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Spectral Properties of Coprime Graphs for Dihedral Groups

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Abstract. For a finite group G, the coprime graph Γ_G of G is defined as the graph with vertex set G, the group itself, and two distinct vertices u, v in Γ_G are adjacent if and only if $gcd(|u|, |v|) = 1$, where $|u|$ is the order of u. This study analyzes the characteristic polynomial of matrices for the dihedral group of order $2n$, where n is a power of a prime number. In addition, this paper examines the characteristic polynomial of the matrices for a power of a prime number n. The energy of the graph is also obtained.

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1. Introduction

In spectral graph theory, specific matrices provide information about graphs, such as adjacency, Laplacian, or signless Laplacian matrices. A graph can be characterized by the spectrum of one of these matrices. In general, the spectra of these various matrices can provide useful information about the graph. Apart from that, graphs can be involved in decision-making theory as seen in [1–3] and more terminologies in [4, 5]. Therefore, in this paper, we discuss a coprime graph of a finite group.

Definition 1. *[6] Let G be a finite group. Coprime graph of G is denoted by Γ_G and G as the set of vertices and $\forall\ a, b \in G$ adjacent whenever $gcd(|a|, |b|) = 1$.*

The dihedral group is denoted by $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ [7]. This research investigates the coprime graph for the non-abelian D_{2n}, where $n \geq 3$ and $n \in \mathbb{N}$, denoted by $\Gamma_{D_{2n}}$.

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The energy of $\Gamma_{D_{2n}}$ is denoted by $E(\Gamma_{D_{2n}})$. Gutman [8] first defined the graph energy in 1978. The graphs on $2n$ vertices with $E(\Gamma_{D_{2n}}) \geq 2n$ is nonhypoenergetic, while if $E(\Gamma_{D_{2n}}) < 2n - 1$ is strongly hypoenergetic [9].

In recent years, numerous papers on spectral graph theory have been published. Romdhini et al. obtained the spectral properties of the non-commuting graph for D_{2n} corresponding with the Sombor matrix [10] and Wiener-Hosoya matrix [11]. Apart from that, the spectral and structural properties of the cubic power graph for D_{2n} can be seen in [12, 13], the equal-square graph is presented by [14], the prime ideal graph [15], and the prime coprime graph [16]. Furthermore, in [17], it is shown that a precise formula can be derived for the calculation of the degree of the vertex in a coprime order graph of group D_{2n}". Additional terminology related to this discussion can be found in [18–21]. They worked on finite groups, and the review on graphs on groups can be seen in [22]. In addition, the discussion on the vertex degree of the coprime graph for dihedral groups has been discussed in [23]. Therefore, this paper examines the characteristic polynomial of a coprime graph associated with the adjacency, Laplacian, and signless Laplacian matrices. The definition of them refers to a book from [24].

The basic definitions and notational conventions relevant to this research are summarized in the following table.

Definition 2. *[24] An $n \times n$ adjacency (A) matrix of $\Gamma_{D_{2n}}$ is denoted by $A(\Gamma_{D_{2n}}) = [a_{ij}]$, where*

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3. *[24] An $n \times n$ diagonal degree matrix of $\Gamma_{D_{2n}}$ is $D(\Gamma_{D_{2n}}) = [d_{ij}]$ whose (i, j)-th entry*

$$d_{ij} = \begin{cases} deg(v_i), & \text{if } v_i = v_j \\ 0, & \text{otherwise,} \end{cases}$$

where $deg(v_i)$ represent the degree of v_i.

Definition 4. *An $n \times n$ Laplacian (L) matrix of $\Gamma_{D_{2n}}$ is $L(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) - A(\Gamma_{D_{2n}})$.*

Definition 5. *[24] An $n \times n$ signless Laplacian (SL) matrix of $\Gamma_{D_{2n}}$ is $SL(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) + A(\Gamma_{D_{2n}})$.*

The energy of $\Gamma_{D_{2n}}$ [8] associated with $A(\Gamma_{D_{2n}})$ is defined as

$$E_A(\Gamma_{D_{2n}}) = \Sigma_{i=1}^{n} |\lambda_i|,$$

and $A-$spectral radius of $\Gamma_{D_{2n}}$ are defined as

$$\rho_A(\Gamma_{D_{2n}}) = max\{|\lambda| : \lambda \in Spec_A(\Gamma_{D_{2n}})\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ are eigenvalues of $A(\Gamma_{D_{2n}})$ as the roots of characteristic polynomial $P_{A(\Gamma_{D_{2n}})}(\lambda) = |\lambda I_{2n} - A(\Gamma_{D_{2n}})| = 0$, and $Spec_A(\Gamma_{D_{2n}})$ is the spectrum of $A(\Gamma_{D_{2n}})$. These notations also apply for $L(\Gamma_{D_{2n}})$ and $SL(\Gamma_{D_{2n}})$.

Some previous results on the degree of a vertex in $\Gamma_{D_{2n}}$ are presented as follows.

Table 1: Notation and its Definition

Symbol	Definition
G	group
Γ_G	coprime graph of G
$\|u\|$	order of u in G
D_{2n}	dihedral group of order $2n$
$\Gamma_{D_{2n}}$	coprime graph for D_{2n}
$deg(u)$	degree of vertex u
$A(\Gamma_{D_{2n}})$	adjacency matrix of $\Gamma_{D_{2n}}$
$D(\Gamma_{D_{2n}})$	diagonal degree matrix of $\Gamma_{D_{2n}}$
$L(\Gamma_{D_{2n}})$	Laplacian matrix of $\Gamma_{D_{2n}}$
$SL(\Gamma_{D_{2n}})$	signless Laplacian matrix of $\Gamma_{D_{2n}}$
$P_{A(\Gamma_{D_{2n}})}(\lambda)$	characteristic polynomial of $A(\Gamma_{D_{2n}})$
$P_{L(\Gamma_{D_{2n}})}(\lambda)$	characteristic polynomial of $L(\Gamma_{D_{2n}})$
$P_{SL(\Gamma_{D_{2n}})}(\lambda)$	characteristic polynomial of $SL(\Gamma_{D_{2n}})$
λ_i	eigenvalues of the matrix
$E_A(\Gamma_{D_{2n}})$	adjacency energy of $\Gamma_{D_{2n}}$
$E_L(\Gamma_{D_{2n}})$	Laplacian energy of $\Gamma_{D_{2n}}$
$E_{SL}(\Gamma_{D_{2n}})$	signless Laplacian energy of $\Gamma_{D_{2n}}$
$Spec_A(\Gamma_{D_{2n}})$	spectrum of $A(\Gamma_{D_{2n}})$
$\rho_A(\Gamma_{D_{2n}})$	spectral radius of $\Gamma_{D_{2n}}$ associated with $A(\Gamma_{D_{2n}})$
$\rho_L(\Gamma_{D_{2n}})$	spectral radius of $\Gamma_{D_{2n}}$ associated with $L(\Gamma_{D_{2n}})$
$\rho_{SL}(\Gamma_{D_{2n}})$	spectral radius of $\Gamma_{D_{2n}}$ associated with $SL(\Gamma_{D_{2n}})$
R_i	the i-th row of the matrix
C_i	the i-th column of the matrix

Theorem 1. *[23] Let D_{2n} be the dihedral group with n is a prime number or $n = p^k$, $p \neq 2$ for $k \in \mathbb{N}$, then $\Gamma_{D_{2n}}$ is a complete tripartite graph and $deg(e) = 2n-1$, $deg(a^i) = n+1$, $deg(a^ib) = n$ for $1 \leq i \leq n-1$.*

Theorem 2. *[23] Let D_{2n} be the dihedral group with $n = 2^k, k \in \mathbb{N}$, then $\Gamma_{D_{2n}}$ is a complete bipartite graph and $deg(a^i) = deg(a^ib) = 1$ and $deg(e) = 2n-1$.*

Moreover, the determinant properties of a square matrix are useful to ease the characteristic polynomial of $\Gamma_{D_{2n}}$. Now, let $J_{m\times n}$ be an $m \times n$ matrix whose entries are all 1.

Lemma 1. *[25] For real numbers a, b, c, and d, the determinant of*

$$\begin{vmatrix} (\lambda+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\lambda+b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

of size $n_1 + n_2$ can be simplified as

$$(\lambda+a)^{n_1-1}(\lambda+b)^{n_2-1}\left((\lambda-(n_1-1)a)(\lambda-(n_2-1)b) - n_1n_2cd\right).$$

Additionally, we require some row and column operations in our proof. We shall introduce the $i-$th row of a matrix, R_i, and the $i-$th column, C_i.

2. Main results

In this part, we discuss the spectral radius of the coprime graph for the dihedral group, D_{2n}, with n as a prime number or $n = 2^k$ or $n = p^k$, where $p \neq 2$, $p, k \in \mathbb{N}$ corresponding with $A(\Gamma_{D_{2n}})$, $L(\Gamma_{D_{2n}})$, and $SL(\Gamma_{D_{2n}})$.

2.1. Adjacency energy

We first prove the adjacency matrix of the coprime graph for D_{2n}.

Theorem 3. *Let $\Gamma_{D_{2n}}$ be the coprime graph for D_{2n} with n is a prime number or $n = p^k$, $p \neq 2$ for a $k \in \mathbb{N}$, then*

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \lambda^{2n-3}\left(\lambda^3 - \left(n^2 + n - 1\right)\lambda + 2n(n-1)\right).$$

Proof. From Theorem 1, $\Gamma_{D_{2n}}$, for D_{2n} with n is a prime number or $n = p^k$, $p \neq 2$ for a $k \in \mathbb{N}$, is a complete tripartite graph, then we have a $2n \times 2n$ adjacency matrix of $\Gamma_{D_{2n}}$,

$$A(\Gamma_{D_{2n}}) = \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccc} e & a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \end{pmatrix} \end{array}. \quad (1)$$

Moreover, $A(\Gamma_{D_{2n}})$ can be mentioned as

$$A(\Gamma_{D_{2n}}) = \begin{pmatrix} 0 & J_{1\times(n-1)} & J_{1\times n} \\ J_{(n-1)\times 1} & 0_{n-1} & J_{(n-1)\times n} \\ J_{n\times 1} & J_{n\times(n-1)} & 0_n \end{pmatrix}.$$

The characteristic formula of $A(\Gamma_{D_{2n}})$ is given in the following

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda & -J_{1\times(n-1)} & -J_{1\times n} \\ -J_{(n-1)\times 1} & \lambda I_{n-1} & -J_{(n-1)\times n} \\ -J_{n\times 1} & -J_{n\times(n-1)} & \lambda I_n \end{vmatrix}.$$

The proof follows from the following sequential steps:

(i) For $i = 1, 2, \ldots, n-1$, we replace elements in the $n+1+i$-th row by subtracting the corresponding element of the $n+1+i$-th row from the element in the $n+1$-th row, or in other words, $R_{n+1+i} \longrightarrow R_{n+1+i} - R_{n+1}$.

(ii) We replace the elements of the $n+1$-th column with the sum of the corresponding elements in columns numbered $n+1$, $n+2$, ..., and $2n$. It is written by $C_{n+1} \longrightarrow C_{n+1} + C_{n+2} + \ldots + C_{2n}$.

(iii) We replace elements in the $n+1$-th row by subtraction between those elements and the elements in the first row, or equivalently, $R_{n+1} \longrightarrow R_{n+1} - R_1$.

(iv) We replace the first column by adding its own elements and $\frac{\lambda+1}{\lambda+n}$ times the elements of the $n+1$-th column, or in other words, $C_1 \longrightarrow C_1 + \left(\frac{\lambda+1}{\lambda+n}\right) C_{n+1}$.

(v) For $i = 1, 2, \ldots, n-2$, the $2+i$-th row is replaced by subtraction between the elements of the $2+i$th row and the second row, and is expressed as $R_{2+i} \longrightarrow R_{2+i} - R_2$.

(vi) The second column is replaced by the sum of the second, third, fourth,..., and n-th columns, and is stated as $C_2 \longrightarrow C_2 + C_3 + \ldots + C_n$.

Hence,

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \lambda^{2n-3} \left(\lambda^3 - \left(n^2 + n - 1\right) \lambda + 2n(n-1)\right).$$

Theorem 4. *Let $\Gamma_{D_{2n}}$ be the coprime graph for D_{2n} with $n = 2^k, k \in \mathbb{N}$, then*

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \lambda^{2n-2} \left(\lambda - \sqrt{2n-1}\right) \left(\lambda + \sqrt{2n-1}\right).$$

Proof. According to Theorem 2, $\Gamma_{D_{2n}}$, for D_{2n} with $n = 2^k, k \in \mathbb{N}$, is a complete bipartite graph, then we provide a $2n \times 2n$ adjacency matrix of $\Gamma_{D_{2n}}$.

$$A(\Gamma_{D_{2n}}) = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Observe that $A(\Gamma_{D_{2n}})$ is

$$A(\Gamma_{D_{2n}}) = \begin{pmatrix} 0 & J_{1\times(2n-1)} \\ J_{(2n-1)\times 1} & 0_{2n-1} \end{pmatrix}.$$

It can be seen that

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda & -J_{1\times(2n-1)} \\ -J_{(2n-1)\times 1} & \lambda I_{2n-1} \end{vmatrix}.$$

Based on Lemma 1 with $a = b = 0$, $c = d = 1$, $n_1 = 1$, and $n_2 = 2n-1$, then we obtain

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \lambda^{2n-2}\left(\lambda - \sqrt{2n-1}\right)\left(\lambda + \sqrt{2n-1}\right).$$

2.2. Laplacian energy

In the next two theorems, we focus on the Laplacian matrix of $\Gamma_{D_{2n}}$.

Theorem 5. *Let $\Gamma_{D_{2n}}$ be the coprime graph for D_{2n} with n is a prime number or $n = p^k$, $p \neq 2$ for a $k \in \mathbb{N}$, then*

$$P_{L(\Gamma_{D_{2n}})}(\lambda) = \lambda(\lambda - n)^{n-1}(\lambda - 2n)(\lambda - (n+1))^{n-2}(\lambda - 2n).$$

Proof. Since $\Gamma_{D_{2n}}$, for D_{2n} with n is a prime number or $n = p^k$, $p \neq 2$ for a $k \in \mathbb{N}$, has $deg(e) = 2n-1$, $deg(a^i) = n+1$, and $deg(a^i b) = n$ corforming from Theorem 1, then we have

$$D(\Gamma_{D_{2n}}) = \begin{pmatrix} 2n-1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & n+1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & n+1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n+1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & n & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & n \end{pmatrix}. \quad (2)$$

According to Equation 1 and following Definition 4, then we obtain

$$\begin{aligned} L(\Gamma_{D_{2n}}) &= D(\Gamma_{D_{2n}}) - A(\Gamma_{D_{2n}}) \\ &= \begin{pmatrix} 2n-1 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\ -1 & n+1 & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\ -1 & 0 & n+1 & \dots & 0 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & n+1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 & n & 0 & \dots & 0 \\ -1 & -1 & -1 & \dots & -1 & 0 & n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 & 0 & 0 & \dots & n \end{pmatrix}. \end{aligned}$$

Hence, it is

$$L(\Gamma_{D_{2n}}) = \begin{pmatrix} 2n-1 & -J_{1\times(n-1)} & -J_{1\times n} \\ -J_{(n-1)\times 1} & (n+1)I_{n-1} & -J_{(n-1)\times n} \\ -J_{n\times 1} & -J_{n\times(n-1)} & nI_n \end{pmatrix}.$$

The characteristic equation of $L(\Gamma_{D_{2n}})$ is given below

$$P_{L(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda-(2n-1) & J_{1\times(n-1)} & J_{1\times n} \\ J_{(n-1)\times 1} & (\lambda-(n+1))I_{n-1} & J_{(n-1)\times n} \\ J_{n\times 1} & J_{n\times(n-1)} & (\lambda-n)I_n \end{vmatrix}.$$

We follow the following operational steps in the same manner as the proof of Theorem 3:

(i) $R_{n+1+i} \longrightarrow R_{n+1+i} - R_{n+1}$, for $i = 1, 2, \ldots, n-1$.

(ii) $C_{n+1} \longrightarrow C_{n+1} + C_{n+2} + \ldots + C_{2n}$.

(iii) $R_{n+1} \longrightarrow R_{n+1} - R_1$.

(iv) $C_1 \longrightarrow C_1 + C_{n+1}$.

(v) For $i = 1, 2, \ldots, n-2$, $R_{2+i} \longrightarrow R_{2+i} - R_2$.

(vi) $C_2 \longrightarrow C_2 + C_{2+1} + \ldots + C_n$.

Therefore,

$$P_{L(\Gamma_{D_{2n}})}(\lambda) = \lambda(\lambda-n)^{n-1}(\lambda-2n)(\lambda-(n+1))^{n-2}(\lambda-2n).$$

Theorem 6. *Let $\Gamma_{D_{2n}}$ be the coprime graph for D_{2n} with $n = 2^k, k \in \mathbb{N}$, then*

$$P_{L(\Gamma_{D_{2n}})}(\lambda) = (\lambda-1)^{2n-2}(\lambda+1)(\lambda-(2n-1)).$$

Proof. Based on Theorem 2, $\Gamma_{D_{2n}}$, for D_{2n} with $n = 2^k, k \in \mathbb{N}$, has $deg(a^i) = deg(a^ib) = 1$ and $deg(e) = 2n-1$, then we can provide a $2n \times 2n$ degree matrix of $\Gamma_{D_{2n}}$.

$$D(\Gamma_{D_{2n}}) = \begin{pmatrix} 2n-1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \quad (3)$$

Based on Equation 1 and by Definition 4, then

$$
\begin{aligned}
L(\Gamma_{D_{2n}}) =& D(\Gamma_{D_{2n}}) - A(\Gamma_{D_{2n}}) \\
=& \begin{pmatrix}
2n-1 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\
-1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
-1 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\
-1 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\
-1 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1
\end{pmatrix}.
\end{aligned}
$$

Hence, we have

$$
L(\Gamma_{D_{2n}}) = \begin{pmatrix} 2n-1 & -J_{1\times(2n-1)} \\ -J_{(2n-1)\times 1} & I_{2n-1} \end{pmatrix}.
$$

Furthermore, as $P_{L(\Gamma_{D_{2n}})}(\lambda) = |L(\Gamma_{D_{2n}}) - \lambda I_{2n}|$, then

$$
P_{L(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda - (2n-1) & J_{1\times(2n-1)} \\ J_{(2n-1)\times 1} & (\lambda-1)I_{2n-1} \end{vmatrix}.
$$

We follow the following operational steps

(i) $R_{2+i} \longrightarrow R_{2+i} - R_2$, for $i = 1, 2, \dots, 2n-2$.

(ii) $C_2 \longrightarrow C_2 + C_3 + \dots + C_{2n}$.

Therefore,

$$
P_{L(\Gamma_{D_{2n}})}(\lambda) = \lambda(\lambda-1)^{2n-2}(\lambda - 2n).
$$

2.3. Signless Laplacian energy

The next theorems are the results of the signless Laplacian matrix of $\Gamma_{D_{2n}}$.

Theorem 7. *Let $\Gamma_{D_{2n}}$ be the coprime graph for D_{2n} with n is a prime number or $n = p^k$, $p \neq 2$ for a $k \in \mathbb{N}$, then*

$$
P_{SL(\Gamma_{D_{2n}})}(\lambda) = (\lambda - n)^{n-1}(\lambda - (n+1))^{n-2}\left(\lambda^3 - 4n\lambda^2 + 4n^2\lambda - 4n(n-1)\right).
$$

Proof. As the same argument with Theorem 5 and according to Equation 2 and Definition 5, then SL-matrix of $\Gamma_{D_{2n}}$ is

$$SL(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) + A(\Gamma_{D_{2n}}) = \begin{pmatrix} 2n-1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & n+1 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & n+1 & \dots & 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & n+1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & n & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 & n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & n \end{pmatrix}.$$

Thus, $SL(\Gamma_{D_{2n}})$ is a partitioned matrix,

$$SL(\Gamma_{D_{2n}}) = \begin{pmatrix} 2n-1 & J_{1\times(n-1)} & J_{1\times n} \\ J_{(n-1)\times 1} & (n+1)I_{n-1} & J_{(n-1)\times n} \\ J_{n\times 1} & J_{n\times(n-1)} & nI_n \end{pmatrix}.$$

Since $P_{SL(\Gamma_{D_{2n}})}(\lambda) = |SL(\Gamma_{D_{2n}}) - \lambda I_{2n}|$, then

$$P_{SL(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda-(2n-1) & -J_{1\times(n-1)} & -J_{1\times n} \\ -J_{(n-1)\times 1} & (\lambda-(n+1))I_{n-1} & -J_{(n-1)\times n} \\ -J_{n\times 1} & -J_{n\times(n-1)} & (\lambda-n)I_n \end{vmatrix}.$$

By the same argument of the proof of Theorem 3, we apply the following steps:

(i) $R_{n+1+i} \longrightarrow R_{n+1+i} - R_{n+1}$, for $i = 1, 2, \dots, n-1$.

(ii) $C_{n+1} \longrightarrow C_{n+1} + C_{n+2} + \dots + C_{2n}$.

(iii) $R_{n+1} \longrightarrow R_{n+1} - R_1$.

(iv) $C_1 \longrightarrow C_1 + \left(\frac{\lambda-2(n-1)}{\lambda}\right) C_{n+1}$.

(v) For $i = 1, 2, \dots, n-2$, $R_{2+i} \longrightarrow R_{2+i} - R_2$.

(vi) $C_2 \longrightarrow C_2 + C_{2+1} + \dots + C_n$.

Hence, we obtain

$$P_{SL(\Gamma_{D_{2n}})}(\lambda) = (\lambda-n)^{n-1}(\lambda-(n+1))^{n-2}\left(\lambda^3 - 4n\lambda^2 + 4n^2\lambda - 4n(n-1)\right).$$

Theorem 8. *Let $\Gamma_{D_{2n}}$ be the coprime graph for D_{2n} with $n = 2^k, k \in \mathbb{N}$, then*

$$P_{SL(\Gamma_{D_{2n}})}(\lambda) = (\lambda - 1)^{2n-2} (\lambda + 1) (\lambda - (2n - 1)).$$

Proof. Based on the adjacency matrix in Equation 1, the degree matrix in Equation 3, and Definition 5, then we can produce a $2n \times 2n$ signless Laplacian matrix of $\Gamma_{D_{2n}}$,

$$\begin{aligned} SL(\Gamma_{D_{2n}}) =& D(\Gamma_{D_{2n}}) + A(\Gamma_{D_{2n}}) \\ =& \begin{pmatrix} 2n-1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \end{aligned}$$

In this case, $SL(\Gamma_{D_{2n}})$ is

$$SL(\Gamma_{D_{2n}}) = \begin{pmatrix} 2n-1 & J_{1\times(2n-1)} \\ J_{(2n-1)\times 1} & I_{2n-1} \end{pmatrix}.$$

Hence, using $|SL(\Gamma_{D_{2n}}) - \lambda I_{2n}|$, we have

$$P_{SL(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda - (2n-1) & -J_{1\times(2n-1)} \\ -J_{(2n-1)\times 1} & (\lambda - 1) I_{2n-1} \end{vmatrix}.$$

We follow the following operational steps

(i) $R_{2+i} \longrightarrow R_{2+i} - R_2$, for $i = 1, 2, \dots, 2n - 2$.

(ii) $C_2 \longrightarrow C_2 + C_3 + \dots + C_{2n}$.

Then we obtain

$$P_{SL(\Gamma_{D_{2n}})}(\lambda) = \lambda(\lambda - 1)^{2n-2} (\lambda - 2n).$$

3. Further discussions

In comparing the result from Theorems 6 and 8, we derive the following fact:

Corollary 1. *Let $\Gamma_{D_{2n}}$ be the coprime graph for D_{2n} with $n = 2^k, k \in \mathbb{N}$, then*

$$P_{L(\Gamma_{D_{2n}})}(\lambda) = P_{SL(\Gamma_{D_{2n}})}(\lambda).$$

According to Theorems 4, 6, and 8, we can determine the energy of $\Gamma_{D_{2n}}$ as presented in Theorems 9 and 10.

Theorem 9. *Let* $\Gamma_{D_{2n}}$ *be the coprime graph for* D_{2n} *with* $n = 2^k, k \in \mathbb{N}$*, then*

$$E_A(\Gamma_{D_{2n}}) = 2\sqrt{2n-1} = 2\rho_A(\Gamma_{D_{2n}}).$$

Proof. Theorem 4 gives the formula of $P_{A(\Gamma_{D_{2n}})}(\lambda)$. The roots of this polynomial are $\lambda_1 = 0$ with multiplicity $2n-2$, and $\lambda_{2,3} = \pm\sqrt{2n-1}$. Thus, the spectral radius and energy of $\Gamma_{D_{2n}}$ corresponds with adjacency matrix is

$$\begin{aligned} &\rho_A(\Gamma_{D_{2n}}) = \sqrt{2n-1}, \\ &E_A(\Gamma_{D_{2n}}) = (2n-2)|0| + \left|\pm\sqrt{2n-1}\right| = 2\sqrt{2n-1}. \end{aligned}$$

Theorem 10. *Let* $\Gamma_{D_{2n}}$ *be the coprime graph for* D_{2n} *with* $n = 2^k, k \in \mathbb{N}$*, then*

$$E_L(\Gamma_{D_{2n}}) = E_{SL}(\Gamma_{D_{2n}}) = 2(2n-1) = 2\rho_L(\Gamma_{D_{2n}}) = 2\rho_{SL}(\Gamma_{D_{2n}}).$$

Proof. Theorem 1 gives $P_{L(\Gamma_{D_{2n}})}(\lambda) = P_{SL(\Gamma_{D_{2n}})}(\lambda)$. The roots of this polynomial are $\lambda_1 = 1$ with multiplicity $2n-2$, and a single $\lambda_2 = 0$, $\lambda_3 = 2n$. Therefore, the spectral radius and the energy of $\Gamma_{D_{2n}}$ corresponds with Laplacian and signless Laplacian matrices is

$$\begin{aligned} &\rho_L(\Gamma_{D_{2n}}) = \rho_{SL}(\Gamma_{D_{2n}}) = 2n, \\ &E_L(\Gamma_{D_{2n}}) = E_{SL}(\Gamma_{D_{2n}}) = (2n-2)|1| + (1)|0| + (1)|2n| = 2(2n-1). \end{aligned}$$

According to the two previous theorems, we can conclude that $\Gamma_{D_{2n}}$ with $n = 2^k, k \in \mathbb{N}$ is always twice their spectral radius. Moreover, it is shown that the energy of $\Gamma_{D_{2n}}$ is never an odd integer and strongly hypoenergetic, meanwhile the L and SL-energies are always even integers and are nonhypoenergetic. These results are consistent with the previous results of Bapat and Pati [26].

4. Conclusion

This paper presented the spectral properties of the coprime graph for dihedral groups. The spectral radius and the energy of $\Gamma_{D_{2n}}$ have been studied. We also found a correlation between them and the graph classification based on the obtained energies.

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