EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6314
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Rational-Type Contractive Mappings in Bi-Complex Valued Control Metric Spaces and Applications

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Abstract. This manuscript studies some unique and common fixed point results in the context of bi-complex valued control metric space(BVCMS) using rational-type inequalities. The presented work explains the idea of BVCMS and then shows the necessary criteria for a pair of contractive type mappings in this space to have common fixed points. To show how applicable our results are, we also give an example. Finally, the existence of solutions of a system of fractional differential equations has been studied using the obtained results.

2020 Mathematics Subject Classifications: 47H10, 54H25, 34A08

Key Words and Phrases: Bi-complex valued controlled metric spaces, Fixed Point, Rational type contractions, Fractional Differential Equations

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1. Introduction and Preliminaries

The notion of conventional differential equations may be extended to non-integer orders using fractional differential equations (FDEs), an excellent mathematical tool. Unlike fractional calculus, which introduces the idea of fractional derivatives that may be derived for non-integer orders, conventional differential equations only deal with integer-order derivatives. Numerous scientific fields, including physics, engineering, economics, biology, and more, have begun to pay close attention to the study of fractional differential equations. This is because FDEs offer a more precise and adaptable method for simulating

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6314

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complicated processes that display non-local and memory-dependent behavior. Multiple applications of fractional differential equations (FDEs) may be found in physics, engineering, economics, and biology. They faithfully represent complex systems with anomalous diffusion, long-range interactions, and memory dependence. Control systems, the behavior of viscoelastic materials, asset pricing, image processing, and biomedicine are all improved by FDEs. In the research of FDEs, [1–4] have outstanding data.

Furthermore, fixed point theorems offer helpful methods for demonstrating that solutions to specific fractional differential equations exist, supporting the mathematical study and real-world use of these equations in various scientific and technical fields. Since the proof of the well-known Banach contraction theorem, the metric fixed point theorem has appeared. Since then, there have been numerous discoveries relating to maps satisfying various contractive requirements and different metric spaces.
In the context of non-linear analysis the theory of fixed point gained extraordinary importance. After the famous Banach presented the very first result in metric fixed point theory [5], many researchers in this direction put forward the generalized structure of the contraction principle. One of the recent generalizations in this sequel was the introduction of b-metric space by Bakhtin [6]. In generalizing the metrix space the triangular inequality was introduced in a different manner by introducing $s \geq 1$, a constant multiple. This generalization leads to some important developments notably, extended b-metric spaces, which was developed by Kamran et al. [7]. Likewise, Mlaiki et al.[8] presented controlled metric spaces in 2018. Moreover, researcher expanded many results in this space such as [9, 10]

In this direction, the concept of complex-valued metric spaces was first presented by Azam et al. [11], and they also developed some fixed point results for pairs of mappings that fulfill the contraction requirement for rational expressions. Furthermore, Segre [12] established a base for bi-complex numbers and supported a commutative replacement for the skew field of quaternions. These numbers more strongly and explicitly generalized and extended the complex numbers to quaternions. Bi-complex valued metric spaces (BCVMS) were first proposed by Choi et al. in 2017 [13], who connected the two ideas, bi-complex numbers and complex-valued metric spaces. They developed common fixed-point outcomes for weakly compatible mappings.

A contractive type common fixed point for two maps in bicomplex-valued metric spaces was established by [14]. Later, several researchers used this notion to describe their results; see [15–22]. Guechi [23] first discussed the idea of optimum control for Hilfer fractional equations and demonstrated fixed-point outcomes. Similarly, inspired by the above work, G.Mani and S.Haque at [24] investigate the existence of a unique solution of the fractional differential equation in a bi-complex, controlled metric space.

In the realm of fixed point theory, the exploration of results within bi-complex valued metric spaces holds substantial significance due to the enriched algebraic and topological structures these spaces exhibit in comparison to classical and even complex-valued metric spaces. Bi-complex numbers, which generalize complex numbers by incorporating two imaginary units, provide a more flexible and comprehensive framework for analyzing various mathematical models. This generalization is particularly beneficial in domains involving multidimensional or hyper-complex systems, such as quantum mechanics, signal processing, and dynamic systems analysis.

The inherent complexity of bi-complex valued metric spaces allows researchers to investigate more intricate contraction mappings and convergence behaviors that may not be adequately represented in traditional settings. This adaptability not only broadens the applicability of fixed point results but also facilitates the development of novel theorems and techniques tailored for more abstract and challenging mathematical problems.

Moreover, fixed point results in bi-complex settings serve to unify and generalize existing results in real, complex, and complex-valued metric spaces. As a result, they contribute to a deeper understanding and a more universally applicable theoretical foundation within the scope of fixed point theory.

The current manuscript illustrates some unique common Fixed point results on (BCVMS). Then, we provide an application to identify the unique common solution for the fractional differential equation (FDE) system.

$$\begin{cases} {}^{\epsilon}D^{\beta}\aleph(y) + \vartheta(y,\Lambda(y)) = 0, \ \ {}^{\epsilon}D^{\beta}\Omega(y) + \upsilon(y,\chi(y)) = 0, \ \ 1 < \epsilon \leq 2, y \in [0,1]. \\ \lambda(0) = \omega(0) = \ell, \lambda(1) = \omega(1) = \jmath, where \ \ \ell \ \ and \ \ \jmath \ \ are \ \ constant. \end{cases}$$

Where ${}^{\epsilon}D^{\beta}$ represent the order of β as the Caputo fractional derivatives and $\Lambda.\Omega : [0,1] \times [0,+\infty) \to [0,+\infty)$.

In the subsequent sections, we will review fundamental definitions and notations derived from existing literature, which will be employed throughout the remainder of this work.

Throughout the manuscript, we represent the sets of real, complex, and bi-complex numbers by $\mathbb{C}_0$, $\mathbb{C}_1$, and $\mathbb{C}_2$, respectively. Segre [12] provided the following list of complex numbers.

$$\gamma = \wp_1 + \wp_2 i_1,$$

where $\wp_1, \wp_2 \in \mathbb{C}_0$, $i_1^2 = -1$.
$\mathbb{C}_1$ is represented as follows:

$$\mathbb{C}_1 = \{\gamma : \gamma = \wp_1 + \wp_2 i_1, \wp_1, \wp_2 \in \mathbb{C}_0\}.$$

Let $\gamma \in \mathbb{C}_1$, then $|\gamma| = (\wp_1^2 + \wp_2^2)^{1/2}$. All entries in $\mathbb{C}_1$ with a real-valued positive norm function $\|\cdot\| : \mathbb{C}_1 \to \mathbb{C}_0^+$ is defined by

$$\|\gamma\| = (\wp_1^2 + \wp_2^2)^{1/2}.$$

Segre [12] described the bi-complex number (BCN) as:

$$\chi = \wp_1 + \wp_2 i_1 + \wp_3 i_2 + \wp_4 i_1 i_2,$$

where $\wp_1, \wp_2, \wp_3, \wp_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 satisfy $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We represent the BCN set $\mathbb{C}_2$ as:

$$\mathbb{C}_2 = \{\chi : \chi = \wp_1 + \wp_2 i_1 + \wp_3 i_2 + \wp_4 i_1 i_2, \quad \wp_1, \wp_2, \wp_3, \wp_4 \in \mathbb{C}_0\},$$

that is,

$$\mathbb{C}_2 = \{\chi : \chi = \gamma_1 + i_2\gamma_2, \quad \gamma_1, \gamma_2 \in \mathbb{C}_1\},$$

where $\gamma_1 = \wp_1 + \wp_2 i_1 \in \mathbb{C}_1$ and $\gamma_2 = \wp_3 + \wp_4 i_1 \in \mathbb{C}_1$. If $\chi = \gamma_1 + i_2\gamma_2$ and $\nu = \omega_1 + i_2\omega_2$ are any two BCNs, then their sum is

$$\begin{aligned} \chi \pm \nu &= (\gamma_1 + i_2\gamma_2) \pm (\omega_1 + i_2\omega_2) \\ &= \gamma_1 \pm \omega_1 + i_2(\gamma_2 \pm \omega_2) \end{aligned}$$

and the product is

$$\begin{aligned} \chi.\nu &= (\gamma_1 + i_2\gamma_2)(\omega_1 + i_2\omega_2) \\ &= (\gamma_1\omega_1 - \gamma_2\omega_2) + i_2(\gamma_1\omega_2 + \gamma_2\omega_1). \end{aligned}$$

In $\mathbb{C}_2$, there exist four idempotent elements, they are 0, 1, $\varepsilon_1 = \frac{1+i_1i_2}{2}$, $\varepsilon_2 = \frac{1-i_1i_2}{2}$ of which ε_1 and ε_2 are non-trivial, such that $\varepsilon_1 + \varepsilon_2 = 1$ and $\varepsilon_1\varepsilon_2 = 0$. Every BCN $\gamma_1 + i_2\gamma_2$ may be written in a specific way as a combination of ε_1 and ε_2. Namely,

$$\chi = \gamma_1 + i_2\gamma_2 = (\gamma_1 - i_1\gamma_2)\varepsilon_1 + (\gamma_1 + i_1\gamma_2)\varepsilon_2.$$

The complex components $\chi_1 = (\gamma_1 - i_1\gamma_2)$ and $\chi_2 = (\gamma_1 + i_1\gamma_2)$ are referred to as the idempotent components of the BCN χ, and this representation of χ is known as the idempotent representation of a BCN.
Each element in $\mathbb{C}_2$ with a positive real-valued norm function $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\chi\| &= \|\gamma_1 + i_2\gamma_2\| = \left\{\|\gamma_1\|^2 + \|\gamma_2\|^2\right\}^{1/2} \\ &= \left[\frac{|\gamma_1 - i_1\gamma_2|^2 + |\gamma_1 + i_1\gamma_2|^2}{2}\right]^{1/2} \\ &= (\wp_1^2 + \wp_2^2 + \wp_3^2 + \theta_4^2)^{1/2} \end{aligned}$$

where $\chi = \wp_1 + \wp_2 i_1 + \wp_3 i_2 + \wp_4 i_1 i_2 = \gamma_1 + i_2\gamma_2 \in \mathbb{C}_2$.
The linear space $\mathbb{C}_2$ with respect to a defined norm is a normed linear space, and $\mathbb{C}_2$ is complete. Therefore, $\mathbb{C}_2$ is a Banach space. If $\chi, \nu \in \mathbb{C}_2$, then $\|\chi\nu\| \leq \sqrt{2}\|\chi\|\|\nu\|$ holds instead of $\|f\nu\| \leq \|\chi\|\|\nu\|$, and therefore $\mathbb{C}_2$ is not a Banach algebra. For any two BCNs $\chi, \nu \in \mathbb{C}_2$, then

(i) $\chi \preceq \nu \Leftrightarrow \|\chi\| \leq \|\nu\|$;

(ii) $\|\chi + \nu\| \leq \|\chi\| + \|\nu\|$;

(iii) $\|\wp\chi\| = |\wp|\|\chi\|$, where $\wp$ is in $\mathbb{C}_0$;

(iv) $\|\chi\nu\| \leq \sqrt{2}\|\chi\|\|\nu\|$, and $\|\chi\nu\| = \sqrt{2}\|\chi\|\|\nu\|$ holds if just one of χ or ν is degenerated;

(v) $\|\chi^{-1}\| = \|\chi\|^{-1}$, if χ is degenerated with $\chi \succ 0$;

(vi) $\|\frac{\chi}{\nu}\| = \frac{\|\chi\|}{\|\nu\|}$, if ν is a degenerated BCN.

The relation $\preceq$ (partial order) is defined on $\mathbb{C}_2$ as given below. Let $\mathbb{C}_2$ be a set of BCNs and $\chi = \gamma_1 + i_2\gamma_2$ and $\nu = \omega_1 + i_2\omega_2 \in \mathbb{C}_2$. Then, $\chi \preceq \nu$ if and only if $\gamma_1 \preceq \omega_1$ and $\gamma_2 \preceq \omega_2$, i.e., $\chi \preceq \nu$, if one of the following conditions are fulfilled:

(i) $\gamma_1 = \omega_1$, $\gamma_2 = \omega_2$;

(ii) $\gamma_1 \preceq \omega_1$, $\gamma_2 = \omega_2$;

(iii) $\gamma_1 = \omega_1$, $\gamma_2 \preceq \omega_2$;

(iv) $\gamma_1 \preceq \omega_1$, $\gamma_2 \preceq \omega_2$.

It is obvious that we can write $\chi \precnsim \nu$ if $\chi \preceq \nu$ and $\chi \neq \nu$, i.e., if 2, 3, or 4 are fulfilled, and we will write $\chi \preceq \nu$ if only 4 is satisfied.

Definition 1. *[8] Let $S \neq \varnothing$ and $\vartheta : S \times S \to [1, +\infty)$. The functional $m_c : S \times S \to [0, +\infty)$ is called controlled-type metric (CM) if:*

$$\begin{aligned} &(CM1) \quad m_c(\varsigma, \alpha) = 0 \Longleftrightarrow \varsigma = \alpha, \\ &(CM2) \quad m_c(\varsigma, \alpha) = m_c(\alpha, \varsigma), \\ &(CM3) \quad m_c(\varsigma, \beta) \leq \vartheta(\varsigma, \alpha) m_c(\varsigma, \alpha) + \vartheta(\alpha, \beta) m_c(\alpha, \beta), \end{aligned}$$

for all $\varsigma, \alpha, \beta \in S$. Then, the doublet (S, m_c) is called a CM space.

Example 1. *[8] Choose $S = \{1, 2, \ldots, \}$. Take $m_c : S \times S \longrightarrow [0, +\infty)$ such that*

$$m_c(\varsigma, \alpha) = \begin{cases} 0 & \text{if and only if } \varsigma = \alpha \\ \frac{1}{\varsigma} & \text{if } \varsigma = 2n \text{ and } \alpha = 2n+1, \\ \frac{1}{\alpha} & \text{if } \varsigma = 2n+1 \text{ and } \alpha = 2n, \\ 1 & \text{otherwise.} \end{cases}$$

Consider $\gamma : S \times S \longrightarrow [1, +\infty)$ as

$$\gamma(\varsigma, \alpha) = \begin{cases} \varsigma & \text{if } \varsigma = 2n \text{ and } \alpha = 2n+1, \\ \alpha & \text{if } \varsigma = 2n+1 \text{ and } \alpha = 2n, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that condition (CM1) and (CM2) are satisfied.
Now, we have to investigate condition (CM3)
***case 1**. If $\beta = \varsigma$ or $\beta = \alpha$, (d3) is satisfied.*
***case 2**. If $\beta \neq \varsigma$ and $\beta \neq \alpha$, (CM3) is true when $\varsigma = \alpha$. Now, we may assume that $\varsigma \neq \alpha$. Then, we have $\varsigma \neq \alpha \neq \beta$. It is clear that (CM3) holds in all of the following possible subcases:*

(i) α *is odd, and* ς, β *are even.*

(ii) α, β *are odd, and* ς *is even.*

(iii) α *is even, and* ς, β *are odd.*

(iv) ς, α, β *are even.*

(v) β *is odd, and* ς, α *are even.*

(vi) β *is even, and* ς, α *are odd.*

(vii) ς, α, β *are odd.*

Thus, m_c *is a controlled metric type.*

Definition 2. *[25] Let* $S \neq \varnothing$ *and* $\vartheta : S \times S \to [1, +\infty)$*. The functional* $d_b : S \times S \to \mathbb{C}_2$ *is termed the briefly bicomplex valued controlled-type metric (BCVMS) if:*

$$\begin{aligned}
&(BVCM1) \quad d_b(\varsigma, \alpha) \precsim 0,\\
&(BVCM2) \quad d_b(\varsigma, \alpha) = 0 \Longleftrightarrow \varsigma = \alpha,\\
&(BVCM3) \quad d_b(\varsigma, \alpha) = d_b(\alpha, \varsigma),\\
&(BVCM4) \quad d_b(\varsigma, \beta) \precsim \vartheta(\varsigma, \alpha) d_b(\varsigma, \alpha) + \vartheta(\alpha, \beta) d_b(\alpha, \beta),
\end{aligned}$$

for all $\varsigma, \alpha, \beta \in S$*. Then, the pair* (S, d_b) *is termed as a BVCM space.*

Example 2. *[24] Let* $S = [0, 1]$ *and define the function*

$$d_b : S \times S \to \mathbb{C}^2 \quad by \quad d_b(\sigma, v) = |\sigma - v|^2 + i^2 |\sigma - v|^2.$$

Then, (S, d_b) *is a complete bi-complex b-metric space with* $\vartheta(\sigma, v) = 2$*.*

Remark 1. *[24] Every bicomplex-valued b-metric space is a BVCM space.*

Example 3. *[25] Let* $S = \{1, 2, 3\}$ *and* $b : S \times S \to \mathbb{C}_2$ *be defined as follows:*

$$\begin{aligned}
d_b(1,1) &= d_b(2,2) = d_b(3,3) = 0,\\
d_b(2,1) &= d_b(1,2) = 4 + 4i_2,\\
d_b(3,2) &= d_b(2,3) = 1 + 2i_2,\\
d_b(3,1) &= d_b(1,3) = 1 - i_2.
\end{aligned}$$

Also, let $\vartheta : S \times S \to [1, +\infty)$ *be defined as follows:*

$$\begin{aligned}
\vartheta(1,1) &= \vartheta(2,2) = \vartheta(3,3) = 3,\\
\vartheta(1,2) &= \vartheta(2,1) = 2,\\
\vartheta(2,3) &= \vartheta(3,2) = 4,\\
\vartheta(1,3) &= \vartheta(3,1) = 1.
\end{aligned}$$

It is obvious that the conditions (BVCM1) and (BVCM3) fulfilled. Now,
case 1*. If $\varsigma = \beta$ then the condition (BVCM3) fulfilled.*
case 2*. If $\varsigma = 1$ and $\beta = 3$ (same as $\beta = 1$ and $\varsigma = 3$) and $\alpha = 2$,*

$$\begin{aligned} d_b(\varsigma, \beta) &= |d_b(1,3)| = |1 - i_2| \precsim |12 + 16i_2| \\ &= |2(4 + 4i_2) + 4(1 + 2i_2)| \precsim 2\,|4 + 4i_2| + 4\,|1 + 2i_2| \\ &= \vartheta(1,2)d_b(1,2) + \vartheta(2,3)d_b(2,3) = \vartheta(\varsigma, \alpha)d_b(\varsigma, \alpha) + \vartheta(\alpha, \beta)d_b(\alpha, \beta). \end{aligned}$$

case 3*. If $\varsigma = 1$ and $\beta = 2$ (same as $\beta = 1$ and $\varsigma = 2$) and $\alpha = 3$,*

$$\begin{aligned} d_b(\varsigma, \beta) &= |d_b(1,2)| = |4 + 4i_2| \precsim |5 + 7i_2| \\ &= |1(1 - i_2) + 4(1 + 2i_2)| \precsim |1 - i_2| + 4|1 + 2i_2| \\ &= \vartheta(1,3)d_b(1,3) + \vartheta(3,2)d_b(3,2) = \vartheta(\varsigma, \alpha)d_b(\varsigma, \alpha) + \vartheta(\alpha, \beta)d_b(\alpha, \beta). \end{aligned}$$

case 4*. If $\varsigma = 2$ and $\beta = 3$ (same as $\beta = 3$ and $\varsigma = 2$) and $\alpha = 1$,*

$$\begin{aligned} d_b(\varsigma, \beta) &= |d_b(2,3)| = |1 + 2i_2| \precsim |9 + 7i_2| \\ &= |2(4 + 4i_2) + 1(1 - i_2)| \precsim |4 + 4i_2| + 1|1 - i_2| \\ &= \vartheta(2,1)d_b(2,1) + \vartheta(1,3)d_b(1,3) = \vartheta(\varsigma, \alpha)d_b(\varsigma, \alpha) + \vartheta(\alpha, \beta)d_b(\alpha, \beta). \end{aligned}$$

Then, (S, d_b) is a (BCVMS).

Definition 3. *[25] Let (S, d_b) be a (BCVMS) with a sequence $\{\varkappa_{\mathfrak{j}}\}$ in S and $\varkappa \in S$. Then,*
[i.]

(i) A sequence $\{\varkappa_{\mathfrak{j}}\}$ in S is convergent to $\varkappa \in S$ if for all $0 \precsim \alpha \in \mathbb{C}_2$, there exists a natural number $\mathbb{N}$ such that $d_b(\varkappa_{\mathfrak{j}}, \varkappa) \precsim \alpha$ for each $\mathfrak{j} \geq \mathbb{N}$. Then, $\lim_{\mathfrak{j} \to +\infty} \varkappa_{\mathfrak{j}} = \varkappa$ or $\varkappa_{\mathfrak{j}} \to \varkappa$ as $\mathfrak{j} \to +\infty$.

(ii) If, for each $0 \precsim \alpha$ where $\alpha \in \mathbb{C}_2$, there exists a natural number $\mathbb{N}$ such that $d_b(\varkappa_{\mathfrak{j}}, \varkappa_{\mathfrak{j}+\mathfrak{m}}) \precsim \alpha$ for each $\mathfrak{m} \in \mathbb{N}$ and $\mathfrak{j} > \mathbb{N}$. Then, $\{\varkappa_{\mathfrak{j}}\}$ is referred to as a Cauchy sequence in (S, d_b).

(iii) If each Cauchy sequence is convergent in S, the (BCVMS) $(S, \pounds_{bvc})$ is said to be complete.

Theorem 1. *[25] Suppose that (S, d_b) is (BCVMS) which is complete and $\psi : S \to S$ is a map, therefore*

$$d_b(\psi\varsigma, \psi\alpha) \precsim \omega d_b(\varsigma, \alpha),$$

for all $\varsigma, \alpha \in S$, where $0 < \omega < 1$. For $\varsigma_0 \in S$, we denote $\varsigma_m = \psi^m \varsigma_0$. Suppose that

$$\max_{m \geq 1} \lim_{i \to +\infty} \frac{\vartheta(\varsigma_{i+1}, \varsigma_{i+2})\vartheta(\varsigma_{i+1}, \varsigma_m)}{\vartheta(\varsigma_i, \varsigma_{i+1})} < \frac{1}{\omega}.$$

In addition, for each $\varsigma \in S$,

$$\lim_{\eta \to +\infty} \vartheta(\varsigma_\eta, \varsigma) \quad \textit{and} \quad \lim_{\eta \to +\infty} \vartheta(\varsigma, \varsigma_\eta) \exists \textit{ and is finite.}$$

Then, ψ has a UFP.

Theorem 2. *[25] Suppose that (S, d_b) is (BCVMS) which is complete and $\psi : S \to S$ is a map, therefore*

$$d_b(\psi\varsigma, \psi\mathcal{I}) \precsim_{i2} \kappa(d_b(\psi\varsigma, \varsigma) + d_b(\psi\alpha, \alpha))$$

. for all $\varkappa, \sigma \in S$, where $0 \leq \omega < \frac{1}{2}$. For $\varsigma_0 \in S$, we denote $\varsigma_m = \psi^m \varsigma_0$. Suppose that

$$\max_{m \geq 1} \lim_{i \to +\infty} \frac{\vartheta(\varsigma_{i+1}, \varsigma_{i+2})\vartheta(\varsigma_i, \varsigma_m)}{\vartheta(\varsigma_i, \varsigma_{i+1})} < \frac{1}{\omega}.$$

where $\omega = \frac{\kappa}{1-\kappa}$
In addition, for each $\varsigma \in S$,

$$\lim_{\eta \to +\infty} \vartheta(\varsigma_\eta, \varsigma) \quad \textit{and} \quad \lim_{\eta \to +\infty} \vartheta(\varsigma, \varsigma_\eta) \textit{exists and is finite.}$$

Then, ψ has a UFP.

2. MAIN RESULTS

In this section, we provide the proof of the unique and common fixed point theorem in bi-complex valued controlled metric space . On the basis of the theorems, we also offer examples and applications. The following is the first theorem.

Theorem 3. *Let (S, d_b) be a (BCVMS) which is complete and $\psi : S \longrightarrow S$ be such that there are $\mu, \nu, \gamma \in (0, 1)$ with $\omega = \frac{\mu+\nu}{1-\gamma} < 1$, such that*

$$d_b(\psi_\varkappa, \psi_\sigma) \precsim \mu d_b(\varkappa, \sigma) + \nu d_b(\varkappa, \psi_\varkappa) + \gamma d_b(\sigma, \psi_\sigma), \tag{1}$$

for all $\varkappa, \sigma \in S$, where $0 \leq \omega < 1$. For $\varsigma_0 \in S$, we assume that $\varsigma_m = \psi^m \varsigma_0$. Let

$$\max_{m \geq 1} \lim_{i \to +\infty} \frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2})\vartheta(\varkappa_i, \varkappa_m)}{\vartheta(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\omega}. \tag{2}$$

Suppose that,

$$\lim_{\eta \to +\infty} \vartheta(\varkappa_\eta, \varkappa) \quad \textit{and} \lim_{\eta \to +\infty} \vartheta(\varkappa, \varkappa_\eta)$$

exist and are finite, and $\gamma \lim_{\eta \to +\infty} \vartheta(\varkappa_\eta, \varkappa) < 1$ for every $\varkappa \in S$, then ψ have a UFP.

Proof. The examined sequence $(\varkappa_n)$ verifies $\varkappa_{n+1} = \Psi(\varkappa_n)$ for all $n \in \mathbb{N}$. Clearly, if there exists $n_0 \in \mathbb{N}$ for which $\varkappa_{n_0+1} = \varkappa_{n_0}$, then $\Psi(\varkappa_{n_0}) = \varkappa_{n_0}$, and the proof is complete. Thus, we assume that $\varkappa_{n+1} = \varkappa_n$ for every $n \in \mathbb{N}$.
Thus by (1), we have

$$d_b(\varkappa_n, \varkappa_{n+1}) \precsim d_b(\Psi(\varkappa_{n-1}), \Psi(\varkappa_n)) \precsim \mu d_b(\varkappa_{n-1}, \varkappa_n) + \nu d_b(\varkappa_{n-1}, \Psi(\varkappa_{n-1})) + \gamma d_b(\varkappa_n, \Psi(\varkappa_n)),$$

which implies that

$$d_b(\varkappa_n, \varkappa_{n+1}) \precsim \mu d_b(\varkappa_{n-1}, \varkappa_n) + \nu d_b(\varkappa_{n-1}, \varkappa_n) + \gamma d_b(\varkappa_n, \varkappa_{n+1}),$$

$$d_b(\varkappa_n, \varkappa_{n+1}) - \gamma d_b(\varkappa_n, \varkappa_{n+1}) \precsim \mu d_b(\varkappa_{n-1}, \varkappa_n) + \nu d_b(\varkappa_{n-1}, \varkappa_n),$$

$$(1-\gamma) d_b(\varkappa_n, \varkappa_{n+1}) \precsim (\mu+\nu) d_b(\varkappa_{n-1}, \varkappa_n),$$

$$d_b(\varkappa_n, \varkappa_{n+1}) \precsim \frac{(\mu+\nu)}{(1-\gamma)} d_b(\varkappa_{n-1}, \varkappa_n),$$

$$d_b(\varkappa_n, \varkappa_{n+1}) \precsim \frac{(\mu+\nu)}{(1-\gamma)} d_b(\varkappa_{n-1}, \varkappa_n) = \omega d_b(\varkappa_{n-1}, \varkappa_n).$$

Thus, we have

$$d_b(\varkappa_n, \varkappa_{n+1}) \precsim \omega d_b(\varkappa_{n-1}, \varkappa_n) \precsim \omega^2 d_b(\varkappa_{n-2}, \varkappa_{n-1}) \precsim \cdots \precsim \omega^n d_b(\varkappa_0, \varkappa_1).$$

For all $n, m \in \mathbb{N}$ $(n < m)$*, we have*

$$\begin{aligned} d_b(\varkappa_n, \varkappa_m) &\precsim \vartheta(\varkappa_n, \varkappa_{n+1}) d_b(\varkappa_n, \varkappa_{n+1}) + \vartheta(\varkappa_{n+1}, \varkappa_m) d_b(\varkappa_{n+1}, \varkappa_m) \\ &\precsim \vartheta(\varkappa_n, \varkappa_{n+1}) d_b(\varkappa_n, \varkappa_{n+1}) + \vartheta(\varkappa_{n+1}, \varkappa_m) \vartheta(\varkappa_{n+1}, \varkappa_{n+2}) \\ &\quad \times d_b(\varkappa_{n+1}, \varkappa_{n+2}) + \vartheta(\varkappa_{n+1}, \varkappa_m) \vartheta(\varkappa_{n+2}, \varkappa_m) d_b(\varkappa_{n+2}, \varkappa_m) \\ &\precsim \vartheta(\varkappa_n, \varkappa_{n+1}) d_b(\varkappa_n, \varkappa_{n+1}) + \vartheta(\varkappa_{n+1}, \varkappa_m) \vartheta(\varkappa_{n+1}, \varkappa_{n+2}) \\ &\quad \times d_b(\varkappa_{n+1}, \varkappa_{n+2}) + \vartheta(\varkappa_{n+1}, \varkappa_m) \vartheta(\varkappa_{n+2}, \varkappa_m) \vartheta(\varkappa_{n+2}, \varkappa_{n+3}) \\ &\quad \times d_b(\varkappa_{n+3}, \varkappa_m) \\ &\precsim \cdots \\ &\precsim \vartheta(\varkappa_n, \varkappa_{n+1}) d_b(\varkappa_n, \varkappa_{n+1}) + \sum_{i=n+1}^{m-2} \left[\prod_{j=n+1}^{i} \vartheta(\varkappa_j, \varkappa_m) \right] \\ &\quad \times \vartheta(\varkappa_i, \varkappa_{i+1}) d_b(\varkappa_i, \varkappa_{i+1}) + \prod_{i=n+1}^{m-1} \vartheta(\varkappa_i, \varkappa_m) d_b(\varkappa_{m-1}, \varkappa_m). \end{aligned} \tag{3}$$

This implies that

$$\begin{aligned} d_b(\varkappa_n, \varkappa_m) &\precsim \vartheta(\varkappa_n, \varkappa_{n+1}) d_b(\varkappa_n, \varkappa_{n+1}) + \sum_{i=n+1}^{m-2} \left[\prod_{j=n+1}^{i} \vartheta(\varkappa_j, \varkappa_m) \right] \\ &\quad \times \vartheta(\varkappa_i, \varkappa_{i+1}) d_b(\varkappa_i, \varkappa_{i+1}) + \left[\prod_{i=n+1}^{m-1} \vartheta(\varkappa_j, \varkappa_m) \right] \vartheta(\varkappa_{m-1}, \varkappa_m) d_b(\varkappa_{m-1}, \varkappa_m) \\ &\precsim \vartheta(\varkappa_n, \varkappa_{n+1}) \omega^n d_b(\varkappa_n, \varkappa_{n+1}) + \sum_{i=n+1}^{m-2} \left[\prod_{j=n+1}^{i} \vartheta(\varkappa_j, \varkappa_m) \right] \vartheta(\varkappa_i, \varkappa_{i+1}) \omega^i d_b(\varkappa_0, \varkappa_1) \end{aligned}$$

$$+\left[\prod_{i=n+1}^{m-1}\vartheta(\varkappa_j,\varkappa_m)\right]\vartheta(\varkappa_{m-1},\varkappa_m)\omega^{m-1}d_b(\varkappa_0,\varkappa_1)$$

$$=\vartheta(\varkappa_n,\varkappa_{n+1})\omega^n d_b(\varkappa_0,\varkappa_1)+\sum_{i=n+1}^{m-1}\left[\prod_{j=n+1}^{i}\vartheta(\varkappa_j,\varkappa_m)\right]\vartheta(\varkappa_i,\varkappa_{i+1})\omega^i d_b(\varkappa_0,\varkappa_1). \tag{4}$$

Let

$$\Upsilon_\ell=\sum_{i=0}^{\ell}\left[\prod_{j=0}^{i}\vartheta(\varkappa_j,\varkappa_m)\right]\vartheta(\varkappa_i,\varkappa_{i+1})\omega^i d_b(\varkappa_0,\varkappa_1). \tag{5}$$

Consider

$$\Lambda_i=\left[\prod_{j=0}^{i}\vartheta(\varkappa_j,\varkappa_m)\right]\vartheta(\varkappa_i,\varkappa_{i+1})\omega^i d_b(\varkappa_0,\varkappa_1), \tag{6}$$

we have

$$\frac{\Lambda_{i+1}}{\Lambda_i}=\vartheta(\varkappa_{i+1},\varkappa_m)\frac{\vartheta(\varkappa_{i+1},\varkappa_{i+2})}{\vartheta(\varkappa_i,\varkappa_{i+1})}\omega. \tag{7}$$

We make sure that the series $\sum_i \Lambda_i$ converges in the context of the condition (2) and ratio test. Therefore, there is $\lim_{n\to+\infty}\Upsilon_\ell$. Thus the Υ_ℓ is cauchy as a result. Now, using (4), we get

$$d_b(\varkappa_n,\varkappa_m)\precsim d_b(\varkappa_0,\varkappa_1)[\omega^n\vartheta(\varkappa_i,\varkappa_{i+1})+(\Upsilon_{m-1}-\Upsilon_n)]. \tag{8}$$

Above, we used $\vartheta(\varkappa,\sigma)\geq 1$. Letting $n,m\to+\infty$ in (8) we obtain

$$\lim_{n,m\to+\infty}d_b(\varkappa_n,\varkappa_m)=0. \tag{9}$$

Thus, the sequence $\{\varkappa_n\}$ is a Cauchy in BCVMS (S,d_b).For some $\varkappa^\star\in S$ so that

$$\lim_{n\to+\infty}d_b(\varkappa_n,\varkappa^\star)=0, \tag{10}$$

that is $\varkappa_n\to\varkappa^\star$ as $n\to+\infty$.
We shall now demonstrate that $\varkappa^\star$ is a fixed point of S. By applying condition (iii) and using (1), we obtain

$$\begin{aligned}
d_b(\varkappa^\star,\psi\varkappa^\star)&\precsim\vartheta(\varkappa^\star,\varkappa_{n+1})d_b(\varkappa^\star,\varkappa_{n+1})+\vartheta(\varkappa_{n+1},\psi\varkappa^\star)d_b(\varkappa_{n+1},\psi\varkappa^\star)\\
&=\vartheta(\varkappa^\star,\varkappa_{n+1})d_b(\varkappa^\star,\varkappa_{n+1})+\vartheta(\varkappa_{n+1},\psi\varkappa^\star)d_b(\psi\varkappa_n,\psi\varkappa^\star)\\
&\precsim\vartheta(\varkappa^\star,\varkappa_{n+1})d_b(\varkappa^\star,\varkappa_{n+1})+\vartheta(\varkappa_{n+1},\psi\varkappa^\star)\left[\mu d_b(\varkappa_n,\varkappa^\star)\right.\\
&\quad\left.+\nu d_b(\varkappa_n,\psi\varkappa_n)+\gamma d_b(\varkappa^\star,\psi\varkappa^\star)\right]\\
&=\vartheta(\varkappa^\star,\varkappa_{n+1})d_b(\varkappa^\star,\varkappa_{n+1})+\vartheta(\varkappa_{n+1},\psi\varkappa^\star)\left[\mu d_b(\varkappa_n,\varkappa^\star)\right.\\
&\quad\left.+\nu d_b(\varkappa_n,\varkappa_{n+1})+\gamma d_b(\varkappa^\star,\psi\varkappa^\star)\right]
\end{aligned} \tag{11}$$

Employing the limit, $n \to +\infty$ and utilizing (3),(4) and the fact that $\lim_{\eta\to+\infty} \vartheta(\varkappa_\eta, \varkappa)$ and $\lim_{\eta\to+\infty} \vartheta(\varkappa, \varkappa_\eta)$ exist and are finite, we would have

$$d_b(\varkappa^\star, \psi\varkappa^\star) \precsim \big[\gamma \lim_{n\to+\infty} \vartheta(\varkappa^\star, \varkappa_\psi\varkappa^\star)\big] d_b(\varkappa^\star, \varkappa_\psi\varkappa^\star). \tag{12}$$

Suppose that $\varkappa^\star \neq \psi\varkappa^\star$, having in mind that $\big[\gamma \lim_{n\to+\infty} \vartheta(\varkappa^\star, \varkappa_\psi\varkappa^\star)\big] < 1$, so

$$0 \prec_{i_2} d_b(\varkappa^\star, \psi\varkappa^\star) \precsim \big[\gamma \lim_{n\to+\infty} \vartheta(\varkappa^\star, \varkappa_\psi\varkappa^\star)\big] d_b(\varkappa^\star, \varkappa_\psi\varkappa^\star) \prec_{i_2} d_b(\varkappa^\star, \varkappa_\psi\varkappa^\star). \tag{13}$$

It is a contradiction. This yields that $\varkappa^\star = \psi\varkappa^\star$.
Uniqueness:
Next, we need to justify that $\varkappa^\star$ is a unique fixed point of ψ. Suppose that there is one more fixed point $\varkappa^\bullet$ that is $\varkappa^\bullet = \psi\varkappa^\bullet$ it follows that;

$$d_b(\varkappa^\star, \varkappa^\bullet) = d_b(\psi\varkappa^\star, \psi\varkappa^\bullet) \precsim \mu d_b(\varkappa^\star, \varkappa^\bullet) + \nu d_b(\varkappa^\star, \varkappa^\star) + \gamma d_b(\varkappa^\bullet, \varkappa^\bullet)]$$

$$d_b(\varkappa^\star, \varkappa^\bullet) \precsim \mu d_b(\varkappa^\star, \varkappa^\bullet).$$

Since $\mu \in (0,1)$, so we have $d_b(\varkappa^\star, \varkappa^\bullet)$. Therefore, we have $\varkappa^\star = \varkappa^\bullet$ and thus $\varkappa^\star$ is a unique fixed point of ψ.

Theorem 4. *Let (S, ϑ, d_b) be (BCVMS) which is complete and $\Phi, \Psi : S \to S$. If there exist $\mu, \nu : S \to [0,1)$ such that:*

(i) $\mu(\Phi\varkappa) \leq \mu(\varkappa)$ *and* $\nu(\Phi\varkappa) \leq \nu(\varkappa)$;

(ii) $\mu(\Psi\varkappa) \leq \mu(\varkappa)$ *and* $\nu(\Psi\varkappa) \leq \nu(\varkappa)$;

(iii) $(\mu + \nu)(\varkappa) < 1$;

(iv)
$$d_b(\Phi\varkappa, \Psi\sigma) \precsim \mu(\varkappa) d_b(\varkappa, \sigma) + \nu(\varkappa) \frac{d_b(\varkappa, \Phi\varkappa) d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)} \tag{14}$$

for all $\varkappa, \sigma \in S$.
For $\varkappa_0 \in S$, we set $\frac{\mu(\varkappa_0)}{1-\nu(\varkappa_0)} = \omega$. Suppose that:

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2})\vartheta(\varkappa_{i+1}, \varkappa_m)}{\vartheta(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\omega} \tag{15}$$

where $\varkappa_{2n+1} = \Phi\varkappa_{2n}$ and $\varkappa_{2n+2} = \Psi\varkappa_{2n+1}$ for each $n \geq 0$.
Assume further, that for every $\varkappa \in S$, we have $\lim_{n\to+\infty} \vartheta(\varkappa_n, \varkappa)$ and $\lim_{n\to+\infty} \vartheta(\varkappa, \varkappa_n)$, which exist and are finite. Then, Φ and Ψ have a UCFP.

Proof. *Suppose $\mu_0 \in S$. We find $\{\varkappa_n\}$ in S by $\varkappa_{2n+1} = \Phi\varkappa_{2n}$ and $\varkappa_{2n+2} = \Psi\varkappa_{2n+1}$ for each $n \geq 0$. From hypothesis and (14) we get:*

$$d_b(\varkappa_{2n+1}, \varkappa_{2n+2}) \;=\; d_b(\Phi\varkappa_{2n}, \Psi\varkappa_{2n+1})$$

$$
\begin{aligned}
&\precsim \mu(\varkappa_{2n})d_b(\varkappa_{2n},\varkappa_{2n+1})+\nu(\varkappa_{2n})\frac{d_b(\varkappa_{2n},\Phi\varkappa_{2n})d_b(\varkappa_{2n+1},\Psi\varkappa_{2n+1})}{1+d_b(\varkappa_{2n},\varkappa_{2n+1})}\\
&= \mu(\varkappa_{2n})d_b(\varkappa_{2n},\varkappa_{2n+1})+\nu(\varkappa_{2n})\frac{d_b(\varkappa_{2n},\varkappa_{2n+1})d_b(\varkappa_{2n+1},\varkappa_{2n+2})}{1+d_b(\varkappa_{2n},\varkappa_{2n+1})}\\
&\precsim \mu(\varkappa_{2n})d_b(\varkappa_{2n},\varkappa_{2n+1})+\nu(\varkappa_{2n})d_b(\varkappa_{2n+1},\varkappa_{2n+2})\\
&= \mu(\Psi\varkappa_{2n-1})d_b(\varkappa_{2n},\varkappa_{2n+1})+\nu(\Psi\varkappa_{2n-1})d_b(\varkappa_{2n+1},\varkappa_{2n+2})\\
&\precsim \mu(\varkappa_{2n-1})d_b(\varkappa_{2n},\varkappa_{2n+1})+\nu(\varkappa_{2n-1})d_b(\varkappa_{2n+1},\varkappa_{2n+2})\\
&= \mu(\Phi\varkappa_{2n-2})d_b(\varkappa_{2n},\varkappa_{2n+1})+\nu(\Phi\varkappa_{2n-2})d_b(\varkappa_{2n+1},\varkappa_{2n+2})\\
&\precsim \mu(\varkappa_{2n-2})d_b(\varkappa_{2n},\varkappa_{2n+1})+\nu(\varkappa_{2n-2})d_b(\varkappa_{2n+1},\varkappa_{2n+2})\\
&.\\
&.\\
&.\\
&\precsim \mu(\varkappa_0)d_b(\varkappa_{2n},\varkappa_{2n+1})+\nu(\varkappa_0)d_b(\varkappa_{2n+1},\varkappa_{2n+2}).
\end{aligned}
$$

Which implies that,

$$
d_b(\varkappa_{2n+1},\varkappa_{2n+2})\precsim\left(\frac{\mu(\varkappa_0)}{1-\nu(\varkappa_0)}\right)d_b(\varkappa_{2n},\varkappa_{2n+1}).
$$

Similarly,

$$
\begin{aligned}
d_b(\varkappa_{2n+2},\varkappa_{2n+3}) &= d_b(\Psi\varkappa_{2n+1},\Phi\varkappa_{2n+2})=d_b(\Phi\varkappa_{2n+2},\Psi\varkappa_{2n+1})\\
&\precsim \mu(\varkappa_{2n+2})d_b(\varkappa_{2n+2},\varkappa_{2n+1})\\
&\quad +\nu(\varkappa_{2n+2})\frac{d_b(\varkappa_{2n+2},\Phi\varkappa_{2n+2})d_b(\varkappa_{2n+1},\Psi\varkappa_{2n+1})}{1+d_b(\varkappa_{2n+2},\varkappa_{2n+1})}\\
&= \mu(\varkappa_{2n+2})d_b(\varkappa_{2n+2},\varkappa_{2n+1})\\
&\quad +\nu(\varkappa_{2n+2})\frac{d_b(\varkappa_{2n+2},\varkappa_{2n+3})d_b(\varkappa_{2n+1},\varkappa_{2n+2})}{1+d_b(\varkappa_{2n+2},\varkappa_{2n+1})}\\
&= \mu(\varkappa_{2n+2})d_b(\varkappa_{2n+2},\varkappa_{2n+1})+\nu(\varkappa_{2n+2})d_b(\varkappa_{2n+3},\varkappa_{2n+2})\\
&= \mu(\Psi\varkappa_{2n+1})d_b(\varkappa_{2n+2},\varkappa_{2n+1})+\nu(\Psi\varkappa_{2n+1})d_b(\varkappa_{2n+3},\varkappa_{2n+2})\\
&\quad \mu(\Psi\varkappa_{2n+1})d_b(\varkappa_{2n+2},\varkappa_{2n+1})+\nu(\Psi\varkappa_{2n+1})d_b(\varkappa_{2n+3},\varkappa_{2n+2})\\
&\precsim \mu(\varkappa_{2n+1})d_b(\varkappa_{2n+2},\varkappa_{2n+1})+\nu(\varkappa_{2n+1})d_b(\varkappa_{2n+3},\varkappa_{2n+2})\\
&= \mu(\Phi\varkappa_{2n})d_b(\varkappa_{2n+2},\varkappa_{2n+1})+\nu(\Phi\varkappa_{2n})d_b(\varkappa_{2n+3},\varkappa_{2n+2})\\
&\precsim \mu(\varkappa_{2n})d_b(\varkappa_{2n+2},\varkappa_{2n+3})+\nu(\varkappa_{2n})d_b(\varkappa_{2n+3},\varkappa_{2n+2})\\
&.\\
&.\\
&.\\
&\precsim \mu(\varkappa_0)d_b(\varkappa_{2n+2},\varkappa_{2n+1})+\nu(\varkappa_0)d_b(\varkappa_{2n+3},\varkappa_{2n+2}).
\end{aligned}
$$

Which implies that

$$d_b(\varkappa_{2n+3},\varkappa_{2n+2}) \precsim \left(\frac{\mu(\varkappa_0)}{1-\nu(\varkappa_0)}\right) d_b(\varkappa_{2n+2},\varkappa_{2n+1}) = \omega d_b(\varkappa_{2n+2},\varkappa_{2n+1}.$$

To continue the process in this direction, we obtain,

$$d_b(\varkappa_n,\varkappa_{n+1}) \precsim \omega d_b(\varkappa_{n-1},\varkappa_n) \precsim \omega^2 d_b(\varkappa_{n-2},\varkappa_{n-1}) \precsim \cdots \omega^n d_b(\varkappa_0,\varkappa_1).$$

Thus,

$$d_b(\varkappa_n,\varkappa_{n+1}) \precsim \omega^n d_b(\varkappa_0,\varkappa_0). \tag{16}$$

For all $n,m \in \mathbb{N}$ $(n<m)$, we have

$$\begin{aligned}
d_b(\varkappa_n,\varkappa_m) &\precsim \vartheta(\varkappa_n,\varkappa_{n+1}) d_b(\varkappa_n,\varkappa_{n+1}) + \vartheta(\varkappa_{n+1},\varkappa_m) d_b(\varkappa_{n+1},\varkappa_m)\\
&\precsim \vartheta(\varkappa_n,\varkappa_{n+1}) d_b(\varkappa_n,\varkappa_{n+1}) + \vartheta(\varkappa_{n+1},\varkappa_m)\vartheta(\varkappa_{n+1},\varkappa_{n+2})\\
&\quad \times d_b(\varkappa_n+1,\varkappa_{n+2}) + \vartheta(\varkappa_{n+1},\varkappa_m)\vartheta(\varkappa_{n+2},\varkappa_m) d_b(\varkappa_{n+2},\varkappa_m)\\
&\precsim \vartheta(\varkappa_n,\varkappa_{n+1}) d_b(\varkappa_n,\varkappa_{n+1}) + \vartheta(\varkappa_{n+1},\varkappa_m)\vartheta(\varkappa_{n+1},\varkappa_{n+2})\\
&\quad \times d_b(\varkappa_{n+1},\varkappa_{n+2}) + \vartheta(\varkappa_{n+1},\varkappa_m)\vartheta(\varkappa_{n+2},\varkappa_m)\vartheta(\varkappa_{n+2},\varkappa_{n+3})\\
&\quad \times d_b(\varkappa_{n+2},\varkappa_{n+3}) + \vartheta(\varkappa_{n+1},\varkappa_m)\vartheta(\varkappa_{n+2},\varkappa_m)\\
&\quad \times \vartheta(\varkappa_{n+3},\varkappa_m) d_b(\varkappa_{n+3},\varkappa_m)\\
&\precsim \cdots\\
&\precsim \vartheta(\varkappa_n,\varkappa_{n+1}) d_b(\varkappa_n,\varkappa_{n+1}) + \sum_{i=n+1}^{m-2}\left[\prod_{j=n+1}^{i}\vartheta(\varkappa_j,\varkappa_m)\right]\\
&\quad \times \vartheta(\varkappa_i,\varkappa_{i+1}) d_b(\varkappa_i,\varkappa_{i+1}) + \prod_{i=n+1}^{m-1}\vartheta(\varkappa_i,\varkappa_m) d_b(\varkappa_{m-1},\varkappa_m)
\end{aligned} \tag{17}$$

This implies that

$$\begin{aligned}
d_b(\varkappa_n,\varkappa_m) &\precsim \vartheta(\varkappa_n,\varkappa_{n+1}) d_b(\varkappa_n,\varkappa_{n+1}) + \sum_{i=n+1}^{m-2}\left[\prod_{j=n+1}^{i}\vartheta(\varkappa_j,\varkappa_m)\right]\\
&\quad \times \vartheta(\varkappa_i,\varkappa_{i+1}) d_b(\varkappa_i,\varkappa_{i+1}) + \left[\prod_{i=n+1}^{m-1}\vartheta(\varkappa_j,\varkappa_m)\right]\\
&\quad \times \vartheta(\varkappa_{m-1},\varkappa_m) d_b(\varkappa_{m-1},\varkappa_m)\\
&\precsim \vartheta(\varkappa_n,\varkappa_{n+1})\omega^n d_b(\varkappa_n,\varkappa_{n+1}) + \sum_{i=n+1}^{m-2}\left[\prod_{j=n+1}^{i}\vartheta(\varkappa_j,\varkappa_m)\right]\\
&\quad \times \vartheta(\varkappa_i,\varkappa_{i+1})\omega^i d_b(\varkappa_0,\varkappa_1) + \left[\prod_{i=n+1}^{m-1}\vartheta(\varkappa_j,\varkappa_m)\right]
\end{aligned}$$

$$
\begin{aligned}
&\times \vartheta(\varkappa_{m-1}, \varkappa_m)\omega^{m-1} d_b(\varkappa_0, \varkappa_1) \\
&= \vartheta(\varkappa_n, \varkappa_{n+1})\omega^n d_b(\varkappa_0, \varkappa_1) + \sum_{i=n+1}^{m-1} \left[\prod_{j=n+1}^{i} \vartheta(\varkappa_j, \varkappa_m) \right] \\
&\times \vartheta(\varkappa_i, \varkappa_{i+1})\omega^i d_b(\varkappa_0, \varkappa_1)
\end{aligned} \tag{18}
$$

Let

$$
\Upsilon_\ell = \sum_{i=0}^{\ell} \left[\prod_{j=0}^{i} \vartheta(\varkappa_j, \varkappa_m) \right] \vartheta(\varkappa_i, \varkappa_{i+1})\omega^i d_b(\varkappa_0, \varkappa_1). \tag{19}
$$

Consider

$$
\Lambda_i = \left[\prod_{j=0}^{i} \vartheta(\varkappa_j, \varkappa_m) \right] \vartheta(\varkappa_i, \varkappa_{i+1})\omega^i d_b(\varkappa_0, \varkappa_1), \tag{20}
$$

we have

$$
\frac{\Lambda_{i+1}}{\Lambda_i} = \vartheta(\varkappa_{i+1}, \varkappa_m) \frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2})}{\vartheta(\varkappa_i, \varkappa_{i+1})} \omega. \tag{21}
$$

We make sure that the series $\sum_i \Lambda_i$ converges in the context of the condition (15) and ratio test. Therefore, there is $\lim_{n\to+\infty} \Upsilon_\ell$. Thus the sequence Υ_ℓ is cauchy as a result. Now, using (18), we get

$$
d_b(\varkappa_n, \varkappa_m) \precsim d_b(\varkappa_0, \varkappa_1)[\omega^n \vartheta(\varkappa_i, \varkappa_{i+1}) + (\Upsilon_{m-1} - \Upsilon_n)]. \tag{22}
$$

Above, we used $\vartheta(\varkappa, \sigma) \geq 1$. Letting $n, m \to +\infty$ in (22) we obtain

$$
\lim_{n,m\to+\infty} d_b(\varkappa_n, \varkappa_m) = 0. \tag{23}
$$

Thus, the sequence $\{\varkappa_n\}$ is a Cauchy in (BCVMS) (S, d_b, ϑ). Thus for all $\varkappa^\star \in S$ such that

$$
\lim_{n\to+\infty} d_b(\varkappa_n, \varkappa^\star) = 0, \tag{24}
$$

that is $\varkappa_n \to \varkappa^\star$ as $n \to +\infty$.
Now, we'll show that $\varkappa^\star$ is a fixed point of S. By using (14) and condition (iii), we get

$$
\begin{aligned}
d_b(\varkappa^\star, \Phi\varkappa^\star) &\precsim \vartheta(\varkappa^\star, \varkappa_{2n+2}) d_b(\varkappa^\star, \varkappa_{2n+2}) + \vartheta(\varkappa_{2n+2}, \Phi\varkappa^\star) d_b(\varkappa_{2n+2}, \Phi\varkappa^\star) \\
&= \vartheta(\varkappa^\star, \varkappa_{2n+2}) d_b(\varkappa^\star, \varkappa_{2n+2}) + \vartheta(\varkappa_{2n+2}, \Phi\varkappa^\star) d_b(\Psi\varkappa_{2n+1}, \Phi\varkappa^\star) \\
&= \vartheta(\varkappa^\star, \varkappa_{2n+2}) d_b(\varkappa^\star, \varkappa_{2n+2}) + \vartheta(\varkappa_{2n+2}, \Phi\varkappa^\star) d_b(\Phi\varkappa^\star, \psi\varkappa_{2n+1}) \\
&= \vartheta(\varkappa^\star, \varkappa_{2n+2}) d_b(\varkappa^\star, \varkappa_{2n+2}) + \vartheta(\varkappa_{2n+2}, \Phi\varkappa^\star) \Big[\mu(\varkappa^\star) d_b(\varkappa^\star, \varkappa_{2n+1}) \\
&\quad + \nu(\varkappa^\star) \frac{d_b(\varkappa^\star, \Phi\varkappa^\star) d_b(\varkappa_{2n+1}, \varkappa_{2n+2})}{1 + d_b(\varkappa^\star, \varkappa_{2n+2})} \Big]
\end{aligned} \tag{25}
$$

Letting $n \to +\infty$ and by using (24), there arise contradiction to $d_b(\varkappa^\star, \Phi\varkappa^\star) \prec_{i_2} 0$. Thus, $d_b(\varkappa^\star, \Phi\varkappa^\star) = 0$. This implies that $\varkappa^\star = \Phi\varkappa^\star$. Similarly one can show that $\varkappa^\star =$

$\Psi\varkappa^\star$. Therefore, $\varkappa^\star$ is common fixed point of Φ and Ψ.
Uniqueness:
Now we have to show that $\varkappa^\star$ is a unique fixed point of Ψ and Φ. Assume that there exists another common fixed point $\varkappa^\bullet$ that is $\varkappa^\bullet = \Psi\varkappa^\bullet = \Phi\varkappa^\bullet$. It follows that:

$$d_b(\varkappa^\star, \varkappa^\bullet) = d_b(\Phi\varkappa^\star, \Psi\varkappa^\bullet) \precsim \mu(\varkappa^\star)d_b(\varkappa^\star, \varkappa^\bullet) + \nu(\varkappa^\star)\frac{d_b(\varkappa^\star, \Phi\varkappa^\star)d_b(\varkappa^\bullet, \Psi\varkappa^\bullet)}{1 + d_b(\varkappa^\star, \varkappa^\bullet)}$$

$$d_b(\varkappa^\star, \varkappa^\bullet) \precsim \mu(\varkappa^\star)d_b(\varkappa^\star, \varkappa^\bullet).$$

Since $\mu \in [0, 1)$, so we have $d_b(\varkappa^\star, \varkappa^\bullet)$. Therefore, we have $\varkappa^\star = \varkappa^\bullet$ and thus $\varkappa^\star$ is a unique common fixed point ofΦ and Ψ.

Corollary 1. *Let (S, ϑ, d_b) be a complete controlled metric space and $\Phi, \Psi : S \to S$. If there exist $\mu, \nu : S \to [0, 1)$ such that:*

$$d_b(\Phi\varkappa, \Psi\nu) \precsim \mu d_b(\varkappa, \sigma) + \nu\frac{d_b(\varkappa, \Phi\varkappa)d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)}$$

for all $\varkappa, \sigma \in S$.
For $\varkappa_0 \in S$, we set $\frac{\mu(\varkappa)}{1-\nu(\varkappa)} = \omega$. Suppose that,

$$\sup_{m\geq 1}\lim_{i\to+\infty}\frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2})\vartheta(\varkappa_{i+1}, \varkappa_m)}{\vartheta(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\omega}$$

where $\varkappa_{2n+1} = \Phi\varkappa_{2n}$ and $\varkappa_{2n+2} = \Psi\varkappa_{2n+1}$ for each $n \geq 0$.
Assume further, that for every $\varkappa \in S$, we have $\lim_{n\to+\infty}\vartheta(\varkappa_n, \varkappa)$ and $\lim_{n\to+\infty}\vartheta(\varkappa, \varkappa_n)$, which are finite and exist. Then, Φ and Ψ have a UCFP.

Proof. The proof is trivial by setting $\mu(\varkappa) = \mu$ and $\nu(\varkappa) = \nu$ in the Theorem (4).

Corollary 2. *Let (S, ϑ, d_b)be a (BCVMS) which is complete and $\Psi : S \to S$. If there exist $\mu, \nu : S \to [0, 1)$ such that:*

(i) $\mu(\Psi\varkappa) \leq \mu(\varkappa)$ *and* $\nu(\Psi\varkappa) \leq \nu(\varkappa)$;

(ii) $(\mu + \nu)(\varkappa) < 1$;

(iii)

$$d_b(\Psi\varkappa, \Psi\nu) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Psi\varkappa)d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)}$$

for all $\varkappa, \sigma \in S$. For $\varkappa_0 \in S$, we set $\frac{\mu(\varkappa_0)}{1-\nu(\varkappa_0)} = \omega$. Suppose that,

$$\sup_{m\geq 1}\lim_{i\to+\infty}\frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2})\vartheta(\varkappa_{i+1}, \varkappa_m)}{\vartheta(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\omega}$$

where $\varkappa_{n+1} = \Psi\varkappa_n$ *and for each* $n \geq 0$.
Assume further, that for every $\varkappa \in S$, *we have* $\lim_{n\to+\infty} \vartheta(\varkappa_n, \varkappa)$ *and* $\lim_{n\to+\infty} \vartheta(\varkappa, \varkappa_n)$, *which exist and are finite. Then,*Ψ *have a UFP.*

Proof. *The proof is trivial by setting* $\Psi = \Phi$ *in the Theorem (4) this result can be obtained.*

Corollary 3. *Let* (S, ϑ, d_b)*be a (BCVMS) which is complete and* $\Psi : S \to S$. *If there exist* $\mu, \nu : S \to [0, 1)$ *such that:*

(i) $\mu(\Psi\varkappa) \leq \mu(\varkappa)$;

(ii)

$$d_b(\Psi\varkappa, \Psi\nu) \precsim \mu(\varkappa)d_b(\varkappa, \sigma)$$

for all $\varkappa, \sigma \in S$.
For $\varkappa_0 \in S$. *Suppose that:*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2})\vartheta(\varkappa_{i+1}, \varkappa_m)}{\vartheta(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\mu(\varkappa_0)}$$

where $\varkappa_{n+1} = \Psi\varkappa_n$ *and for each* $n \geq 0$.
Assume further, that for every $\varkappa \in S$, *we have* $\lim_{n\to+\infty} \vartheta(\varkappa_n, \varkappa)$ *and* $\lim_{n\to+\infty} \vartheta(\varkappa, \varkappa_n)$, *which exist and are finite. Then,*Ψ *have a UFP.*

Proof. *This result can be obtained by putting* $\nu(\varkappa) = 0$ *in the Corollary (2).*

Corollary 4. *Let* (S, ϑ, d_b)*be a (BCVMS) which is complete and* $\Psi : S \to S$. *If there exist* $\mu, \nu : S \to [0, 1)$ *such that:*

$$d_b(\Psi\varkappa, \Psi\nu) \precsim \mu d_b(\varkappa, \sigma) + \nu\frac{d_b(\varkappa, \Psi\varkappa)d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)}$$

for all $\varkappa, \sigma \in S$.
For $\varkappa_0 \in S$, *we set* $\frac{\mu(\varkappa_0)}{1-\nu(\varkappa_0)} = \omega$. *Suppose that,*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2})\vartheta(\varkappa_{i+1}, \varkappa_m)}{\vartheta(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\omega}$$

where $\varkappa_{n+1} = \Psi\varkappa_n$ *and for each* $n \geq 0$.
Assume further, that for every $\varkappa \in S$, *we have* $\lim_{n\to+\infty} \vartheta(\varkappa_n, \varkappa)$ *and* $\lim_{n\to+\infty} \vartheta(\varkappa, \varkappa_n)$, *which exist and are finite. Then,*Ψ *have a UFP.*

Proof. *By setting* $\mu(\varkappa) = \mu$ *and* $\nu(\varkappa) = \nu$ *in the Corollary (2) this result can be obtained.*

Corollary 5. *Let* (S, ϑ, d_b)*be a (BCVMS) which is complete and* $\Psi : S \to S$. *If there exist* $\mu, \nu : S \to [0, 1)$ *such that:*

$$d_b(\Psi\varkappa, \Psi\nu) \precsim \mu d_b(\varkappa, \sigma)$$

for all $\varkappa, \sigma \in S$.
For $\varkappa_0 \in S$. *Suppose that,*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2})\vartheta(\varkappa_{i+1}, \varkappa_m)}{\vartheta(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\mu}$$

where $\varkappa_{n+1} = \Psi\varkappa_n$ *and for each* $n \geq 0$.
Assume further, that for every $\varkappa \in S$, *we have* $\lim_{n\to+\infty} \vartheta(\varkappa_n, \varkappa)$ *and* $\lim_{n\to+\infty} \vartheta(\varkappa, \varkappa_n)$, *which exist and are finite. Then,* Ψ *have a UFP.*

Proof. *This result can be obtained by putting* $\mu(\varkappa) = \mu$ *in Corollary (3).*

Remark 2. *Corollary (4) and (5) are the outcomes of paper [24].*

Theorem 5. *Let* (S, ϑ, d_b) *be a (BCVMS) which is complete and* $\Psi : S \to S$. *If there exist* $\mu, \nu : S \to [0, 1)$ *such that:*

(i) $\mu(\Psi^n\varkappa) \leq \mu(\varkappa)$ *and* $\nu(\Psi^n\varkappa) \leq \nu(\varkappa)$;

(ii) $(\mu + \nu)(\varkappa) < 1$;

(iii)

$$d_b(\Psi^n\varkappa, \Psi^n\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Psi^n\varkappa)d_b(\sigma, \Psi^n\sigma)}{1 + d_b(\varkappa, \sigma)} \tag{26}$$

for all $\varkappa, \sigma \in S$.
For $\varkappa_0 \in S$, *we set* $\frac{\mu(\varkappa_0)}{1-\nu(\varkappa_0)} = \omega$. *Suppose that,*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2})\vartheta(\varkappa_{i+1}, \varkappa_m)}{\vartheta(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\omega}$$

where $\varkappa_{n+1} = \Psi\varkappa_n$ *and for each* $n \geq 0$.
Assume further, that for every $\varkappa \in S$, *we have* $\lim_{n\to+\infty} \vartheta(\varkappa_n, \varkappa)$ *and* $\lim_{n\to+\infty} \vartheta(\varkappa, \varkappa_n)$, *which exist and are finite. Then,* Ψ *have a UFP.*

Proof. *By the Corollary (2),* Ψ^n *possess a unique fixed point* $\varkappa^\star$. *It can be deduced from,*

$$\Psi^n(\Psi\varkappa^\star) = \Psi(\Psi^n\varkappa^\star) = \Psi\varkappa^\star$$

that $\Psi\varkappa^\star$ *is a fixed point of* Ψ^n. *Thus* $\Psi\varkappa^\star = \varkappa^\star$ *by the uniqueness of a fixed point of* $\varkappa^n$ *and then* $\varkappa^\star$ *is also fixed point of* Ψ. *As a result, since the fixed point is unique so it is the fixed point of both* Ψ *and* Ψ^n.

Corollary 6. *Let* (S, ϑ, d_b) *be a (BCVMS) which is complete and* $\Psi : S \to S$. *If there exist* $\mu, \nu : S \to [0, 1)$ *such that:*

$$d_b(\Psi^n \varkappa, \Psi^n \sigma) \precsim \varkappa d_b(\varkappa, \sigma) + \nu \varkappa \frac{d_b(\varkappa, \Psi^n \varkappa) d_b(\sigma, \Psi^n \sigma)}{1 + d_b(\varkappa, \sigma)}$$

for all $\varkappa, \sigma \in S$.
For $\varkappa_0 \in S$*, we set* $\frac{\varkappa}{1-\varkappa} = \omega$*. Suppose that,*

$$\sup_{m \geq 1} \lim_{i \to +\infty} \frac{\vartheta(\varkappa_{i+1}, \varkappa_{i+2}) \vartheta(\varkappa_{i+1}, \varkappa_m)}{\vartheta(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\omega}$$

where $\varkappa_{n+1} = \Psi \varkappa_n$ *and for each* $n \geq 0$.
Assume further, that for every $\varkappa \in S$*, we have* $\lim_{n \to +\infty} \vartheta(\varkappa_n, \varkappa)$ *and* $\lim_{n \to +\infty} \vartheta(\varkappa, \varkappa_n)$*, which exist and are finite. Then,*Ψ *have a UFP.*

Proof. This result can be obtained by setting $\mu(\varkappa) = \mu$ *and* $\nu(\varkappa) = \nu$ *in Theorem (5).*

Example 4. *Let* d_b *be a (BCVMS) defined on* $S = \{0, 1, 2\}$ *such that,*

$$d_b(0, 1) = d_b(1, 2) = 1 + i_2, \quad d_b(0, 2) = 4 + 4i_2 \ .$$

Where $\vartheta : S \times S \to [1, +\infty)$ *such that*
$\vartheta(0, 0) = \vartheta(1, 1) = \vartheta(2, 2) = \vartheta(1, 2) = 1, \quad \vartheta(0, 2) = 2, \quad \vartheta(0, 1) = \frac{3}{2}$.
Let $\psi : S \to S$ *as*

$$\psi(0) = 2 \text{ and } \psi(1) = \psi(2) = 1.$$

Consider,

$$\mu = \tfrac{1}{11} \text{ and } \nu = \gamma = \tfrac{3}{11},$$

and $\varkappa_0 = 0$ *, then* $\varkappa_1 = 2$ *and* $\varkappa_n = 1$ *for all* $n \geq 2$.
Clearly, (2) is satisfied. Now, we take different cases to check that (1) is also satisfied.
Case I:
If $\varkappa = \sigma = 0, \quad \varkappa = \sigma = 1, \quad \varkappa = \sigma = 2$.
Then clearly our result can be obtained.
Case II:
If $\varkappa = 0$ *and* $\sigma = 1$ *then we obtain,*

$$d_b(\psi \varkappa, \psi \sigma) = d_b(\psi(0), \psi(1)) = 1 + i_2,$$

$d_b(\varkappa, \sigma) = d_b(0, 1) = 1 + i_2$
$d_b(\varkappa, \psi \varkappa) = d_b(0, \psi(0)) = 4 + 4i_2$,
$d_b(\sigma, \psi \sigma) = d_b(1, \psi(1)) = 0 + 0i_2$.

$$d_b\left(\psi \varkappa, \psi \sigma\right) \precsim \mu d_b(\varkappa, \sigma) + \nu d_b\left(\varkappa, \psi \varkappa\right) + \gamma d_b\left(\sigma, \psi \sigma\right),$$

The above distances contraction condition of Theorem (3) by using the partial order for BCNs, Thus the result is obvious for Case(II).

Case III:
If $\varkappa = 0$ and $\sigma = 2$ then we obtain,

$$d_b(\psi\varkappa, \psi\sigma) = d_b(\psi(0), \psi(2)) = 1 + i_2,$$

$d_b(\varkappa, \sigma) = d_b(0, 2) = 4 + 4i_2,$
$d_b(\varkappa, \psi\varkappa) = d_b(0, \psi(0)) = 4 + 4i_2,$
$d_b(\sigma, \psi\sigma) = d_b(2, \psi(2)) = 1 + i_2.$

$$d_b(\psi\varkappa, \psi\sigma) \precsim \mu d_b(\varkappa, \sigma) + \nu d_b(\varkappa, \psi\varkappa) + \gamma d_b(\sigma, \psi\sigma),$$

The above distances contraction condition of Theorem (3) by using the partial order for BCNs, Thus the result is obvious for Case(III).
Case IV:
If $\varkappa = 1$ and $\sigma = 2$ then we obtain,

$$d_b(\psi\varkappa, \psi\sigma) = d_b(\psi(1), \psi(2)) = 0 + 0i_2,$$

$d_b(\varkappa, \sigma) = d_b(1, 2) = 1 + i_2,$
$d_b(\varkappa, \psi\varkappa) = d_b(1, \psi(1)) = 0 + 0i_2,$
$d_b(\sigma, \psi\sigma) = d_b(2, \psi(2)) = 1 + i_2.$

$$d_b(\psi\varkappa, \psi\sigma) \precsim \mu d_b(\varkappa, \sigma) + \nu d_b(\varkappa, \psi\varkappa) + \gamma d_b(\sigma, \psi\sigma),$$

The above distances satisfies contraction condition of Theorem (3) by using the partial order for BCNs, Thus the result is also obvious for Case(IV).
Therefore, all the necessities of the Theorem (3) are true for all the cases so ψ has a unique fixed point.

Example 5. *Let d_b be a (BCVMS) defined on $S = \{0, 1, 2\}$ such that,*

$$d_b(0, 1) = 60 + 60i_2, \quad d_b(1, 2) = 1 + i_2 \text{ and } \quad d_b(0, 2) = 90 + 90i_2 .$$

Where $\vartheta : S \times S \to [1, +\infty)$ such that,
$\vartheta(0, 0) = \vartheta(1, 1) = \vartheta(2, 2) = \vartheta(1, 2) = 1, \quad \vartheta(0, 2) = 2, \quad \vartheta(0, 1) = \frac{3}{2}.$
Let $\Phi, \Psi : S \to S$ as

$$\Psi(0) = \Phi(0) = 1 \text{ and } \Psi(1) = \Phi(1) = \Psi(2) = \Phi(2) = 2.$$

Consider,

$$\mu(\varkappa) = \tfrac{(\varkappa-3)^2}{30} \text{ and } \nu(\varkappa) = \tfrac{(\varkappa-3)^2}{20},$$

and
$\varkappa_0 = 0$, *then $\varkappa_1 = 2$ and $\varkappa_n = 1$ for all $n \geq 2$.*
Clearly, condition(i),(ii),(iii) and (15) are satisfied.
Now, we take different cases to check that (14) is also satisfied.
Case I:

If $\varkappa = \sigma = 0, \quad \varkappa = \sigma = 1, \quad \varkappa = \sigma = 2.$
Then clearly our result can be obtained.
Case II:
If $\varkappa = 0$ *and* $\sigma = 1$ *then we obtain*
$d_b(\Phi\varkappa, \Psi\sigma) = d_b(\Phi(0), \Psi(1)) = 1 + i_2,$
$d_b(\varkappa, \sigma) = d_b(0, 1) = 60 + 60i_2,$
$d_b(\sigma, \Psi\sigma) = d_b(1, \Psi(1)) = 1 + i_2,$
$d_b(\varkappa, \phi\varkappa) = d_b(0, \Phi(0)) = 60 + 60i_2.$

$$d_b(\Phi\varkappa, \Psi\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Phi\varkappa)d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies the contraction for Theorem (4) *by using type partial order for BCNs, Thus the result is obvious for Case(II).*
Case III:
If $\varkappa = 0$ *and* $\sigma = 2$ *then we obtain*
$d_b(\Phi\varkappa, \Psi\sigma) = d_b(\Phi(0), \Psi(2)) = 1 + i_2,$
$d_b(\varkappa, \sigma) = d_b(0, 2) = 90 + 90i_2,$
$d_b(\varkappa, \Phi\varkappa) = d_b(0, \Phi(0)) = 60 + 60i_2,$
$d_b(\sigma, \psi\sigma) = d_b(2, \psi(2)) = 0 + 0i_2.$

$$d_b(\Phi\varkappa, \Psi\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Phi\varkappa)d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies the contraction for Theorem (4) *by using type partial order for BCNs, Thus the result is obvious for Case(III).*
Case IV:
If $\varkappa = 1$ *and* $\sigma = 2$ *then we obtain,* $d_b(\Phi\varkappa, \Psi\sigma) = d_b(\Phi(1), \Psi(2)) = 0 + 0i_2,$
$d_b(\varkappa, \sigma) = d_b(1, 2) = 1 + i_2,$
$d_b(\varkappa, \Phi\varkappa) = d_b(1, \Phi(1)) = 1 + i_2,$
$d_b(\sigma, \Psi\sigma) = d_b(2, \Psi(2)) = 0 + 0i_2.$

$$d_b(\Phi\varkappa, \Psi\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Phi\varkappa)d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies the contraction for Theorem (4) *by using type partial order for BCNs, Thus the result is obvious for Case(IV).*
Case V:
If $\varkappa = 1$ *and* $\sigma = 0$ *then we obtain,* $d_b(\Phi\varkappa, \Psi\sigma) = d_b(\Phi(1), \Psi(0)) = 1 + 1i_2,$
$d_b(\varkappa, \sigma) = d_b(1, 0) = 60 + 60i_2,$
$d_b(\varkappa, \Phi\varkappa) = d_b(1, \Phi(1)) = 1 + i_2,$
$d_b(\sigma, \Psi\sigma) = d_b(0, \Psi(0)) = 60 + 60i_2.$

$$d_b(\Phi\varkappa, \Psi\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Phi\varkappa)d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies the contraction for Theorem (4) by using type partial order for BCNs, Thus the result is obvious for Case(V).

Case VI:

If $\varkappa = 2$ *and* $\sigma = 0$ *then we obtain,* $d_b(\Phi\varkappa, \Psi\sigma) = d_b(\Psi(2), \Phi(0)) = 1 + i_2$,
$d_b(\varkappa, \sigma) = d_b(2, 0) = 90 + 90i_2$,
$d_b(\varkappa, \Phi\varkappa) = d_b(2, \Phi(2)) = 0 + 0i_2$,
$d_b(\sigma, \Psi\sigma) = d_b(0, \Psi(0)) = 1 + i_2$.

$$d_b(\Phi\varkappa, \Psi\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Phi\varkappa)d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies the contraction for Theorem (4) by using type partial order for BCNs, Thus the result is obvious for Case(VI).

Case VII:

If $\varkappa = 2$ *and* $\sigma = 1$ *then we obtain,* $d_b(\Phi\varkappa, \Psi\sigma) = d_b(\Psi(2), \Phi(1)) = 0 + 0i_2$,
$d_b(\varkappa, \sigma) = d_b(2, 1) = 1 + i_2$,
$d_b(\varkappa, \Phi\varkappa) = d_b(2, \Phi(2)) = 0 + 0i_2$,
$d_b(\sigma, \Psi\sigma) = d_b(2, \Psi(1)) = 0 + 0i_2$.

$$d_b(\Phi\varkappa, \Psi\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Phi\varkappa)d_b(\sigma, \Psi\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies the contraction for Theorem (4) by using type partial order for BCNs, Thus the result is obvious for Case(VII).
Therefore all the axioms of Theorem (4) are fulfilled for all the cases, So Φ *and* Ψ *have a unique common fixed point.*

Example 6. *Let* d_b *be a BCCM defined on* $S = \{0, 1, 2\}$ *such that;*

$$d_b(0, 1) = 60 + 60i_2, \quad d_b(1, 2) = 1 + i_2 \text{ and } \quad d_b(0, 2) = 90 + 90i_2.$$

Where $\vartheta : S \times S \to [1, +\infty)$ *such that*
$\vartheta(0, 0) = \vartheta(1, 1) = \vartheta(2, 2) = \vartheta(1, 2) = 1, \quad \vartheta(0, 2) = 2, \quad \vartheta(0, 1) = \frac{3}{2}$.
Let $\Psi : S \to S$ *as*

$$\Psi^n(0) = 1 \text{ and } \Psi^n(1) = \Psi^n(2) = 2.$$

Consider,

$$\mu(\varkappa) = \tfrac{(\varkappa-3)^2}{30} \text{ and } \nu(\varkappa) = \tfrac{(\varkappa-3)^2}{20}.$$

and
$\varkappa_0 = 0$ *, then* $\varkappa_1 = 2$ *and* $\varkappa_n = 1$ *for all* $n \geq 2$.
Clearly, condition(i),(ii),(iii) and (5) are satisfied.
Now, we take different cases to check that (26) is also satisfied.

Case I:

If $\varkappa = \sigma = 0, \quad \varkappa = \sigma = 1, \quad \varkappa = \sigma = 2$.

Then clearly our result can be obtained.

Case II:

If $\varkappa = 0$ *and* $\sigma = 1$ *then we obtain,*
$d_b(\Psi^n\varkappa, \Psi^n\sigma) = d_b(\Psi^n(0), \Psi^n(1)) = 1 + i_2,$
$d_b(\varkappa, \sigma) = d_b(0, 1) = 60 + 60i_2,$
$d_b(\sigma, \Psi^n\sigma) = d_b(1, \Psi^n(1)) = 1 + i_2,$
$d_b(\varkappa, \Psi^n\varkappa) = d_b(0, \Psi^n(0)) = 60 + 60i_2.$

$$d_b(\Psi^n\varkappa, \Psi^n\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Psi^n\varkappa)d_b(\sigma, \Psi^n\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies contraction condition for Theorem (5) *by using type partial order for BCNs, Thus the result is obvious for Case(II).*

Case III:

If $\varkappa = 0$ *and* $\sigma = 2$ *then we obtain*
$d_b(\Psi^n\varkappa, \Psi^n\sigma) = d_b(\Psi^n(0), \Psi^n(2)) = 1 + i_2,$
$d_b(\varkappa, \sigma) = d_b(0, 2) = 90 + 90i_2,$
$d_b(\varkappa, \Psi^n\varkappa) = d_b(0, \Psi^n(0)) = 60 + 60i_2,$
$d_b(\sigma, \Psi^n\sigma) = d_b(2, \Psi^n(2)) = 0 + 0i_2.$

$$d_b(\Psi^n\varkappa, \Psi^n\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Psi^n\varkappa)d_b(\sigma, \Psi^n\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies contraction condition for Theorem (5) *by using type partial order for BCNs, Thus the result is obvious for Case(III).*

Case IV:

If $\varkappa = 1$ *and* $\sigma = 2$ *then we obtain*
$d_b(\Psi^n\varkappa, \Psi^n\sigma) = d_b(\Psi^n(1), \Psi^n(2)) = 0 + 0i_2,$
$d_b(\varkappa, \sigma) = d_b(1, 2) = 1 + i_2,$
$d_b(\varkappa, \Psi^n\varkappa) = d_b(1, \Psi^n(1)) = 1 + i_2,$
$d_b(\sigma, \Psi^n\sigma) = d_b(2, \Psi^n(2)) = 0 + 0i_2.$

$$d_b(\Psi^n\varkappa, \Psi^n\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Psi^n\varkappa)d_b(\sigma, \Psi^n\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies contraction condition for Theorem (5) *by using type partial order for BCNs, Thus the result is obvious for Case(IV).*

Case V:

If $\varkappa = 1$ *and* $\sigma = 0$ *then we obtain*
$d_b(\Psi^n\varkappa, \Psi^n\sigma) = d_b(\Psi^n(1), \Psi^n(0)) = 1 + 1i_2,$
$d_b(\varkappa, \sigma) = d_b(1, 0) = 60 + 60i_2,$
$d_b(\varkappa, \Psi^n\varkappa) = d_b(1, \Psi^n(1)) = 1 + i_2,$
$d_b(\sigma, \Psi^n\sigma) = d_b(0, \Psi^n(0)) = 60 + 60i_2.$

$$d_b(\Psi^n\varkappa, \Psi^n\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Psi^n\varkappa)d_b(\sigma, \Psi^n\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies contraction condition for Theorem (5) by using type partial order for BCNs, Thus the result is obvious for Case(V).

Case VI:

If $\varkappa = 2$ and $\sigma = 0$ then we obtain
$d_b(\Psi^n\varkappa, \Psi^n\sigma) = d_b(\Psi^n(2), \Psi^n(0)) = 1 + i_2$,
$d_b(\varkappa, \sigma) = d_b(2, 0) = 90 + 90i_2$,
$d_b(\varkappa, \Psi^n\varkappa) = d_b(2, \Psi^n(2)) = 0 + 0i_2$,
$d_b(\sigma, \Psi^n\sigma) = d_b(0, \Psi^n(0)) = 1 + i_2$.

$$d_b(\Psi^n\varkappa, \Psi^n\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Psi^n\varkappa)d_b(\sigma, \Psi^n\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies contraction condition for Theorem (5) by using type partial order for BCNs, Thus the result is obvious for Case(VI).

Case VII:

If $\varkappa = 2$ and $\sigma = 1$ then we obtain
$d_b(\Psi^n\varkappa, \Psi^n\sigma) = d_b(\Psi^n(2), \Psi^n(1)) = 0 + 0i_2$,
$d_b(\varkappa, \sigma) = d_b(2, 1) = 1 + i_2$,
$d_b(\varkappa, \Psi^n\varkappa) = d_b(2, \Psi^n(2)) = 0 + 0i_2$,
$d_b(\sigma, \Psi^n\sigma) = d_b(2, \Psi^n(1)) = 0 + 0i_2$.

$$d_b(\Psi^n\varkappa, \Psi^n\sigma) \precsim \mu(\varkappa)d_b(\varkappa, \sigma) + \nu(\varkappa)\frac{d_b(\varkappa, \Psi^n\varkappa)d_b(\sigma, \Psi^n\sigma)}{1 + d_b(\varkappa, \sigma)}$$

The above distances satisfies contraction condition for Theorem (5) by using type partial order for BCNs, Thus the result is obvious for Case(VII).
Therefore all the axioms of Theorem (5) are fulfilled for all the cases, Thus Ψ has a UFP.

3. Application

In this section we solve the system of fractional differential equation with the help of Theorem(4)

$$\begin{cases} {}^{\epsilon}D^{\beta}\aleph(y) + \vartheta(y, \Lambda(y)) = 0,\ {}^{\epsilon}D^{\beta}\Omega(y) + \upsilon(y, \chi(y)) = 0,\ \ 1 < \epsilon \leq 2, y \in [0, 1]. \\ \lambda(0) = \omega(0) = \ell, \lambda(1) = \omega(1) = \jmath, \textit{where}\ \ \ell \textit{ and } \jmath \textit{ are constant.} \end{cases} \tag{27}$$

Where ${}^{\epsilon}D^{\beta}$ represent the order of β as the Caputo fractional derivatives and $\Lambda.\Omega : [0, 1] \times [0, +\infty) \to [0, +\infty)$.
By using the result of [26] we have ,

$$I^{\beta}[D^{\beta}\aleph(y)] = \aleph(y) + \mathcal{C}_0 + y\mathcal{C}_1 + y^2\mathcal{C}_2 + ... + y^{n-} + \mathcal{C}_{n-1}y^{n-1},$$

where $n = [\beta] + 1$ and $\mathcal{C} \in \mathbb{R}^+$ and I^{β} is the integral operator of fractional order.The system of integral equations then provides the solution to (27);

$$\begin{cases} \aleph(y) = \ell + y(\jmath - \ell) + \int_0^1 \Im(y, s)\Delta(s, \varkappa(s))ds, \\ \omega(y) = \ell + y(\jmath - \ell) + \int_0^1 \Im(y, s)\Theta(s, \sigma(s))ds \end{cases} \tag{28}$$

where

$$\Im(y,s)=\frac{1}{\beth(\beta)}\begin{cases} y(1-s)^{\beta-1}-(y-s)^{\beta-1}, & 0\le s\le y\le 1,\\ y(1-s)^{\beta-1}, & 0\le s\le y\le 1.\end{cases}$$

In the above system (28), let us denote
$\Gamma(y)=\ell+y(\jmath-\ell)$, $\quad R_{\varkappa}(y)=\int_0^1\Im(y,s)\Delta(s,\varkappa(s))ds, \quad S_{\sigma}(y)=\int_0^1\Im(y,s)\Theta(s,\sigma(s))ds.$
Consider $\mathcal{C}([0,1],\mathbb{R})=S$ is a space described by $[0,1]$, and $d_b: S\times S\to\mathbb{C}_2$ is a (BCVMS), such that;

$$d_b(\varkappa,\sigma)=\sup_{m\in[0,1]}\mid\varkappa(m)-\sigma(m)\mid^2+i_2\sup_{m\in[0,1]}\mid\varkappa(m)-\sigma(m)\mid^2,$$

for all $\varkappa,\sigma\in S$. Let $\vartheta: S\times S\to[1,+\infty)$ be defined by $\vartheta(\varkappa,\sigma)=2$,
for all $\varkappa,\sigma\in S$. Then, (S,d_b) is (BCVMS).

Theorem 6. *Consider a system of non-linear fractional differential equations (27). Assume that the following claims are verified:*
If for all $y\in[0,1]$ there exist $\mu,\nu: S\to[0,1)$ such that:

(i) $\mu(R_{\varkappa}+\Gamma(y))\le\mu(\varkappa)$ *and* $\nu(R_{\varkappa}+\Gamma(y))\le\nu(\varkappa)$*;*

(ii) $\mu(S_{\sigma}+\Gamma(y))\le\mu(\sigma)$ *and* $\nu(S_{\sigma}+\Gamma(y))\le\nu(\sigma)$*;*

(iii) $(\mu+\nu)(\varkappa)<1$ *;*

(iv) $\parallel R_{\varkappa}(y)-S_{\sigma}(y)\parallel^2\precsim\mu(\varkappa)\digamma_1(\varkappa,\sigma)(y)+\nu(\varkappa)\digamma_2(\varkappa,\sigma)(y)$

(v) $\sup_{y\in[0,1]}\int_0^1\Im(y,s)ds<1.$

For all $\varkappa,\sigma\in S$, where:

$$\digamma_1(\varkappa,\sigma)(y)=\parallel\varkappa(y)-\sigma(y)\parallel^2,$$

and

$$\digamma_2(\varkappa,\sigma)(y)=\frac{\parallel R_{\varkappa}(y)+\Gamma(y)-\varkappa(y)\parallel^2\parallel S_{\sigma}(y)+\Gamma(y)-\sigma(y)\parallel^2}{1+\parallel\varkappa(y)-\sigma(y)\parallel}.$$

Then the system of FDE (27) has a unique common solution.

Proof. *Let us define $\Phi,\Psi: S\to S$ by*

$$\Phi\varkappa=R_{\varkappa}+\Gamma,$$

and

$$\Psi\sigma=S_{\sigma}+\Gamma.$$

Then

$$\begin{aligned} d_b(\Phi_{\varkappa},\Psi_{\sigma}) &= \sup_{y\in[0,1]}\Bigg(\parallel R_{\varkappa}(y)-S_{\sigma}(y)+\Gamma(y)-\Gamma(y)\parallel^2\Bigg)(1+i_2)\\ &= \sup_{y\in[0,1]}\Bigg(\parallel R_{\varkappa}(y)-S_{\sigma}(y)\parallel^2\Bigg)(1+i_2), \end{aligned}$$

$$d_b(\varkappa,\sigma) = \sup_{y\in[0,1]}\Bigg(\parallel \varkappa(y) - \sigma(y) \parallel^2 \Bigg)(1+i_2),$$

$$d_b(\varkappa,\Phi\varkappa) = \sup_{y\in[0,1]}\Bigg(\parallel R_\varkappa(y) + \Gamma(y) - \varkappa(y) \parallel^2 \Bigg)(1+i_2),$$

and

$$d_b(\sigma,\Phi\sigma) = \sup_{y\in[0,1]}\Bigg(\parallel S_\sigma(y) + \Gamma(y) - \sigma(y) \parallel^2 \Bigg)(1+i_2).$$

Now by (14) of Theorem (4) we obtain,

$$\begin{aligned}
&\parallel R_\varkappa(y) - S_\sigma(y) \parallel^2 \\
&\quad\precsim \mu(\varkappa) \parallel \varkappa(y) - \sigma(y) \parallel^2 \\
&\quad+ \nu(\varkappa)\frac{\parallel R_\varkappa(y) + \Gamma(y) - \varkappa(y) \parallel^2 \parallel S_\sigma(y) + \Gamma(y) - \sigma(y) \parallel^2 \ (1+i_2)}{1+ \parallel \varkappa(y) - \sigma(y) \parallel^2 \ (1+i_2)} \\
&\quad\precsim \mu(\varkappa) \parallel \varkappa(y) - \sigma(y) \parallel^2 \\
&\quad+ \nu(\varkappa)\frac{\parallel R_\varkappa(y) + \Gamma(y) - \varkappa(y) \parallel^2 \parallel S_\sigma(y) + \Gamma(y) - \sigma(y) \parallel^2}{1+ \parallel \varkappa(y) - \sigma(y) \parallel^2} \\
&\quad= \mu(\varkappa)\digamma_1(\varkappa,\sigma)(y) + \nu(\varkappa)\digamma_2(\varkappa,\sigma)(y).
\end{aligned}$$

Thus, it implies that:

(i) $\mu(\Phi\varkappa) \le \mu(\varkappa)$ *and* $\nu(\Phi\varkappa) \le \nu(\varkappa)$;

(ii) $\mu(\Psi\varkappa) \le \mu(\varkappa)$ *and* $\nu(\Psi\varkappa) \le \nu(\varkappa)$;

(iii) $(\mu+\nu)(\varkappa) < 1$;

(iv)

$$d_b(\Phi\varkappa,\Psi\sigma) \precsim \mu(\varkappa)d_b(\varkappa,\sigma) + \nu(\varkappa)\frac{d_b(\varkappa,\Phi\varkappa)d_b(\sigma,\Psi\sigma)}{1+d_b(\varkappa,\sigma)}, \tag{29}$$

for all $\varkappa,\sigma \in S$*. Therefore, by the Theorem (4), we obtain that* Φ *and* Ψ *have a unique common fixed point. Thus we claim that the system of FDE (27) have a unique common solution.*

4. Conclusion

Fractional differential equations (FDEs) have emerged as powerful tools for modeling a variety of complex, real-world processes encountered in physics, engineering, biology, and finance. Their ability to incorporate non-integer order derivatives makes them particularly effective in capturing memory effects and hereditary properties inherent in many dynamic systems. Analyzing such equations often involves transforming them into equivalent integral formulations, which in turn require robust mathematical frameworks to examine the existence, uniqueness, and stability of their solutions. Fixed point theory, especially in the context of generalized metric spaces and non-traditional contraction principles, provides a foundational approach in this analytical endeavor. In particular, rational-type contractive conditions have proven to be especially useful in establishing strong convergence results and solution behaviors.

In this study, we investigated unique and common fixed point results within the setting of bi-complex valued control metric spaces using rational-type inequalities. The use of bi-complex numbers, which extend complex analysis through the introduction of two imaginary units, provides a richer and more flexible algebraic and topological structure. This enhanced framework enables the examination of more generalized and complex contractive mappings that cannot be adequately addressed within conventional real or complex-valued metric spaces. The theoretical results obtained not only contribute meaningfully to the broader field of fixed point theory but also have direct applications in the analysis and solution of fractional differential equations. Overall, this work opens up new avenues for the study of intricate mathematical models and offers innovative tools for addressing the analytical challenges presented by fractional systems in higher-dimensional and abstract settings.

Acknowledgements

The authors M. Sarwar, N. Fatima and K. Abodayeh would like to thank Prince Sultan University for APC and for the support of this work through TAS research lab.

Author's contributions

All authors contribute equally to the writing of this manuscript. All authors reads and approved the final version.

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