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Mangasarian-Type Higher-Order Duality in Multi-Objective Programming with Complementarity Constraints

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Abstract. This article establishes Mangasarian-type second and higher-order dual formulations for a multi-objective mathematical program with complementarity constraints (MMPCC) by utilizing μ-stationary conditions. Subsequently, we present several duality theorems for the second and higher-order duals of (MMPCC) under υ-bonvexity assumptions. These results are further illustrated with appropriate examples.

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Key Words and Phrases: Multi-objective programming, mathematical programs with complementary constraints, generalized convexity, second-order and higher-order type I duality

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1. Introduction

In the past thirty years, researchers have made a lot of progress in solving multi-objective problems, especially in understanding the conditions that ensure good solutions and in developing related duality theories. These problems are important in areas like psychology, economics, and operations research. What makes them challenging is the need to optimize more than one objective at the same time. Often, improving one goal can worsen another, creating a conflict between the objectives. Because of these challenges, a lot of work has been done to build better theories and tools to deal with such problems (see [1–3] for more information).

Another area that has received attention is mathematical programming with complemen-

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tarity constraints (MPCC). These problems show up in real-world applications like engineering systems and economic models [4]. An MPCC is a kind of optimization problem that has both equality and inequality constraints, and its structure resembles the conditions needed for solving constrained optimization problems. However, MPCCs are tricky because they usually don't satisfy the common assumptions needed to apply standard solution techniques, like the Mangasarian-Fromovitz constraint qualification [5]. Despite these challenges, MPCCs are widely used in practice-for instance, in natural gas network modeling [6], electricity market operations [7], and optimal control tasks [8].

From a theoretical and computational standpoint, MPCCs are hard to solve. The special structure of complementarity constraints means that the usual optimality conditions, like the Karush-Kuhn-Tucker (KKT) conditions, often don't apply directly refer to [9, 10]. Because of this, researchers have proposed new types of conditions tailored to MPCCs e.g., [11–13].

Scheel and Scholtes [14] pointed out that progress in theory and numerical methods for MPCCs go hand in hand. Techniques like using stationarity conditions or penalty functions have helped in solving these problems. For example, Scholtes [15] studied how a sequence of approximations behaves when using a nonlinear programming approach to regularize complementarity constraints. Ahmad et al. [16] introduced a generalized class of higher-order (F, α, ρ, d)-type I functions and developed a Mond–Weir type higher-order dual model for nondifferentiable multiobjective programming problems, establishing several higher-order duality relations. Further, Ahmad et al. [17] proposed a unified higher-order dual formulation for nondifferentiable minimax programming problems, where weak, strong, and strict converse duality theorems were derived using the framework of generalized higher-order (F, α, ρ, d)-type I functions.

Later, Fletcher and Leyffer [18] handled MPCCs by converting them into nonlinear programming (NLP) problems that can be solved with existing tools. Anitescu [19] proposed sufficient conditions under which solutions still exist even when standard assumptions break down. They also showed that using an elastic mode approach can lead to a valid solution under certain penalty settings.

In classical multi-objective optimization, several dual and efficiency frameworks, such as the Wolfe-type and Mond-Weir-type models, have been widely used to characterize optimality under convexity assumptions [20, 21]. These approaches, however, are generally limited to problems without complementarity structures and rely on first or second-order differentiability. Later developments, including the second-order Mangasarian dual model, incorporated quadratic curvature information to tighten the dual bounds. Building on these, the present study introduces Mangasarian-type second and higher-order dual formulations for multi-objective mathematical programs with complementarity constraints (MMPCC), integrating μ-stationarity and υ-bonvexity assumptions to capture both curvature and complementarity effects in a unified framework. This provides a stronger characterization of Pareto-optimal solutions and generalizes existing models.

This paper aims to build on this work by developing Mangasarian-type second and higher-order dual models for a multi-objective problem with complementarity constraints (MMPCC), using μ-stationary conditions. As far as we know, this approach has not yet been studied in the literature.

The structure of the paper is as follows: Section 2 gives the basic definitions. Sections 3 and 4 introduce the second and higher-order dual models for MMPCC and provide duality results. Section 5 explores special cases, and Section 6 concludes the paper.

2. Preliminaries

For $c, \lambda \in \mathbb{R}^n$, we define the following ordering relations for vectors in $\mathbb{R}^n$:

$$\begin{aligned} &c \geqq \lambda \Longleftrightarrow c_p \geqq \lambda_p, \forall p \in 1, 2, ..., n; \\ &c \geq \lambda \Longleftrightarrow c_p \geqq \lambda_p, \forall p \in 1, 2, ..., n, \text{but } c \neq \lambda; \\ &c > \lambda \Longleftrightarrow c_p > \lambda_p, \forall p \in 1, 2, ..., n. \end{aligned}$$

The negation of $c \geq \lambda$ is denoted by $c \ngeq \lambda$. The index sets are $\xi = \{1, 2, ..., f_0\}$, $\mathbb{W} = \{1, 2, ..., g\}$, $\mathbb{D} = \{1, 2, ..., h\}$ and $\mathbb{M} = \{1, 2, ..., m_0\}$, for each $t \in \xi$, $\xi_t = \xi - \{t\}$. Let $\nabla\Psi(\varepsilon)$ and $\nabla^2\Psi(\varepsilon)$ be the gradient and $n \times n$ symmetric Hessian matrix of Ψ at ε. For a given mapping $\psi : R^n \to R^m$, $\nabla\psi(\varepsilon)$ stands for the transposed Jacobian of ψ at ε. For a multi-variable function $\mathbb{H}$, let $\nabla\mathbb{H}$ and $\nabla\mathbb{H}^2$ represent the gradient and the Hessian matrix of $\mathbb{H}$ with respect to the first argument, respectively. Let η^T be the transpose of the vector η, and $\eta \geqq 0$ implies that the vector η is component-wise nonnegative. Given a symmetric matrix $\mathbb{E}$, the matrix $\mathbb{E}$ is positive semi-definite (positive definite) if $\mathbb{E} \succeq 0$ ($\succ 0$). Let $Im(\mathbb{E})$ denote the image of the matrix $\mathbb{E}$. Let $\mathbb{R}^n_+$ be a non-negative orthant of an n-dimensional Euclidean space $\mathbb{R}^n$.

Let us study the below given multi-objective mathematical program involving complementarity constraints:

$$\begin{aligned} \text{(MMPCC)} \quad \min_{\varepsilon \in \mathbb{R}^n} \quad & \Psi(\varepsilon) = (\Psi_1(\varepsilon), \Psi_2(\varepsilon), ..., \Psi_{f_0}(\varepsilon)) \\ & \text{subject to} \\ & \varphi(\varepsilon) \leqq 0, \\ & \Phi(\varepsilon) = 0, \\ & 0 \leqq \Pi(\varepsilon) \perp \Theta(\varepsilon) \geqq 0, \end{aligned}$$

where $\Psi_t : \mathbf{R}^n \to \mathbb{R}$, $t = 1, 2, ..., f_0$, $\varphi : \mathbb{R}^n \to \mathbb{R}^g$, $\Phi : \mathbb{R}^n \to \mathbb{R}^h$ and $\Pi, \Theta : \mathbb{R}^n \to \mathbb{R}^{m_0}$ are twice differentiable functions, and $\Pi(\varepsilon) \perp \Theta(\varepsilon)$ implies that the vectors $\Pi(\varepsilon)$ and $\Theta(\varepsilon)$ are perpendicular namely, $\Pi(\varepsilon)^T\Theta(\varepsilon) = 0$.

Let us study the below given nonlinear multi-objective programming problem:

$$\begin{aligned} \text{(MP)} \quad \min_{\varepsilon \in \mathbb{R}^n} \quad & \Psi(\varepsilon) = (\Psi_1(\varepsilon), \Psi_2(\varepsilon), ..., \Psi_{f_0}(\varepsilon)) \\ & \text{subject to} \end{aligned}$$

$$\varphi(\varepsilon) \leqq 0.$$

where $\Psi_t,\ t = 1, 2, ..., f_0,$ and φ are differentiable functions from $\mathbb{R}^n \to \mathbb{R}$ and $\mathbb{R}^{m_0}$, respectively. From the sub-problems of two algorithms in [22, 23], Mangasarian [24], we consider the below mentioned second-order dual problem for (MP) :

$$\text{(MD)} \quad \max_{\varepsilon,\varsigma,\delta\in\mathbb{R}^n} \Big(\Psi_1(\varepsilon) + \delta^T\varphi(\varepsilon) - \frac{1}{2}\,\varsigma^T[\nabla^2\Psi_1(\varepsilon) + \nabla^2\varphi(\varepsilon)]\varsigma,\ .\ .\ .\ .\ ,$$

$$\Psi_{f_0}(\varepsilon) + \delta^T\varphi(\varepsilon) - \frac{1}{2}\,\varsigma^T[\nabla^2\Psi_{f_0}(\varepsilon) + \nabla^2\varphi(\varepsilon)]\varsigma\Big)$$

subject to

$$\sum_{t=1}^{f_0}\Lambda_t[\nabla\Psi_t(\varepsilon) + \nabla^2\Psi_t(\varepsilon)\varsigma] + [\nabla\delta^T\varphi(\varepsilon) + \nabla^2\delta^T\varphi(\varepsilon)\varsigma] = 0,$$

$$\Lambda_t > 0, \sum_{t=1}^{f_0}\Lambda_t = 1, \delta \geqq 0.$$

On the lines of Mangasarian [24], we present the below given higher-order dual problem for (MP):

$$\text{(HD1)} \quad \max_{\varepsilon,\varsigma,\delta\in\mathbb{R}^n} \Big(\Psi_1(\varepsilon) + \xi_1(\varepsilon,\varsigma) + \delta^T\varphi(\varepsilon) + \delta^T\xi_\varphi(\varepsilon,\varsigma),\ .\ .\ .\ ,$$

$$\Psi_{f_0}(\varepsilon) + \xi_{f_0}(\varepsilon,\varsigma) + \delta^T\varphi(\varepsilon) + \delta^T\xi_\varphi(\varepsilon,\varsigma)\Big)$$

subject to

$$\sum_{t=1}^{f_0}\Lambda_t\nabla_\varsigma\xi_t(\varepsilon,\varsigma) + \nabla_\varsigma\delta^T\xi_\varphi(\varepsilon,\varsigma) = 0,$$

$$\Lambda_t > 0, \sum_{t=1}^{f_0}\Lambda_t = 1, \delta \geqq 0.$$

by employing approximations $\Psi_t(\varepsilon)+\xi_t(\varepsilon,\varsigma)$ to Ψ_t, $t = 1, 2, ..., f_0$, and $\varphi(\varepsilon)+\xi_\varphi(\varepsilon,\varsigma)$ to φ, respectively, where $\xi_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $t = 1, 2, ..., f_0$, $\xi_\varphi = (\xi_{\varphi_1}, ..., \xi_{\varphi_g})^T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^g$ are differentiable functions and $\nabla_\varsigma\xi_t(\varepsilon,\varsigma)$ denotes the gradient of $\xi_t(\varepsilon,\varsigma)$ with respect to ς, and various duality theorems are proved by using some assumptions. Throughout this article, we represent the feasible set of (MMPCC) by $\mathbf{Y}$.

Definition 1. *An element $\varepsilon_0 \in \mathbf{Y}$ is termed as an efficient (Pareto optimal, noninferior, nondominated) solution of (MMPCC), if there exists no other $\varepsilon \in \mathbf{Y}$ such that*

$$\Psi_t(\varepsilon) \leqq \Psi_t(\varepsilon_0),\ \text{for all}\ \ t \in \xi,$$

$$\Psi_\ell(\varepsilon) < \Psi_\ell(\varepsilon_0),\ \text{for atleast one}\ \ell \in \xi_t.$$

For a given feasible solution $\varepsilon^0 \in \mathbf{Y}$, we define the subsequent index sets:

$$\begin{aligned}\mathbf{J}_\varphi(\varepsilon^0) =& \{\jmath : \varphi_\jmath(\varepsilon^0) = 0\},\\ \mathbf{P}(\varepsilon^0) =& \{\jmath : \Pi_\jmath(\varepsilon^0) = 0 < \Theta_\jmath(\varepsilon^0)\},\\ \mathbf{Q}(\varepsilon^0) =& \{\jmath : \Pi_\jmath(\varepsilon^0) = 0 = \Theta_\jmath(\varepsilon^0)\},\\ \mathbf{U}(\varepsilon^0) =& \{\jmath : \Pi_\jmath(\varepsilon^0) > 0 = \Theta_\jmath(\varepsilon^0)\}.\end{aligned}$$

It is obvious that $\mathbf{P}(\varepsilon^0), \mathbf{Q}(\varepsilon^0), \mathbf{U}(\varepsilon^0)$ is a partition of $1, 2, ..., m_0$. The Lagrangian function of (MMPCC) is defined in the following way:

$$\Omega_0(\varepsilon, \Lambda, \delta, \sigma, \psi_a, \psi_b) = \sum_{t=1}^{f_0} \Lambda_t \Psi_t(\varepsilon) + \delta^T \varphi(\varepsilon) + \sigma^T \Phi(\varepsilon) - {\psi_a}^T \Pi(\varepsilon) - {\psi_b}^T \Theta(\varepsilon), \tag{1}$$

$$\nabla\Omega_0(\varepsilon, \Lambda, \delta, \sigma, \psi_a, \psi_b) = \sum_{t=1}^{f_0} \Lambda_t \nabla\Psi_t(\varepsilon) + \nabla\delta^T \varphi(\varepsilon) + \nabla\sigma^T \Phi(\varepsilon) - \nabla{\psi_a}^T \Pi(\varepsilon) - \nabla{\psi_b}^T \Theta(\varepsilon),$$

$$\begin{aligned}\nabla^2\Omega_0(\varepsilon, \Lambda, \delta, \sigma, \psi_a, \psi_b) = \sum_{t=1}^{f_0} \Lambda_t \nabla^2\Psi_t(\varepsilon) + \nabla^2\delta^T \varphi(\varepsilon) + \nabla^2\sigma^T \Phi(\varepsilon) - \nabla^2{\psi_a}^T \Pi(\varepsilon)\\ -\nabla^2{\psi_b}^T \Theta(\varepsilon).\end{aligned}$$

On the lines of Zhang and Lin [25], we recall the μ-stationary criteria for (MMPCC) and definitions of υ-bonvex and higher-order type I functions.

Definition 2. *Let $\varepsilon^* \in \mathbf{Y}$ be a μ-stationary point of (MMPCC). If there exists a multiplier vector $(\Lambda, \delta, \sigma, \psi_a, \psi_b) \in \mathbb{R}^{(f_0)} \times \mathbb{R}^g \times \mathbb{R}^h \times \mathbb{R}^{m_0} \times \mathbb{R}^{m_0}$ satisfying*

$$\begin{aligned}&\sum_{t=1}^{f_0} \Lambda_t \nabla\Psi_t(\varepsilon^*) + \nabla\delta^T \varphi(\varepsilon^*) + \nabla\sigma^T \Phi(\varepsilon^*) - \nabla{\psi_a}^T \Pi(\varepsilon^*) - \nabla{\psi_b}^T \Theta(\varepsilon^*) = 0,\\ &\delta \geqq 0, \qquad \varphi(\varepsilon^*)^T \delta = 0,\\ &(\psi_a)_\jmath = 0, \qquad \jmath \in \mathbf{U}(\varepsilon^*),\\ &(\psi_b)_\jmath = 0, \qquad \jmath \in \mathbf{P}(\varepsilon^*),\\ &(\psi_a)_\jmath = 0,\ (\psi_b)_\jmath = 0, \quad \jmath \in \mathbf{Q}(\varepsilon^*).\end{aligned}$$

Definition 3. ([26]) *A real valued twice-differentiable function $\Psi_t : \mathbf{Y} \to \mathbb{R}$ on an open set $\mathbf{Y} \subseteq \mathbb{R}^n$ is said to be υ-bonvex at $\varpi \in \mathbf{Y}$, if there exists $\upsilon(\varepsilon, \varpi) : \mathbf{Y} \times \mathbf{Y} \to \mathbb{R}^n$ such that for all $\varsigma \in \mathbb{R}^n$,*

$$\Psi_t(\varepsilon) - \Psi_t(\varpi) \geq [\upsilon(\varepsilon, \varpi)]^T [\nabla\Psi_t(\varpi) + \nabla^2\Psi_t(\varpi)\varsigma] - \frac{1}{2}\,\varsigma^T \nabla^2\Psi_t(\varpi)\varsigma,\ \forall\varepsilon \in \mathbf{Y}$$

If the strict inequality holds for all $\varepsilon \neq \varpi$ in the above definition, the function is said to be strictly υ-bonvex.
The following result gives a judgment about υ-bonvex functions.

Proposition 1. ([26]) *A function Ψ is υ-bonvex at ϖ if and only if $\nabla^2\Psi(\varpi) \succeq 0$ and either $\nabla\Psi(\varpi) \notin Im\nabla^2\Psi(\varpi)$, or $\nabla\Psi(\varpi) \in Im\nabla^2\Psi(\varpi)$ and $\Psi(\varepsilon) \geq \Psi(\varpi) - \dfrac{1}{2}\langle \Xi, \nabla^2\Psi(\varpi)\Xi\rangle$ for every $\varepsilon \in \mathbf{Y}$, where $\nabla^2\Psi(\varpi)\Xi = \nabla\Psi(\varpi)$.*

Replacing $\varepsilon \in \mathbf{Y}$ by $\varepsilon \in \mathbf{Y} \setminus \varpi$ and $\geq$ by $>$, respectively, in the above result, one gets the characterization of strictly υ-bonvex functions.
We next give a strictly convex quadratic function that is strictly υ-bonvex and will be used later.

Corollary 1. ([25]) *Suppose that $\Psi : \mathbb{R}^n \to \mathbb{R}$ with $\Psi(\varepsilon) = \dfrac{1}{2}\langle \varepsilon, \mathbb{E}\varepsilon\rangle + \langle b, \varepsilon\rangle + \omega$, where $\mathbb{E} \succ 0$, $b \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$. Then Ψ is υ-bonvex at each $\varepsilon \in \mathbb{R}^n$ and strictly υ-bonvex at every $\varepsilon \in \mathbb{R}^n \setminus \{-\mathbb{E}^{-1}b\}$.*

Definition 4. ([21]) *Let $\Psi(\varepsilon)$ be a differentiable function on an open set $\mathbf{Y} \subseteq \mathbb{R}^n$. The function Ψ is said to be higher-order type I at $\varepsilon^* \in \mathbf{Y}$ if there exists $\upsilon(\varepsilon, \varpi) : \mathbf{Y} \times \mathbf{Y} \to \mathbb{R}^n$ such that for all $\varsigma \in \mathbb{R}^n$ and $\varepsilon \in \mathbf{Y}$,*

$$\Psi(\varepsilon) - \Psi(\varepsilon^*) \geq [\upsilon(\varepsilon, \varepsilon^*)]^T \nabla_\varsigma \xi(\varepsilon^*, \varsigma) + \xi(\varepsilon^*, \varsigma) - \varsigma^T \nabla_\varsigma \xi(\varepsilon^*, \varsigma),$$

where $\Psi(\varepsilon) + \xi(\varepsilon, \varsigma)$ is an approximation of $\Psi(\varepsilon)$. If the strict inequality holds for all $\varepsilon \neq \varpi$ in the above definition, the function is said to be strictly higher-order type I.

3. Mangasarian-type second-order duality

The present section deals with the Mangasarian-type second-order dual problem for (MMPCC) and proves theorems of duality based on υ-bonvexity assumptions. The Mangasarian-type second-order dual problem is a powerful approach in mathematical programming, particularly in multi-objective and complementarity-constrained optimization. It is useful in scenarios where convexity and second-order conditions play a crucial role in stability and optimality.

Example 1. (Application of the Mangasarian-Type Second-Order Dual Problem:) *A multinational retail company operates a supply chain network involving multiple warehouses and retail stores. The objective is to minimize operational costs while maximizing service level across different regions. However, the supply chain system follows market equilibrium constraints, where supply and demand must balance at optimal prices.*
Mathematical Formulation:

$$\text{(MMPCC-1)} \quad \min_{\varepsilon \in \mathbb{R}^n} \quad \Psi(\varepsilon) = (\Psi_1(\varepsilon), \Psi_2(\varepsilon))$$

subject to

$$\varphi(\varepsilon) \leqq 0,$$
$$\Phi(\varepsilon) = 0,$$
$$0 \leqq \Pi(\varepsilon) \perp \Theta(\varepsilon) \geqq 0,$$

where $\Psi_1(\varepsilon)$ represent the total cost of operations (transportation, inventory, and production costs) and $\Psi_2(\varepsilon)$ is the service level index (e.g., customer demand fulfillment). The constraints include:
- *Capacity constraints: $\varphi(\varepsilon) \leqq 0$ (e.g., warehouse limits, transportation limits).*
- *Market equilibrium constraints: $\Phi(\varepsilon) = 0$ (e.g., supply-demand matching).*
- *Complementarity constraints: $0 \leqq \Pi(\varepsilon) \perp \Theta(\varepsilon) \geqq 0$ (e.g., pricing effects on supply-demand interactions).*

Mangasarian-Type Second-Order Dual Formulation

$$\text{(MSD)-1} \quad \max_{(\varpi,\varsigma,\delta,\sigma,\psi_a,\psi_b)} \Big(\Psi_1(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi),$$
$$\Psi_2(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi)\Big)$$
$$-\frac{1}{2}\varsigma^T\Big(\nabla^2\Psi_1(\varpi) + \nabla^2\delta^T\varphi(\varpi) + \nabla^2\sigma^T\Phi(\varpi) - \nabla^2{\psi_a}^T\Pi(\varpi) - \nabla^2{\psi_b}^T\Theta(\varpi),$$
$$\nabla^2\Psi_2(\varpi) + \nabla^2\delta^T\varphi(\varpi) + \nabla^2\sigma^T\Phi(\varpi) - \nabla^2{\psi_a}^T\Pi(\varpi) - \nabla^2{\psi_b}^T\Theta(\varpi)\Big)\varsigma,$$

where ϖ, $\varsigma \in \mathbb{R}^n$, and $\delta \in \mathbb{R}^{m_0}$, $\sigma \in \mathbb{R}^h, \psi_a \in \mathbb{R}^{m_0}$, and $\psi_b \in \mathbb{R}^{m_0}$. This dual problem helps identify optimal supply chain strategies while ensuring that product prices and inventory levels are balanced to avoid excess stock or shortages, with complementarity constraints. Additionally, it considers second-order curvature effects, capturing sensitivity to small changes in inventory, transport, and pricing.

We give the subsequent first-order Wolfe-type dual problem for (MMPCC) along the lines of Guo et al. [20].

$$\text{(MFD)} \quad \max_{(\varepsilon,\delta,\sigma,\psi_a,\psi_b)} \Big(\Psi_1(\varepsilon) + \delta^T\varphi(\varepsilon) + \sigma^T\Phi(\varepsilon) - {\psi_a}^T\Pi(\varepsilon) - {\psi_b}^T\Theta(\varepsilon),$$
$$, \ \ldots, \Psi_{f_0}(\varepsilon) + \delta^T\varphi(\varepsilon) + \sigma^T\Phi(\varepsilon) - {\psi_a}^T\Pi(\varepsilon) - {\psi_b}^T\Theta(\varepsilon)\Big)$$

subject to

$$\sum_{t=1}^{f_0}\Lambda_t\nabla\Psi_t(\varepsilon) + \nabla\delta^T\varphi(\varepsilon) + \nabla\sigma^T\Phi(\varepsilon) - \nabla{\psi_a}^T\Pi(\varepsilon) - \nabla{\psi_b}^T\Theta(\varepsilon) = 0,$$
$$\Lambda > 0, \sum_{t=1}^{f_0}\Lambda_t = 1, \delta \geqq 0, \ \psi_a \geqq 0, \ \psi_b \geqq 0.$$

The second-order dual problem of (MMPCC) on the lines of Guo et al. [20] and Mangasarian [24] is defined as follows:

$$\text{(MSD)} \quad \max_{(\varpi,\varsigma,\delta,\sigma,\psi_a,\psi_b)} \Bigg(\Psi_1(\varpi)+\delta^T\varphi(\varpi)+\sigma^T\Phi(\varpi)-{\psi_a}^T\Pi(\varpi)-{\psi_b}^T\Theta(\varpi),$$

$$, \ \ldots, \Psi_{f_0}(\varpi)+\delta^T\varphi(\varpi)+\sigma^T\Phi(\varpi)-{\psi_a}^T\Pi(\varpi)-{\psi_b}^T\Theta(\varpi)\Bigg)$$

$$-\frac{1}{2}\,\varsigma^T\Bigg(\nabla^2\Psi_1(\varpi)+\nabla^2\delta^T\varphi(\varpi)+\nabla^2\sigma^T\Phi(\varpi)-\nabla^2{\psi_a}^T\Pi(\varpi)-\nabla^2{\psi_b}^T\Theta(\varpi)$$

$$, \ \ldots, \nabla^2\Psi_{f_0}(\varpi)+\nabla^2\delta^T\varphi(\varpi)+\nabla^2\sigma^T\Phi(\varpi)-\nabla^2{\psi_a}^T\Pi(\varpi)-\nabla^2{\psi_b}^T\Theta(\varpi)\Bigg)\varsigma$$

subject to

$$\sum_{t=1}^{f_0}\Lambda_t\Big(\nabla\Psi_t(\varpi)+\nabla^2\Psi_t(\varpi)\varsigma\Big)+\nabla\delta^T\varphi(\varpi)+\nabla^2\delta^T\varphi(\varpi)\varsigma+\nabla\sigma^T\Phi(\varpi)+\nabla^2\sigma^T\Phi(\varpi)\varsigma$$

$$-\nabla{\psi_a}^T\Pi(\varpi)-\nabla^2{\psi_a}^T\Pi(\varpi)\varsigma-\nabla{\psi_b}^T\Theta(\varpi)-\nabla^2{\psi_b}^T\Theta(\varpi)\varsigma=0, \tag{2}$$

$$\Lambda>0,\sum_{t=1}^{f_0}\Lambda_t=1,\ \delta\geqq 0,\ \psi_a\geqq 0,\ \psi_b\geqq 0,$$

where $\varpi,\varsigma\in\mathbb{R}^n$, $t\in\xi$ and $\delta\in\mathbb{R}^{m_0}$. We represent the feasible region of (MSD) by $\mathbb{F}_{MSD}$. Let $\mathbb{F}_{MSD}(\varpi)=\{\varpi:(\varpi,\varsigma,\Lambda,\delta,\sigma,\psi_a,\psi_b)\in\mathbb{F}_{MSD}\}$. Observe that the above duality problem is associated to the μ-stationary criteria of (MMPCC). The basic notion in the proof of weak duality result is similar to Theorem 1 in Guo et al. [20]

Theorem 1. (Weak duality) *Let ε and $(\varpi,\varsigma,\Lambda,\delta,\sigma,\psi_a,\psi_b)$ be feasible solutions for (MMPCC) and (MSD), respectively. Suppose that functions $\Psi_t, t\in\xi$, $\varphi_\jmath, \jmath\in\mathbb{W}$, Φ_ν, $-\Phi_\nu,\nu\in\mathbb{D}$, $-\Pi_\gamma$ and $-\Theta_\gamma,\gamma\in\mathbb{M}$ are υ-bonvex functions at ϖ. Then, the following cannot hold:*

$$\Psi_t(\varepsilon)\leqq\Psi_t(\varpi)+\delta^T\varphi(\varpi)+\sigma^T\Phi(\varpi)-{\psi_a}^T\Pi(\varpi)-{\psi_b}^T\Theta(\varpi)$$

$$-\frac{1}{2}\,\varsigma^T\Big(\nabla^2\Psi_t(\varpi)+\nabla^2\delta^T\varphi(\varpi)+\nabla^2\sigma^T\Phi(\varpi)-\nabla^2{\psi_a}^T\Pi(\varpi)-\nabla^2{\psi_b}^T\Theta(\varpi)\Big)\varsigma,\ \text{for all } t\in\xi, \tag{3}$$

and

$$\Psi_\ell(\varepsilon)<\Psi_\ell(\varpi)+\delta^T\varphi(\varpi)+\sigma^T\Phi(\varpi)-{\psi_a}^T\Pi(\varpi)-{\psi_b}^T\Theta(\varpi)$$

$$-\frac{1}{2}\,\varsigma^T\Big(\nabla^2\Psi_t(\varpi)+\nabla^2\delta^T\varphi(\varpi)+\nabla^2\sigma^T\Phi(\varpi)-\nabla^2{\psi_a}^T\Pi(\varpi)-\nabla^2{\psi_b}^T\Theta(\varpi)\Big)\varsigma, \tag{4}$$

for atleast one $\ell\in\xi_t$.

Proof. Suppose to the contrary that (3) and (4) hold, that is,

$$\Psi_t(\varepsilon) \leqq \Psi_t(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi)$$

$$-\frac{1}{2}\,\varsigma^T\Big(\nabla^2\Psi_t(\varpi)+\nabla^2\delta^T\varphi(\varpi)+\nabla^2\sigma^T\Phi(\varpi)-\nabla^2{\psi_a}^T\Pi(\varpi)-\nabla^2{\psi_b}^T\Theta(\varpi)\Big)\varsigma,\ \text{for all } t\in\xi,$$

and

$$\Psi_\ell(\varepsilon) < \Psi_\ell(\varpi)) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi)$$

$$-\frac{1}{2}\,\varsigma^T\Big(\nabla^2\Psi_t(\varpi)+\nabla^2\delta^T\varphi(\varpi)+\nabla^2\sigma^T\Phi(\varpi)-\nabla^2{\psi_a}^T\Pi(\varpi)-\nabla^2{\psi_b}^T\Theta(\varpi)\Big)\varsigma,\ \text{for atleast one } \ell\in\xi_t.$$

Because $\Lambda > 0$ and $\sum_{t=1}^{f_0}\Lambda_t = 1$, the above inequalities yield

$$\sum_{t=1}^{f_0}\Lambda_t\Psi_t(\varepsilon) < \sum_{t=1}^{f_0}\Big(\Lambda_t\Psi_t(\varpi) - \frac{1}{2}\,\varsigma^T\nabla^2\Psi_t(\varpi)\varsigma\Big) + \delta^T\varphi(\varpi) - \frac{1}{2}\,\varsigma^T\nabla^2\delta^T\varphi(\varpi)\varsigma + \sigma^T\Phi(\varpi) -$$

$$\frac{1}{2}\,\varsigma^T\nabla^2\sigma^T\Phi(\varpi))\varsigma - {\psi_a}^T\Pi(\varpi) + \frac{1}{2}\,\varsigma^T\nabla^2{\psi_a}^T\Pi(\varpi)\varsigma - {\psi_b}^T\Theta(\varpi) - \frac{1}{2}\,\varsigma^T\nabla^2{\psi_b}^T\Theta(\varpi)\varsigma. \quad (5)$$

We introduce the index sets in the following manner:

$$\begin{aligned}
&\Gamma^+ = \{\nu : \sigma_\nu \geq 0\}, && \Gamma^- = \{\nu : \sigma_\nu \leq 0\},\\
&\vartheta^+ = \{\gamma : (\psi_a)_\gamma > 0, (\psi_b)_\gamma > 0\}, && \vartheta^0 = \{\gamma : (\psi_a)_\gamma = 0, (\psi_b)_\gamma = 0\},\\
&\vartheta^+_\Theta = \{\gamma : (\psi_a)_\gamma = 0, (\psi_b)_\gamma > 0\}, && \vartheta^+_\Pi = \{\gamma : (\psi_a)_\gamma > 0, (\psi_b)_\gamma = 0\}.
\end{aligned}$$

Since Ψ, $\varphi_\jmath$, Φ_ν, $-\Phi_\nu$, $-\Pi_\gamma$ and $-\Theta_\gamma$ are υ-bonvex functions at ϖ, for any $\varepsilon \in \mathbf{Y}$ and $\varpi \in \mathbb{F}_{MSD}(\varpi)$, we have

$$\Psi_t(\varepsilon) - \Psi_t(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T[\nabla\Psi_t(\varpi) + \nabla^2\Psi_t(\varpi)\varsigma] - \frac{1}{2}\,\varsigma^T\nabla^2\Psi_t(\varpi)\varsigma, \quad (6)$$

$$\varphi_\jmath(\varepsilon) - \varphi_\jmath(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T[\nabla\varphi_\jmath(\varpi) + \nabla^2\varphi_\jmath(\varpi)\varsigma] - \frac{1}{2}\,\varsigma^T\nabla^2\varphi_\jmath(\varpi)\varsigma,\ \forall\jmath\in\{1,2,...,g\}, \quad (7)$$

$$\Phi_\nu(\varepsilon) - \Phi_\nu(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T[\nabla\Phi_\nu(\varpi) + \nabla^2\Phi_\nu(\varpi)\varsigma] - \frac{1}{2}\,\varsigma^T\nabla^2\Phi_\nu(\varpi)\varsigma,\ \forall\nu\in\Gamma^+, \quad (8)$$

$$-\Phi_\nu(\varepsilon) + \Phi_\nu(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T[-\nabla\Phi_\nu(\varpi) - \nabla^2\Phi_\nu(\varpi)\varsigma] + \frac{1}{2}\,\varsigma^T\nabla^2\Phi_\nu(\varpi)\varsigma,\ \forall\nu\in\Gamma^-, \quad (9)$$

$$-\Pi_\gamma(\varepsilon) + \Pi_\gamma(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T[-\nabla\Pi_\gamma(\varpi) - \nabla^2\Pi_\gamma(\varpi)\varsigma] + \frac{1}{2}\,\varsigma^T\nabla^2\Pi_\gamma(\varpi)\varsigma,\ \forall\gamma\in\vartheta^+\cup\vartheta^+_\Pi, \quad (10)$$

$$-\Theta_\gamma(\varepsilon) + \Theta_\gamma(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T[-\nabla\Theta_\gamma(\varpi) - \nabla^2\Theta_\gamma(\varpi)\varsigma] + \frac{1}{2}\,\varsigma^T\nabla^2\Theta_\gamma(\varpi)\varsigma,\ \forall\gamma\in\vartheta^+\cup\vartheta^+_\Theta. \quad (11)$$

Multiplying inequalities (6)–(11) by $\Lambda_t > 0$, $t \in \xi$, $\delta_\jmath(\jmath \in \{1,2,...,g\})$, $\sigma_\nu(\nu \in \Gamma^+)$, $-\sigma_\nu(\nu \in \Gamma^-)$, $(\psi_a)_\gamma(\gamma \in \vartheta^+ \cup \vartheta^+_\Pi)$, and $(\psi_b)_\gamma(\gamma \in \vartheta^+ \cup \vartheta^+_\Theta)$, respectively, and adding, then we obtain, the subsequent inequality

$$\sum_{t=1}^{f_0} \Lambda_t \bigg(\Psi_t(\varepsilon) - \Psi_t(\varpi) + \frac{1}{2}\varsigma^T \nabla^2 \Psi_t(\varpi)\varsigma \bigg) + \delta^T \varphi(\varepsilon) - \delta^T \varphi(\varpi) + \frac{1}{2}\varsigma^T \nabla^2 \delta^T \varphi(\varpi)\varsigma$$

$$\sigma^T \Phi(\varepsilon) - \sigma^T \Phi(\varpi) + \frac{1}{2}\varsigma^T \nabla^2 \sigma^T \Phi(\varpi)\varsigma - {\psi_a}^T \Pi(\varepsilon) + {\psi_a}^T \Pi(\varpi) - \frac{1}{2}\varsigma^T \nabla^2 {\psi_a}^T \Pi(\varpi)\varsigma - {\psi_b}^T \Theta(\varpi)$$

$$+{\psi_b}^T \Theta(\varpi) - \frac{1}{2}\varsigma^T \nabla^2 {\psi_b}^T \Theta(\varpi)\varsigma \geqq [\upsilon(\varepsilon,\varpi)]^T \Bigg[\sum_{t=1}^{f_0} \Lambda_t \bigg(\nabla \Psi_t(\varpi) + \nabla^2 \Psi_t(\varpi)\varsigma \bigg) + \nabla \delta^T \varphi(\varpi)$$

$$+\nabla^2 \delta^T \varphi(\varpi)\varsigma + \nabla \sigma^T \Phi(\varpi) + \nabla^2 \sigma^T \Phi(\varpi)\varsigma - \nabla {\psi_a}^T \Pi(\varpi) - \nabla^2 {\psi_a}^T \Pi(\varpi)\varsigma$$

$$-\nabla {\psi_b}^T \Theta(\varpi) - \nabla^2 {\psi_b}^T \Theta(\varpi)\varsigma \Bigg]. \tag{12}$$

Since ε and $(\varpi, \varsigma, \delta, \sigma, \psi_a, \psi_b)$ are feasible solutions for (MMPCC) and (MSD), respectively, we imply that $\varphi(\varepsilon) \leqq 0$, $\Phi(\varepsilon) = 0$, $\Pi(\varepsilon) \geqq 0$, $\Theta(\varepsilon) \geqq 0$, $\delta \geqq 0$, $\psi_a \geqq 0$, $\psi_b \geqq 0$, whence

$$\delta^T \varphi(\varepsilon) + \sigma^T \Phi(\varepsilon) - {\psi_a}^T \Pi(\varepsilon) - {\psi_b}^T \Theta(\varepsilon) \leqq 0. \tag{13}$$

The inequality (12), in view of (5) and (13), gives

$$\sum_{t=1}^{f_0} \Lambda_t \bigg(\nabla \Psi_t(\varpi) + \nabla^2 \Psi_t(\varpi)\varsigma \bigg) + \nabla \delta^T \varphi(\varpi) + \nabla^2 \delta^T \varphi(\varpi)\varsigma + \nabla \sigma^T \Phi(\varpi) +$$

$$\nabla^2 \sigma^T \Phi(\varpi)\varsigma - \nabla {\psi_a}^T \Pi(\varpi) - \nabla^2 {\psi_a}^T \Pi(\varpi)\varsigma - \nabla {\psi_b}^T \Theta(\varpi) - \nabla^2 {\psi_b}^T \Theta(\varpi)\varsigma < 0,$$

which is a contradiction to (2). Thus, the proof of the theorem is validated.

Now, we provide an algorithm for the Mangasarian-type second-order weak duality theorem in the following way.

Algorithm 1: Algorithm of Mangasarian-type second-order weak duality

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Input:

- Objective function of the primal problem:

$$\min_{\varepsilon \in \mathbb{R}^n} \quad \Psi(\varepsilon) = (\Psi_1(\varepsilon), \Psi_2(\varepsilon), ..., \Psi_s(\varepsilon))$$

- Set of constraints of the primal problem:

$$\varphi(\varepsilon) \leqq 0,$$

$$\Phi(\varepsilon) = 0,$$
$$0 \leqq \Pi(\varepsilon) \perp \Theta(\varepsilon) \geqq 0.$$

• Feasible point for primal problem:

$$\mathbf{Y} = \{\varepsilon \in \mathbb{R}^n : \varphi(\varepsilon) \leqq 0, \Phi(\varepsilon) = 0, 0 \leqq \Pi(\varepsilon) \perp \Theta(\varepsilon) \geqq 0\}.$$

• Set of self data of the primal problem:

$$\Psi_t : \mathbf{R}^n \to \mathbb{R}, t = 1, 2, ..., f_0, \varphi : \mathbb{R}^n \to \mathbb{R}^g, \Phi : \mathbb{R}^n \to \mathbb{R}^h \text{ and } \Pi, \Theta : \mathbb{R}^n \to \mathbb{R}^{m_0}$$

are twice differentiable functions, and $\Pi(\varepsilon) \perp \Theta(\varepsilon)$ implies that the vectors $\Pi(\varepsilon)$ and $\Theta(\varepsilon)$ are perpendicular namely, $\Pi(\varepsilon)^T \Theta(\varepsilon) = 0$.

• Objective function of the Mangasarian-type second-order dual problem:

$$\max_{(\varpi,\varsigma,\delta,\sigma,\psi_a,\psi_b)} \Bigg(\Psi_1(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi),$$
$$, \ . \ . \ . \ , \Psi_{f_0}(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi)\Bigg)$$
$$-\frac{1}{2}\varsigma^T\Bigg(\nabla^2\Psi_1(\varpi) + \nabla^2\delta^T\varphi(\varpi) + \nabla^2\sigma^T\Phi(\varpi) - \nabla^2{\psi_a}^T\Pi(\varpi) - \nabla^2{\psi_b}^T\Theta(\varpi)$$
$$, \ . \ . \ . \ , \nabla^2\Psi_{f_0}(\varpi) + \nabla^2\delta^T\varphi(\varpi) + \nabla^2\sigma^T\Phi(\varpi) - \nabla^2{\psi_a}^T\Pi(\varpi) - \nabla^2{\psi_b}^T\Theta(\varpi)\Bigg)\varsigma.$$

• Set of constraints of the Mangasarian-type second-order dual problem:

$$\sum_{t=1}^{f_0}\Lambda_t\Bigg(\nabla\Psi_t(\varpi) + \nabla^2\Psi_t(\varpi)\varsigma\Bigg) + \nabla\delta^T\varphi(\varpi) + \nabla^2\delta^T\varphi(\varpi)\varsigma + \nabla\sigma^T\Phi(\varpi) + \nabla^2\sigma^T\Phi(\varpi)\varsigma$$
$$-\nabla{\psi_a}^T\Pi(\varpi) - \nabla^2{\psi_a}^T\Pi(\varpi)\varsigma - \nabla{\psi_b}^T\Theta(\varpi) - \nabla^2{\psi_b}^T\Theta(\varpi)\varsigma = 0,$$
$$\Lambda > 0, \sum_{t=1}^{f_0}\Lambda_t = 1, \ \delta \geqq 0, \ \psi_a \geqq 0, \ \psi_b \geqq 0,$$

where ϖ, $\varsigma \in \mathbb{R}^n$, $t \in \xi$ and $\delta \in \mathbb{R}^{m_0}$.

• Feasible point for Mangasarian-type second-order dual problem:

$$\mathbb{F}_{MSD}(\varpi) = \{\varpi : (\varpi, \varsigma, \Lambda, \delta, \sigma, \psi_a, \psi_b) \in \mathbb{F}_{MSD}\}.$$

• Set of self data of the Mangasarian-type second-order dual problem:
$\Psi_t, t \in \xi$, $\varphi_\jmath, \jmath \in \mathbb{W}$, Φ_ν, $-\Phi_\nu, \nu \in \mathbb{D}$, $-\Pi_\gamma$ and $-\Theta_\gamma, \gamma \in \mathbb{M}$ are υ-bonvex functions at ϖ.

• Primal objective function:

$$\Psi_1(\varepsilon), \Psi_2(\varepsilon), \ . \ . \ . \ , \Psi_s(\varepsilon).$$

- Mangasarian-type second-order dual objective function:

$$\Psi_1(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi)$$

$$, \ . \ . \ . \ , \Psi_{f_0}(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi).$$

- Mangasarian-type second-order weak duality condition:

$$\Psi_t(\varepsilon) \leqq \Psi_t(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi)$$

$$-\frac{1}{2}\,\varsigma^T\left(\nabla^2\Psi_t(\varpi) + \nabla^2\delta^T\varphi(\varpi) + \nabla^2\sigma^T\Phi(\varpi) - \nabla^2{\psi_a}^T\Pi(\varpi) - \nabla^2{\psi_b}^T\Theta(\varpi)\right)\varsigma, \text{ for all } t \in \xi$$

and

$$\Psi_\ell(\varepsilon) < \Psi_\ell(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi)$$

$$-\frac{1}{2}\,\varsigma^T\left(\nabla^2\Psi_t(\varpi) + \nabla^2\delta^T\varphi(\varpi) + \nabla^2\sigma^T\Phi(\varpi) - \nabla^2{\psi_a}^T\Pi(\varpi) - \nabla^2{\psi_b}^T\Theta(\varpi)\right)\varsigma,$$

for atleast one $\ell \in \xi_t$ cannot hold.

Output:

Mangasarian-type second-order weak duality holds if the Mangasarian-type second-order weak duality condition holds: Mangasarian-type second-order weak duality does not hold if the Mangasarian-type second-order weak duality condition does not holds:

Begin

- Selection Stage: Select feasible points ε and $(\varpi, \varsigma, \Lambda, \delta, \sigma, \psi_a, \psi_b)$
  of the problems (MMPC) and (MSD) ;
  if $\Psi_t, t \in \xi, {\varphi_0}^{\mathbb{U}}, \varphi_\jmath, \jmath \in \mathbb{W}, -\Phi_\nu, \nu \in \mathbb{D}, -\Pi_\gamma, -\Theta_\gamma, \gamma \in \mathbb{M},$ are $\upsilon-$
  bonvex at ϖ;
  then Proceed to Screening Stage;
  else Stop;
  end if

- Screening Stage: detecting twice differentiable functions on $\mathbb{R}^n$
  if the self data is satisfied;
  then Proceed to Conclusive Stage;
  else Stop;
  end if

- Conclusive Stage: **if** $\Psi_t(\varepsilon) \leqq \Psi_t(\varpi) + \delta^T\varphi(\varpi) + \sigma^T\Phi(\varpi) - {\psi_a}^T\Pi(\varpi) - {\psi_b}^T\Theta(\varpi)$

$$-\frac{1}{2}\varsigma^T\Big(\nabla^2\Psi_t(\varpi)+\nabla^2\delta^T\varphi(\varpi)+\nabla^2\sigma^T\Phi(\varpi)-\nabla^2{\psi_a}^T\Pi(\varpi)-$$

$$\nabla^2{\psi_b}^T\Theta(\varpi)\Big)\varsigma,\ \text{for all } t\in\xi \text{ and}$$

$$\Psi_\ell(\varepsilon)<\Psi_\ell(\varpi)+\delta^T\varphi(\varpi)+\sigma^T\Phi(\varpi)-{\psi_a}^T\Pi(\varpi)-{\psi_b}^T\Theta(\varpi)$$

$$-\frac{1}{2}\varsigma^T\Big(\nabla^2\Psi_t(\varpi)+\nabla^2\delta^T\varphi(\varpi)+\nabla^2\sigma^T\Phi(\varpi)$$

$$-\nabla^2{\psi_a}^T\Pi(\varpi)-\nabla^2{\psi_b}^T\Theta(\varpi)\Big)\varsigma,\ \text{for atleast one } \ell\in\xi_t \text{ cannot hold ;}$$

then Mangasarian-type second-order weak duality holds;
else Stop;
end if

End

If a point ε satisfies the Linear Independence Constraint Qualification (LICQ), then the Kuhn–Tucker conditions are necessary for ε to be an efficient solution of problem (MMPCC) [14]. In the following, we derive the *Strong Duality Theorem* for the proposed model by utilizing these Kuhn–Tucker conditions, which characterize the necessary optimality criteria for an efficient solution of problem (MMPCC).

Theorem 2. (Strong duality) *Let ε be an efficient solution of (MMPCC) which satisfies the Kuhn-Tucker conditions with the multiplier $(\Lambda,\delta,\sigma,\psi_a,\psi_b)$ and $(\psi_a,\psi_b)\geq 0$. Suppose that functions $\Psi_t(t=1,...,f_0)$, $\varphi_\jmath(\jmath=1,...,g)$, Φ_ν, $-\Phi_\nu(\nu=1,...,h)$, $-\Pi_\gamma$ and $-\Theta_\gamma(\gamma=1,...,m_0)$ are υ-bonvex functions at $\varpi\in\mathbb{F}_{MSD}(\varpi)$. Then $(\varepsilon,\varsigma=0,\Lambda,\delta,\sigma,\psi_a,\psi_b)$ is a globally efficient solution of (MSD) and the objective values of $(MMPCC)$ and (MSD) are equal.*

Proof. By the assumption, we get the following system:

$$\sum_{t=1}^{f_0}\Lambda_t\nabla\Psi_t(\varepsilon)+\nabla\delta^T\varphi(\varepsilon)+\nabla\sigma^T\Phi(\varepsilon)-\nabla{\psi_a}^T\Pi(\varepsilon)-\nabla{\psi_b}^T\Theta(\varepsilon)=0, \tag{14}$$

$$\delta^T\varphi(\varepsilon)=0,{\psi_a}^T\Pi(\varepsilon)=0,{\psi_b}^T\Theta(\varepsilon)=0, \tag{15}$$

$$\Lambda>0,\sum_{t=1}^{f_0}\Lambda_t=1,\delta\geqq 0,\psi_a\geqq 0,\psi_b\geqq 0. \tag{16}$$

From (14) and (16), it follows that $(\varepsilon,\varsigma=0,\Lambda,\delta,\sigma,\psi_a,\psi_b)$ is a feasible solution of (MSD). By using equation (15) together with $\varsigma=0$ and $\Phi_\nu(\varepsilon)=0,\nu\in\{1,2,...,d\}$ we obtain that the objective values of (MPCC) and (MSD) are equal. Thus $(\varepsilon, \varsigma = 0, \Lambda, \delta, \sigma, \psi_a, \psi_b)$ is a globally efficient solution of (MSD) by Theorem 1.

Theorem 3. (Strict converse duality) *Let ε^* and $(\varpi^*, \varsigma^*, \Lambda^*, \delta^*, \sigma^*, {\psi_a}^*, {\psi_b}^*)$ be feasible solutions for (MMPCC) and (MSD), respectively, such that*

$$\sum_{t=1}^{f_0} \Lambda_t^* \Psi_t(\varepsilon^*) \leqq \sum_{t=1}^{f_0} \left(\Lambda_t^* \Psi_t(\varpi^*) - \frac{1}{2} \varsigma^{*T} \nabla^2 \Psi_t(\varpi^*) \varsigma^* \right) + \delta^{*T} \varphi(\varpi^*) - \frac{1}{2} \varsigma^{*T} \nabla^2 \delta^{*T} \varphi(\varpi^*) \varsigma^*$$

$$+\sigma^{*T} \Phi(\varpi^*) - \frac{1}{2} \varsigma^{*T} \nabla^2 \sigma^{*T} \Phi(\varpi^*) \varsigma^* - \psi_a^{*T} \Pi(\varpi^*) + \frac{1}{2} \varsigma^{*T} \nabla^2 \psi_a^{*T} \Pi(\varpi^*) \varsigma^* - \psi_b^{*T} \Theta(\varpi^*)$$

$$-\frac{1}{2} \varsigma^{*T} \nabla^2 \psi_b^{*T} \Theta(\varpi^*) \varsigma^*. \tag{17}$$

Suppose that $\Psi_t, t \in \xi$ is strictly bonvex function at ϖ^*, $\varphi_\jmath, \jmath \in \mathbb{W}$, Φ_ν, $-\Phi_\nu, \nu \in \mathbb{D}$, $-\Pi_\gamma$ and $-\Theta_\gamma, \gamma \in \mathbb{M}$ are υ-bonvex functions at ϖ^*, then $\varepsilon^* = \varpi^*$.

Proof. We assume that $\varepsilon^* \neq \varpi^*$ and exhibit a contradiction. The strict υ-bonvexity of $\Psi_t, t \in \xi$ at ϖ^* and υ-bonvexity of $\varphi_\jmath, \jmath \in \mathbb{W}$, Φ_ν, $-\Phi_\nu, \nu \in \mathbb{D}$, $-\Pi_\gamma$ and $-\Theta_\gamma, \gamma \in \mathbb{M}$ at ϖ^* from Weak duality Theorem 1, imply

$$\Psi_t(\varepsilon^*) - \Psi_t(\varpi^*) > [\upsilon(\varepsilon^*, \varpi^*)]^T [\nabla \Psi_t(\varpi^*) + \nabla^2 \Psi_t(\varpi^*) \varsigma^*] - \frac{1}{2} \varsigma^{*T} \nabla^2 \Psi_t(\varpi^*) \varsigma^*, \tag{18}$$

$$\varphi_\jmath(\varepsilon^*) - \varphi_\jmath(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T [\nabla \varphi_\jmath(\varpi^*) + \nabla^2 \varphi_\jmath(\varpi^*) \varsigma^*] - \frac{1}{2} \varsigma^{*T} \nabla^2 \varphi_\jmath(\varpi^*) \varsigma^*, \ \forall \jmath \in \{1, 2, ..., g\}, \tag{19}$$

$$\Phi_\nu(\varepsilon^*) - \Phi_\nu(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T [\nabla \Phi_\nu(\varpi^*) + \nabla^2 \Phi_\nu(\varpi^*) \varsigma^*] - \frac{1}{2} \varsigma^{*T} \nabla^2 \Phi_\nu(\varpi^*) \varsigma^*, \ \forall \nu \in \Gamma^+, \tag{20}$$

$$-\Phi_\nu(\varepsilon^*) + \Phi_\nu(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T [-\nabla \Phi_\nu(\varpi^*) - \nabla^2 \Phi_\nu(\varpi^*) \varsigma^*] + \frac{1}{2} \varsigma^{*T} \nabla^2 \Phi_\nu(\varpi^*) \varsigma^*, \ \forall \nu \in \Gamma^-, \tag{21}$$

$$-\Pi_\gamma(\varepsilon^*) + \Pi_\gamma(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T [-\nabla \Pi_\gamma(\varpi^*) - \nabla^2 \Pi_\gamma(\varpi^*) \varsigma^*] + \frac{1}{2} \varsigma^{*T} \nabla^2 \Pi_\gamma(\varpi^*) \varsigma^*,$$
$$\forall \nu \in \vartheta^+ \cup \vartheta^+_\Pi, \tag{22}$$

$$-\Theta_\gamma(\varepsilon^*) + \Theta_\gamma(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T [-\nabla \Theta_\gamma(\varpi^*) - \nabla^2 \Theta_\gamma(\varpi^*) \varsigma^*] + \frac{1}{2} \varsigma^{*T} \nabla^2 \Theta_\gamma(\varpi^*) \varsigma^*,$$
$$\forall \nu \in \vartheta^+ \cup \vartheta^+_\Theta. \tag{23}$$

Multiplying inequalities (18)–(23) by $\Lambda_t^* > 0$, $t \in \xi$, $\delta_\jmath^*(\jmath \in \{1, 2, ..., g\})$, $\sigma_\nu^*(\nu \in \Gamma^+)$, -$\sigma_\nu^*(\nu \in \Gamma^-)$, $(\psi_a)_\gamma^*(\gamma \in \vartheta^+ \cup \vartheta^+_\Pi)$, and $(\psi_b)_\gamma^*(\gamma \in \vartheta^+ \cup \vartheta^+_\Theta)$, respectively, and adding, we then get the following inequality

$$\sum_{t=1}^{f_0} \Lambda_t^* \Psi_t(\varepsilon^*) - \sum_{t=1}^{f_0} \Lambda_t^* \left(\Psi_t(\varpi^*) - \frac{1}{2} \varsigma^{*T} \nabla^2 \Psi_t(\varpi^*) \varsigma^* \right) + (\delta^*)^T \varphi(\varepsilon^*) - (\delta^*)^T \varphi(\varpi^*) +$$

$$\frac{1}{2}\,\varsigma^{*T}\nabla^2(\delta^*)^T\varphi(\varpi^*)\varsigma^* + (\sigma^*)^T\Phi(\varepsilon^*) - (\sigma^*)^T\Phi(\varpi^*) + \frac{1}{2}\,\varsigma^{*T}\nabla^2(\sigma^{*T})\Phi(\varpi^*)\varsigma^*$$

$$-({\psi_a}^*)^T\Pi(\varepsilon^*) + ({\psi_a}^*)^T\Pi(\varpi^*) - \frac{1}{2}\,\varsigma^{*T}\nabla^2({\psi_a}^*)^T\Pi(\varpi^*)\varsigma^* - ({\psi_b}^*)^T\Theta(\varepsilon^*) + ({\psi_b}^*)^T\Theta(\varpi^*)$$

$$-\frac{1}{2}\,\varsigma^{*T}\nabla^2({\psi_b}^*)^T\Theta(\varpi^*)\varsigma^* > [\upsilon(\varepsilon^*,\varpi^*)]^T\Bigg[\sum_{t=1}^{f_0}\Lambda_t^*\Big(\nabla\Psi_t(\varpi^*) + \nabla^2\Psi_t(\varpi^*)\varsigma^*\Big)+$$

$$\nabla(\delta^*)^T\varphi(\varpi^*) + \nabla^2(\delta^*)^T\varphi(\varpi^*)\varsigma^* + \nabla(\sigma^*)^T\Phi(\varpi^*) + \nabla^2(\sigma^*)^T\Phi(\varpi^*)\varsigma^* -$$

$$\nabla({\psi_a}^*)^T\Pi(\varpi^*) - \nabla^2({\psi_a}^*)^T\Pi(\varpi^*)\varsigma^* - \nabla({\psi_b}^*)^T\Theta(\varpi^*) - \nabla^2({\psi_b}^*)^T\Theta(\varpi^*)\varsigma^*\Bigg] \tag{24}$$

Since ε^* and $(\varpi^*, \varsigma^*, \Lambda^*, \delta^*, \sigma^*, {\psi_a}^*, {\psi_b}^*)$ are feasible solutions for (MMPCC) and (MSD), respectively, we imply that $\varphi(\varepsilon^*) \leqq 0$, $\Phi(\varepsilon^*) = 0$, $\Pi(\varepsilon^*) \geqq 0$, $\Theta(\varepsilon^*) \geqq 0$, $\delta^* \geqq 0$, ${\psi_a}^* \geqq 0$, ${\psi_b}^* \geqq 0$, hence

$$(\delta^*)^T\varphi(\varepsilon^*) + (\sigma^*)^T\Phi(\varepsilon^*) - ({\psi_a}^*)^T\Pi(\varepsilon^*) - ({\psi_b}^*)^T\Theta(\varepsilon^*) \leqq 0. \tag{25}$$

The inequality (24) on using (2) and (25), gives

$$\sum_{t=1}^{f_0}\Lambda_t^*\Psi_t(\varepsilon^*) > \sum_{t=1}^{f_0}\Big(\Lambda_t^*\Psi_t(\varpi^*) - \frac{1}{2}\,\varsigma^{*T}\nabla^2\Psi_t(\varpi^*)\varsigma^*\Big) + (\delta^*)^T\varphi(\varpi^*)$$

$$-\frac{1}{2}\,\varsigma^{*T}\nabla^2(\delta^*)^T\varphi(\varpi^*)\varsigma^* + (\sigma^*)^T\Phi(\varpi^*) - \frac{1}{2}\,\varsigma^{*T}\nabla^2(\sigma^*)^T\Phi(\varpi^*)\varsigma^* - (\psi_a^*)^T\Pi(\varpi^*)$$

$$+\frac{1}{2}\,\varsigma^{*T}\nabla^2(\psi_a^*)^T\Pi(\varpi^*)\varsigma^* - (\psi_b^*)^T\Theta(\varpi^*) - \frac{1}{2}\,\varsigma^{*T}\nabla^2(\psi_b^*)^T\Theta(\varpi^*)\varsigma^*, \tag{26}$$

which is a contradiction to (17). Hence, $\varepsilon^* = \varpi^*$.

The Theorems 1-3 are justified with an illustration mentioned below.

Example 2. *Consider the problem:*

$$\text{(P1)}\quad \min_{\varepsilon\in\mathbb{R}^n}\quad (\Psi_1(\varepsilon), \Psi_2(\varepsilon)) = \Big[(\varepsilon_1-1)^2 + (\varepsilon_2+1)^2, (\varepsilon_1-3)^2 + (\varepsilon_2+3)^2\Big]$$

subject to

$$\Phi(\varepsilon) = \varepsilon_1 + \varepsilon_2 - 1 = 0,$$
$$\Pi(\varepsilon) = -(\varepsilon_1-1)^2 + \varepsilon_2 \geq 0,$$
$$\Theta(\varepsilon) = -(\varepsilon_1+\varepsilon_2)^2 + 4(\varepsilon_1+\varepsilon_2) \geq 0,$$
$$\Pi(\varepsilon)^T\Theta(\varepsilon) = 0.$$

Clearly, $(1,0)$ *is an efficient solution of (P1). One can easily verify that* $(\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\Lambda}_1, \bar{\Lambda}_2, \bar{\sigma}, \bar{\psi}_a, \bar{\psi}_b) = (1, 0, 1/2, 1/2, 0, 0, 7)$ *is an* μ*-stationary point of (P1) and* $(\bar{\psi}_a, \bar{\psi}_b) \geq 0$.

We next prove that Ψ, Φ, $-\Phi$, $-\Theta$, $-\Pi$ *are* υ*-bonvex functions at each* $\varepsilon \in \mathbb{R}^2$*. For* $\Psi(\varepsilon)$ *and* $\bar{\varepsilon} \in \mathbb{R}^2$*, it holds*

$\nabla\Psi_1(\varepsilon) = \begin{pmatrix} 2(\varepsilon_1 - 1) \\ \\ 2(\varepsilon_2 + 1) \end{pmatrix}$, $\nabla^2\Psi_1(\varepsilon) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Thus $\nabla^2\Psi_1(\bar{\varepsilon}) \succ 0$. $\nabla\Psi_2(\varepsilon) = \begin{pmatrix} 2(\varepsilon_1 - 3) \\ \\ 2(\varepsilon_2 + 3) \end{pmatrix}$, $\nabla^2\Psi_2(\varepsilon) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Thus $\nabla^2\Psi_2(\bar{\varepsilon}) \succ 0$.

For $\Phi(\varepsilon)$*, we get* $\nabla\Phi(\varepsilon) = \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix}$, $\nabla^2\Phi(\varepsilon) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Then $\nabla^2\Phi(\varepsilon) \succeq 0$ *and* $\nabla\Phi(\varepsilon) \notin Im[\nabla^2\Phi(\varepsilon)]$.

For $-\Pi(\varepsilon)$*, we have*

$-\nabla\Pi(\varepsilon) = \begin{pmatrix} 2(\varepsilon_1 - 1) \\ \\ (2(\varepsilon_1 - 1), -1) \end{pmatrix}$, $-\nabla^2\Pi(\varepsilon) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$,

hence $-\nabla^2\Pi(\varepsilon) \succeq 0$ *and* $-\nabla\Pi(\varepsilon) \notin Im[-\nabla^2\Pi(\varepsilon)]$.

For $-\Theta(\varepsilon)$ *and* $\bar{\varepsilon} \in R^2$*, it satisfies*

$-\nabla\Theta(\varepsilon) = \begin{pmatrix} 2(\varepsilon_1 + \varepsilon_2) - 4 \\ \\ 2(\varepsilon_1 + \varepsilon_2) - 4) \end{pmatrix}$, $-\nabla^2\Theta(\varepsilon) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$,

hence $-\nabla^2\Theta(\varepsilon) \succeq 0$ *and* $-\nabla\Theta(\varepsilon) \in Im[-\nabla^2\Theta(\varepsilon)]$*, and*

$$(\varepsilon_1 + \varepsilon_2)^2 - 2(\varepsilon_1 + \varepsilon_2) = (\varepsilon_1 + \varepsilon_2 - 1)^2 - 1 > -7 = (\bar{\varepsilon}_1 + \bar{\varepsilon}_2)^2 - 2(\bar{\varepsilon}_1 + \bar{\varepsilon}_2)$$

$$-\frac{1}{2}(2\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2, -4) \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2 \\ \\ -4 \end{pmatrix}$$

Thus, Ψ *is* υ*-bonvex function for any* $\bar{\varepsilon} \in \mathbb{R}^2$ *by Corollary 1 and* Φ, $-\Phi$, $-\Theta$, $-\Pi$ *are* υ*-bonvex functions at each* $\varepsilon \in \mathbb{R}^2$ *by Proposition 1.*

For $\Omega_0(., \Lambda_1, \Lambda_2, \delta, \sigma, \psi_a, \psi_b)$ *and* $\bar{\varepsilon} \in R^2$*, we have*

$$\nabla\Omega_0(\varepsilon_1, \varepsilon_2, \Lambda_1, \Lambda_2, \delta, \sigma, \psi_a, \psi_b) =$$

$$\begin{pmatrix} 2\Lambda_1(\varepsilon_1 - 1) + 2\Lambda_2(\varepsilon_1 - 3) + \sigma + 2\psi_a(\varepsilon_1 - 1) + 2\psi_b(\varepsilon_1 + \varepsilon_2) - 4\psi_b \\ \\ 2\Lambda_2(\varepsilon_2 + 1) + 2\Lambda_2(\varepsilon_2 + 3) + \sigma - \psi_a + 2\psi_b(\varepsilon_1 + \varepsilon_2) - 4\psi_b \end{pmatrix},$$

$$\nabla\Omega_0(\varepsilon_1, \varepsilon_2, \Lambda_1, \Lambda_2, \delta, \sigma, \psi_a, \psi_b) =$$

$$\begin{pmatrix} 2(1/2)(\varepsilon_1 - 1) + 2(1/2)(\varepsilon_1 - 3) + \sigma + 2\psi_a(\varepsilon_1 - 1) + 2\psi_b(\varepsilon_1 + \varepsilon_2) - 4\psi_b \\ \\ 2(1/2)(\varepsilon_2 + 1) + 2(1/2)(\varepsilon_2 + 3) + \sigma - \psi_a + 2\psi_b(\varepsilon_1 + \varepsilon_2) - 4\psi_b \end{pmatrix},$$

$$\nabla^2\Omega_0(\varepsilon_1,\varepsilon_2,\Lambda_1,\Lambda_2,\delta,\sigma,\psi_a,\psi_b)=\begin{pmatrix}2+2\psi_a+2\psi_b & 2\psi_b\\ 2\psi_b & 2+2\psi_b\end{pmatrix}.$$

Then $\nabla^2\Omega_0(\varepsilon_1,\varepsilon_2,\Lambda_1,\Lambda_2,\delta,\sigma,\psi_a,\psi_b)\succ 0$ *because of* $\psi_a,\psi_b\geq 0$. *Thus, we obtain that* $\Omega_0(.,\Lambda_1,\Lambda_2,\delta,\sigma,\psi_a,\psi_b)$ *is strictly* υ*-bonvex function at* $\bar{\varepsilon}$ *by Corollary 1, where the infimum of* $\Omega_0(.,\Lambda_1,\Lambda_2,\delta,\sigma,\psi_a,\psi_b)$ *cannot be attained at* $\bar{\varepsilon}$. *Hence,* $(1,0,1/2,1/2,0,0,7)$ *is a globally efficient solution of (P1) by Theorems 1-2, and Theorem 3 also holds for the (P1).*

Next we prove that the second-order dual offers a tighter lower bound, we compare the objective function values given by the approximation $(\varepsilon_1,\varepsilon_2)=(0,0)$ *in the (MSD) with that in (MFD). Note that the objective value of (MMPCC) at* $(0,0)$ *is* $\Psi=(\Psi_1,\Psi_2)=(2,18)$. *Suppose* $(0,0)$ *is a feasible solution of (MFD), it should satisfy* $\nabla\Omega_0(\varepsilon,\Lambda_1,\Lambda_2,\delta,\sigma,\psi_a,\psi_b)=0$, *i.e.,*

$$-4+\sigma-2\psi_a-4\psi_b=0, \tag{27}$$

$$4+\sigma-\psi_a-4\psi_b=0. \tag{28}$$

and $\psi_a\geq 0,\psi_b\geq 0$. *From (27) and (28), we get* $-8-\psi_a=0$, *which yields* $\psi_a=-8$, *contradicting* $\psi_a\geq 0$. *Thus,* $(0,0)$ *is not a feasible solution of (MFD). If the functions* Ψ, Φ, Π *and* Θ *satisfy the constraints of (MSD) at* $(0,0)$, *then the subsequent system holds*

$$\nabla\Omega_0(\varpi_1,\varpi_2,\Lambda,\delta,\sigma,\psi_a,\psi_b)+\nabla^2\Omega_0(\varpi_1,\varpi_2,\Lambda,\delta,\sigma,\psi_a,\psi_b)\varsigma=0,\psi_a\geq 0,\psi_b\geq 0,$$

namely,

$$-4+\sigma-2\psi_a-4\psi_b+(2+2\psi_a+2\psi_b)\varsigma_1+2\psi_b\varsigma_2=0,$$

$$4+\sigma-\psi_a-4\psi_b+2\psi_b\varsigma_1+(2+2\psi_b)\varsigma_2=0,\psi_a\geq 0,\psi_b\geq 0.$$

Choosing $\varsigma_1=2$, $\varsigma_2=-2$, *we obtain,* $\sigma=4$, $\psi_a=0$, $\psi_b=1$. *We get, a feasible solution of* $(0,0,2,-2,1/2,1/2,4,0,1)$ *of (MSD). The corresponding objective value of (MSD) is*

$$\Big((\varpi_1-1)^2+(\varpi_2+1)^2+\delta^T.0+\sigma^T(\varpi_1+\varpi_2-1)-{\psi_a}^T[(\varpi_1-1)^2-\varpi_2]-{\psi_b}^T[(\varpi_1+\varpi_2)^2,$$

$$-4(\varpi_1+\varpi_2)](\varpi_1-3)^2+(\varpi_2+3)^2+\delta^T.0+\sigma^T(\varpi_1+\varpi_2-1)-{\psi_a}^T[(\varpi_1-1)^2-\varpi_2]-{\psi_b}^T[(\varpi_1+\varpi_2)^2$$

$$-4(\varpi_1+\varpi_2)]\Big)=\Big((0-1)^2+(0+1)^2-4,(0-3)^2+(0+3)^2-4\Big)=(-2,14)$$

which provides a lower bound for (MMPCC).

4. Mangasarian-type higher-order duality

The present section deals with the formulation of Mangasarian-type higher-order dual problem for (MMPCC) and proves the results of duality based on the higher-order type I condition.

Remark 1. (Key Benefits of Using Higher-Order Duality) *While second-order duality captures quadratic approximations using the Hessian, higher-order duality extends this to include cubic and higher-degree polynomial approximations, leading to more accurate solutions in highly nonlinear and nonconvex problems. Additionally, higher-order duality helps in proving stronger optimality results, especially in cases where second-order conditions are insufficient to guarantee global optimality.*

Example 3. (Application of Higher-Order Duality in a Supply Chain Model) *Consider a multi-stage supply chain model where a firm optimizes its production, inventory, and transportation costs while ensuring customer demand satisfaction. The objective function includes cost components such as production cost $\Psi_1(\varepsilon)$, holding cost $\Psi_2(\varepsilon)$, and transportation cost $\Psi_3(\varepsilon)$, leading to the following multi-objective mathematical program with complementarity constraints:*

$$
\begin{aligned}
\text{(MMPCC-2)} \quad \min_{\varepsilon \in \mathbb{R}^n} \quad & \Psi(\varepsilon) = (\Psi_1(\varepsilon), \Psi_2(\varepsilon), \Psi_3(\varepsilon)) \\
& \text{subject to} \\
& \varphi(\varepsilon) \leqq 0, \\
& \Phi(\varepsilon) = 0, \\
& 0 \leqq \Pi(\varepsilon) \perp \Theta(\varepsilon) \geqq 0,
\end{aligned}
$$

Here, ε represents decision variables like production levels, shipment quantities, and inventory levels. The complementarity constraints capture supply-demand balance and warehouse capacity limitations.

Using Mangasarian-type higher-order duality, we introduce additional derivatives to analyze supply chain sensitivity and robustness. Unlike second-order duality, which only uses Hessians, higher-order duality incorporates third and higher-degree derivatives to improve accuracy in nonconvex settings. The higher-order dual problem takes the form:

$$
\text{(MHD-1)} \quad \max \quad \Omega_0(\varpi, \varsigma, \Lambda, \delta, \sigma, \psi_a, \psi_b) = \Big((\Omega_0)_1(\varpi, \varsigma, \Lambda, \delta, \sigma, \psi_a, \psi_b)
$$

$$
(\Omega_0)_2(\varpi, \varsigma, \Lambda, \delta, \sigma, \psi_a, \psi_b), \ (\Omega_0)_3(\varpi, \varsigma, \Lambda, \delta, \sigma, \psi_a, \psi_b) \Big)
$$

where

$$
(\Omega_0)_t(\varpi, \varsigma, \Lambda, \delta, \sigma, \psi_a, \psi_b) = \Psi_t(\varpi) + \{\xi_t(\varpi, \varsigma) - \varsigma^T \nabla_\varsigma \xi_t(\varpi, \varsigma)\} + \sum_{\jmath=1}^{g} \delta_\jmath \{\varphi_\jmath(\varpi)
$$

$$
+\xi_{\varphi_\jmath}(\varpi, \varsigma) - \varsigma^T \nabla_\varsigma \xi_{\varphi_\jmath}(\varpi, \varsigma)\} + \sum_{\nu=1}^{h} \sigma_\nu \{\Phi_\nu(\varpi) + \xi_{\Phi_\nu}(\varpi, \varsigma) - \varsigma^T \nabla_\varsigma \xi_{\Phi_\nu}(\varpi, \varsigma)\}
$$

$$-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\Pi_\gamma(\varpi)+\xi_{\pi_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\Theta_\gamma(\varpi)+\xi_{\Theta_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\},\ t=1,2,3.$$

Let $\xi_t:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $t=1,2,3$, $\xi_\varphi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^g$, $\xi_\Phi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^h$, $\xi_\Pi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{m_0}$, $\xi_\Theta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{m_0}$ *are differentiable functions.*

Mangasarian-type higher-order duality provides a robust mathematical framework for solving complex supply chain optimization problems. By incorporating higher-degree derivatives, this approach enhances cost approximation, risk assessment, and decision-making accuracy, leading to more efficient and resilient supply chain models.

The higher-order dual problem of (MMPCC) is defined in the following way:

$$\text{(MHD)}\quad \max\quad \Omega_0(\varpi,\varsigma,\Lambda,\delta,\sigma,\psi_a,\psi_b)=\Big((\Omega_0)_1(\varpi,\varsigma,\Lambda,\delta,\sigma,\psi_a,\psi_b)$$

$$,\ \ldots,\ (\Omega_0)_{f_0}(\varpi,\varsigma,\Lambda,\delta,\sigma,\psi_a,\psi_b)\Big)$$

subject to

$$\sum_{t=1}^{f_0}\Lambda_t\{\nabla_\varsigma\xi_t(\varpi,\varsigma)\}+\sum_{\jmath=1}^{g}\delta_\jmath\{\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\}+\sum_{\nu=1}^{h}\sigma_\nu\{\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\}=0, \tag{29}$$

$$\Lambda>0,\sum_{t=1}^{f_0}\Lambda_t=1,\delta\geqq 0,\psi_a\geqq 0,\psi_b\geqq 0,$$

where

$$(\Omega_0)_t(\varpi,\varsigma,\Lambda,\delta,\sigma,\psi_a,\psi_b)=\Psi_t(\varpi)+\{\xi_t(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_t(\varpi,\varsigma)\}+\sum_{\jmath=1}^{g}\delta_\jmath\{\varphi_\jmath(\varpi)$$

$$+\xi_{\varphi_\jmath}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\}+\sum_{\nu=1}^{h}\sigma_\nu\{\Phi_\nu(\varpi)+\xi_{\Phi_\nu}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\Pi_\gamma(\varpi)+\xi_{\pi_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\Theta_\gamma(\varpi)+\xi_{\Theta_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\},\ t=1,2,...,f_0.$$

Let $\xi_t:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R},\ t=1,2,...,f_0,\ \xi_\varphi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^g,\ \xi_\Phi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^h$, $\xi_\Pi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{m_0}$, $\xi_\Theta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{m_0}$ are differentiable functions.

Remark 2. (i) *If* $\xi_t(\varpi,\varsigma)=\varsigma^T\nabla\Psi_t(\varpi)+\frac{1}{2}\varsigma^T\nabla^2\Psi_t(\varpi)\varsigma(t=1,2...,f_0)$, $\xi_{\varphi_\jmath}(\varpi,\varsigma)=\varsigma^T\nabla\varphi_\jmath(\varpi)+\frac{1}{2}\varsigma^T\nabla^2\varphi_\jmath(\varpi)\varsigma$ $(\jmath=1,2...,g)$, $\xi_{\Phi_\nu}(\varpi,\varsigma)=\varsigma^T\nabla\Phi_\nu(\varpi)+\frac{1}{2}\varsigma^T\nabla^2\Phi_\nu(\varpi)\varsigma$ $(\nu=1,2...,h)$, $\xi_{\Pi_\gamma}(\varpi,\varsigma)=\varsigma^T\nabla\Pi_\gamma(\varpi)+\frac{1}{2}\varsigma^T\nabla^2\Pi_\gamma(\varpi)\varsigma$ $(\gamma=1,2...,m_0)$ *and* $\xi_{\Theta_\gamma}(\varpi,\varsigma)=\varsigma^T\nabla\Theta_\gamma(\varpi)+\frac{1}{2}\varsigma^T\nabla^2\Theta_\gamma(\varpi)\varsigma$ $(\gamma=1,2...,m_0)$ *then* (MHD) *reduces to* (MSD).
(ii) *If* $\xi_t(\varpi,\varsigma)=\varsigma^T\nabla\Psi_t(\varpi)$ $(t=1,2...,f_0)$, $\xi_{\varphi_\jmath}(\varpi,\varsigma)=\varsigma^T\nabla\varphi_\jmath(\varpi)$ $(\jmath=1,2...,g)$, $\xi_{\Phi_\nu}(\varpi,\varsigma)=\varsigma^T\nabla\Phi_\nu(\varpi)$ $(\nu=1,2...,h)$, $\xi_{\Pi_\gamma}(\varpi,\varsigma)=\varsigma^T\nabla\Pi_\gamma(\varpi)$ $(\gamma=1,2...,m_0)$ *and* $\xi_{\Theta_\gamma}(\varpi,\varsigma)=\varsigma^T\nabla\Theta_\gamma(\varpi)$ $(\gamma=1,2...,m_0)$ *then* (MHD) *reduces to* (MFD).

We use $\mathbb{F}_{MHD}$ to denote the feasible region of Problem (MHD). Let $\mathbb{F}_{MHD}(\varpi)=\{\varpi:(\varpi,\varsigma,\Lambda,\delta,\sigma,\psi_a,\psi_b)\in\mathbb{F}_{MHD}\}$.
We next establish several higher order duality theorems for (MMPCC).

Theorem 4. (Weak duality) *Let* ε *and* $(\varpi,\varsigma,\Lambda,\delta,\sigma,\psi_a,\psi_b)$ *be feasible solutions for (MMPCC) and (MSD), respectively. Suppose that functions* $\Psi_t(t=1,...,f_0)$, $\varphi_\jmath(\jmath=1,...,g)$, Φ_ν, $-\Phi_\nu(\nu=1,...,h)$, $-\Pi_\gamma$ *and* $-\Theta_\gamma(\gamma=1,...,m_0)$ *are higher-order type I at* ϖ, *then the following cannot hold:*

$$\Psi_t(\varepsilon)\leqq\Psi_t(\varpi)+\{\xi_t(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_t(\varpi,\varsigma)\}+\sum_{\jmath=1}^{g}\delta_\jmath\{\varphi_\jmath(\varpi)+\xi_{\varphi_\jmath}(\varpi,\varsigma))-\varsigma^T\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\}$$
$$+\sum_{\nu=1}^{h}\sigma_\nu\{\Phi_\nu(\varpi)+\xi_{\Phi_\nu}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}$$
$$-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\Pi_\gamma(\varpi)+\xi_{\Pi_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}$$
$$-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\Theta_\gamma(\varpi)+\xi_{\Theta_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\},\ \text{for all } t\in\xi, \tag{30}$$

and

$$\Psi_\ell(\varepsilon)<\Psi_\ell(\varpi)+\{\xi_t(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_t(\varpi,\varsigma)\}+\sum_{\jmath=1}^{q}\delta_\jmath\{\varphi_\jmath(\varpi)+\xi_{\varphi_\jmath}(\varpi,\varsigma))-\varsigma^T\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\}$$

$$+\sum_{\nu=1}^{h}\sigma_\nu\{\Phi_\nu(\varpi)+\xi_{\Phi_\nu}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\Pi_\gamma(\varpi)+\xi_{\Pi_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\Theta_\gamma(\varpi)+\xi_{\Theta_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\},\ \text{for atleast one } \ell\in\xi_t. \tag{31}$$

Proof. Suppose to the contrary that (30) and (31) hold, that is,

$$\Psi_t(\varepsilon)\leqq\Psi_t(\varpi)+\{\xi_t(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_t(\varpi,\varsigma)\}+\sum_{\jmath=1}^{g}\delta_\jmath\{\varphi_\jmath(\varpi)+\xi_{\varphi_\jmath}(\varpi,\varsigma))-\varsigma^T\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\}$$

$$+\sum_{\nu=1}^{h}\sigma_\nu\{\Phi_\nu(\varpi)+\xi_{\Phi_\nu}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\Pi_\gamma(\varpi)+\xi_{\Pi_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\Theta_\gamma(\varpi)+\xi_{\Theta_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\},\ \text{for all } t\in\xi,$$

and

$$\Psi_\ell(\varepsilon)<\Psi_\ell(\varpi)+\{\xi_t(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_t(\varpi,\varsigma)\}+\sum_{\jmath=1}^{g}\delta_\jmath\{\varphi_\jmath(\varpi)+\xi_{\varphi_\jmath}(\varpi,\varsigma))-\varsigma^T\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\}$$

$$+\sum_{\nu=1}^{h}\sigma_\nu\{\Phi_\nu(\varpi)+\xi_{\Phi_\nu}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\Pi_\gamma(\varpi)+\xi_{\Pi_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\Theta_\gamma(\varpi)+\xi_{\Theta_\gamma}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\},\ \text{for atleast one } \ell\in\xi_t.$$

Because $\Lambda>0$ and $\sum_{t=1}^{f_0}\Lambda_t=1$, the above inequalities yield:

$$\sum_{t=1}^{f_0}\Lambda_t\Psi_t(\varepsilon)<\sum_{t=1}^{f_0}\Lambda_t\{\Psi_t(\varpi)+\xi_t(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_t(\varpi,\varsigma)\}+\sum_{\jmath=1}^{g}\delta_\jmath\{\varphi_\jmath(\varpi)+\xi_{\varphi_\jmath}(\varpi,\varsigma))-$$

$$\varsigma^T\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\} + \sum_{\nu=1}^{h}\sigma_\nu\{\Phi_\nu(\varpi) + \xi_{\Phi_\nu}(\varpi,\varsigma) - \varsigma^T\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}$$
$$-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\Pi_\gamma(\varpi) + \xi_{\Pi_\gamma}(\varpi,\varsigma) - \varsigma^T\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}$$
$$-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\Theta_\gamma(\varpi) + \xi_{\Theta_\gamma}(\varpi,\varsigma) - \varsigma^T\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\}. \tag{32}$$

We introduce the subsequent index sets:

$$\Gamma^+ = \{\nu : \sigma_\nu \geq 0\}, \qquad \Gamma^- = \{\nu : \sigma_\nu \leq 0\},$$
$$\vartheta^+ = \{\gamma : (\psi_a)_\gamma > 0, (\psi_b)_\gamma > 0\}, \qquad \vartheta^0 = \{\gamma : (\psi_a)_\gamma = 0, (\psi_b)_\gamma = 0\},$$
$$\vartheta^+_\Theta = \{\gamma : (\psi_a)_\gamma = 0, (\psi_b)_\gamma > 0\}, \qquad \vartheta^+_\Pi = \{\gamma : (\psi_a)_\gamma > 0, (\psi_b)_\gamma = 0\}.$$

Since $\Psi_t(t = 1, ..., f_0)$, $\varphi_\jmath$, Φ_ν, $-\Phi_\nu$, $-\Pi_\gamma$ and $-\Theta_\gamma$ are higher-order type I at ϖ, we have, for any $\varepsilon \in \mathbf{Y}$ and $\varpi \in \mathbb{F}_{MHD}(\varpi)$,

$$\Psi_t(\varepsilon) - \Psi_t(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T\nabla_\varsigma\xi_t(\varpi,\varsigma) + \xi_t(\varpi,\varsigma) - \varsigma^T(\nabla_\varsigma\xi_t(\varpi,\varsigma)), \tag{33}$$

$$\varphi_\jmath(\varepsilon)-\varphi_\jmath(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)+\xi_{\varphi_\jmath}(\varpi,\varsigma)-\varsigma^T(\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)), \forall\jmath \in \{1,2,...,g\}, \tag{34}$$

$$\Phi_\nu(\varepsilon) - \Phi_\nu(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma) + \xi_{\Phi_\nu}(\varpi,\varsigma) - \varsigma^T(\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)), \forall\nu \in \Gamma^+, \tag{35}$$

$$-\Phi_\nu(\varepsilon)+\Phi_\nu(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T[-\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)]-\xi_{\Phi_\nu}(\varpi,\varsigma)+\varsigma^T(\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)), \forall\nu \in \Gamma^-, \tag{36}$$

$$-\Pi_\gamma(\varepsilon)+\Pi_\nu(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T[-\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)]-\xi_{\Pi_\gamma}(\varpi,\varsigma)+\varsigma^T(\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)), \forall\gamma \in \vartheta^+\cup\vartheta^+_\Pi, \tag{37}$$

$$-\Theta_\gamma(\varepsilon)+\Theta_\nu(\varpi) \geqq [\upsilon(\varepsilon,\varpi)]^T[-\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)]-\xi_{\Theta_\gamma}(\varpi,\varsigma)+\varsigma^T(\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)), \forall\gamma \in \vartheta^+\cup\vartheta^+_\Theta. \tag{38}$$

Multiplying inequalities (33)–(38) by $\Lambda_t > 0$, $t \in \xi$, $\delta_\jmath(\jmath \in \{1,2,...,g\})$, $\sigma_\nu(\nu \in \Gamma^+)$, $-\sigma_\nu(\nu \in \Gamma^-)$, $(\psi_a)_\gamma(\gamma \in \vartheta^+ \cup \vartheta^+_\Pi)$, and $(\psi_b)_\gamma(\gamma \in \vartheta^+ \cup \vartheta^+_\Theta)$, respectively, and adding, we then get the following inequality

$$\sum_{t=1}^{f_0}\Lambda_t\Psi_t(\varepsilon)-\sum_{t=1}^{f_0}\Lambda_t\{\Psi_t(\varpi)+\xi_t(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_t(\varpi,\varsigma)\}+\sum_{\jmath=1}^{g}\delta_\jmath\varphi_\jmath(\varepsilon)-\sum_{\jmath=1}^{g}\delta_\jmath\{\varphi_\jmath(\varpi)+\xi_{\varphi_\jmath}(\varpi,\varsigma))$$

$$-\varsigma^T\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\}+\sum_{\nu=1}^{h}\sigma_\nu\Phi_\nu(\varepsilon)-\sum_{\nu=1}^{h}\sigma_\nu\{\Phi_\nu(\varpi)+\xi_{\Phi_\nu}(\varpi,\varsigma)-\varsigma^T\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\Pi_\gamma(\varepsilon)$$

$$+\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\Pi_\gamma(\varpi) + \xi_{\Pi_\gamma}(\varpi,\varsigma) - \varsigma^T\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\} - \sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\Theta_\gamma(\varepsilon) + \sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\Theta_\gamma(\varpi)+$$

$$\xi_{\Theta_\gamma}(\varpi,\varsigma) - \varsigma^T\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\} \geqq [\upsilon(\varepsilon,\varpi)]^T\sum_{t=1}^{f_0}\Lambda_t\{\nabla_\varsigma\xi_t(\varpi,\varsigma)\} + \sum_{\jmath=1}^{g}\delta_\jmath\{\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\}+$$

$$\sum_{\nu=1}^{h}\sigma_\nu\{\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\}. \tag{39}$$

Since ε and $(\varpi,\varsigma,\delta,\sigma,\psi_a,\psi_b)$ are feasible solutions for (MMPCC) and (MSD), respectively, we imply that $\varphi(\varepsilon)\leqq 0$, $\Phi(\varepsilon)=0$, $\Pi(\varepsilon)\geqq 0$, $\Theta(\varepsilon)\geqq 0$, $\delta\geqq 0$, $\psi_a\geqq 0$, $\psi_b\geqq 0$, whence

$$\sum_{\jmath=1}^{g}\delta_\jmath\varphi_\jmath(\varepsilon)+\sum_{\nu=1}^{h}\sigma_\nu\Phi_\nu(\varepsilon)-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\Pi_\gamma(\varepsilon)-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\Theta_\gamma(\varepsilon)\leqq 0. \tag{40}$$

The inequality (39), in view of (32) and (40) gives

$$\sum_{t=1}^{f_0}\Lambda_t\{\nabla_\varsigma\xi_t(\varpi,\varsigma)\}+\sum_{\jmath=1}^{g}\delta_\jmath\{\nabla_\varsigma\xi_{\varphi_\jmath}(\varpi,\varsigma)\}+\sum_{\nu=1}^{h}\sigma_\nu\{\nabla_\varsigma\xi_{\Phi_\nu}(\varpi,\varsigma)\}$$
$$-\sum_{\gamma=1}^{m_0}(\psi_a)_\gamma\{\nabla_\varsigma\xi_{\Pi_\gamma}(\varpi,\varsigma)\}-\sum_{\gamma=1}^{m_0}(\psi_b)_\gamma\{\nabla_\varsigma\xi_{\Theta_\gamma}(\varpi,\varsigma)\}<0, \tag{41}$$

which contradicts (29).

Theorem 5. (Strong duality) *Let ε_0 be an efficient solution of (MMPCC) which satisfies the Kuhn–Tucker conditions with the multiplier $(\Lambda_0,\delta_0,\sigma_0,(\psi_a)_0,(\psi_b)_0)$ and $((\psi_a)_0,(\psi_b)_0)\geq 0$. For $t=1,...,f_0$, $\jmath=1,...,g$, $\nu=1,...,h$, $\gamma=1,...,m_0$, let*

$$\begin{cases}\xi_t(\varepsilon_0,0)=0,\xi_\varphi(\varepsilon_0,0)=0,\xi_\Phi(\varepsilon_0,0)=0,\xi_\Pi(\varepsilon_0,0)=0,\xi_\Theta(\varepsilon_0,0)=0,\\ \nabla_\varsigma\xi_t(\varepsilon_0,0)=\nabla_\varsigma\Psi_t(\varepsilon_0),\nabla_\varsigma\xi_{\varphi_\jmath}(\varepsilon_0,0)=\nabla_\varsigma\varphi_\jmath(\varepsilon_0),\nabla_\varsigma\xi_{\Phi_\nu}(\varepsilon_0,0)=\nabla_\varsigma\Phi_\nu(\varepsilon_0),\\ \nabla_\varsigma\xi_{\Pi_\gamma}(\varepsilon_0,0)=\nabla_\varsigma\Pi_\gamma(\varepsilon_0),\nabla_\varsigma\xi_{\Theta_\gamma}(\varepsilon_0,0)=\nabla_\varsigma\Theta_\gamma(\varepsilon_0).\end{cases} \tag{42}$$

Then $(\varepsilon_0,\varsigma=0,\delta_0,\sigma_0,(\psi_a)_0,(\psi_b)_0)$ is feasible for (MHD), and the corresponding objective values of (MMPCC) and (MHD) are equal. If, in addition, functions $\Psi_t(t=1,...,f_0)$, $\varphi_\jmath(\jmath=1,...,g)$, Φ_ν, $-\Phi_\nu(\nu=1,...,h)$, $-\Pi_\gamma$ and $-\Theta_\gamma(\gamma=1,...,m_0)$ are higher-order type I at $\varpi\in\mathbb{F}_{MSD}(\varpi)$, then $(\varepsilon_0,\varsigma=0,\Lambda_0,\delta_0,\sigma_0,(\psi_a)_0,(\psi_b)_0)$ is a globally efficient solution of (MHD).

Proof. Since ε_0 be an efficient solution of (MMPCC) which satisfies the Kuhn–Tucker conditions with the multiplier $(\delta_0,\sigma_0,(\psi_a)_0,(\psi_b)_0)$ and $((\psi_a)_0,(\psi_b)_0)\geq 0$, we have

$$\sum_{t=1}^{f_0}(\Lambda_0)_t\nabla\Psi_t(\varepsilon_0)+\nabla{\delta_0}^T\varphi(\varepsilon_0)+\nabla{\sigma_0}^T\Phi(\varepsilon_0)-\nabla(\psi_a)_0^T\Pi(\varepsilon_0)-\nabla(\psi_b)_0^T\Theta(\varepsilon_0)=0, \tag{43}$$

$${\delta_0}^T\varphi(\varepsilon_0)=0,{(\psi_a)_0}^TG(\varepsilon_0)=0,{(\psi_b)_0}^T\Theta(\varepsilon_0)=0, \tag{44}$$

$$\Lambda_0>0,\sum_{t=1}^{f_0}(\Lambda_0)_t=1,\delta_0\geqq 0,(\psi_a)_0\geqq 0,(\psi_b)_0\geqq 0. \tag{45}$$

From (42), (43) and using the feasibility of ε_0 for (MMPCC), it follows that

$$\sum_{t=1}^{f_0}(\Lambda_0)_t\{\nabla_\varsigma\xi_t(\varepsilon_0,\varsigma)\}+\sum_{\jmath=1}^{g}(\delta_0)_\jmath\{\nabla_\varsigma\xi_{\varphi_\jmath}(\varepsilon_0,\varsigma)\}+\sum_{\nu=1}^{h}(\sigma_0)_\nu\{\nabla_\varsigma\xi_{\Phi_\nu}(\varepsilon_0,\varsigma)\}$$

$$-\sum_{\gamma=1}^{m_0}((\psi_a)_0)_\gamma\{\nabla_\varsigma\xi_{\Pi_\gamma}(\varepsilon_0,\varsigma)\}-\sum_{\gamma=1}^{m_0}((\psi_b)_0)_\gamma\{\nabla_\varsigma\xi_{\Theta_\gamma}(\varepsilon_0,\varsigma)\}=$$

$$\sum_{t=1}^{f_0}(\Lambda_0)_t\nabla\Psi_t(\varepsilon_0)+\nabla{\delta_0}^T\varphi(\varepsilon_0)+\nabla{\sigma_0}^T\Phi(\varepsilon_0)-\nabla(\psi_a)_0^T\Pi(\varepsilon_0)-\nabla(\psi_b)_0^T\Theta(\varepsilon_0)=0. \quad (46)$$

Hence (45) and (46) imply that $(\varepsilon_0,\varsigma=0,\Lambda_0,\delta_0,\sigma_0,(\psi_a)_0,(\psi_b)_0)$ is a feasible solution for (MHD). The assumption (42) together with (44) and $\Phi(\varepsilon_0)=0$ shows that

$$(\Psi_1(\varepsilon_0),\Psi_2(\varepsilon_0),...,\Psi_t(\varepsilon_0))=\Omega_0(\varepsilon_0,\varsigma=0,\Lambda_0,\delta_0,\sigma_0,(\psi_a)_0,(\psi_b)_0),$$

hence the corresponding objective values of (MMPCC) and (MHD) are equal. Note that functions $\Psi_t(t=1,...,f_0)$, $\varphi_\jmath(\jmath=1,...,g)$, Φ_ν, $-\Phi_\nu(\nu=1,...,h)$, $-\Pi_\gamma$ and $-\Theta_\gamma(\gamma=1,...,m_0)$ are higher-order type I at $\varpi\in\mathbb{F}_{MSD}(\varpi)$. Then $(\varepsilon_0,\varsigma=0,\Lambda_0,\delta_0,\sigma_0,(\psi_a)_0,(\psi_b)_0)$ is a globally efficient solution of (MHD) by Theorem 4.

Theorem 6. (Strict converse duality) *Let ε^* and $(\varpi^*,\varsigma^*,\Lambda^*,\delta^*,\sigma^*,{\psi_a}^*,{\psi_b}^*)$ be feasible solutions for (MMPCC) and (MHD), respectively, such that*

$$\sum_{t=1}^{f_0}\Lambda_t^*\Psi_t(\varepsilon^*)\leqq\sum_{t=1}^{f_0}\Lambda_t^*\{\Psi_t(\varpi^*)+\xi_t(\varpi^*,\varsigma^*)-{\varsigma^*}^T\nabla_{\varsigma^*}\xi_t(\varpi^*,\varsigma^*)\}+\sum_{\jmath=1}^{g}{\delta^*}_\jmath\{\varphi_\jmath(\varpi^*)$$

$$+\xi_{\varphi_\jmath}(\varpi^*,\varsigma^*))-{\varsigma^*}^T\nabla_{\varsigma^*}\xi_{\varphi_\jmath}(\varpi^*,\varsigma^*)\}+\sum_{\nu=1}^{h}\sigma_\nu^*\{\Phi_\nu(\varpi^*)+\xi_{\Phi_\nu}(\varpi^*,\varsigma^*)$$

$$-{\varsigma^*}^T\nabla_{\varsigma^*}\xi_{\Phi_\nu}(\varpi^*,\varsigma^*)\}-\sum_{\gamma=1}^{m_0}({\psi_a}^*)_\gamma\{\Pi_\gamma(\varpi^*)+\xi_{\Pi_\gamma}(\varpi^*,\varsigma^*)-{\varsigma^*}^T\nabla_{\varsigma^*}\xi_{\Pi_\gamma}(\varpi^*,\varsigma^*)\}$$

$$-\sum_{\gamma=1}^{m_0}({\psi_b}^*)_\gamma\{\Theta_\gamma(\varpi^*)+\xi_{\Theta_\gamma}(\varpi^*,\varsigma^*)-{\varsigma^*}^T\nabla_{\varsigma^*}\xi_{\Theta_\gamma}(\varpi^*,\varsigma^*)\}. \quad (47)$$

Suppose that $\Psi_t(t=1,...,f_0)$ is strictly higher-order type I at ϖ^*, $\varphi_\jmath(\jmath=1,...,g)$, Φ_ν, $-\Phi_\nu(\nu=1,...,h)$, $-\Pi_\gamma$ and $-\Theta_\gamma(\gamma=1,...,m_0)$ are higher-order type I at ϖ^*, then $\varepsilon^*=\varpi^*$.

Proof. We assume that $\varepsilon^*\neq\varpi^*$ and exhibit a contradiction. Since $\Psi_t(t=1,...,f_0)$ is strictly higher-order type I at ϖ^*, $\varphi_\jmath$, Φ_ν, $-\Phi_\nu$, $-\Pi_\gamma$ and $-\Theta_\gamma$ are higher-order type I at ϖ^*, we have, for any $\varepsilon^*\in\mathbf{Y}$ and $\varpi^*\in\mathbb{F}_{MHD}(\varpi^*)$,

$$\Psi_t(\varepsilon^*)-\Psi_t(\varpi^*)>[\upsilon(\varepsilon^*,\varpi^*)]^T\nabla_{\varsigma^*}\xi_t(\varpi^*,\varsigma^*)+\xi_t(\varpi^*,\varsigma^*)-{\varsigma^*}^T(\nabla_{\varsigma^*}\xi_t(\varpi^*,\varsigma^*)), \quad (48)$$

$$\varphi_\jmath(\varepsilon^*) - \varphi_\jmath(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T \nabla_{\varsigma^*} \xi_{\varphi_\jmath}(\varpi^*, \varsigma^*) + \xi_{\varphi_\jmath}(\varpi^*, \varsigma^*) \\ - \varsigma^{*T}(\nabla_{\varsigma^*} \xi_{\varphi_\jmath}(\varpi^*, \varsigma^*)), \forall \jmath \in \{1, 2, ..., g\}, \tag{49}$$

$$\Phi_\nu(\varepsilon^*) - \Phi_\nu(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T \nabla_{\varsigma^*} \xi_{\Phi_\nu}(\varpi^*, \varsigma^*) + \xi_{\Phi_\nu}(\varpi^*, \varsigma^*) - \\ \varsigma^{*T}(\nabla_{\varsigma^*} \xi_{\Phi_\nu}(\varpi^*, \varsigma^*)), \forall \nu \in \Gamma^+, \tag{50}$$

$$-\Phi_\nu(\varepsilon^*) + \Phi_\nu(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T [-\nabla_{\varsigma^*} \xi_{\Phi_\nu}(\varpi^*, \varsigma^*)] - \xi_{\Phi_\nu}(\varpi^*, \varsigma^*) \\ + \varsigma^{*T}(\nabla_{\varsigma^*} \xi_{\Phi_\nu}(\varpi^*, \varsigma^*)), \forall \nu \in \Gamma^-, \tag{51}$$

$$-\Pi_\gamma(\varepsilon^*) + \Pi_\nu(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T [-\nabla_{\varsigma^*} \xi_{\Pi_\gamma}(\varpi^*, \varsigma^*)] - \xi_{\Pi_\gamma}(\varpi^*, \varsigma^*) \\ + \varsigma^{*T}(\nabla_{\varsigma^*} \xi_{\Pi_\gamma}(\varpi^*, \varsigma^*)), \forall \gamma \in \vartheta^+ \cup \vartheta^+_\Pi, \tag{52}$$

$$-\Theta_\gamma(\varepsilon^*) + \Theta_\nu(\varpi^*) \geqq [\upsilon(\varepsilon^*, \varpi^*)]^T [-\nabla_{\varsigma^*} \xi_{\Pi_\gamma}(\varpi^*, \varsigma^*)] - \xi_{\Theta_\gamma}(\varpi^*, \varsigma^*) \\ + \varsigma^{*T}(\nabla_{\varsigma^*} \xi_{\Theta_\gamma}(\varpi^*, \varsigma^*)), \forall \gamma \in \vartheta^+ \cup \vartheta^+_\Theta. \tag{53}$$

Multiplying inequalities (48)–(53) by $\Lambda^*_t > 0$, $t \in \xi$, $\delta^*_\jmath(\jmath \in \{1, 2, ..., g\})$, $\sigma^*_\nu(\nu \in \Gamma^+)$, -$\sigma^*_\nu(\nu \in \Gamma^-)$, $({\psi_a}^*)_\gamma(\gamma \in \vartheta^+ \cup \vartheta^+_\Pi)$, and $({\psi_b}^*)_\gamma(\gamma \in \vartheta^+ \cup \vartheta^+_\Theta)$, respectively, and adding, we then get the following inequality

$$\sum_{t=1}^{f_0} \Lambda^*_t \Psi_t(\varepsilon^*) - \sum_{t=1}^{f_0} \Lambda^*_t \{\Psi_t(\varpi^*) + \xi_t(\varpi^*, \varsigma^*) - \varsigma^{*T} \nabla_{\varsigma^*} \xi_t(\varpi^*, \varsigma^*)\} + \sum_{\jmath=1}^{g} {\delta^*}_\jmath \varphi_\jmath(\varepsilon^*) - \sum_{\jmath=1}^{g} {\delta^*}_\jmath \{\varphi_\jmath(\varpi^*) \\ + \xi_{\varphi_\jmath}(\varpi^*, \varsigma^*)) - \varsigma^{*T} \nabla_{\varsigma^*} \xi_{\varphi_\jmath}(\varpi^*, \varsigma^*)\} + \sum_{\nu=1}^{h} \sigma^*_\nu \Phi_\nu(\varepsilon^*) - \sum_{\nu=1}^{h} \sigma^*_\nu \{\Phi_\nu(\varpi^*) + \xi_{\Phi_\nu}(\varpi^*, \varsigma^*) - \\ \varsigma^{*T} \nabla_{\varsigma^*} \xi_{\Phi_\nu}(\varpi^*, \varsigma^*)\} - \sum_{\gamma=1}^{m_0} ({\psi_a}^*)_\gamma \Pi_\gamma(\varepsilon^*) + \sum_{\gamma=1}^{m_0} ({\psi_a}^*)_\gamma \{\Pi_\gamma(\varpi^*) + \xi_{\Pi_\gamma}(\varpi^*, \varsigma^*) - \varsigma^{*T} \nabla_{\varsigma^*} \xi_{\Pi_\gamma}(\varpi^*, \varsigma^*)\} \\ - \sum_{\gamma=1}^{m_0} ({\psi_b}^*)_\gamma \Theta_\gamma(\varepsilon^*) + \sum_{\gamma=1}^{m_0} ({\psi_b}^*)_\gamma \{\Theta_\gamma(\varpi^*) + \xi_{\Theta_\gamma}(\varpi^*, \varsigma^*) - \varsigma^{*T} \nabla_{\varsigma^*} \xi_{\Theta_\gamma}(\varpi^*, \varsigma^*)\} > [\upsilon(\varepsilon^*, \varpi^*)]^T \\ \sum_{t=1}^{f_0} \Lambda^*_t \{\nabla_{\varsigma^*} \xi_t(\varpi^*, \varsigma^*)\} + \sum_{\jmath=1}^{g} {\delta^*}_\jmath \{\nabla_{\varsigma^*} \xi_{\varphi_\jmath}(\varpi^*, \varsigma^*)\} + \sum_{\nu=1}^{h} \sigma^*_\nu \{\nabla_{\varsigma^*} \xi_{\Phi_\nu}(\varpi^*, \varsigma^*)\} - \\ \sum_{\gamma=1}^{m_0} ({\psi_a}^*)_\gamma \{\nabla_{\varsigma^*} \xi_{\Pi_\gamma}(\varpi^*, \varsigma^*)\} - \sum_{\gamma=1}^{m_0} ({\psi_b}^*)_\gamma \{\nabla_{\varsigma^*} \xi_{\Theta_\gamma}(\varpi^*, \varsigma^*)\}. \tag{54}$$

Since ε^* and $(\varpi^*, \varsigma^*, \Lambda^*, \delta^*, \sigma^*, {\psi_a}^*, {\psi_b}^*)$ are feasible solutions for (MMPCC) and (MHD), respectively, we imply that $\varphi(\varepsilon^*) \leqq 0$, $\Phi(\varepsilon^*) = 0$, $\Pi(\varepsilon^*) \geqq 0$, $\Theta(\varepsilon^*) \geqq 0$, $\delta^* \geqq 0$, ${\psi_a}^* \geqq 0$, ${\psi_b}^* \geqq 0$, whence

$$\sum_{\jmath=1}^{g} {\delta^*}_\jmath \varphi_\jmath(\varepsilon^*) + \sum_{\nu=1}^{h} \sigma^*_\nu \Phi_\nu(\varepsilon^*) - \sum_{\gamma=1}^{m_0} ({\psi_a}^*)_\gamma \Pi_\gamma(\varepsilon^*) - \sum_{\gamma=1}^{m_0} ({\psi_b}^*)_\gamma \Theta_\gamma(\varepsilon^*) \leqq 0. \tag{55}$$

The inequality (54) on using (29) and (55), gives

$$\sum_{t=1}^{f_0} \Lambda_t^* \Psi_t(\varepsilon^*) > \sum_{t=1}^{f_0} \Lambda_t^* \{\Psi_t(\varpi^*) + \xi_t(\varpi^*, \varsigma^*) - \varsigma^{*T} \nabla_{\varsigma^*} \xi_t(\varpi^*, \varsigma^*)\} + \sum_{\jmath=1}^{g} \delta^*{}_\jmath \{\varphi_\jmath(\varpi^*)$$

$$+\xi_{\varphi_\jmath}(\varpi^*, \varsigma^*)) - \varsigma^{*T} \nabla_{\varsigma^*} \xi_{\varphi_\jmath}(\varpi^*, \varsigma^*)\} + \sum_{\nu=1}^{h} \sigma_\nu^* \{\Phi_\nu(\varpi^*) + \xi_{\Phi_\nu}(\varpi^*, \varsigma^*)$$

$$-\varsigma^{*T} \nabla_{\varsigma^*} \xi_{\Phi_\nu}(\varpi^*, \varsigma^*)\} - \sum_{\gamma=1}^{m_0} (\psi_a{}^*)_\gamma \{\Pi_\gamma(\varpi^*) + \xi_{\Pi_\gamma}(\varpi^*, \varsigma^*) - \varsigma^{*T} \nabla_{\varsigma^*} \xi_{\Pi_\gamma}(\varpi^*, \varsigma^*)\}$$

$$-\sum_{\gamma=1}^{m_0} (\psi_b{}^*)_\gamma \{\Theta_\gamma(\varpi^*) + \xi_{\Theta_\gamma}(\varpi^*, \varsigma^*) - \varsigma^{*T} \nabla_{\varsigma^*} \xi_{\Theta_\gamma}(\varpi^*, \varsigma^*)\}, \tag{56}$$

which contradicts (47).

5. Conclusion

In this study, we proposed Mangasarian-type second and higher-order dual formulations for multi-objective mathematical programs with complementarity constraints (MMPCC), incorporating μ-stationarity conditions. By invoking υ-bonvexity and higher-order type I assumptions, we derived weak, strong, and converse duality theorems for the respective dual models. The illustrative examples demonstrate that higher-order duals can yield tighter optimality bounds than their first-order counterparts, thus underscoring their theoretical and practical relevance.

Future research may aim to extend the proposed duality framework to nonconvex and nonsmooth settings, develop efficient computational algorithms for large-scale problems, and explore stochastic or fuzzy extensions to address uncertainty. It may also consider alternative notions of second-order generalized convexity, such as those discussed by Zualinescu [26], and investigate effective computational strategies for solving the proposed dual formulations in practical scenarios.

The proposed Mangasarian-type second- and higher-order duality framework strengthens theoretical insights into multi-objective problems with complementarity constraints but is limited by its reliance on smoothness assumptions, higher-order differentiability, and υ-bonvexity conditions, which may restrict use in nonsmooth or highly nonconvex cases. Despite these limitations, the approach has broad potential applications in economics, engineering, supply chain optimization, and data science, where multi-criteria decision-making and equilibrium constraints naturally arise.

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Conflicts of interest or competing interests

The authors declare that they have no conflicts of interest.

Data and code Availability

No data were used to support this study

Supplementary information

Not Applicable

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors

Informed Consent

The authors are fully aware and satisfied with the contents of the article.

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