth the idea of fuzzy sets [1], which handle vagueness and unpredictability in set theory. Zadeh [2] explored fuzzy relations and their underlying mechanisms, aiming to address the inherent vagueness in human reasoning. His work laid the groundwork for applying fuzzy logic to diverse fields such as abstraction, information technology, and communication. Inspired by his notion of fuzzy sets, Azriel Rosenfeld [3] instilled fuzzy set theory into fuzzy graph theory in 1975. This state of the art of graph theory allowed us to convey the strength of relationships within the interval $[0, 1]$. Different mathematicians were given the freedom to experiment after being inspired by

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Rosenfeld's papers. In fuzzy graphs, he developed fuzzy relations on fuzzy sets and verified several fundamental concepts. Recent works such as Platil and Tanaka [4] developed multi-criteria evaluation frameworks for intuitionistic fuzzy sets using set-relations, which may serve as a useful theoretical backdrop for the spectral approaches. It was Gary Chartrand, Gamy L. Johns, and Songlin Tian who presented the notion of detour distance in graphs [5]. Later, Rosenfeld, who developed a metric called the μ distance in fuzzy graphs. Several other researchers, including P. S. Nair [6], Mathew and Mordeson [7, 8] presented geodetic distance and other properties in fuzzy graphs, Vijayakumar and Sunitha [9], and Rajeshkumar and Anto [10, 11], have conducted significant studies within the frameworks of fuzzy and intuitionistic fuzzy graph theory. Nagoorgani and Umamaheswari presented the notion of fuzzy detour μ-distance and some of its properties in [12].

The concept of graph energy was introduced by I. Gutman in 1978 [13] in a concise and accessible form. As discussed in [14–18], several energy-related concepts have been proposed by various researchers, expanding the scope of graph energy studies. However, most of the existing studies on graph energy have focused primarily on crisp graphs, with limited extensions into fuzzy graphs. Even within the fuzzy graph domain, energy-related measures have largely overlooked the roles of alternative distance metrics such as geodetic and detour distances. The concept of fuzzy graph energy, in particular, introduces additional complexity, as the fuzziness of both vertices and edges must be integrated into spectral computations. Anjali and Mathew [19] were among the first to explore this area, proposing a method to derive spectral properties from fuzzy adjacency matrices and generalizing the classical notion of graph energy to accommodate fuzzy structures. Gutman and Zhou explored the Laplacian energy in classical graph theory [20], while S. Rahimi Sharbaf and F. Fayazi introduced the concept of Laplacian energy for fuzzy graphs [21]. M. Nath and S. Paul have investigated the distance Laplacian spectra of graphs [22]. Furthermore, there has been little exploration of how these distance-based energies interact with structural properties of fuzzy graphs, especially in terms of spectral measures like the Laplacian. The theoretical foundation of the present study draws inspiration from these key works.

The application-oriented motivation for this study stems from the foundational work of J. N. Mordeson and S. Mathew on distance measures and energy concepts in fuzzy graphs [8], as well as the study by Binu, Mathew, and Mordeson, which applied the fuzzy Wiener index to model illegal immigration networks [23]. Furthermore, a recent investigation into the intuitionistic fuzzy geodetic Wiener index for global human trading analysis [24] has highlighted the relevance of fuzzy distance-based measures in addressing complex real-world problems. These works collectively inspired the present study to explore novel parameters for evaluating uncertain and dynamic network structures. Additional insights from related literature [25–27] have also enriched the conceptual and methodological development of this work.

This study addresses these gaps by extending the concept of fuzzy graph energy to include fuzzy geodetic energy, fuzzy detour energy, and, most notably, fuzzy geodetic-Laplacian energy and fuzzy detour-Laplacian energy. These new energy constructs provide a more comprehensive framework for analyzing fuzzy graphs, as they integrate distance-based information and spectral properties. The novelty of the proposed fuzzy geodetic-Laplacian and detour-Laplacian energies lies in their ability to reflect both the topological variation and fuzzy connectivity of the graph—features that classical energy measures do not fully capture. Compared to traditional graph energies, these fuzzy spectral energies offer greater flexibility and descriptive power, especially for applications involving uncertainty and imprecision.

This paper systematically extends spectral concepts to the domain of fuzzy graphs by introducing novel distance-based energies and exploring their structural implications. The core contributions are organized across seven sections. Section 2 introduces foundational definitions and notations relevant to fuzzy graph theory. Section 3 presents the concept of fuzzy geodetic energy, extending classical spectral ideas by defining the geodetic spectrum and computing energy based on fuzzy distances. In addition, the study also outlines an algorithm for computing geodetic paths in fuzzy graphs, thereby complementing the theoretical developments with computational techniques that ultimately lead to the determination of the fuzzy geodetic energy. Section 4 investigates fuzzy detour energy derived from the spectrum of the fuzzy detour distance matrix, offering new structural insights using long-path distance measures. Section 5 defines fuzzy geodetic-Laplacian energy, a novel spectral invariant based on the fuzzy geodetic-distance matrix, representing a new innovation within the framework of Laplacian energy concepts.. Section 6 introduces fuzzy detour-Laplacian energy by applying spectral analysis to the detour-distance Laplacian matrix, bridging classical Laplacian energy and detour-based metrics. Finally, Section 7 explores the application of fuzzy geodetic-Laplacian energy in optimizing decision-making processes, particularly in areas such as modern-day slavery [28] prevention and law enforcement modeling.

2. Preliminaries

This section presents the foundational definitions necessary for the study. Fuzzy graphs and fuzzy relations were first introduced by Rosenfeld [3]. The algebraic underpinnings of fuzzy graph structures may be further contextualized by fuzzy Γ-semimodule theory, as discussed by Platil and Petalcorin [29], which enriches the mathematical modeling of generalized fuzzy relationships. In this work, we adopt the notion of fuzzy graphs (FGs) as discussed in [7, 8], along with other related concepts from existing literature.

Definition 1. *[7] A fuzzy graph* $\mathcal{G} : (\mathcal{V}, \sigma, \mu)$ consists of a non-empty set $\mathcal{V}$ along with a pair of functions $\sigma : \mathcal{V} \rightarrow [0, 1]$ and $\mu : E \rightarrow [0, 1]$ such that $\forall\, u,\ v \in \mathcal{V}$, with $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$. Where the elements in $\mathcal{V}$ are the vertices and the elements in E are edges of the fuzzy graph.

Definition 2. *[8] Let $\mathcal{G} : (\mathcal{V}, \sigma,\ \mu)$ is a fuzzy graph, then $\mathcal{H} : (\mathcal{V}, \tau,\ \vartheta\ )$ is called a partial fuzzy subgraph of $\mathcal{G}$ if $\tau \subseteq\ \sigma$ and $\vartheta \subseteq \mu$. Similarly, a fuzzy subgraph $\mathcal{H} : (\mathcal{V}', \tau, \vartheta)$ of $\mathcal{G}$ is induced by $\mathcal{V}'$ if $\mathcal{V}' \subseteq \mathcal{V}$, $\tau\,(u) = \sigma(u)\ \forall\ u\ \epsilon\, \mathcal{V}'$ and $\vartheta\,(u,\ v) = \mu(u, v)\ \forall\, u, v\, \epsilon\ \mathcal{P}$.*

Definition 3. *[8] The support of σ is described as, Supp $(\sigma) = u\,\epsilon\ \mathcal{V}$: $\sigma\,(u) > 0$ and is denoted as σ^*. The definitions and additional findings that we utilize later in our* *discussion are listed below.*

Definition 4. *[12] In a fuzzy graph $\mathcal{G} : (\mathcal{V},\ \sigma,\ \mu)$ a path $\mathcal{P}$ is a chain of distinct vertices $v_0,\ v_1,\ \ldots\ldots v_n$ such that $(v_{i-1}, v_i) > 0,\ 1 \le i \le n$, where n denotes the length of the path.*

Definition 5. *[8] Every path has a strength and is defined to be $\bigwedge_{i=1}^{n}\{\mu\,(v_{i-1}, v_i)\}$. The symbol $\bigwedge$ denotes the minimum.*

Definition 6. *[7] The strength of connectedness between two vertices is the maximum strength among all paths connecting them and is represented as $\mu^{\infty}(v_{i-1}, v_i)$.*

Definition 7. *[8] A fuzzy graph $\mathcal{G} : (\mathcal{V},\ \sigma,\ \mu)$ is defined to be a complete fuzzy graph if $\mu\,(v_i, v_j) = \sigma\,(v_i) \wedge \sigma(v_j)\ \forall\ v_i\ , v_j\ \epsilon\ \sigma^*$.*

Definition 8. *[7] A connected fuzzy graph is called a fuzzy tree if it has a fuzzy spanning subgraph $\mathcal{H}$:$(\mathcal{V}, \sigma,\ \tau)$,* which is a tree where for all v_iv_j not in $\mathcal{H}$, $\mu\,(v_i,\ v_j) <\ \tau^{\infty}(v_i,\ v_j)$.

Definition 9. *[8] In a fuzzy graph (FG), the fuzzy distance between any two nodes is defined and denoted as $d_f(u,\ v) = \bigwedge_{\mathcal{P}}\{l(\mathcal{P}) * S(\mathcal{P})\}$, where the length of the path is denoted as $l(\mathcal{P})$ and the strength of the path is denoted as $S(\mathcal{P})$, while the symbol $\bigwedge$ denotes the minimum over all such paths. Moreover, a $v_i - v_j$ path is called a fuzzy $v_i - v_j$ geodesic if $d_f(v_i,\ v_j) = l(\mathcal{P}) * S(\mathcal{P})$.*

Definition 10. *[12] A $u - v$ path is called a fuzzy detour if its length is $\Delta\,(u, v)$, where $\Delta\,(u, v)$ is the fuzzy detour μ – distance between pair of vertices u and v and is defined by the maximum μ-length of any $u - v$ path, where the μ- length of a path $\mathcal{P} : u = v_0,\ v_1,\ \ldots\ldots, v_n = v$ is $l\,(P) = \sum_{i=1}^{n} \frac{1}{\mu(v_{i-1},\ v_i)}$.*

Definition 11. *[20] An $m \times m$ matrix $FD_{\mathcal{G}} = [a_{ij}]$, where $a_{ij} = d_{FG}(s_i)$, if $i = j$ and 0 otherwise, is called the degree matrix of fuzzy graph $\mathcal{G} : (\mathcal{V},\ \sigma,\ \mu)$.*

Definition 12. *[20] Let $\mathcal{G} : (\mathcal{V},\ \sigma,\ \mu)$ be an FG with $|\mathcal{V}| = n$ and $\mu_1 \ge \mu_2 \ge \ldots\ldots \ge \mu_n$ be the Laplacian eigenvalues. Then $LE(\mathcal{G}) = |\mu_i - \frac{2\sum_{1 \le i < j \le n} \mu(v_i,\ v_j)}{n}|$.*

Theorem 1. *[18] Let $\mathcal{G}$ be a weighted graph of order n each of whose edges has nonzero weight and $e_1, \ldots, e_m$ be all the edges of $\mathcal{G}$. Then $E(\mathcal{G}) \le 2\sum_{i=1}^{m} |w(e_i)|$, where equality holds if and only if each connected component of $\mathcal{G}$ has at most two vertices.*

Theorem 2. *[8] Let $\mathcal{G} : (\mathcal{V},\ \sigma,\ \mu)$ be FG with $|\mathcal{V}| = n$ and $\mu^* = \{e_1, ..., e_m\}$. Then, $E(\mathcal{G}) \le 2\sum_{i=1}^{m} \mu(e_i)$.*

Theorem 3. *[8] Let $\mathcal{H} : (\mathcal{V},\ \sigma,\ \mu)$ be a complete fuzzy subgraph of $\mathcal{G}$. Then, $\mathcal{H}$ is a fuzzy geodetic block (FGB) if and only if $\{v_iv_j\}$ is geodetic convex for any two vertices.*

The key symbols, notations and abbreviations employed throughout this paper are summarized in Table 1.

Table 1: Symbols, notations and abbreviations.

Symbols/notations	**Definition**
$\mathcal{G} : (\mathcal{V},\ \sigma,\ \mu)$	A fuzzy graph (FG) with membership functions σ and μ.
$\mathcal{H} : (\mathcal{V}, \tau,\ \vartheta\ )$	Partial fuzzy subgraph of an FG.
μ	fuzzy edge membership function.
σ	fuzzy vertex membership function.
$\mathcal{P}$	A path in an FG.
$\bigwedge$	Symbol for minimum.
min	Minimum.
max	Maximum.
FGB	Fuzzy geodetic block.
$d_f(v_i,\ v_j)$	Fuzzy $v_i - v_j$ geodesic
$\Delta\,(u, v)$	Fuzzy detour μ – distance.
$E(\mathcal{G})$	Energy of an FG.
$LE(\mathcal{G})$	Laplacian energy of an FG.
$\mathcal{F}_g E(\mathcal{G}))$	Fuzzy geodetic energy ($\mathcal{F}_g$-energy) .
$\mathcal{F}_d E(\mathcal{G}))$	Fuzzy detour energy ($\mathcal{F}_d$-energy) .
$GLE(\mathcal{G})$	Geodetic Laplacian energy.
$DLE(\mathcal{G})$	Detour Laplacian energy.
$\mathcal{M}_{\mathcal{F}_{gd}}$	Fuzzy geodetic distance matrix ($\mathcal{F}_g$-matrix).
$T_{\mathcal{F}_{gd}}$	Fuzzy geodetic transmission matrix.
$\mathcal{M}_{\mathcal{F}_{gld}}$	Fuzzy geodetic-Laplacian distance matrix.
$\mathcal{M}_{\mathcal{F}_{dd}}$	Fuzzy detour distance matrix ($\mathcal{F}_d$-matrix).
$T_{\mathcal{F}_{dd}}$	Fuzzy detour transmission matrix.
$\mathcal{M}_{\mathcal{F}_{dld}}$	Fuzzy detour Laplacian-distance matrix .
$\mathcal{F}_g S_r$	Fuzzy geodetic spectral radius.
$\mathcal{F}_d S_r$	Fuzzy detour spectral radius.

3. Fuzzy Geodetic Spectrum

In classical graph theory, the distance spectrum focuses on the lengths of paths between pairs of vertices. In recent years, researchers have extended spectral concepts to fuzzy graphs, where uncertainty in connections must be taken into account. The concept of fuzzy graph energy by Anjali and Mathew in [19], involves deriving spectral properties from fuzzy adjacency matrices to extend classical graph energy to fuzzy structures. Motivated by the mathematical richness of graph energy and its variants, we take the helm in this context by introducing the fuzzy geodetic spectrum and investigating its mathematical properties. Furthermore, we define and compute the fuzzy geodetic energy of fuzzy graphs, thereby extending classical spectral concepts into the fuzzy domain.

Definition 13. *Let $\mathcal{G}$ be a fuzzy graph with vertex set $\sigma^* = v_1, v_2, \ldots, v_m$. The fuzzy geodetic distance matrix ($\mathcal{F}_g$-matrix) of $\mathcal{G}$ is the $n \times n$ matrix $\mathcal{M}_{\mathcal{F}_{gd}} = [bij]$, where $b_{ij} = d_f(v_i, v_j)$ denotes the fuzzy geodetic distance between v_i and v_j. The set of eigenvalues of*

$\mathcal{M}_{\mathcal{F}_{gd}}$ is referred to as the fuzzy geodetic spectrum ($\mathcal{F}_g$-spectrum) of $\mathcal{G}$.

Definition 14. *The fuzzy geodetic spectral radius ($\mathcal{F}_g S_r$) of the matrix $\mathcal{M}_{\mathcal{F}_g}$ is defined as the maximum eigenvalue in the $\mathcal{F}_g$-spectrum of $\mathcal{G}$.*

Definition 15. *The fuzzy geodetic energy ($\mathcal{F}_g$-energy or $\mathcal{F}_g E(\mathcal{G})$) is defined as the sum of the absolute values of the $\mathcal{F}_g$-eigenvalues of the matrix $\mathcal{M}_{\mathcal{F}_{gd}}$.*

Example 1. *Consider the FG $\mathcal{G} = (\mathcal{V},\ \sigma,\ \mu)$ as shown in Figure 1. Let the $\mathcal{F}_g$-matrix of $\mathcal{G}$, corresponding to the vertex set $\sigma^* = \{v_1,\ v_2,\ v_3,\ v_4\}$, be given as follows:*

$$\begin{pmatrix} 0.0 & 0.2 & 0.1 & 0.2 \\ 0.2 & 0.0 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.0 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.0 \end{pmatrix}.$$

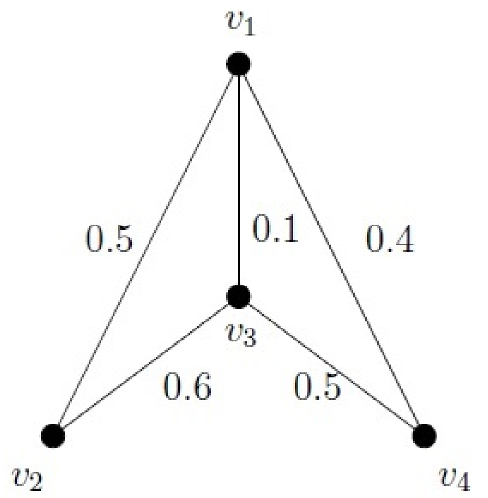

Figure 1: A fuzzy graph with $\mathcal{F}_g$-spectrum $\{0.6123,\ -0.3,\ -0.2123,\ -0.1\}$.

To compute the $\mathcal{F}_g$-spectrum of $\mathcal{G}$, we employ the classical matrix theory approach: the eigenvalues of the $\mathcal{F}_g$-matrix $\mathcal{M}_{\mathcal{F}_{gd}}$ are obtained by solving the characteristic polynomial equation

$$\det(\lambda I - \mathcal{M}_{\mathcal{F}_{gd}}) = 0, \tag{1}$$

where I denotes the identity matrix of order $|\sigma^*|$.

For the present case, solving (1) yields the eigenvalues,

$$\lambda_1 = 0.6123, \quad \lambda_2 = -0.3, \quad \lambda_3 = -0.2123, \quad \lambda_4 = -0.1.$$

Thus, the $\mathcal{F}_g$-spectrum of $\mathcal{G}$ is $\{0.6123,\ -0.3,\ -0.2123,\ -0.1\}$. Accordingly, the $\mathcal{F}_g$-energy of $\mathcal{G}$ is evaluated as,

$$\mathcal{F}_g E(\mathcal{G}) = \sum_{i=1}^{4} |\lambda_i| = 0.6123 + 0.3 + 0.2123 + 0.1 = 1.2246.$$

Theorem 4. *Let $\mathcal{G}$ be an FG with $|\mathcal{V}| = m$ and $|E| = n$. Suppose the fuzzy geodetic distance between vertices v_i and v_j is given by $d_f(v_i, v_j) = l(\mathcal{P}) \times S(\mathcal{P}) = d_{ij}$, and let $\mathcal{M}_{\mathcal{F}_{gd}}$ denote the fuzzy geodetic distance matrix. Then, the fuzzy geodetic energy $\mathcal{F}_g E(\mathcal{G})$ satisfies the following bounds:*

$$\sqrt{2\sum_{i<j}(d_{ij})^2 + m(m-1)|\mathcal{M}_{\mathcal{F}_{gd}}|^{\frac{2}{m}}} \leq \mathcal{F}_g E(\mathcal{G}) \leq \sqrt{2\left(\sum_{i<j}(d_{ij})^2\right)m}\,.$$

Proof. By applying the Cauchy–Schwarz inequality to the number of vertices and the set of absolute values of the eigenvalues,

$$\sum_{i=1}^{m}|\theta_i| \leq \sqrt{m}\sqrt{\sum_{i=1}^{m}|\theta_i|^2}\,. \tag{2}$$

$$\left(\sum_{i=1}^{m}\theta_i\right)^2 = \sum_{i=1}^{m}|\theta_i|^2 + 2\sum_{i<j}\theta_i\theta_j\,. \tag{3}$$

Next, we analyze the coefficients of $\prod_{i=1}^{m}(\theta - \theta_i) = |\mathcal{M}_{\mathcal{F}_{gd}} - \theta I|$ implies

$$\sum_{i<j}\theta_i\theta_j = -\sum_{i<j}d_{ij}^2\,. \tag{4}$$

Now, in light of Equation (3)

$$\sum_{i=1}^{m}|\theta_i|^2 = 2\sum_{i<j}(d_{ij})^2\,. \tag{5}$$

Substituting Equation (5) into Equation (2) yields

$$\sum_{i=1}^{m}|\theta_i| \leq \sqrt{m}\sqrt{2\sum_{i<j}d_{ij}^2} = \sqrt{2\left(\sum_{i<j}d_{ij}^2\right)m}$$

$$\mathcal{F}_g E(\mathcal{G}) \leq \sqrt{2\left(\sum_{i<j}(d_{ij})^2\right)m}\,.$$

Regarding the second inequality, we have

$$
\begin{aligned}
[\mathcal{F}_g E(\mathcal{G})]^2 &= \left(\sum_{i=1}^{m} \theta_i\right)^2 \\
&= \sum_{i=1}^{m} |\theta_i|^2 + 2\sum_{i<j} \theta_i\theta_j \\
&\geq 2\sum_{i<j} (d_{ij})^2 + m(m-1)GM\{|\theta_i\theta_j|\} \\
\implies \mathcal{F}_g E(\mathcal{G}) &\geq \sqrt{2\sum_{i<j} (d_{ij})^2 + m(m-1)GM\{|\theta_i\theta_j|\}}\,.
\end{aligned}
$$

But,

$$
GM\{|\theta_i\theta_j|\} = \left(\prod_{i<j} |\theta_i\theta_j|\right)^{\frac{2}{m(m-1)}} = \left(\prod_{i<j} |\theta_i|\right)^{\frac{2}{m}} = |\mathcal{M}_{d_f}|^{\frac{2}{m}}.
$$

Now,

$$
\mathcal{F}_g E(\mathcal{G}) \geq \sqrt{2\sum_{i<j} (d_{ij})^2 + m(m-1)|\mathcal{M}_{\mathcal{F}_{gd}}|^{\frac{2}{m}}}\,.
$$

This completes the proof.

Upper bounds for the $\mathcal{F}_g$-energy of an FG can be derived using the spectral radius.

Theorem 5. *Let $\mathcal{G}$ be an FG with m vertices, and let r_i denote the sum of the entries in the i^{th} row of the $\mathcal{F}_g$-matrix. Then, $\mathcal{F}_g E(\mathcal{G}) \leq m \max\{r_i\}, 1 \leq i \leq m$.*

Proof. The $\mathcal{F}_g$-matrix is a non-negative square matrix with zero diagonal entries. By the properties of non-negative matrices, the spectral radius $\mathcal{F}_g S_r$ satisfies

$$
\min_{1\leq i\leq m} r_i \leq \mathcal{F}_g S_r \leq \max_{1\leq i\leq m} r_i,
$$

where r_i is the sum of the i^{th} row of the $\mathcal{F}_g$-matrix.

Thus,

$$
\mathcal{F}_g S_r \leq \max_{1\leq i\leq m} r_i.
$$

Since $\mathcal{F}_g E(\mathcal{G})$ is the sum of the absolute values of the eigenvalues, and the spectral radius bounds the largest absolute eigenvalue, it follows that

$$
\mathcal{F}_g E(\mathcal{G}) \leq m \max_{1\leq i\leq m} r_i,
$$

completing the proof.

To establish a lower bound for $\mathcal{F}_g E(\mathcal{G})$, a similar approach can be applied. The following theorem presents the equivalence of adjacency energy and geodetic energy within fuzzy geodetic blocks.

Theorem 6. *Let $\mathcal{G}$ be a fuzzy geodetic block graph (FGB). Then, the spectra of the $\mathcal{F}_g$-matrix and the adjacency matrix A of $\mathcal{G}$ coincide, and consequently, their energies are equal.*

Proof. Assume $\mathcal{G}$ is an FGB. By definition, the pair $\{v_i, v_j\}$ is fuzzy geodetic convex for any two vertices v_i and v_j. Thus, every edge corresponds to a unique fuzzy geodesic, implying that all vertex pairs are adjacent. Therefore, the adjacency matrix A and the fuzzy geodetic distance matrix $\mathcal{M}_{\mathcal{F}_{gd}}$ coincide. This establishes the equivalence of their spectra and hence their energies.

3.1. Geodesic Path Identification and Fuzzy Geodetic Energy Algorithm

The pseudocode presented below offers a clear and systematic description of the geodesic path identification process, along with its application in constructing the fuzzy geodetic distance matrix, spectrum, and energy.

Input: A fuzzy graph $\mathcal{G} = (\mathcal{V}, \sigma, \mu)$ with vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ and fuzzy edge weights μ
Output: Fuzzy geodesic paths, fuzzy geodetic distance matrix $\mathcal{M}_{\mathcal{F}_{gd}}$, fuzzy geodetic spectrum, and fuzzy geodetic energy

(i) Initialize an $n \times n$ matrix $\mathcal{M}_{\mathcal{F}_{gd}}$ with zeros.

(ii) For each unique unordered vertex pair (v_i, v_j):

   (a) Generate all possible paths $\mathcal{P}$ between v_i and v_j.

   (b) For each path $\mathcal{P}$:

      i. Compute the path length $l(\mathcal{P})$ = number of edges in $\mathcal{P}$.

      ii. Compute the path strength $S(\mathcal{P}) = \min\{\mu(e) : e \in \mathcal{P}\}$.

   (c) Determine the fuzzy geodetic distance:

$$d_f(v_i, v_j) = \min_{\mathcal{P}}\{l(\mathcal{P}) \times S(\mathcal{P})\}.$$

   (d) Assign $\mathcal{M}_{\mathcal{F}_{gd}}[i][j] \leftarrow d_f(v_i, v_j)$.

(iii) After processing every pair, $\mathcal{M}_{\mathcal{F}_{gd}}$ represents the fuzzy geodetic distance matrix.

(iv) Evaluate the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $\mathcal{M}_{\mathcal{F}_{gd}}$.

(v) Define the fuzzy geodetic spectrum as:

$$\mathcal{F}_g\text{-spectrum} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

(vi) Determine the fuzzy geodetic energy:

$$\mathcal{F}_g E(\mathcal{G}) = \sum_{i=1}^{n} |\lambda_i|.$$

(vii) Display the identified geodesic paths, the constructed $\mathcal{F}_g$-matrix, its spectrum, and the computed energy.

4. Fuzzy Detour Spectrum

In fuzzy graph theory, Nagoorgani and Umamaheswari introduced the fuzzy detour μ-distance and examined its fundamental properties. Later, Rajesh and Anto explored detour convexity properties. Extending this framework, we investigate the fuzzy detour energy derived from the spectrum of the fuzzy detour distance matrix. This development enriches the spectral analysis of fuzzy graphs by incorporating long-path distance measures under fuzziness, leading to new bounds and structural insights.

Definition 16. *The fuzzy detour distance matrix ($\mathcal{F}_d$ - matrix) of $\mathcal{G}$ with $\sigma^* = \{v_1,\ v_2, ...$ .. $v_m\}$ is an $m \times m$ matrix $\mathcal{M}_{\mathcal{F}_{dd}} = [c_{ij}]$ where $c_{ij} = \Delta(v_i,\ v_j)$. The fuzzy detour energy ($\mathcal{F}_d$-Energy) is the sum of the absolute values of its $\mathcal{F}_d$- eigenvalues of $\mathcal{M}_{\mathcal{F}_{dd}}$. The set of $\mathcal{F}_d$- eigenvalues is called the fuzzy detour-spectrum ($\mathcal{F}_d -$ spectrum) of $\mathcal{G}$. The fuzzy detour-spectral radius ($\mathcal{F}_d S_r$) of $\mathcal{M}$ is the maximum of its $\mathcal{F}_d$- eigenvalues.*

Example 2. *For instance, consider the $\mathcal{F}_d$-matrix (distance has been corrected to two decimal places) of $\mathcal{G}$ in Figure 2 with $\sigma^* = \{v_1,\ v_2,\ v_3,\ v_4\}$.*

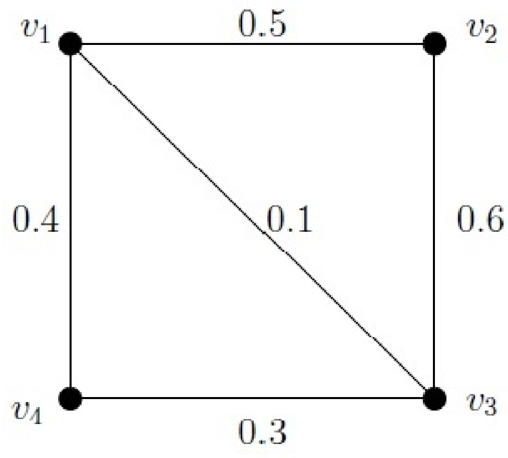

Figure 2: A fuzzy graph with fuzzy detour-spectral radius 36.9236.

$$\begin{pmatrix} 0.00 & 11.66 & 10.00 & 13.33 \\ 11.66 & 0.00 & 12.00 & 14.66 \\ 10.00 & 12.00 & 0.00 & 12.50 \\ 13.33 & 14.16 & 12.50 & 0.00 \end{pmatrix}.$$

Here, the $\mathcal{F}_d$-spectrum is $\{36.9236,\ -14.6074,\ -12.3964,\ -9.91979\}$*, and thus the* $\mathcal{F}_dE(\mathcal{G}) = 73.84719$*. The* $\mathcal{F}_dS_r = 36.9236$*.*

Theorem 7. *Let $\mathcal{G}$ be an FG. Then,*

$$\mathcal{F}_dE(\mathcal{G}) \leq 2\sum_{i,j} \Delta(v_i, v_j), \quad \forall\, i, j.$$

Proof. Consider the fuzzy detour distance matrix $\mathcal{M}_{\mathcal{F}_{dd}}$, which is a non-negative, symmetric square matrix. The trace of $\mathcal{M}^2_{\mathcal{F}_{dd}}$ is given by

$$\mathrm{Tr}(\mathcal{M}^2_{\mathcal{F}_{dd}}) = \sum_{i=1}^{m}\sum_{j=1}^{m} \Delta(v_i, v_j)\Delta(v_j, v_i).$$

Since $\mathcal{M}_{\mathcal{F}_{dd}}$ is symmetric, we have $\Delta(v_i, v_j) = \Delta(v_j, v_i)$, and thus

$$\mathrm{Tr}(\mathcal{M}^2_{\mathcal{F}_{dd}}) = \sum_{i=1}^{m}\sum_{j=1}^{m} \Delta(v_i, v_j)^2.$$

Using Theorem 1 and properties of symmetric matrices, it follows that

$$\mathcal{F}_dE(\mathcal{G}) \leq 2\sum_{i,j} \Delta(v_i, v_j), \quad \forall\, i, j,$$

as required.

Theorem 8. *Let $\mathcal{G}$ be a fuzzy graph, and let r_i denote the sum of the entries in the i^{th} row of the $\mathcal{F}_d$-matrix. Then,*

$$\mathcal{F}_dE(\mathcal{G}) \leq n \max_{1\leq i\leq n} r_i.$$

Proof. The $\mathcal{F}_d$-matrix is a non-negative square matrix with zero diagonal entries. Thus, the spectral radius $\mathcal{F}_dS_r$ satisfies

$$\min_{1\leq i\leq n} r_i \leq \mathcal{F}_dS_r \leq \max_{1\leq i\leq n} r_i.$$

Hence,

$$\mathcal{F}_dS_r \leq \max_{1\leq i\leq n} r_i.$$

Since the fuzzy detour energy $\mathcal{F}_dE(\mathcal{G})$ is the sum of the absolute values of the eigenvalues, we have

$$\mathcal{F}_dE(\mathcal{G}) \leq n \cdot \max_{1\leq i\leq n} r_i,$$

as required.

To derive the lower bound for $\mathcal{F}_dE(\mathcal{G})$, a similar proof is possible.

Theorem 9. *Let $\mathcal{G}$ be a FG with $|\mathcal{V}| = m$ and $|E| = n$. If $c_{ij} = \Delta(v_i, v_j)$ denotes the fuzzy detour distance and $\mathcal{M}_{\mathcal{F}_{dd}}$ is the fuzzy detour distance matrix, then*

$$\mathcal{F}_d E(\mathcal{G}) \leq \sqrt{(m-1)\left[\kappa - \theta_i^2\right]}, \text{ where } \kappa = \sum_{i<j} c_{ij}^2.$$

Proof. By the Cauchy–Schwarz inequality, we have:

$$\sum_{i=2}^{m} |\theta_i| \leq \sqrt{(m-1)\sum_{i=2}^{m} \theta_i^2}. \tag{6}$$

Suppose that the total sum of squared eigenvalues satisfies:

$$\sum_{i=1}^{m} \theta_i^2 \leq \kappa = \sum_{i<j} c_{ij}^2.$$

Then,

$$\theta_1^2 + \sum_{i=2}^{m} \theta_i^2 \leq \kappa \quad \Rightarrow \quad \sum_{i=2}^{m} \theta_i^2 \leq \kappa - \theta_1^2.$$

Substituting into (6), we get:

$$\sum_{i=2}^{m} |\theta_i| \leq \sqrt{(m-1)(\kappa - \theta_1^2)}.$$

Therefore, the fuzzy detour energy satisfies:

$$\mathcal{F}_d E(\mathcal{G}) = \sum_{i=1}^{m} |\theta_i| \leq \theta_1 + \sqrt{(m-1)(\kappa - \theta_1^2)},$$

as required.

Another, potentially more satisfying, upper and lower bound is established in the next theorem.

Theorem 10. *Let $\mathcal{G}$ be a FG with $|\mathcal{V}| = m$ and $|E| = n$. If $c_{ij} = \Delta(v_i,\ v_j)$ denotes the fuzzy detour distance and $\mathcal{M}_{\mathcal{F}_{dd}}$ is the fuzzy detour distance matrix, then*

$$\sqrt{2\sum_{i<j}(c_{ij})^2 + m(m-1)|\mathcal{M}_{\mathcal{F}_{dd}}|^{\frac{2}{m}}} \leq \mathcal{F}_d E(\mathcal{G}) \leq \sqrt{2\left(\sum_{i<j}(c_{ij})^2\right)m}.$$

Proof. The proof follows similarly to that of Theorem 4.

5. Fuzzy Geodetic-Laplacian Energy

In classical graph theory, distance Laplacian spectra were introduced by Milan Nath and Somnath Paul. In fuzzy graphs, the ordinary and Laplacian energies, studied in [19] and [21], share several spectral similarities with their crisp counterparts. However, to extend Laplacian-based energy to geodetic frameworks, we define the fuzzy geodetic-Laplacian energy, derived from the spectrum of the fuzzy geodetic-distance matrix, offering a novel spectral invariant for fuzzy graphs.

Definition 17. *For a connected FG $\mathcal{G} = (\mathcal{V},\ \sigma,\ \mu)$ with $|\mathcal{V}| = n$, let $\mathcal{M}_{\mathcal{F}_{gd}}$ denote the $\mathcal{F}_g$-matrix. Then the fuzzy geodetic transmission matrix, $T_{\mathcal{F}_{gd}}$, is an $n \times n$ diagonal matrix, where each diagonal entry t_{ii} is defined as*

$$t_{ii} = \sum_{j=1}^{n} d_f(v_i,\ v_j).$$

Definition 18. *Let $\mathcal{M}_{\mathcal{F}_{gd}}$ and $T_{\mathcal{F}_{gd}}$ be the $\mathcal{F}_g$-matrix and the fuzzy geodetic transmission matrix, respectively, of an FG $\mathcal{G} = (\mathcal{V},\ \sigma,\ \mu)$. Their difference is called the fuzzy geodetic-Laplacian matrix, represented by $\mathcal{M}_{\mathcal{F}_{gld}}$, and is defined as*

$$\mathcal{M}_{\mathcal{F}_{gld}} = T_{\mathcal{F}_{gd}} - \mathcal{M}_{\mathcal{F}_{gd}}.$$

Definition 19. *Let $\mathcal{G} = (\mathcal{V},\ \sigma,\ \mu)$ be an FG with $|\mathcal{V}| = n$, and let $\gamma_{gl_1} \geq \gamma_{gl_2} \geq \cdots \geq \gamma_{gl_n}$ be the fuzzy geodetic-Laplacian eigenvalues. Then, the geodetic-Laplacian energy is defined as*

$$GLE(\mathcal{G}) = \sum_{i=1}^{n} |\psi_i|,$$

where

$$|\psi_i| = \left| \gamma_{gl_i} - \frac{2\sum_{1\leq i<j\leq n} d_f(v_i, v_j)}{n} \right|.$$

Example 3. *Figure 3 illustrates the fuzzy geodetic-Laplacian matrix and its eigenvalues for the FG $\mathcal{G}$.*

The associated fuzzy geodetic-Laplacian matrix is:

$$\begin{pmatrix} 1.4 & -0.6 & -0.5 & -0.3 \\ -0.6 & 1.7 & -0.5 & -0.6 \\ -0.5 & -0.5 & 1.3 & -0.3 \\ -0.3 & -0.6 & -0.3 & 1.2 \end{pmatrix}.$$

Here, $\gamma_{gl_1} = 2.2842,\ \gamma_{gl_2} = 1.8384,\ \gamma_{gl_3} = 1.4774$, and $\gamma_{gl_4} = 0.0$, then $GLE(\mathcal{G}) = 2.8$.

The following theorems establish the upper and lower bounds for the geodetic-Laplacian energy.

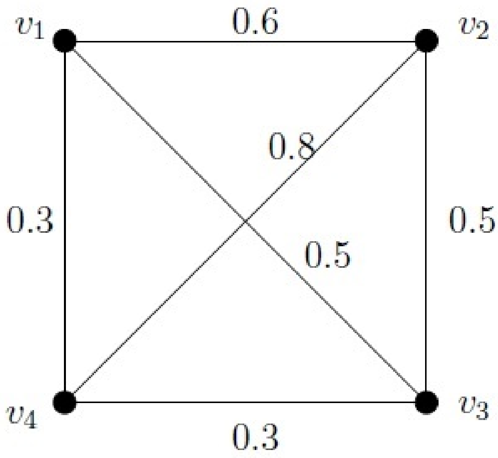

Figure 3: Illustration of the fuzzy geodetic–Laplacian matrix

Theorem 11. *Let $\mathcal{G} = (\mathcal{V}, \sigma, \mu)$ be an FG with fuzzy geodetic-Laplacian matrix $\mathcal{M}_{\mathcal{F}_{gld}}$ whose diagonal entries are denoted as $d_{\mathcal{M}_{\mathcal{F}_{gld}}}(t_i)$. Then, the geodetic-Laplacian energy satisfies:*

$$GLE(\mathcal{G}) \leq \sqrt{n\left(2\sum_{1\leq i<j\leq n} d_{ij}^2 + \sum_{i=1}^{n}\left(d_{\mathcal{M}_{\mathcal{F}_{gld}}}(t_i) - \frac{2\sum d_{ij}}{n}\right)^2\right)}.$$

Proof. Applying the Cauchy–Schwarz inequality to the vector $(|\psi_1|, |\psi_2|, \ldots, |\psi_n|)$, we have

$$\left|\sum_{i=1}^{n}\psi_i\right|^2 \leq n\sum_{i=1}^{n}|\psi_i|^2,$$

where

$$\psi_i = \gamma_{gl_i} - \frac{2\sum_{1\leq i<j\leq n} d_f(v_i, v_j)}{n}.$$

By Definition 19,

$$GLE(\mathcal{G}) = \sum_{i=1}^{n}|\psi_i| \leq \sqrt{n\sum_{i=1}^{n}|\psi_i|^2} = \sqrt{2Mn},$$

where

$$M = \sum_{1\leq i<j\leq n} d_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mathcal{M}_{\mathcal{F}_{gld}}}(t_i) - \frac{2\sum d_{ij}}{n}\right)^2.$$

Thus,

$$GLE(\mathcal{G}) \leq \sqrt{n\left(2\sum_{1\leq i<j\leq n} d_{ij}^2 + \sum_{i=1}^{n}\left(d_{\mathcal{M}_{\mathcal{F}_{gld}}}(t_i) - \frac{2\sum d_{ij}}{n}\right)^2\right)}.$$

Theorem 12. *Let $\mathcal{G} = (\mathcal{V},\ \sigma,\ \mu)$ be an FG with fuzzy geodetic-Laplacian matrix $\mathcal{M}_{\mathcal{F}_{gld}}$, and let its diagonal entries be represented by $d_{\mathcal{M}_{\mathcal{F}_{gld}}}(t_i)$. Then, the geodetic-Laplacian energy satisfies the inequality:*

$$GLE(\mathcal{G}) \geq 2\sqrt{\left(\sum_{1\leq i<j\leq n} d_{ij}^2 + \tfrac{1}{2}\sum_{i=1}^{n}\left(d_{\mathcal{M}_{\mathcal{F}_{gld}}}(t_i) - \tfrac{2\sum d_{ij}}{n}\right)^2\right)}.$$

Proof. By Definition 19, we have

$$(GLE(\mathcal{G}))^2 = (\textstyle\sum_{i=1}^{n} |\psi_i|)^2 = \sum_{i=1}^{n} |\psi_i|^2 + 2\sum_{1\leq i<j\leq n} |\psi_i||\psi_j| \geq 4M.$$

Thus we get,

$$GLE(\mathcal{G}) \geq 2\sqrt{\left(\sum_{1\leq i<j\leq n} d_{ij}^2 + \tfrac{1}{2}\sum_{i=1}^{n}\left(d_{\mathcal{M}_{\mathcal{F}_{gld}}}(t_i) - \tfrac{2\sum d_{ij}}{n}\right)^2\right)}.$$

Theorem 13. *Let $\mathcal{G} = (\mathcal{V}, \sigma, \mu)$ be an FG. If $\mathcal{G}$ is a geodetic block, then $LE(\mathcal{G}) = GLE(\mathcal{G})$.*

Proof. Since $\mathcal{G}$ is a geodetic block, every edge corresponds to a distinct fuzzy geodesic. Consequently, the fuzzy geodetic-Laplacian matrix $\mathcal{M}_{\mathcal{F}_{gld}}$ is identical to the standard Laplacian matrix. Hence, the spectra of both matrices are the same, and it follows that $LE(\mathcal{G}) = GLE(\mathcal{G})$.

6. Fuzzy Detour-Laplacian Energy

In fuzzy graph theory, detour distance has been studied for its role in path analysis, although detour-based spectral properties have been explored in earlier sections, the extension of Laplacian energy concepts to detour-based frameworks remains largely unaddressed. In this section, we define the fuzzy detour-Laplacian energy by employing the spectrum of the fuzzy detour-distance Laplacian matrix. This formulation offers a new spectral invariant and bridges the gap between classical Laplacian energy and detour-based distance measures in fuzzy graphs.

Definition 20. *Let $\mathcal{G} = (\mathcal{V}, \sigma, \mu)$ be a connected FG with $|\mathcal{V}| = n$, and let $\mathcal{M}_{\mathcal{F}_{dd}}$ denote the $\mathcal{F}_d$-matrix. Then, the detour transmission matrix $T_{\mathcal{F}_{dd}}$ is an $n \times n$ diagonal matrix whose diagonal entries are given by*

$$t_{ii} = \sum_{j=1}^{n} \Delta(v_i, v_j),$$

where $\Delta(v_i, v_j)$ denotes the fuzzy detour distance between vertices v_i and v_j.

Definition 21. *Let $\mathcal{G} = (\mathcal{V}, \sigma, \mu)$ be an FG. If $\mathcal{M}_{\mathcal{F}_{dd}}$ and $T_{\mathcal{F}_{dd}}$ denote the $\mathcal{F}_d$-matrix and the detour transmission matrix of $\mathcal{G}$, respectively, then their difference is defined as the fuzzy detour-Laplacian matrix $\mathcal{M}_{\mathcal{F}_{dld}}$, given by*

$$\mathcal{M}_{\mathcal{F}_{dld}} = T_{\mathcal{F}_{dd}} - \mathcal{M}_{\mathcal{F}_{dd}}.$$

Definition 22. *Let $\mathcal{G} = (\mathcal{V}, \sigma, \mu)$ be an FG with $|\mathcal{V}| = n$, and let $\gamma_{dl_1} \geq \gamma_{dl_2} \geq \cdots \geq \gamma_{dl_n}$ denote the fuzzy detour-Laplacian eigenvalues of $\mathcal{G}$. Then the fuzzy detour-Laplacian energy is defined as*

$$DLE(\mathcal{G}) = \sum_{i=1}^{n} |\chi_i|,$$

where

$$\chi_i = \gamma_{dl_i} - \frac{2}{n} \sum_{1 \leq i < j \leq n} \Delta(v_i, v_j).$$

Example 4. *To comprehend the fuzzy detour-Laplacian matrix and its eigenvalues, consider the graph $\mathcal{G}$ shown in Figure 2. The associated fuzzy detour-Laplacian matrix $\mathcal{M}_{\mathcal{F}_{dld}}$ is given by:*

$$\begin{pmatrix} 34.99 & -11.66 & -10.00 & -13.33 \\ -11.66 & 38.32 & -12.00 & -14.66 \\ -10.00 & -12.00 & 34.5 & -12.50 \\ -13.33 & -14.16 & -12.50 & 39.99 \end{pmatrix}.$$

Here, $\gamma_{dl_1} = 54.0166, \gamma_{dl_2} = 49.0913, \gamma_{dl_3} = 44.6921$ and $\gamma_{dl_4} = 0.0000$, $DLE(\mathcal{G}) = 148.8$.

The following theorems establish upper and lower bounds for the detour-Laplacian energy.

Theorem 14. *Let $\mathcal{G} : (\mathcal{V}, \sigma, \mu)$ be an FG with fuzzy detour-Laplacian matrix $\mathcal{M}_{\mathcal{F}_{dld}}$, and let its diagonal entries be denoted as $d_{\mathcal{M}_{\mathcal{F}_{dld}}}(t_i)$. Then,*

$$DLE(\mathcal{G}) \leq \sqrt{\left(2 \sum_{1 \leq i < j \leq n} c_{ij}^2 + \sum_{i=1}^{n} \left(d_{\mathcal{M}_{\mathcal{F}_{dld}}}(t_i) - \frac{2 \sum c_{ij}}{n}\right)^2\right) n}.$$

Proof. Applying the Cauchy–Schwarz inequality to the vector $(|\chi_1|,\ |\chi_2|,\ \ldots,\ |\chi_n|)$ yields the result. The argument proceeds analogously to Theorem 11 and is thus omitted for brevity.

Theorem 15. *Let $\mathcal{G} : (\mathcal{V}, \sigma, \mu)$ be an FG with fuzzy detour-Laplacian matrix $\mathcal{M}_{\mathcal{F}_{dld}}$, and let its diagonal entries be denoted as $d_{\mathcal{M}_{\mathcal{F}_{dld}}}(t_i)$. Then,*

$$DLE(\mathcal{G}) \geq 2 \sqrt{\left(\sum_{1 \leq i < j \leq n} c_{ij}^2 + \frac{1}{2} \sum_{i=1}^{n} \left(d_{\mathcal{M}_{\mathcal{F}_{dld}}}(t_i) - \frac{2 \sum c_{ij}}{n}\right)^2\right)},$$

where $c_{ij} = \Delta(v_i, v_j)$.

Proof. By Definition 22, we have

$$(DLE(\mathcal{G}))^2 = \left(\sum_{i=1}^{n} |\chi_i|\right)^2 = \sum_{i=1}^{n} |\chi_i|^2 + 2 \sum_{1\leq i<j\leq n} |\chi_i||\chi_j|.$$

Since both terms are non-negative, it follows that

$$(DLE(\mathcal{G}))^2 \geq 4M,$$

where

$$M = \sum_{1\leq i<j\leq n} c_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mathcal{M}_{\mathcal{F}_{dld}}}(t_i) - \frac{2\sum c_{ij}}{n}\right)^2.$$

Taking square roots on both sides yields the desired inequality.

7. Applications of Fuzzy Geodetic-Laplacian Energy

This section explores the use of geodetic-Laplacian energy in decision-making processes aimed at addressing modern-day slavery. Additionally, the study highlights an effective mathematical model for its application in modern law enforcement, particularly in the areas of efficient crime detection and combating illegal activities.

7.1. Analyzing the Illicit Routes of Human Trading Networks

Human trafficking often operates through complex networks of individuals and organizations. Given the inherent ambiguity and uncertainty in these connections, Fuzzy Graph (FG) theory provides a valuable framework for examining and visualizing these networks. This strategy could be helpful to law enforcement and governments in identifying the structure and dynamics of trafficking routes. In this study, we focus on the major illicit human trafficking routes into the United States, as outlined in studies [23]. Our primary aim is to evaluate the fuzzy geodetic Laplacian energy as a comparative measure, in contrast to the various fuzzy graph metrics discussed in previous researches [23–25].

The Global Slavery Index, as referenced in [28], offers a detailed assessment of the susceptibility to modern slavery across 150+ countries. Over the past two decades, global institutions like the United Nations Office on Drugs and Crime (UNODC) and the International Labour Organization (ILO) have significantly advanced efforts to quantify the scale of modern slavery worldwide. Various methods have been applied to fuzzy graphs in order to assess the likelihood of illegal immigration along different routes from countries of origin to the United States.

Notably, Binu et al. [23] explored the use of the Wiener index on fuzzy graphs to examine pathways of illegal immigration, providing a novel approach to studying human trafficking networks. Additionally, studies referenced in [24] have introduced the use of the Intuitionistic Fuzzy Geodetic Wiener Index, a method that proves effective in modeling illicit trafficking by capturing migration flows and network centralities within fuzzy relational frameworks.

Building upon these earlier works, this paper adopts the fuzzy graph structure employed by Binu et al.[23] to test the applicability and accuracy of the fuzzy geodetic Laplacian energy as a metric for analyzing human trafficking networks. By comparing this approach with existing fuzzy graph measures, we aim to enhance our understanding of the dynamics and complexities of trafficking networks, ultimately contributing to more effective measures for combating human exploitation. Let $d_f(\mathcal{P}_{u\to v})$ represent the fuzzy geodetic distance relation matrix for various path relations $\mathcal{P}_{u\to v}$, and let $FD_{\mathcal{G}}(\mathcal{P}_{u\to v})$ denote the relation degree matrix.

Figure 4: Main routes of illicit trade in humans to the U.S.

In Figure 4, consider the different paths to the United States. First, we evaluate $d_f(\mathcal{P}_{u\to v})$ for any one of the paths—specifically, say $d_f(\mathcal{P}_{\text{China}\to\text{U. S.}})$. Now we have

$$d_f(\mathcal{P}_{\text{China}\to\text{U. S.}}) = \begin{pmatrix} 0.0 & 0.19 & 0.38 & 0.38 & 0.57 \\ 0.19 & 0.0 & 0.3 & 0.3 & 0.6 \\ 0.38 & 0.3 & 0.0 & 0.32 & 0.64 \\ 0.38 & 0.3 & 0.32 & 0.0 & 0.47 \\ 0.57 & 0.6 & 0.64 & 0.47 & 0.0 \end{pmatrix}.$$

Also, we have

$$T_{\mathcal{F}gd}(\mathcal{P}_{\text{China}\to\text{U. S.}}) = \begin{pmatrix} 1.52 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.39 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.64 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.47 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.28 \end{pmatrix}.$$

Thus, the corresponding fuzzy geodetic-Laplacian matrix, $\mathcal{M}_{\mathcal{F}_{gld}}(\mathcal{P}_{\text{China}\to\text{U. S.}})$ is as follows:

$$\mathcal{M}_{\mathcal{F}_{gld}}(\mathcal{P}_{\text{China}\to\text{U. S.}}) = \begin{pmatrix} 1.52 & -0.19 & -0.38 & -0.38 & -0.57 \\ -0.19 & 1.39 & -0.3 & -0.3 & -0.6 \\ -0.38 & -0.3 & 1.64 & -0.32 & -0.64 \\ -0.38 & -0.3 & -0.32 & 1.47 & -0.47 \\ -0.57 & -0.6 & -0.64 & -0.47 & 2.28 \end{pmatrix}.$$

The eigenvalues are $\gamma_{gl_1} = 2.8692$, $\gamma_{gl_2} = 1.9900$, $\gamma_{gl_3} = 1.8317$, $\gamma_{gl_4} = 1.6090$, and $\gamma_{gl_5} = 0.0000$. Then,

$$GLE(\mathcal{P}_{\text{China}\to\text{U. S.}}) = \left|\gamma_{gl_i} - \frac{2\sum_{1\leq i<j\leq n} d_f(\mathcal{P}_{\text{China}\to\text{U. S.}})}{n}\right|.$$

$$\begin{aligned} GLE(\mathcal{P}_{\text{China}\to\text{U. S.}}) &= |2.8692 - 1.66| + |1.9900 - 1.66| \\ &\quad + |1.8317 - 1.66| + |1.6090 - 1.66| \\ &\quad + |0.0000 - 1.66| \\ &= 3.4219. \end{aligned}$$

Likewise, along the $\mathcal{P}_{\text{India}\to\text{U. S.}}$ route, we have:

$$d_f(\mathcal{P}_{\text{India}\to\text{U. S.}}) = \begin{pmatrix} 0.0 & 0.26 & 0.52 & 0.78 \\ 0.26 & 0.0 & 0.32 & 0.64 \\ 0.52 & 0.32 & 0.0 & 0.47 \\ 0.78 & 0.64 & 0.47 & 0.0 \end{pmatrix}$$

and

$$T_{\mathcal{F}gd}(\mathcal{P}_{\text{India}\to\text{U. S.}}) = \begin{pmatrix} 1.56 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.22 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.31 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.89 \end{pmatrix}.$$

Thus

$$\mathcal{M}_{\mathcal{F}_{gld}}(\mathcal{P}_{\text{India}\to\text{U. S.}}) = \begin{pmatrix} 1.56 & -0.26 & -0.52 & -0.78 \\ -0.26 & 1.22 & -0.32 & -0.64 \\ -0.52 & -0.32 & 1.31 & -0.47 \\ -0.78 & -0.64 & -0.47 & 1.89 \end{pmatrix}.$$

Then we obtain, $GLE(\mathcal{P}_{\text{India}\to\text{U. S.}}) = 2.9789$. Using similar computations for the $\mathcal{P}_{\text{Ethiopia}\to\text{U. S.}}$, we obtain $GLE(\mathcal{P}_{\text{Ethiopia}\to\text{U. S.}}) = 5.026$. Along $\mathcal{P}_{\text{Somalia}\to\text{U. S.}}$ route, $GLE(\mathcal{P}_{\text{Somalia}\to\text{U. S.}}) = 5.855$. In the route $\mathcal{P}_{\text{Nigeria}\to\text{U. S.}}$, $GLE(\mathcal{P}_{\text{Nigeria}\to\text{U. S.}}) = 5.518$.

Now we analyze the human trafficking from the continents of Africa and Asia to enter the U.S. illegally over the Mexican border. Let's first evaluate $\mathcal{M}_{\mathcal{F}_{gld}}(\mathcal{P}_{\text{Asia}\to\text{U. S.}})$. Thus

$$\mathcal{M}_{\mathcal{F}_{gld}}(\mathcal{P}_{\text{Asia}\to\text{U. S.}}) = \begin{pmatrix} 0.78 & 0.0 & 0.0 & 0.0 & -0.78 \\ 0.0 & 0..57 & 0.0 & 0.0 & -0.57 \\ 0.0 & 0.0 & 0.47 & -0.17 & -0.3 \\ 0.0 & 0.0 & -0.17 & 0.41 & -0.24 \\ -0.78 & -0.57 & -0.3 & -0.24 & 1.89 \end{pmatrix}.$$

The eigenvalues are $\gamma_{gl_1} = 1.545$, $\gamma_{gl_2} = 1.281$, $\gamma_{gl_3} = 0.612$, $\gamma_{gl_4} = 0.491$, and $\gamma_{gl_5} = 0.191$. Therefore, we have

$$GLE(\mathcal{P}_{\text{Asia}\to\text{U. S.}}) = 2.356.$$

Similarly, along the $\mathcal{P}_{\text{Africa}\to\text{U. S.}}$ route, we have

$$\mathcal{M}_{\mathcal{F}_{gld}}(\mathcal{P}_{\text{Africa}\to\text{U. S.}}) = \begin{pmatrix} 1.05 & 0.00 & 0.00 & -1.05 \\ 0.00 & 0.36 & 0.00 & -0.36 \\ 0.00 & 0.00 & 0.55 & -0.55 \\ -1.05 & -0.36 & -0.55 & 1.96 \end{pmatrix}.$$

Then we obtain:

$$GLE(\mathcal{P}_{\text{Africa}\to\text{U. S.}}) = 3.608.$$

Based on the methodology employed in [23] to assess the likelihood of illegal immigration from countries of origin to the United States, as well as the approach used in this study, Somalia emerges as the most vulnerable country of origin for migration routes leading to the United States. Similar findings regarding Somalia's high vulnerability are reported in [23–25]. Moreover, our results indicate that the continent of Africa exhibits a higher level of vulnerability compared to Asia. To quantify such vulnerability, this study introduces the concept of fuzzy geodetic-Laplacian energy as a potential metric. This work demonstrates that fuzzy geodetic-Laplacian energy can effectively analyze complex relational structures, providing a valuable tool for identifying patterns of migratory risk. The results validate the applicability of this approach, suggesting that it offers enhanced insights into regional vulnerability assessments and supports informed decision-making.

7.2. Role of Different Agencies in Decision Making for Combating Crimes

In contemporary law enforcement, effective crime detection and prevention require coordinated collaboration across multiple agencies. In this study, we propose a mathematical model that employs fuzzy geodetic-Laplacian energy to facilitate informed decision-making. We consider a hypothetical team formation comprising four police departments—Alpha (A), Bravo (B), Charlie (C), and Delta (D)—each authorized to detect, investigate, and take remedial action against reported crimes. These departments interact across three key relational dimensions: law enforcement coordination (Ψ_1), joint investigative involvement (Ψ_2), and engagement with external agencies (Ψ_3). As illustrated in Figure 5, the relationships among Alpha, Bravo, Charlie, and Delta are clearly depicted. To analyze these interactions, we first compute the geodetic-Laplacian energies for each relational aspect Ψ_i. Let $d_f(\Psi_i)$ denote the fuzzy geodetic distance relation matrix corresponding to the relations Ψ_i, and $FD_{\mathcal{G}}(\Psi_i)$ represent the relation degree matrix.

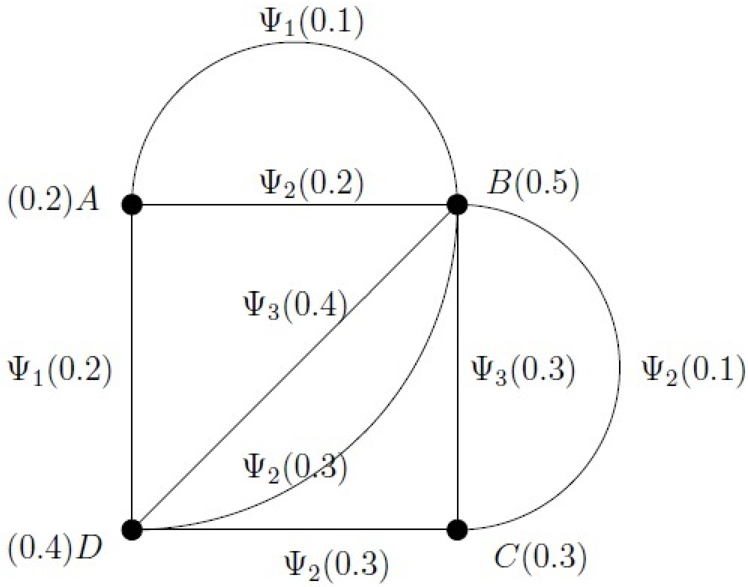

Figure 5: Relational aspects of different agencies.

Now we evaluate GLE for all Ψ_i; from Figure 5 we have

$$d_f(\Psi_1) = \begin{pmatrix} 0.0 & 0.1 & 0.0 & 0.2 \\ 0.1 & 0.0 & 0.0 & 0.2 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.2 & 0.2 & 0.0 & 0.0 \end{pmatrix}.$$

$$T_{\mathcal{F}_{gd}}(\Psi_1) = \begin{pmatrix} 0.3 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.4 \end{pmatrix}.$$

Thus, the corresponding fuzzy geodetic-Laplacian matrix for Ψ_1, denoted $\mathcal{M}_{\mathcal{F}_{gld}}(\Psi_1)$, is

$$\mathcal{M}_{\mathcal{F}_{gld}}(\Psi_1) = \begin{pmatrix} 0.3 & -0.1 & 0.0 & -0.2 \\ -0.1 & 0.1 & 0.0 & -0.2 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ -0.2 & -0.2 & 0.0 & 0.2 \end{pmatrix}.$$

The eigenvalues are $\gamma_{gl_1}(\Psi_1) = 0.6$, $\gamma_{gl_2}(\Psi_1) = 0.4$, $\gamma_{gl_3}(\Psi_1) = 0$, and $\gamma_{gl_4}(\Psi_1) = 0$, yielding

$$GLE(\Psi_1) = 1.$$

Now, according to the relationship Ψ_2, we have

$$d_f(\Psi_2) = \begin{pmatrix} 0.0 & 0.1 & 0.2 & 0.2 \\ 0.1 & 0.0 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.0 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.0 \end{pmatrix}.$$

and

$$T_{\mathcal{F}_{gd}}(\Psi_2) = \begin{pmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.6 \end{pmatrix}.$$

Then,

$$\mathcal{M}_{\mathcal{F}_{gld}}(\Psi_2) = \begin{pmatrix} 0.5 & -0.1 & -0.2 & -0.2 \\ -0.1 & 0.4 & -0.1 & -0.2 \\ -0.2 & -0.1 & 0.5 & -0.2 \\ -0.2 & -0.2 & -0.2 & 0.6 \end{pmatrix}.$$

After computing the eigenvalues,

$$GLE(\Psi_2) = 1.$$

In a similar fashion, for the relationship Ψ_3,

$$\mathcal{M}_{\mathcal{F}_{gld}}(\Psi_3) = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.7 & -0.3 & -0.4 \\ 0.0 & -0.3 & 0.9 & -0.6 \\ 0.0 & -0.4 & -0.6 & 1.0 \end{pmatrix}.$$

Thus,

$$GLE(\Psi_3) = 2.599.$$

Therefore,

$$GLE(\mathcal{G}) = \{1.000,\ 1.000,\ 2.599\}.$$

Key Findings:

- The highest fuzzy geodetic-Laplacian energy is associated with maintaining relationships with external agencies (Ψ_3). This indicates that Bravo (B), Charlie (C), and Delta (D) are primarily responsible for the complex task of coordinating and leveraging support from outside organizations. The significant energy allocation underscores the critical role these external collaborations play in enhancing the overall effectiveness of crime prevention and control.

- The energy related to mutual involvement in investigations (Ψ_2) ranks second. This reflects a strong cooperative dynamic among Alpha (A), Bravo (B), Charlie (C), and Delta (D) in investigative activities, demonstrating effective teamwork in pursuing criminal cases.

- The energy devoted to enforcing law and order (Ψ_1) is comparable to that of mutual investigations (Ψ_2). This suggests that the foundational collaboration between departments is well established, facilitating smoother and more efficient investigative processes.

Overall, to further enhance crime prevention and resolution outcomes, these departments should prioritize strengthening their partnerships with external agencies.

8. Conclusion

This study introduced and examined the fuzzy geodetic spectrum and fuzzy detour spectrum as novel extensions of classical spectral graph theory to the fuzzy domain. We defined the fuzzy geodetic-Laplacian energy and detour-Laplacian energy, derived their respective expressions, and established upper and lower bounds. Notably, it was shown that for geodetic blocks, the geodetic-Laplacian energy coincides with the standard Laplacian energy, highlighting a structural consistency within fuzzy graph frameworks.

The practical applicability of fuzzy geodetic-Laplacian energy was illustrated through a case analysis of trafficking channels, demonstrating its potential in decision-making contexts such as crime detection and prevention. The findings are consistent with results reported in the existing literature, supporting the validity and relevance of the proposed spectral measures.

Future research may focus on extending these spectral invariants to directed or weighted fuzzy graphs, enhancing robustness under data uncertainty, and applying the proposed energies to infrastructure domains such as water distribution systems and wireless communication networks. Algorithmic development for efficient spectral computation in large fuzzy systems also represents a promising direction.

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