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Advanced Mathematical Approaches for Solving Fractional-Order Korteweg-de Vries Equations

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Abstract. In this work, we investigate the use of two new methods, residual power series natural transform method (RPSNTM) and new iteration natural transform method (NINTM), to tackle the fractional-order Korteweg-de Vries (KdV) equation. Many wave effects in fluid dynamics, plasma physics and traffic flow are described with the help of the fractional-order KdV equation. Traditional techniques used to solve equations generally are not adapted to deal with the complications caused by fractional derivatives. The RPSNTM is presented as a valuable method to approximate how the solution will change with time by converting the problem into a step-by-step series. The NINTM is also used to make the solution more effective and reliable gradually. Applying both approaches offers a strong way to find approximate analytical answers to the KdV fractional-order equation. Numerical data is used to reveal that the presented methods work better and faster than existing methods in terms of precision and speed of convergence. These results create new opportunities to apply fractional calculus in analyzing nonlinear waves.preparation.

2020 Mathematics Subject Classifications: 35A20, 35J35, 74H10, 74H20, 74H25

Key Words and Phrases: Fractional-order KdV Equation, Residual Power Series Natural Transform Method, New Iteration Natural Transform Method

1. Introduction

Fractional partial differential equations (FPDEs) add non-integer derivative terms, which are commonly referred to as "fractional derivatives," to classical partial differential equations. They give us a method to describe anomalous diffusion, memory effects or long-range interactions, situations where traditional integer-order derivatives are not enough. Using fractional derivatives, FPDEs can describe systems that do not remain

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the same everywhere, which gives them value in physics, finance, biology and engineering. They serve to bridge the divide between models for the large-scale and small-scale world, giving us a better picture of things that happen across many scales in nature. Fractional calculus, which covers differentiation and integration beyond whole numbers, forms the mathematical base of FPDEs [1–3]. If a system's behavior is non-Markovian, meaning past events affect future states throughout the system, these fractional operators are beneficial. FPDEs are now being used more often to model systems that recall their past, influence their children genetically, and have different features in different parts of space. Although solving FPDEs is a much more challenging task than classical PDEs, valuable ways to solve them, such as finite difference methods, spectral methods and variational techniques, have been found [4–8].

Many works in the field use the Korteweg-de Vries equation, a simple-looking nonlinear partial differential equation which is respected for its way of balancing nonlinearity and dispersion. It is important because it can reproduce many physical processes from shallow water waves to those found in plasma physics and nonlinear optics [9]. The idea for the equation began with the work of Diederik Korteweg and Gustav de Vries in 1895 as they tried to characterize the movement of water waves that are long and only slightly high in a rectangular channel. They found a way to understand what controls these waves by proving that nonlinearity can neutralize dispersion and cause the creation of solitons which remain at a specific place. Solitons which travel without changing shape and bounce off other solitons with no energy loss, break away from how linear waves are meant to operate [10]. The KdV equation looks easy to understand, but it has a lot of interesting details and answers. Integrability is an unusual trait among nonlinear partial differential equations which means the inverse scattering transform can be applied to precisely solve such equations. The inverse scattering transform, developed by Gardner, Greene, Kruskal and Miura in the 1960s, enables us to solve the KdV equation as a linear problem through standard approaches. The results from the linear solutions can be used to return solutions to the KdV equation, showing the soliton patterns and their interactions with each other. Because the KdV equation is integrable, it has many conserved quantities. These quantities are very important for the stability and extended behavior of its solutions [11–14].

Residual power series natural transform (RPSNT) is a method that allows for problem-solving involving crucial nonlinear and fractional differential equations. Using this technique, solutions to difficult differential equations can be found more easily than with regular methods. The method centres on converting the original problem to easy-to-manage terms with natural transforms and completing the solution using an approximation with a series. The technique is well-suited for addressing nonlinear and memory-related differential equations, along with various other challenging behaviors. Physics, engineering and applied mathematics use this method extensively, as the equations describing those problems often do not have quick, exact solutions. With the help of power series, the RPSNT method allows for fast and helpful findings about the system's performance. Furthermore, because the technique can be integrated with other computation methods, it supports both laboratory and theoretical work [15–19].

NINT is a modern way to solve a wide variety of complex differential equations, most commonly those with nonlinearities or fractional parts. The method connects iterative techniques with natural transforms, for example, the Laplace and Fourier transforms, so that approximating solutions becomes both more accurate and efficient. Applying the natural transform to the differential equation repeatedly and then updating the solution each time forms the main idea behind the NINT method. With this approach, the method can address linear and nonlinear equations, providing a useful solution for equations that are difficult to solve in other ways. A major advantage of the NINT method is that it treats complicated and unpredictable equations commonly used in physics, engineering and mathematics. By carrying out the different parts of the method again and again, the solution gets better and finally comes close to being the true solution. Such a method is most valuable when conventional methods have difficulty with chaotic actions, complex shapes or unusual boundaries. Through the new Iteration Natural Transform technique, numerical analysis is given a flexible and effective way to deal with different kinds of challenging differential equations [20–25].

2. Basic Definitions

2.1. Definition

The fractional Rieman-Liouville integral of order a $p \in \mathbb{R}_+$ of a function $h(\gamma) \in L([0,1],\mathbb{R})$ is given as [26–28]

$$I_0^p h(\gamma) = \frac{1}{\Gamma(p)} \int_0^\sigma (\sigma - s)^{p-1} h(s) ds,$$

provided that the integral on the right side converges.

2.2. Definition

Caputo's fractional order derivative of a function $h \in C_{-1}^n$ with $n \in \mathbb{N} \cup \{0\}$ is expressed as [27, 28]

$$D_\sigma^p h(\sigma) = \begin{cases} I^{n-p} f^{(n)}, & n-1 < p \leq n, n \in \mathbb{N}, \\ \frac{d^n}{\mathrm{d}\sigma^n} h(\sigma), & p = n, n \in \mathbb{N}. \end{cases}$$

2.3. Definition

A two parameter Mittag-Leffler function is given by [27, 28]:

$$E_{p,\beta}(\sigma) = \sum_{k=0}^{\infty} \frac{\sigma^k}{\Gamma(kp + \beta)}.$$

For $p = \beta = 1, E_{1,1}(\sigma) = e^\sigma$ and $E_{1,1}(-\sigma) = e^{-\sigma}$.

2.4. Definition

The natural transform (NT) of a function $v(\upsilon,\sigma)$ for $\sigma \geq 0$ is given as [27, 28]

$$\mathcal{N}[v(\upsilon,\sigma)] = R(\upsilon,s,u) = \int_0^{\infty} e^{-s\sigma} v(\upsilon,u\sigma)\mathrm{d}\sigma,$$

where u and s for the transformation parameter and are assumed to be real and positive.

3. The suggested methods

3.1. residual power series natural transform method

Consider the fractional PDE is given as:

$$\begin{aligned} D_\sigma^p \psi(\upsilon,\sigma) &= N_\upsilon[\psi(\upsilon,\sigma)] \\ \psi(\upsilon,0) &= f(\upsilon) \end{aligned} \tag{1}$$

where N_υ is a nonlinear function depending on υ with degree r, $\upsilon \in I$, $\sigma \geq 0$, $\psi(\upsilon,\sigma)$ is an unknown term and D_σ^p is the p-th fractional Caputo operator for $p \in (0,1]$.

Step 1: Applying the NT on both sides of eq. (1),

$$\begin{gathered} \psi(\upsilon,s) = \frac{f(\upsilon)}{s} - \frac{u^p}{s^p}\mathcal{N}\left\{N_\upsilon[\psi(\upsilon,\sigma)]\right\}, \\ \text{where } \psi(\upsilon,s) = \mathcal{N}[\psi(\upsilon,\sigma)](s), s > \sigma. \end{gathered} \tag{2}$$

Step 2: Consider the following fractional approximate result of the eq. (2):

$$\psi(\upsilon,s) = \frac{f(\upsilon)}{s} + \sum_{n=1}^{\infty} \frac{u^p h_n(\upsilon)}{s^{np+1}}, x \in I, s > \sigma \geq 0, \tag{3}$$

and the k-th Natural series solutions is defined by:

$$\psi_k(\upsilon,s) = \frac{f(\upsilon)}{s} + \sum_{n=1}^{k} \frac{u^p h_n(\upsilon)}{s^{np+1}}, \upsilon \in I, s > \sigma \geq 0. \tag{4}$$

Step 3: The fractional k-th Natural residual function (NRF) of (2) is given as:

$$\mathcal{N}\left(\mathrm{Res}_{\psi_k}(\upsilon,s)\right) = \psi_k(\upsilon,s) - \frac{f(\upsilon)}{s} + \frac{u^p}{s^p}\mathcal{N}\left\{N_\upsilon[\psi(\upsilon,\sigma)]\right\}, \tag{5}$$

and (2) NRF is given as:

$$\lim_{k\to\infty} \mathcal{N}\left(\mathrm{Res}_{\psi_k}(\upsilon,s)\right) = \mathcal{N}\left(\mathrm{Res}_{\psi}(\upsilon,s)\right) = \psi(\upsilon,s) - \frac{f(\upsilon)}{s} + \frac{u^p}{s^p}\mathcal{N}\left\{N_\upsilon[\psi(\upsilon,\sigma)]\right\}. \tag{6}$$

The NRF solution: - $\lim_{k\to\infty} \mathcal{N}\left(\mathrm{Res}_{\psi_k}(\upsilon,s)\right) = \mathcal{N}\left(\mathrm{Res}_{\psi}(\upsilon,s)\right)$, for $\upsilon \in I, s > \sigma \geq 0$. - $\mathcal{N}\left(\mathrm{Res}_{\psi}(\upsilon,s)\right) = 0$, for $\upsilon \in I, s > \sigma \geq 0$. - $\lim_{s\to\infty} s^{kp+1}\mathcal{N}\left(\mathrm{Res}_{\psi_k}(\upsilon,s)\right) = 0$, for

$\upsilon \in I, s > \sigma \geq 0$, and $k = 1, 2, 3, \ldots$
Step 4: Now put the k-th Natural series solution (4) into the k-th Natural fractional residual function of (5).

Step 5: The unknown coefficients $h_k(\upsilon)$, for $k = 1, 2, 3, \ldots$, might be obtained by solving the system $\lim_{s\to\infty} s^{ka+1}\mathcal{N}\left(\mathrm{Res}_{\psi_k}(\upsilon, s)\right) = 0$. The achieved coefficient applying fractional expansion series (4) $\psi_k(\upsilon, s)$.

Step 6: Applying the inverse Natural transform operator to both sides of the Natural series solution to obtain an estimated solution $\psi_k(\upsilon, \sigma)$, of the main Equation (1).

3.2. Idea of the Natural Iterative Transform Method

Consider the general fractional PDE is given as

$$D_\sigma^p \psi(\upsilon, \sigma) = \Phi\Big(\psi(\upsilon, \sigma), D_\upsilon^\sigma \psi(\upsilon, \sigma), D_\upsilon^{2\sigma} \psi(\upsilon, \sigma), D_\upsilon^{3\sigma} \psi(\upsilon, \sigma)\Big), \ 0 < p, \sigma \leq 1, \tag{7}$$

Initial scenarios

$$\psi^{(k)}(\upsilon, 0) = h_k, \ k = 0, 1, 2, \cdots, m-1, \tag{8}$$

Applying the NT to eq. 7; $\psi(\upsilon, \sigma)$ is defined as ψ.

$$\mathcal{N}[\psi(\upsilon, \sigma)] = \frac{1}{s^p}\Big(\sum_{k=0}^{m-1} \frac{\psi^{(k)}(\upsilon, 0)}{s^{2-p+k}} + \mathcal{N}\Big[\Phi\Big(\psi(\upsilon, \sigma), D_\upsilon^\sigma \psi(\upsilon, \sigma), D_\upsilon^{2\sigma} \psi(\upsilon, \sigma), D_\upsilon^{3\sigma} \psi(\upsilon, \sigma)\Big)\Big]\Big), \tag{9}$$

Now we apply inverse NT is defined as

$$\psi(\upsilon, \sigma) = \mathcal{N}^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1} \frac{\psi^{(k)}(\upsilon, 0)}{s^{2-p+k}} + \mathcal{N}\Big[\Phi\Big(\psi(\upsilon, \sigma), D_\upsilon^\sigma \psi(\upsilon, \sigma), D_\upsilon^{2\sigma} \psi(\upsilon, \sigma), D_\upsilon^{3\sigma} \psi(\upsilon, \sigma)\Big)\Big]\Big)\Big]. \tag{10}$$

The iterative procedure is given as

$$\psi(\upsilon, \sigma) = \sum_{i=0}^{\infty} \psi_i. \tag{11}$$

Since $\Phi\Big(\psi, D_\upsilon^\sigma \psi, D_\upsilon^{2\sigma} \psi, D_\upsilon^{3\sigma} \psi\Big)$ are non-linear and linear functions that can be decompose as:

$$\begin{aligned} &\Phi\Big(\psi, D_\upsilon^\sigma \psi, D_\upsilon^{2\sigma} \psi, D_\upsilon^{3\sigma} \psi\Big) = \Phi\Big(\psi_0, D_\upsilon^\sigma \psi_0, D_\upsilon^{2\sigma} \psi_0, D_\upsilon^{3\sigma} \psi_0\Big) \\ &+ \sum_{i=0}^{\infty} \left(\Phi\Big(\sum_{k=0}^{i} \Big(\psi_k, D_\upsilon^\sigma \psi_k, D_\upsilon^{2\sigma} \psi_k, D_\upsilon^{3\sigma} \psi_k\Big)\Big) - \Phi\Big(\sum_{k=1}^{i-1} \Big(\psi_k, D_\upsilon^\sigma \psi_k, D_\upsilon^{2\sigma} \psi_k, D_\upsilon^{3\sigma} \psi_k\Big)\Big)\right). \end{aligned} \tag{12}$$

Eqs. 12 and 11 putting into eq. 10.

$$\sum_{i=0}^{\infty}\psi_i(\upsilon,\sigma) = \mathcal{N}^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\psi^{(k)}(\upsilon,0)}{s^{2-p+k}} + \mathcal{N}[\Phi(\psi_0, D_\upsilon^\sigma\psi_0, D_\upsilon^{2\sigma}\psi_0, D_\upsilon^{3\sigma}\psi_0)]\Big)\Big]$$
$$+\mathcal{N}^{-1}\Big[\frac{1}{s^p}\Big(\mathcal{N}\Big[\sum_{i=0}^{\infty}\Big(\Phi\sum_{k=0}^{i}(\psi_k, D_\upsilon^\sigma\psi_k, D_\upsilon^{2\sigma}\psi_k, D_\upsilon^{3\sigma}\psi_k)\Big)\Big]\Big)\Big] \tag{13}$$
$$-\mathcal{N}^{-1}\Big[\frac{1}{s^p}\Big(\mathcal{N}\Big[\Big(\Phi\sum_{k=1}^{i-1}(\psi_k, D_\upsilon^\sigma\psi_k, D_\upsilon^{2\sigma}\psi_k, D_\upsilon^{3\sigma}\psi_k)\Big)\Big]\Big)\Big]$$

$$\begin{aligned}
&\psi_0(\upsilon,\sigma) = \mathcal{N}^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\psi^{(k)}(\upsilon,0)}{s^{2-p+k}}\Big)\Big],\\
&\psi_1(\upsilon,\sigma) = \mathcal{N}^{-1}\Big[\frac{1}{s^p}\Big(\mathcal{N}[\Phi(\psi_0, D_\upsilon^\sigma\psi_0, D_\upsilon^{2\sigma}\psi_0, D_\upsilon^{3\sigma}\psi_0)]\Big)\Big],\\
&\vdots\\
&\psi_{m+1}(\upsilon,\sigma) = \mathcal{N}^{-1}\Big[\frac{1}{s^p}\Big(\mathcal{N}\Big[\sum_{i=0}^{\infty}\Big(\Phi\sum_{k=0}^{i}(\psi_k, D_\upsilon^\sigma\psi_k, D_\upsilon^{2\sigma}\psi_k, D_\upsilon^{3\sigma}\psi_k)\Big)\Big]\Big)\Big]\\
&\quad -\mathcal{N}^{-1}\Big[\frac{1}{s^p}\Big(\mathcal{N}\Big[\Big(\Phi\sum_{k=1}^{i-1}(\psi_k, D_\upsilon^\sigma\psi_k, D_\upsilon^{2\sigma}\psi_k, D_\upsilon^{3\sigma}\psi_k)\Big)\Big]\Big)\Big],\ m=1,2,\cdots.
\end{aligned} \tag{14}$$

The following eq. 7 yield the semi-analytical result for the i-terms defined as:

$$\psi(\upsilon,\sigma) = \sum_{i=0}^{m-1}\psi_i. \tag{15}$$

3.3. Problem 1

3.3.1. Implementation of RPSNTM

Consider the following fractional dispersive KdV Equation

$$D_\sigma^p\psi + 2\frac{\partial\psi}{\partial\upsilon} + \frac{\partial^3\psi}{\partial\upsilon^3} = 0 \quad \sigma > 0, \quad \text{where} \quad 0 < p \leq 1 \tag{16}$$

Having IC's:

$$\psi(\upsilon,0) = \sin(\upsilon) \tag{17}$$

Using Eq. (17) and applying NT to Eq. (16), we obtain:

$$\psi(\upsilon,0) - \frac{\sin(\upsilon)}{s} + \frac{u^p}{s^p}\frac{\partial^3\psi(\upsilon,0)}{\partial\upsilon^3} + \frac{2u^p}{s^p}\frac{\partial\psi(\upsilon,0)}{\partial\upsilon} = 0 \tag{18}$$

Thus, the term series that are k^{th}-truncated are:

$$\psi(\upsilon, s) = \frac{\sin(\upsilon)}{s} + \sum_{r=1}^{k} \frac{u^{rp+1} f_r(\upsilon, s)}{s^{rp+1}}, \quad r = 1, 2, 3, \cdots . \tag{19}$$

Natural residual functions (NRFs) provided as follows:

$$\mathcal{N}_\sigma Res(\upsilon, s) = \psi(\upsilon, 0) - \frac{\sin(\upsilon)}{s} + \frac{u^p}{s^p} \frac{\partial^3 \psi(\upsilon, 0)}{\partial \upsilon^3} + \frac{2u^p}{s^p} \frac{\partial \psi(\upsilon, 0)}{\partial \upsilon} = 0, \tag{20}$$

Along with the k^{th}-NRFs as:

$$\mathcal{N}_\sigma Res_k(\upsilon, s) = \psi_k(\upsilon, 0) - \frac{\sin(\upsilon)}{s} + \frac{u^p}{s^p} \frac{\partial^3 \psi_k(x, 0)}{\partial \upsilon^3} + \frac{2u^p}{s^p} \frac{\partial \psi_k(\upsilon, 0)}{\partial \upsilon} = 0. \tag{21}$$

To find $f_r(\upsilon, s)$ now, $r = 1, 2, 3, \cdots$ We multiply the resulting equation by s^{rp+1}, substitute the $r^{\sigma h}$-truncated series Eq. (19) into the $r^{\sigma h}$-Natural residual function Eq. (21), and solve the relation $\lim_{s\to\infty}(s^{rp+1}\mathcal{N}_\sigma Res_{\psi,r}(\sigma, s)) = 0$ recursively. $1, 2, 3, \cdots$. Here are the first few of terms:

$$f_1(\upsilon, s) = -\cos(\upsilon), \tag{22}$$

$$f_2(\upsilon, s) = -\sin(\upsilon), \tag{23}$$

$$f_3(\upsilon, s) = \cos(\upsilon) \tag{24}$$

and so on.
In Eq. (19), substitute the values of $f_r(\upsilon, s)$, $r = 1, 2, 3, \cdots$, we get:

$$\psi(\upsilon, s) = \frac{u \sin(\upsilon)}{s} - \frac{u^p \cos(\upsilon)}{s^{p+1}} - \frac{u^p \sin(\upsilon)}{s^{2p+1}} + \frac{u^p \cos(\upsilon)}{s^{3p+1}}, \tag{25}$$

Using inverse Natural Transform,we get

$$\psi(\upsilon, \sigma) = \sin(\upsilon) - \frac{\sigma^p \cos(\upsilon)}{\Gamma(p+1)} - \frac{\sigma^{2p} \sin(\upsilon)}{\Gamma(2p+1)} + \frac{\sigma^{3p} \cos(\upsilon)}{\Gamma(3p+1)} + \cdots . \tag{26}$$

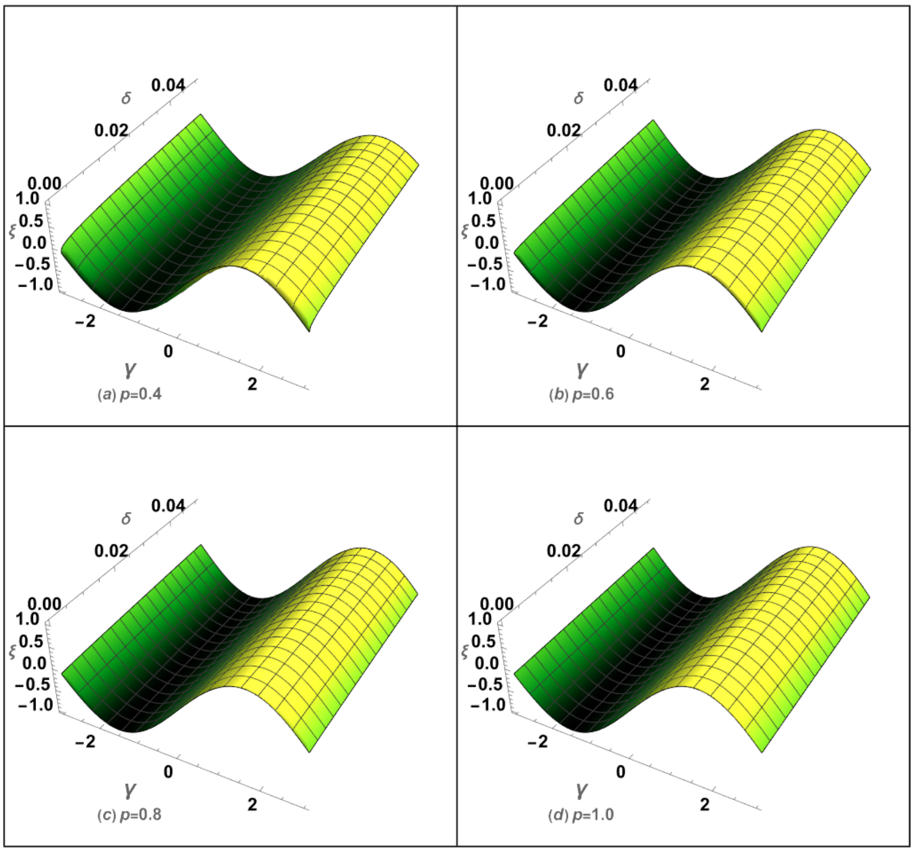

Figure 1: In figure 1, Analysis of fractional order p for $\sigma = 0.05$ of $\psi(\upsilon, \sigma)$ of example 1 using RPSNTM.

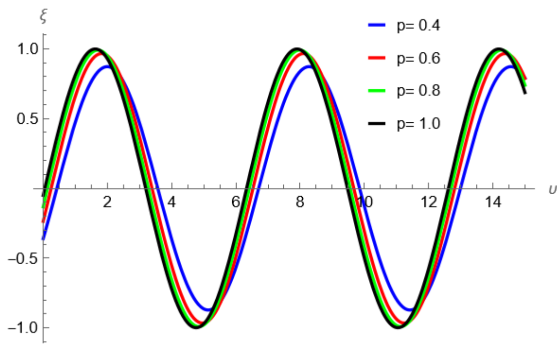

Figure 2: In figure 2, 2D analysis of RPSNTM at $\sigma = 0.05$.

3.3.2. Implementation of NINTM

We derive the corresponding form given below by applying the RL integral on Eq.16:

$$\psi(\upsilon, \sigma) = \sin(\upsilon) - \mathcal{N}_{\sigma}^{p}\Big[- 2\frac{\partial \psi}{\partial \upsilon} - \frac{\partial^3 \psi}{\partial \upsilon^3}\Big]. \tag{27}$$

We obtain the following several terms based on the NINTM procedure:

$$\begin{aligned} \psi_0(\upsilon,\sigma) &= \sin(\upsilon), \\ \psi_1(\upsilon,\sigma) &= -\frac{\sigma^p \cos(\upsilon)}{p\Gamma(p)}, \\ \psi_2(\upsilon,\sigma) &= -\frac{\sigma^{2p} \sin(\upsilon)}{p^2\Gamma(p)^2}, \\ \psi_3(\upsilon,\sigma) &= \frac{\sigma^{3p} \cos(\upsilon)}{p^3\Gamma(p)^3}. \end{aligned} \tag{28}$$

T3he final NINTM algorithm solution is as follows:

$$\psi(\upsilon,\sigma) = \psi_0(\upsilon,\sigma) + \psi_1(\upsilon,\sigma) + \psi_2(\upsilon,\sigma) + \cdots, \tag{29}$$

$$\psi(\upsilon,\sigma) = \sin(\upsilon) - \frac{\sigma^p \cos(\upsilon)}{p\Gamma(p)} - \frac{\upsilon^{2p} \sin(\upsilon)}{p^2\Gamma(p)^2} + \frac{\sigma^{3p} \cos(\upsilon)}{p^3\Gamma(p)^3}. \tag{30}$$

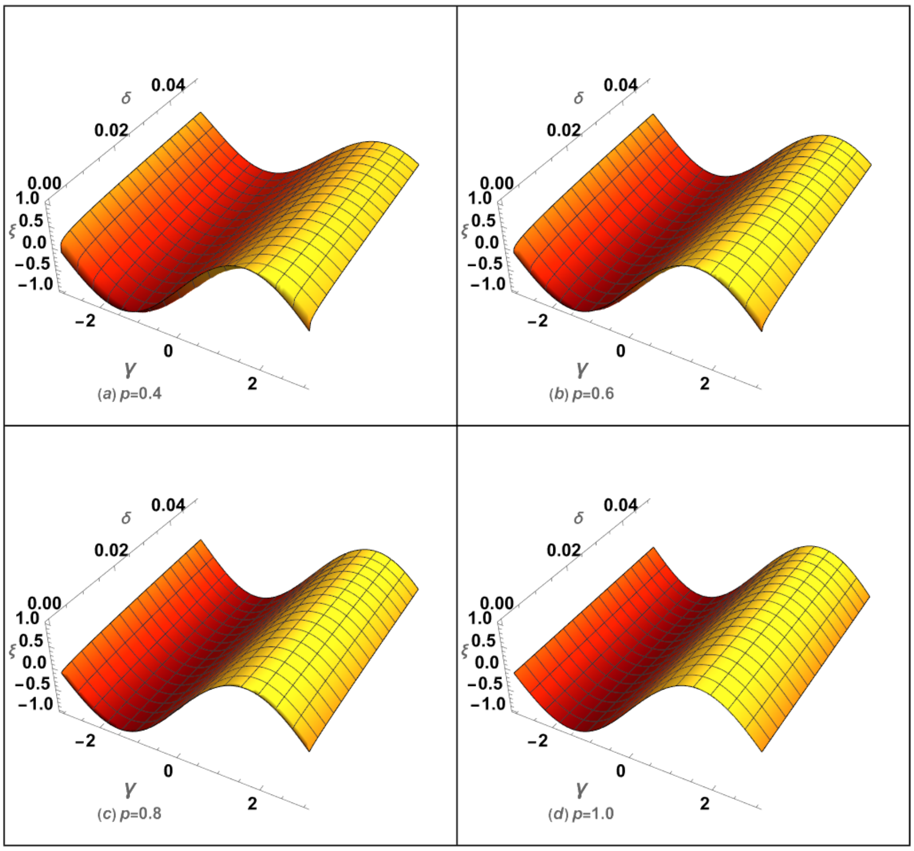

Figure 3: In figure 3, Analysis of fractional order p for $\sigma = 0.05$ of $\psi(\upsilon,\sigma)$ of example 1 using NINTM.

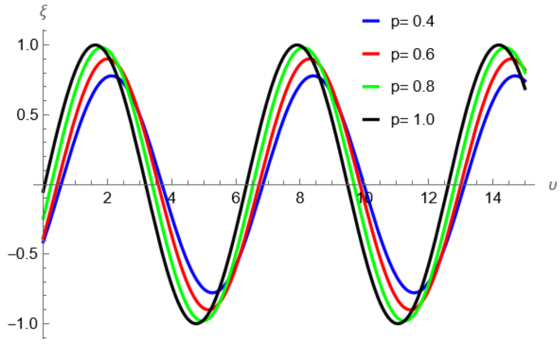

Figure 4: In figure 4, 2D analysis of NINTM at $\sigma = 0.05$.

3.4. Problem 2

3.4.1. Implementation of RPSNTM

Consider the following fractional dispersive KdV Equation

$$D_\sigma^p \psi + \frac{\partial^3 \psi}{\partial \upsilon^3} + \frac{\partial^3 \psi}{\partial \rho^3} = 0 \quad \sigma > 0, \quad \text{where} \quad 0 < p \leq 1 \tag{31}$$

Having IC's are:

$$\psi(\upsilon, 0) = \cos(\upsilon + \rho) \tag{32}$$

Using Eq. (32) and applying NT to Eq. (31), we obtain:

$$\psi(\upsilon, 0) - \frac{u\cos(\psi + \rho)}{s} - \frac{u^p}{s^p}\frac{\partial^3 \psi}{\partial \upsilon^3} + \frac{u^p}{s^p}\frac{\partial^3 \psi}{\partial \rho^3} = 0 \tag{33}$$

Thus, the term series that are k^{th}-truncated are:

$$\psi(\upsilon, s) = \frac{\cos(\upsilon + \rho)}{s} + \sum_{r=1}^{k} \frac{u^{rp+1} f_r(\upsilon, s)}{s^{rp+1}}, \quad r = 1, 2, 3, 4 \cdots . \tag{34}$$

Natural residual functions (NRFs) provided as follows:

$$\mathcal{N}_\sigma Res(\upsilon, s) = \psi(\upsilon, 0) - \frac{u\cos(\upsilon + \rho)}{s} - \frac{u^p}{s^p}\frac{\partial^3 \psi}{\partial \upsilon^3} + \frac{u^p}{s^p}\frac{\partial^3 w}{\partial \rho^3} = 0, \tag{35}$$

Along with the k^{th}-NRFs as:

$$\mathcal{N}_\sigma Res_k(\upsilon, s) = \psi_k(\upsilon, 0) - \frac{\cos(\upsilon + \rho)}{s} - \frac{u^p}{s^p}\frac{\partial^3 \psi_k}{\partial \upsilon^3} + \frac{u^p}{s^p}\frac{\partial^3 \psi_k}{\partial \rho^3} = 0. \tag{36}$$

To find $f_r(\upsilon, s)$ now, $r = 1, 2, 3, \cdots$ We multiply the resulting equation by s^{rp+1}, substitute the $r^{\sigma h}$-truncated series Eq. (34) into the $r^{\sigma h}$-Natural residual function Eq. (36), and solve

the relation
$\lim_{s\to\infty}(s^{rp+1}\mathcal{N}_\sigma Res_{\psi,r}(\sigma,s)) = 0$ recursively. $1, 2, 3, \cdots$. Here are the first few of terms:

$$f1(\upsilon, s) = -2\sin(\upsilon+\rho), \tag{37}$$

$$f_2(\upsilon, s) = -4\cos(\upsilon+\rho), \tag{38}$$

$$f_3(\upsilon, s) = 8\sin(\upsilon+\rho), \tag{39}$$

and so on.
In Eq. (34), substitute the values of $f_r(\upsilon, s)$, $r = 1, 2, 3, \cdots$, we get:

$$\psi(\upsilon, s) = \frac{u\cos(\upsilon+\rho)}{s} - \frac{2u^p\sin(\upsilon+\rho)}{s^{p+1}} - \frac{4u^p\cos(\upsilon+\rho)}{s^{2p+1}} + \frac{8u^p\sin(\upsilon+\rho)}{s^{3p+1}}, \tag{40}$$

With the inverse Natural transform, we obtain:

$$\psi(\upsilon, \sigma) = \cos(\upsilon+\rho) - \frac{2\sigma^p\sin(\upsilon+\rho)}{\Gamma(p+1)} - \frac{4\sigma^{2p}\cos(\upsilon+\rho)}{\Gamma(2p+1)} + \frac{8\sigma^{3p}\sin(\upsilon+\rho)}{\Gamma(3p+1)}. \tag{41}$$

3.4.2. Implementation of NINTM

We derive the corresponding form given below by applying the RL integral on Eq.31:

$$\psi(\upsilon, \sigma) = \sin(\upsilon) - \mathcal{N}_\sigma^p\Big[-2\frac{\partial\psi}{\partial\upsilon} - \frac{\partial^3\psi}{\partial\upsilon^3}\Big]. \tag{42}$$

We obtain the following several terms based on the NINTM procedure:

$$\begin{aligned}
\psi_0(\upsilon, \sigma) &= \sin(\upsilon),\\
\psi_1(\upsilon, \sigma) &= -\frac{\sigma^p\cos(\upsilon)}{p\Gamma(p)},\\
\psi_2(\upsilon, \sigma) &= -\frac{\sigma^{2p}\sin(\upsilon)}{p^2\Gamma(p)^2},\\
\psi_3(\upsilon, \sigma) &= \frac{\sigma^{3p}\cos(\upsilon)}{p^3\Gamma(p)^3}.
\end{aligned} \tag{43}$$

The final NINTM algorithm solution is as follows:

$$\psi(\upsilon, \sigma) = \psi_0(\upsilon, \sigma) + \psi_1(\upsilon, \sigma) + \psi_2(\upsilon, \sigma) + \cdots, \tag{44}$$

$$\psi(\upsilon, \sigma) = \sin(\upsilon) - \frac{\sigma^p\cos(\upsilon)}{p\Gamma(p)} - \frac{\sigma^{2p}\sin(\upsilon)}{p^2\Gamma(p)^2} + \frac{\sigma^{3p}\cos(\upsilon)}{p^3\Gamma(p)^3}. \tag{45}$$

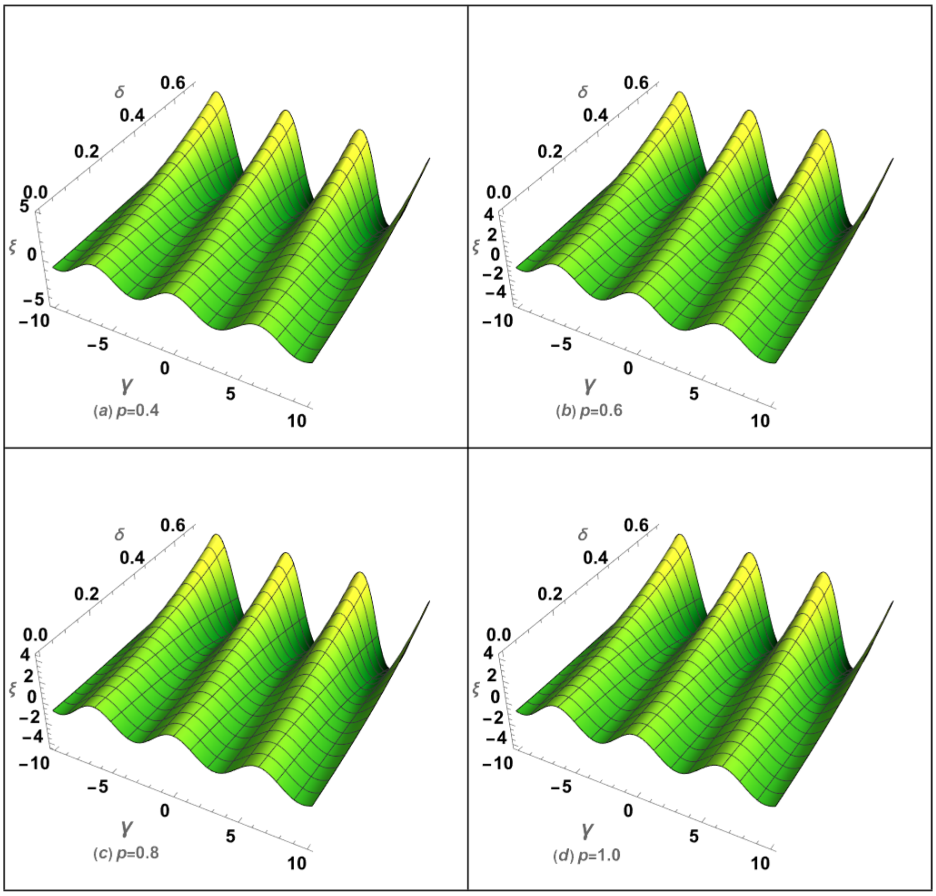

Figure 5: In figure 5, Analysis of fractional order p for $\sigma = 0.7$ of $\psi(\upsilon, \sigma)$ of example 2 using RPSNTM.

3.5. Problem 3

3.5.1. Implementation of RPSNTM

We consider an Fkdv equation with initial condition is given by:

$$D_\sigma^p \psi(\upsilon, \sigma) + \psi \frac{\partial \psi}{\partial \upsilon} - \psi \frac{\partial^3 \psi}{\partial \upsilon^3} + \frac{\partial^5 \psi}{\partial \upsilon^5} = 0, \quad \text{where } \sigma > 0, \quad \psi \in \Re, \quad 0 < p \leq 1 \tag{46}$$

The following ICs are applicable:

$$\psi(\upsilon, 0) = e^\upsilon \tag{47}$$

Using Eq. (47) and applying NT to Eq. (46), we obtain:

$$\begin{aligned} &\psi(\upsilon, s) - \frac{e^\upsilon}{s} + \frac{u^p}{s^p} \mathcal{N}_\sigma [(\mathcal{N}_\sigma^{-1} \psi(\upsilon, \sigma) \frac{\partial (\mathcal{N}_\sigma^{-1} \psi(\upsilon, \sigma)}{\partial \upsilon}] - \\ &\frac{u^p}{s^p} \mathcal{N}_\sigma [(\mathcal{N}_\sigma^{-1} \psi(\upsilon, \sigma) \frac{\partial^3 (\mathcal{N}_\sigma^{-1} \psi(\upsilon, \sigma)}{\partial \upsilon^3}] + \frac{u^p}{s^p} [\frac{\partial^5 \psi(\upsilon, \sigma)}{\partial \upsilon^5}] = 0, \end{aligned} \tag{48}$$

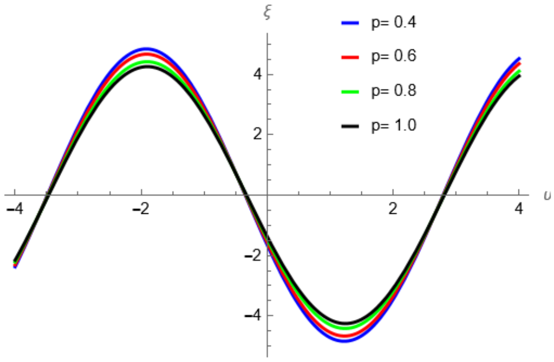

Figure 6: In figure 6, 2D analysis of RPSNTM at $\sigma = 0.7$.

Thus, the term series that are k^{th}-truncated are:

$$\psi(\upsilon, s) = \frac{e^{\upsilon}}{s} + \sum_{r=1}^{k} \frac{u^{rp+1} f_r(\upsilon, s)}{s^{rp+1}}, \quad r = 1, 2, 3, 4 \cdots . \tag{49}$$

Natural residual functions (NRFs) provided as follows:

$$\begin{aligned} \mathcal{N}_\sigma Res(\upsilon, s) = \psi(\upsilon, s) - \frac{e^{\upsilon}}{s} + \frac{u^p}{s^p} \mathcal{N}_\sigma [(\mathcal{N}_\sigma^{-1} \psi(\upsilon, \sigma) \frac{\partial(\mathcal{N}_\sigma^{-1} \psi(\upsilon, \sigma)}{\partial \upsilon}] \\ - \frac{u^p}{s^p} \mathcal{N}_\sigma [(\mathcal{N}_\sigma^{-1} \psi(\upsilon, \sigma) \frac{\partial^3(\mathcal{N}_\sigma^{-1} \psi(\upsilon, \sigma)}{\partial \upsilon^3}] + \frac{u^p}{s^p} [\frac{\partial^5 \psi(\upsilon, \sigma)}{\partial \upsilon^5}] = 0, \end{aligned} \tag{50}$$

Along with the k^{th}-NRFs as

$$\begin{aligned} \mathcal{N}_\sigma Res_k(\upsilon, s) = \psi_k(\upsilon, s) - \frac{e^{\upsilon}}{s} + \frac{u^p}{s^p} \mathcal{N}_\sigma [(\mathcal{N}_\sigma^{-1} \psi_k(\upsilon, \sigma) \frac{\partial(\mathcal{N}_\sigma^{-1} \psi_k(\upsilon, \sigma)}{\partial \upsilon}] \\ - \frac{u^p}{s^p} \mathcal{N}_\sigma [(\mathcal{N}_\sigma^{-1} \psi_k(\upsilon, \sigma) \frac{\partial^3(\mathcal{N}_\sigma^{-1} \psi_k(\upsilon, \sigma)}{\partial \upsilon^3}] + \frac{u^p}{s^p} [\frac{\partial^5 \psi_k(\upsilon, \sigma)}{\partial \upsilon^5}] = 0, \end{aligned} \tag{51}$$

To find $f_r(\upsilon, s)$ now, $r = 1, 2, 3, \cdots$ We multiply the resulting equation by s^{rp+1}, substitute the $r^{\sigma h}$-truncated series Eq. (49) into the $r^{\sigma h}$-Natural residual function Eq. (51), and solve the relation $\lim_{s\to\infty}(s^{rp+1} \mathcal{N}_\sigma Res_{\psi,r}(\sigma, s)) = 0$ recursively. $1, 2, 3, \cdots$. Here are the first few of terms:

$$f1(\upsilon, s) = -e^{\upsilon}, \tag{52}$$

$$f_2(\upsilon, s) = e^{\upsilon}, \tag{53}$$

and so on.
In Eq. (49), substitute the values of $f_r(\upsilon, s)$, $r = 1, 2, 3, \cdots$, we get:

$$\psi(\upsilon, s) = \frac{e^{\upsilon}}{s} - \frac{u^p e^{\upsilon}}{s^{p+1}} + \frac{u^p e^{\upsilon}}{s^{2p+1}}, \tag{54}$$

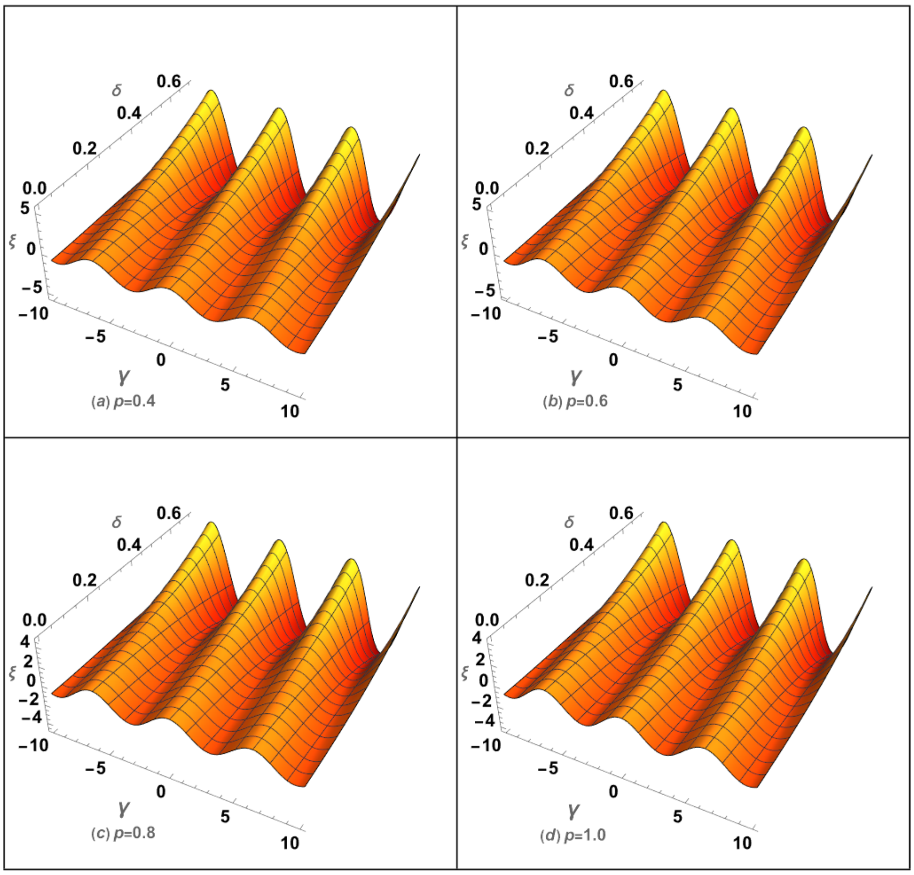

Figure 7: In figure 7, Analysis of various order p for $\sigma = 0.7$ of $\psi(\upsilon, \sigma)$ of example 2 using NINTM.

With the inverse Natural transform, we obtain

$$\psi(\upsilon, \sigma) = e^{\upsilon}\left(1 - \frac{\sigma^p}{\Gamma(p+1)} + \frac{\sigma^{2p}}{\Gamma(2p+1)}\right) + \cdots . \tag{55}$$

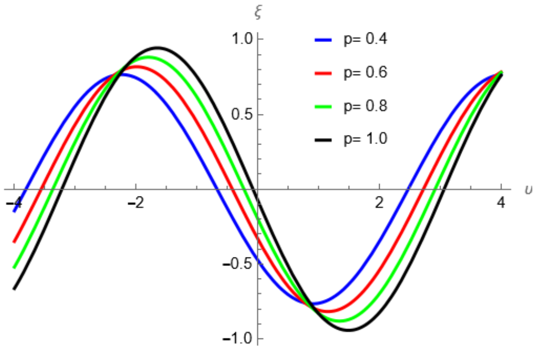

Figure 8: In figure 8, 2D analysis of NINTM at $\sigma = 0.7$.

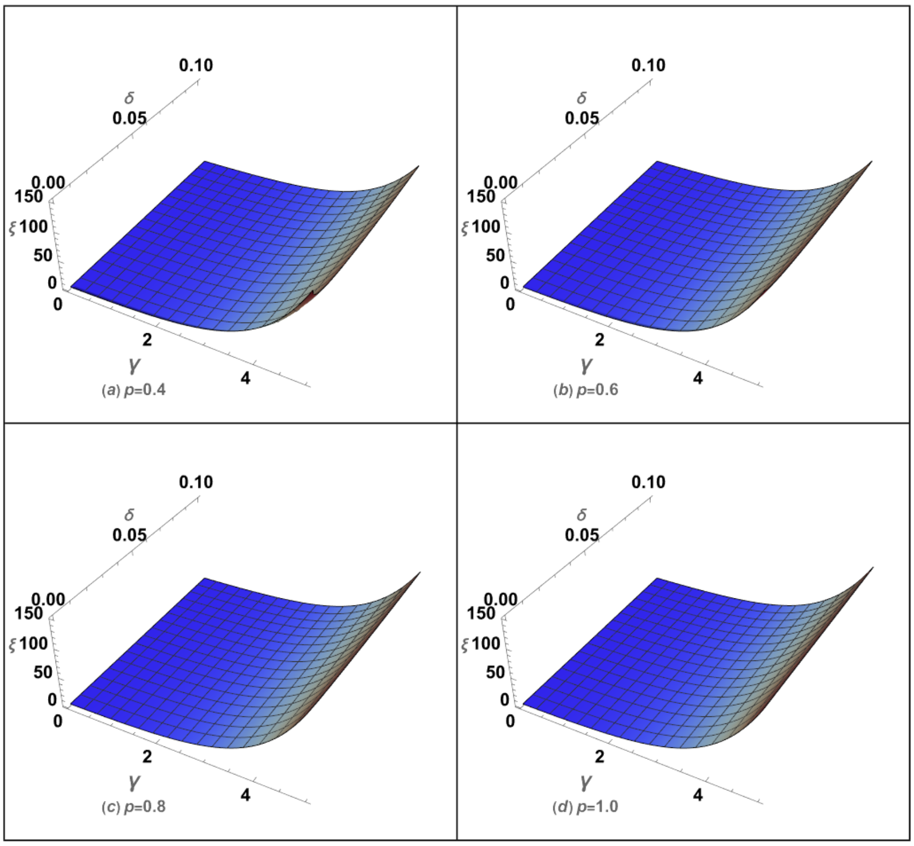

Figure 9: In figure 9, Analysis of fractional order p for $\sigma = 0.1$ of $\psi(\upsilon, \sigma)$ of example 3 using RPSNTM.

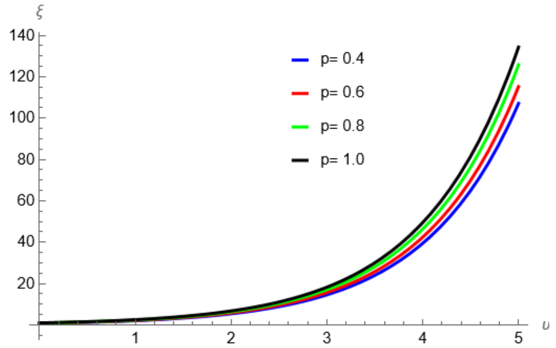

Figure 10: In figure 10, 2D analysis of fractional order p of $\psi(\upsilon, \sigma)$ using RPSNTM for $\sigma = 0.1$

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3.5.2. Implementation of NIM

We derive the corresponding form given below by applying the RL integral on Eq.46:

$$\psi(\upsilon, \sigma) = e^{\upsilon} - \mathcal{N}_{\sigma}^{p}\left[-\psi\frac{\partial\psi}{\partial\upsilon} + \psi\frac{\partial^3\psi}{\partial\upsilon^3} - \frac{\partial^5\psi}{\partial\upsilon^5}\right]. \tag{56}$$

We obtain the following several terms based on the NIM procedure:

$$\begin{aligned} \psi_0(\upsilon, \sigma) &= e^{\upsilon}, \\ \psi_1(\upsilon, \sigma) &= -\frac{e^{\upsilon}\sigma^p}{p\Gamma(p)}, \\ \psi_2(\upsilon, \sigma) &= \frac{e^{\upsilon}\sigma^{2p}}{p^2\Gamma(p)^2}. \end{aligned} \tag{57}$$

The final NIM algorithm solution is as follows:

$$\psi(\upsilon, \sigma) = \psi_0(\upsilon, \sigma) + \psi_1(\upsilon, \sigma) + \psi_2(\upsilon, \sigma) + \cdots, \tag{58}$$

$$\psi(\upsilon, \sigma) = e^{\upsilon} - \frac{e^{\upsilon}\sigma^p}{p\Gamma(p)} + \frac{e^{\upsilon}\sigma^{2p}}{p^2\Gamma(p)^2}. \tag{59}$$

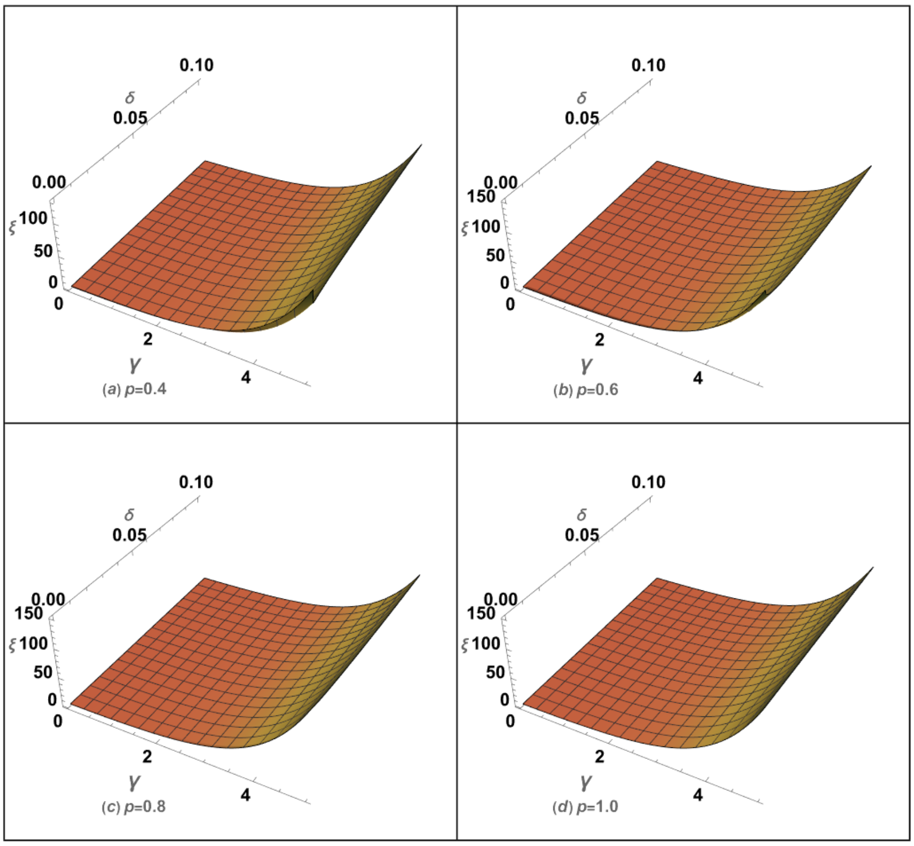

Figure 11: In figure 11, Analysis of fractional order p for $\sigma = 0.1$ of $\psi(\upsilon, \sigma)$ of example 3 using NIM.

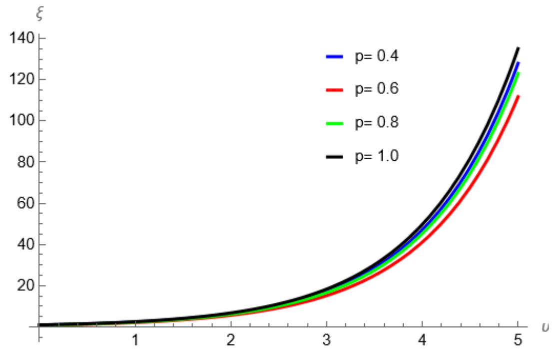

Figure 12: In figure 12, 2D analysis of fractional order p of $\psi(\upsilon, \sigma)$ using NIM for $\sigma = 0.1$

The RPSNTM and NINTM approaches for fractional-order KdV equations in connection with their dispersive effects in a plasma setting. Techniques were reviewed with through graphical drawings and numerical illustrations, showing that they permit approximating solutions to challenging fractional differential equations. Initial solutions were found easily using the RPSNTM and the NINTM improved them by making them more accurate and stable.

Figures 1 and 3 clearly show that the fractional order p has an effect on the solution $\psi(\upsilon, 0.05)$. As shown in Figure 1, there is an impact on the solution's behavior as the order p changes and in Figure 3, using the NINTM, the same analysis is seen with greatly enhanced accuracy. The calculations show that systems such as the fractional-order KdV equations generally converge and deliver more accurate answers with iterative methods such as NINTM. Shown in Figures 2 and 4 are top views of the solutions found by RPSNTM and NINTM when $\sigma = 0.05$. They show the development of waveforms in space and time throughout the plasma area. Although RPSNTM estimates how waves behave, NINTM refines this idea and shows how waves travel more accurately. The distinct images from the two methods reveal the superior accuracy of the NINTM which is useful for dealing with wave dynamics and fractional calculus. Figures 5 and 7 illustrate the findings of using RPSNTM and NINTM on example 2 when $\sigma = 0.7$ and p is changed. We can see from these pictures that a larger value of σ strongly affects both wave spreading and dispersion. For increasing values of fractional order p, larger dispersion appears and the NINTM gives better and more accurate results than the RPSNTM. It demonstrates that repeating the method refines the study of plasma waves and allows for better observations of their behavior. Figures 6 and 8 show the results of the two-dimensional analysis for $\sigma = 0.7$ comparing the RPSNTM and NINTM findings. NINTM demonstrates wave behavior more clearly and accurately than the older method. The experiments prove that the NINTM performs much better in solving nonlinear dispersive equations. In example 3, figures 9 and 11 illustrate once more the role of fractional order in controlling the dispersive properties of the wave. Looking at the results from RPSNTM (Figure 9) and NINTM (Figure 11), it's noticeable that in both cases, changing the value of p changes the pattern of shapes. The NINTM once more gives a more accurate answer. In a similar manner, Figures 10 and 12 show that the NINTM more precisely and clearly represents the spreading wave at $\sigma = 0.1$ than the RPSNTM.

In all, using RPSNTM and NINTM together creates a potent framework for resolving fractional-order KdV equations in connection with dispersive waves in plasma conditions. The graphs show how these methods skillfully handle nonlinear dynamics and NINTM does this more accurately. These methods are enhanced, according to the visual information, by being able to get good approximations for fractional differential equations, covering a large number of scientific and engineering situations. New research might test these methods in nonlinear fractional cases, helping these methods have a stronger impact in mathematical physics and computational science.

4. Conclusion

In this study, we have successfully applied the residual power series natural transform method (RPSNTM) and the natural new iteration method (NINTM) to solve fractional-order Korteweg-de Vries (KdV) equations, particularly focusing on their dispersive properties within a plasma environment. These methods proved to be effective and reliable for deriving approximate analytical solutions to complex fractional differential equations.

The RPSNTM provided a straightforward approach to obtaining initial solutions, while the NINTM further refined these solutions, enhancing their accuracy and convergence. Our analysis demonstrated that these Natural-based methods offer significant advantages in dealing with the nonlinear dynamics of wave propagation in plasma, where traditional methods might struggle with the intricacies of fractional calculus. The numerical examples and graphical illustrations presented in this work underscore the robustness of these methods, making them valuable tools for researchers and engineers working on nonlinear dispersive equations in various scientific and engineering applications. In conclusion, the combination of RPSNTM and NINTM presents a powerful framework for solving fractional-order equations, extending their applicability to a broader range of problems in mathematical physics and engineering, particularly in the study of plasma waves and other complex systems. Future research could explore the extension of these methods to other nonlinear fractional systems, further expanding their utility and impact in applied mathematics and computational physics.

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