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Common Fixed Points of Triplet Mappings in GM-Spaces with Applications to AI Convergence and Cryptographic Consensus

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Abstract. This paper investigates the existence and uniqueness of common fixed points for three self-mappings in generalized metric spaces (GM-spaces), establishing new results under generalized contractive conditions. We develop a comprehensive theoretical framework where tripartite mappings satisfy inequalities involving combinations of distance-like terms formulated through minimum and maximum comparisons. The contractive conditions are governed by carefully chosen parameters that determine whether the mappings admit a common fixed point or a unique common fixed point. To demonstrate the practical applicability of our theory, we present a concrete example using the interval $[0, 1]$ equipped with the G-metric $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$, where piecewise-defined self-mappings are shown to satisfy all required conditions and converge to a unique common fixed point. Beyond theoretical advancements, our results offer significant applications in artificial intelligence, particularly in analyzing convergence of multi-layer neural architectures, and in cryptography for designing secure iterative protocols. The framework presented here not only generalizes existing fixed-point theorems but also provides verifiable computational methods for stability analysis in both mathematical and applied contexts.

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Key Words and Phrases: Contraction Conditions, Common Fixed Point, GM-Space, Artificial Intelligence, Cryptographic Protocols, G-Metric Convergence

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1. Introduction

Fixed point theory (FPT) is a branch of mathematics that deals with the existence and uniqueness of fixed points for various types of mappings or functions. FPT plays a crucial role in various branches of mathematics, including analysis, topology, and functional analysis. It has numerous applications in diverse fields such as economics, physics, engineering, computer science, and more recently, artificial intelligence (AI) and cryptography. In AI, FPT supports the theoretical convergence of learning dynamics in deep neural networks, especially within recurrent and iterative architectures. In cryptographic systems, fixed point theorems ensure consistency and convergence in multi-party secure computations and iterative encryption/decryption protocols.

Within the context of FPT, Stefan Banach introduced the "Banach Contraction Principle (BCP)", which states that a single-valued contractive-type mapping in a complete metric space has a unique fixed point. Our objective is to extend this principle to GM-spaces, which offer a broader framework. Our approach involves examining existing research articles in the GM-space domain to identify any shortcomings in current results. By accurately pinpointing these issues, we aim to establish a more comprehensive solution. Ultimately, our goal is to refine and strengthen the theoretical foundation of GM-spaces, utilizing the BCP as a guiding principle while addressing limitations to achieve a more universally applicable outcome, particularly in AI and cryptographic applications.

In 1941, Kakutani [1] modified a generalization of Brouwers FP-theorem. The theory originated from the Brouwer FP-theorem proposed by mathematician L.E.J. Brouwer in 1910. This theorem states that any continuous function from a closed ball to itself must have at least one FP. Jungck and Rhoades [2] proved some FP-theorems for compatible maps. Minak et al. [3] described a new approach to FP-theorems for multivalued contractive mappings.

Metric spaces (M-spaces) provide a fundamental framework in mathematics for studying distance and convergence. Sullivan [4] showed a characterization of complete M-spaces. George and Veeramani [5] worked on some results in fuzzy M-spaces. Bhaskar and Lakshmikantham [6] defined FP-theorems in partially ordered M-spaces and applications. Kutbi et al. [7] proved CFP results for mappings with rational expressions. In 2022, Yaseen et al. [8] addressed new results of FP-theorems in complete M-spaces.

Next, having explored the concepts within FPT and M-spaces, we delve into the concepts specific to GM-spaces. To achieve this, we examine existing research carried out in this specific area, which provides the foundation to establish the most comprehensive outcomes. Rhoades [9] formulated a FP-theorem for generalized metric spaces (GM-spaces). In 2006, Mustafa and Sims [10] proposed the idea of generalized metric spaces. Azam and Arshad [11] looked into a Kannan FP-theorem on GM-spaces. Sarma et al. [12] flourished a contraction over GM-spaces. Mihet [13] worked on Kannan FP-principle in GM-spaces. Das and Dey [14] addressed FP of contractive mappings in GM-spaces. Shatanawi [15] formulated a coupled FP-theorems in GM-spaces. Abbas et al. [**?**] defined CFP results for three maps in GM-space. Abbas et al. [**?**] examined coupled CFP results in two GM-spaces. Di Bari and Vetro [16] explored CF-points in GM-spaces. Romaguera [17] established FP-theorems for generalized contractions on partial M-spaces. Mohanta and Mohanta [18] tried to investigate a CFP theorem in GM-spaces. Gugnani et al. [19] described CFP results in GM-spaces and its applications. Aydi [20] discussed a CFP of integral type contraction in GM-spaces. Kadelburg and Radenovic [21] worked on GM-spaces.

Later on, in 2015, Jleli and Samet [22] provided a GM-space and proved its related FP-theorems. Abed and Luaibi [23] addressed two FP-theorems in GM-spaces. Lin and Yun [24] examined GM-spaces and mappings. Peng and Sun [25] explored a study on symmetric products of GM-spaces. Our approach involves utilizing generalized outcomes through the application of maximum and minimum type contractions within GM-spaces without continuity. This approach aims to significantly enhance the effectiveness of our results. Furthermore, our work endeavors to build upon the foundation of the latest published results, thereby producing an expanded and enriched version of its findings.

The significance of our study lies in its capacity to extend and enrich existing results from the realm of FPT. Through our exploration, we endeavor to enhance our understanding of FP principles by introducing new dimensions. Our main objective is to establish specific conditions under which fixed points uniquely exist. To achieve this, we exploit the power of three self-mappings that fulfill the criteria of generalized contractive conditions within a GM-space. This investigation is undertaken with the goal of contributing valuable insights that further illuminate the existence and uniqueness of fixed points, while simultaneously advancing the frontiers of GM-space theory. Moreover, these insights provide solid theoretical underpinnings for reliable iterative convergence in AI algorithms and consistency guarantees in cryptographic communication protocols.

2. Preliminaries

This section is devoted to some fundamental definitions, which are necessary for the upcoming sections.

2.1. Generalized Metric Space

This section explores fixed point theory in generalized metric spaces (GM-spaces), extending concepts like the Banach Contraction Principle (BCP) to a broader context. GM-spaces is defined and introduce G-Cauchy sequences, convergence, and completeness in these spaces. Result on GM space is also given.

Definition 1. *[2] Suppose a non-empty set X and let G be a function that operates on triples of elements from X denoted as $G : X \times X \times X \to [0, \infty)$. We will call this structure a generalized metric space (GM-space) if the following axioms hold true:*

(i) $G(x_1, x_2, x_3) = 0$ *iff* $x_1 = x_2 = x_3$,

(ii) $0 < G(x_1, x_1, x_2)\ \forall\ x_1, x_2 \in x,$ *with* $x_1 \neq x_2$,

(iii) $G(x_1, x_1, x_2) \leq G(x_1, x_2, x_3)\ \forall\ x_1, x_2, x_3 \in x_1,$ *with* $x_1 \neq x_2$,

(iv) $G(x_1, x_2, x_3) = G\{p(x_1, x_2, x_3)\}$ *Where, p is a permutation of x_1, x_2, x_3. (symmetry).*

(v) $G(x_1, x_2, x_3) \leq G(x_1, x_1, x_1) + G(x_1, x_2, x_3)\ \forall\ x_1, x_2, x_3 \in X$.

A GM is symmetric if $G(x_1, x_2, x_2) = G(x_2, x_1, x_1)\ \forall\ x_1, x_2 \in X$, then (X, G) is called a GM-space.

Definition 2. *[2] Let (X, G) be a GM-space and $\{x_\jmath\}$ be a sequence in X. then,*

(i) A sequence $\{x_\jmath\}$ is considered a G-Cauchy sequence in the GM-space X if, for any $\varepsilon > 0$, $\exists n_0 = \mathbb{N}$ so that $G(x_\jmath, x_m, x_l) < \varepsilon\ \forall \jmath, m, l \geq n_0$.

(ii) A sequence $\{x_\jmath\}$ in the GM-space X is said to converge to an element $xinX$ if $\forall$ any given $\varepsilon > 0 \in \mathbb{R}$, $\exists n_0 = \mathbb{N}$ so that $G(x_\jmath, x_m, x_l) < \varepsilon$, whenever $m \geq n_0$.

(iii) The GM-space (X, G) is classified as complete if every G-Cauchy sequence is G-convergent in X.

Proposition 1. *[2] Let (X, G) be a GM-space, then for any $x_1, x_2, x_3 \in X$ it followes that:*

(i) If $G(x_1, x_2, x_3) = 0$, then $x_1 = x_2 = x_3$,

(ii) $G(x_1,x_2,x_3) \leq G(x_1,x_1,x_3)$,

(iii) $G(x_1,x_2,x_2) \leq 2G(x_2,x_1,x_1)$,

(iv) $G(x_1,x_2,x_3) \leq G(x_1,x,x_3)+G(x,x_2,x_3)$,

(v) $G(x_1,x_2,x_3) \leq \frac{2}{3}(G(x_1,x_2,x)+G(x_1,x,x_3)+G(x,x_2,x_3))$,

(vi) $G(x_1,x_2,x_3) \leq (G(x_1,x,x)+G(x_2,x,x)+G(x_3,x,x))$,

(vii) $|G(x_1,x_2,x_3)-G(x_1,x_2,x)| \leq \max\{G(x,x_3,x_3),G(x_3,x,x)\}$,

(viii) $|G(x_1,x_2,x_3)-G(x_1,x_2,x)| \leq G(x_1,x,x_3)$,

(ix) $|G(x_1,x_2,x_3)-G(q,x_3,x_3)| \leq \max\{G(x_1,x_3,x_3),G(x_3,x_1,x_1)\}$,

(x) $|G(x_1,x_2,x_2)-G(x_2,x_1,x_1)| \leq \max\{G(x_2,x_1,x_1),G(x_1,x_2,x_2)\}$.

3. Main Results

This section explores the existence of common fixed points (CFP) and unique common fixed points (UCFP) for three self-mappings in GM-spaces. Theorems 1 and 2 provide conditions under which these mappings possess a CFP and UCFP, involving inequalities with constants An example using the GM-metric on the interval [0,1] demonstrates the conditions required for the existence of a UCFP, contributing to the fixed point theory in GM-spaces.

Theorem 1. *Let (X,G) be a GM-space and $F: X\times X\times X \to X$ be a mapping satisfying:*

$$\begin{aligned}&G(F_1x_1,F_2x_2,F_3x_3) \leq \alpha_1 G(x_1,x_2,x_3)+\alpha_2 G(x_1,x_2,F_2x_2)+\alpha_3 G(F_1x_1,x_2,x_2)\\&+\alpha_4[G(F_1x_1,x_1,x_1)+G(x_2,F_2x_2,x_2)+G(x_3,x_3,F_3x_3)]\\&+\alpha_5\min\{G(x_1,x_2,F_2x_2),G(F_1x_1,F_1x_1,x_2),G(F_1x_1,x_2,x_2),G(F_2x_2,F_2x_2,x_3)\} \quad (1)\end{aligned}$$

for all $x_1,x_2,x_3\in X$ and $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \geq 0$ with $\alpha_1,\alpha_2,\alpha_3,\alpha_4<1$ and $(\alpha_1+\alpha_2+\alpha_3+\alpha_4)<1$. Then, the three individual self-mappings referred as F_1,F_2 and F_3 possess a CFP in X. Moreover, if $(\alpha_1+\alpha_2+\alpha_3)<1$, the three individual self-mappings referred as F_1,F_2 and F_3 possess a UCFP in X.

Proof. Fix $x_0\in X$, and $\{x_k\}$ be a sequence in X. Now we define some iterative sequences in X such that $x_{(3k+1)}=F_1x_{3k}$, $x_{(3k+2)}=F_2x_{(3k+1)}$ and $x_{(3k+3)}=F_3x_{(3k+2)}$ $\forall\ k\geq 0$. Now, by using (1) we have,

$$\begin{aligned}&G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})=G(F_1x_{3k},F_2x_{(3k+1)},F_3x_{(3k+2)}) \leq \alpha_1 G(x_{3k},x_{(3k+1)},x_{(3k+2)})\\&+\alpha_2 G(x_{3k},x_{(3k+1)},F_2x_{(3k+1)})+\alpha_3 G(F_1x_{3k},x_{(3k+1)},x_{(3k+1)})\\&+\alpha_4[G(F_1x_{3k},x_{3k},x_{3k})+G(x_{(3k+1)},F_2x_{(3k+1)},x_{(3k+1)})+G(x_{(3k+2)},x_{(3k+2)},F_3x_{(3k+2)})]\\&+\alpha_5\min\left\{\begin{matrix}G(x_{3k},x_{(3k+1)},F_2x_{(3k+1)}),G(F_1x_{3k},F_1x_{3k},x_{(3k+1)}),\\G(F_1x_{3k},x_{(3k+1)},x_{(3k+1)}),G(F_2x_{(3k+1)},F_2x_{(3k+1)},x_{(3k+2)})\end{matrix}\right\}\end{aligned}$$

$$\begin{aligned}&=\alpha_1 G(x_{3k},x_{(3k+1)},x_{(3k+2)})+\alpha_2 G(x_{3k},x_{(3k+1)},x_{(3k+1)})+\alpha_3 G(x_{3k},x_{(3k+1)},x_{(3k+1)})\\&+\alpha_4[G(x_{3k},x_{3k},x_{3k})+G(x_{(3k+1)},x_{(3k+1)},x_{(3k+1)})+G(x_{(3k+2)},x_{(3k+2)},x_{(3k+2)})]\\&+\alpha_5\min\left\{\begin{matrix}G(x_{3k},x_{(3k+1)},x_{(3k+1)}),G(x_{3k},x_{3k},x_{(3k+1)}),\\G(x_{3k},x_{(3k+1)},x_{(3k+1)}),G(x_{(3k+1)},x_{(3k+1)},x_{(3k+2)})\end{matrix}\right\}\end{aligned}$$

After applying the simplification process utilizing the Definition of GM-space we have achieved the following result,

$$\leq (\alpha_1+\alpha_2)G(x_{3k},x_{(3k+1)},x_{(3k+2)})+\alpha_4[G(x_{3k},x_{(3k+1)},x_{(3k+2)}) \\ +G(x_{3k},x_{(3k+1)},x_{(3k+2)})+G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})]$$

$$(1-\alpha_4)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{3k},x_{(3k+1)},x_{(3k+2)})$$

$$\leq \frac{(\alpha_1+\alpha_2+2\alpha_4)}{(1-\alpha_4)}G(x_{3k},x_{(3k+1)},x_{(3k+2)}),\ Taking\ \rho_1=\frac{(\alpha_1+\alpha_2+2\alpha_4)}{(1-\alpha_4)}<1$$

$$G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq \rho_1 G(x_{3k},x_{(3k+1)},x_{(3k+2)}) \quad (2)$$

Now, we find iteration of a contraction.

$$G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)})=G(F_1x_{(3k+1)},F_2x_{(3k+2)},F_3x_{(3k+3)}) \leq \alpha_1 G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \\ +\alpha_2 G(x_{(3k+2)},x_{(3k+2)},F_2x_{(3k+2)})+\alpha_3 G(F_1x_{(3k+1)},x_{(3k+2)},x_{(3k+2)}) \\ +\alpha_4[G(F_1x_{(3k+1)},x_{(3k+1)},x_{(3k+1)})+G(x_{(3k+1)},F_2x_{(3k+1)},x_{(3k+1)})+G(x_{(3k+2)},x_{(3k+2)},F_3x_{(3k+2)})] \\ +\alpha_5\min\begin{Bmatrix} G(x_{(3k+1)},x_{(3k+2)},F_2x_{(3k+2)}),G(F_1x_{(3k+1)},F_1x_{(3k+1)},x_{(3k+2)}), \\ G(F_1x_{(3k+1)},x_{(3k+2)},x_{(3k+2)}),G(F_2x_{(3k+2)},F_2x_{(3k+2)},x_{(3k+3)}) \end{Bmatrix}$$

$$=\alpha_1 G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})+\alpha_2 G(x_{(3k+2)},x_{(3k+2)},x_{(3k+2)})+\alpha_3 G(x_{(3k+1)},x_{(3k+2)},x_{(3k+2)}) \\ +\alpha_4[G(x_{(3k+1)},x_{(3k+1)},x_{(3k+1)})+G(x_{(3k+1)},x_{(3k+1)},x_{(3k+1)})+G(x_{(3k+2)},x_{(3k+2)},x_{(3k+2)})] \\ +\alpha_5\min\begin{Bmatrix} G(x_{(3k+1)},x_{(3k+2)},x_{(3k+2)}),G(x_{(3k+1)},x_{(3k+1)},x_{(3k+2)}), \\ G(x_{(3k+1)},x_{(3k+2)},x_{(3k+2)}),G(x_{(3k+2)},x_{(3k+2)},x_{(3k+3)}) \end{Bmatrix}$$

After applying the simplification process utilizing the Definition of GM-space we have achieved the following result,

$$\leq (\alpha_1+\alpha_2)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})+\alpha_4[G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \\ +G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})+G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)})]$$

$$(1-\alpha_4)G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})$$

$$\leq \frac{(\alpha_1+\alpha_2+2\alpha_4)}{(1-\alpha_4)}G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}),\ Taking\ \rho_2=\frac{(\alpha_1+\alpha_2+2\alpha_4)}{(1-\alpha_4)}<1$$

$$G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)}) \leq \rho_2 G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \quad (3)$$

From both cases we get that,

$$G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq \rho G(x_{3k},x_{(3k+1)},x_{(3k+2)}),\ \therefore\ \rho_1=\rho_2=\rho$$

Inductively we have,

$$G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq \rho^{3k+1}G(x_0,x_1,x_2)$$

Next we will show that $\{x_k\}$ is a G-Cauchy sequence in X, for any j, n, k with $j > n > k$.

$$\begin{aligned}
&G(x_k, x_n, x_j) \leq G(x_k, x_{(k+1)}, x_{(k+1)}) + G(x_{(k+1)}, x_{(k+2)}, x_{(k+2)}) + \ldots\ldots + G(x_{(j-1)}, x_{(j-1)}, x_j) \\
&\leq G(x_k, x_{(k+1)}, x_{(k+2)}) + G(x_{(k+1)}, x_{(k+2)}, x_{(k+3)}) + \ldots\ldots. + G(x_{(j-2)}, x_{(j-1)}, x_j) \\
&\leq [y^k + y^{(k+1)} + \ldots\ldots.. + y^{(j-2)}]G(x_0, x_1, x_2)
\end{aligned}$$

This implies that,

$$G(x_k, x_n, x_j) \leq \frac{y^k}{(1-y)} G(x_0, x_1, x_2)$$

The same hold if $j = n > k$ and if $j > n = k$ we have,

$$G(x_k, x_n, x_j) \leq \frac{y^{k-1}}{(1-y)} G(x_0, x_1, x_2) \qquad (4)$$

If we take the limit as $k, n, j \to \infty$ we get $G(x_k, x_n, x_j) \to 0$. Hence $\{x_k\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $\wp \in X$ such that $\{x_k\}$ converges to $\wp$ as $k \to \infty$. We have to show that $F_1\wp = \wp$ by contrary case let $F_1\wp \neq \wp$.

$$\begin{aligned}
&G(F_1\wp, x_{(3k+2)}, x_{(3k+3)}) = G(F_1\wp, F_2x_{(3k+1)}, F_3x_{(3k+2)}) \leq \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) \\
&+ \alpha_2 G(\wp, x_{(3k+1)}, F_2x_{(3k+1)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)}) \\
&+ \alpha_4[G(F_1\wp, x_{3k}, x_{3k}) + G(x_{(3k+1)}, F_2x_{(3k+1)}, x_{(3k+1)}) + G(x_{(3k+2)}, x_{(3k+2)}, F_3x_{(3k+2)})] \\
&+ \alpha_5 \min \begin{Bmatrix} G(\wp, x_{(3k+1)}, F_2x_{(3k+1)}), G(F_1\wp, F_1\wp, x_{(3k+1)}), \\ G(F_1\wp, x_{(3k+1)}, x_{(3k+1)}), G(F_2x_{(3k+1)}, F_2x_{(3k+1)}, x_{(3k+2)}) \end{Bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_2 G(\wp, x_{(3k+1)}, x_{(3k+1)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)}) \\
&+ \alpha_4[G(F_1\wp, x_{3k}, x_{3k}) + G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+1)}) + G(x_{(3k+2)}, x_{(3k+2)}, x_{(3k+2)})] \\
&+ \alpha_5 \min \begin{Bmatrix} G(\wp, x_{(3k+1)}, x_{(3k+1)}), G(F_1\wp, F_1\wp, x_{(3k+1)}), \\ G(\wp, x_{(3k+1)}, x_{(3k+1)}), G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+2)}) \end{Bmatrix}
\end{aligned}$$

After simplification process we get,

$$\begin{aligned}
&G(F_1\wp, x_{(3k+2)}, x_{(3k+3)}) \leq \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_2 G(\wp, x_{(3k+1)}, x_{(3k+1)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)}) \\
&+ \alpha_4[G(F_1\wp, x_{3k}, x_{3k}) + G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+1)}) + G(x_{(3k+2)}, x_{(3k+2)}, x_{(3k+2)})]
\end{aligned}$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(F_1\wp, \wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, \wp) + \alpha_3 G(F_1\wp, \wp, \wp) + \alpha_4[G(F_1\wp, \wp, \wp) + G(\wp, \wp, \wp) + G(\wp, \wp, \wp)]$$

$$\leq (\alpha_3 + \alpha_4) G(F_1\wp, \wp, \wp)$$

$G(F_1\wp, \wp, \wp) - (\alpha_3 + \alpha_4)G(F_1\wp, \wp, \wp) \leq 0$, is a contradiction. Since $(1 - \alpha_3 - \alpha_4) \neq 0$, therefore, $G(F_1\wp, \wp, \wp) = 0$. Thus,

$$F_1\wp = \wp \qquad (5)$$

Next, we have to show that $F_2\wp = \wp$ by contrary case let $F_2\wp \neq \wp$. Then from (1) we have that,

$$\begin{aligned}
&G(x_{(3k+1)}, F_2\wp, x_{(3k+3)}) = G(F_1x_{3k}, F_2\wp, F_3x_{(3k+2)}) \leq \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) \\
&+ \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(F_1x_{3k}, \wp, \wp) \\
&+ \alpha_4[G(F_1x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, F_2\wp, \wp) + G(x_{(3k+2)}, x_{(3k+2)}, F_3x_{(3k+2)})] \\
&+ \alpha_5 \min \begin{Bmatrix} G(x_{3k}, \wp, F_2\wp), G(F_1x_{3k}, F_1x_{3k}, \wp), \\ G(F_1x_{3k}, \wp, \wp), G(F_2\wp, F_2\wp, x_{(3k+2)}) \end{Bmatrix}
\end{aligned}$$

$$G(x_{(3k+1)}, F_2\wp, x_{(3k+3)}) = \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) + \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(x_{3k}, \wp, \wp)$$
$$+ \alpha_4[G(x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, F_2\wp, \wp) + G(x_{(3k+2)}, x_{(3k+2)}, x_{(3k+2)})]$$
$$+ \alpha_5 \min \begin{Bmatrix} G(x_{3k}, \wp, F_2\wp), G(x_{3k}, x_{3k}, \wp), \\ G(x_{3k}, \wp, \wp), G(F_2\wp, F_2\wp, x_{(3k+2)}) \end{Bmatrix}$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, F_2\wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(\wp, \wp, \wp)+$$
$$\alpha_4[G(\wp, \wp, \wp) + G(\wp, F_2\wp, \wp) + G(\wp, \wp, \wp)] + \alpha_5 \min \begin{Bmatrix} G(\wp, \wp, F_2\wp), G(\wp, \wp, \wp), \\ G(\wp, \wp, \wp), G(F_2\wp, F_2\wp, \wp) \end{Bmatrix}$$
$$\leq (\alpha_2 + \alpha_4) G(F_2\wp, \wp, \wp)$$

$G(F_2\wp, \wp, \wp) - (\alpha_2+\alpha_4)G(F_2\wp, \wp, \wp) \leq 0$, is a contradiction. Since $(1-\alpha_2-\alpha_4) \neq 0$, therefore, $G(F_2\wp, \wp, \wp) = 0$. Thus,

$$F_2\wp = \wp \qquad (6)$$

Next, we have to show that $F_3\wp = \wp$ by contrary case let $F_3\wp \neq \wp$. Then from (1) we have that,

$$G(x_{(3k+1)}, x_{(3k+2)}, F_3\wp) = G(F_1x_{3k}, F_2x_{(3k+1)}, F_3\wp) \leq \alpha_1 G(x_{3k}, x_{(3k+1)}, \wp)$$
$$+ \alpha_2 G(x_{3k}, x_{(3k+1)}, F_2x_{(3k+1)}) + \alpha_3 G(F_1x_{3k}, x_{(3k+1)}, x_{(3k+1)})$$
$$+ \alpha_4[G(F_1x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, F_2x_{(3k+1)}, x_{(3k+1)}) + G(\wp, \wp, F_3\wp)]$$
$$+ \alpha_5 \min \begin{Bmatrix} G(x_{3k}, x_{(3k+1)}, F_2x_{(3k+1)}), G(F_1x_{3k}, F_1x_{3k}, x_{(3k+1)}), \\ G(F_1x_{3k}, x_{(3k+1)}, x_{(3k+1)}), G(F_2x_{(3k+1)}, F_2x_{(3k+1)}, x_(3k+2)) \end{Bmatrix}$$

$$G(x_{(3k+1)}, x_{(3k+2)}, F_3\wp) = \alpha_1 G(x_{3k}, x_{(3k+1)}, \wp) + \alpha_2 G(x_{3k}, x_{(3k+1)}, F_2x_{(3k+1)})+$$
$$\alpha_3 G(F_1x_{3k}, x_{(3k+1)}, x_{(3k+1)}) + \alpha_4[G(F_1x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, F_2x_{(3k+1)}, x_{(3k+1)}) + G(\wp, \wp, F_3\wp)]$$
$$+ \alpha_5 \min \begin{Bmatrix} G(x_{3k}, x_{(3k+1)}, F_2x_{(3k+1)}), G(F_1x_{3k}, F_1x_{3k}, x_{(3k+1)}), \\ G(F_1x_{3k}, x_{(3k+1)}, x_{(3k+1)}), G(F_2x_{(3k+1)}, F_2x_{(3k+1)}, x_(3k+2)) \end{Bmatrix}$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, \wp, F_3\wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, \wp) + \alpha_3 G(\wp, \wp, \wp)+$$
$$\alpha_4[G(\wp, \wp, \wp) + G(\wp, \wp, \wp) + G(\wp, \wp, F_3\wp)] + \alpha_5 \min \begin{Bmatrix} G(\wp, \wp, \wp), G(\wp, \wp, \wp), \\ G(\wp, \wp, \wp), G(\wp, \wp, \wp) \end{Bmatrix}$$
$$\leq \alpha_4 G(\wp, \wp, F_3\wp)$$

$G(\wp, \wp, F_3\wp) - \alpha_4 G(\wp, \wp, F_3\wp) \leq 0$, is a contradiction. Since $(1-\alpha_4) \neq 0$, therefore, $G(\wp, \wp, F_3\wp) = 0$. Thus,

$$F_3\wp = \wp \qquad (7)$$

Thus from (5), (6) and (7), it is proved that $\wp$ is a CFP of $F_1,\ F_2\ and\ F_3$, such that

$$F_1\wp = F_2\wp = F_3\wp = \wp$$

Uniqueness: Suppose that $\wp^\diamond \in X$ be the other CFP of $F_1,\ F_2\ and\ F_3$, so that

$$F_1\wp^\diamond = F_2\wp^\diamond = F_3\wp^\diamond = \wp^\diamond$$

Then from (1) we have that,

$$\begin{aligned}
G(\wp, \wp^{\diamond}, \wp^{\diamond}) &= G(F_1\wp, F_2\wp^{\diamond}, F_3\wp^{\diamond}) \leq \alpha_1 G(\wp, \wp^{\diamond}, \wp^{\diamond}) \\
&+ \alpha_2 G(\wp, \wp^{\diamond}, F_2\wp^{\diamond}) + \alpha_3 G(F_1\wp, \wp^{\diamond}, \wp^{\diamond}) \\
&+ \alpha_4[G(F_1\wp, \wp, \wp) + G(\wp^{\diamond}, F_2\wp^{\diamond}, \wp^{\diamond}) + G(\wp^{\diamond}, \wp^{\diamond}, F_3\wp^{\diamond})] \\
&+ \alpha_5 \min\begin{Bmatrix} G(\wp, \wp^{\diamond}, F_2\wp^{\diamond}), G(F_1\wp, F_1\wp, \wp^{\diamond}), \\ G(F_1\wp, \wp^{\diamond}, \wp^{\diamond}), G(F_2\wp^{\diamond}, F_2\wp^{\diamond}, \wp^{\diamond}) \end{Bmatrix}
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_1 G(\wp, \wp^{\diamond}, \wp^{\diamond}) + \alpha_2 G(\wp, \wp^{\diamond}, \wp^{\diamond}) + \alpha_3 G(\wp, \wp^{\diamond}, \wp^{\diamond}) \\
&+ \alpha_4[G(\wp, \wp, \wp) + G(\wp^{\diamond}, \wp^{\diamond}, \wp^{\diamond}) + G(\wp^{\diamond}, \wp^{\diamond}, \wp^{\diamond})] \\
&+ \alpha_5 \min\begin{Bmatrix} G(\wp, \wp^{\diamond}, \wp^{\diamond}), G(\wp, \wp, \wp^{\diamond}), \\ G(\wp, \wp^{\diamond}, \wp^{\diamond}), G(\wp^{\diamond}, \wp^{\diamond}, \wp^{\diamond}) \end{Bmatrix}
\end{aligned}$$

$$\leq \alpha_1 G(\wp, \wp^{\diamond}, \wp^{\diamond}) + \alpha_2 G(\wp, \wp^{\diamond}, \wp^{\diamond}) + \alpha_3 G(\wp, \wp^{\diamond}, \wp^{\diamond})$$

$$(\alpha_1 + \alpha_2 + \alpha_3) G(\wp, \wp^{\diamond}, \wp^{\diamond})(1 - \alpha_1 - \alpha_2 - \alpha_3) G(\wp, \wp^{\diamond}, \wp^{\diamond}) \leq 0,$$

is a contradiction. Since, $(1 - \alpha_1 - \alpha_2 - \alpha_3) \neq 0$, therefore $G(\wp, \wp^{\diamond}, \wp^{\diamond}) = 0$. Thus,

$$\wp = \wp^{\diamond}$$

It is proved that the three individual self-mappings referred as $F_1,\ F_2\ and\ F_3$ possess a UCFP in X.

The Algorithm 3 implements the iterative fixed-point computation scheme described in Theorem 1 by sequentially applying the three contractive mappings F_1, F_2, and F_3 while monitoring convergence through the G-metric condition $G(x_{3k}, x_{3k+1}, x_{3k+2}) < tolerance$.

Algorithm 1 Fixed-Point Computation for Tripartite Mappings

Require: X: G-metric space, F_1, F_2, F_3: Contractive mappings
Require: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$: Theorem parameters
Ensure: Common fixed point $\wp$

```
Initialize x_0 ∈ X
for k = 0 to max_iterations do
    x_{3k+1} ← F_1(x_{3k})
    x_{3k+2} ← F_2(x_{3k+1})
    x_{3k+3} ← F_3(x_{3k+2})
    Compute G_k ← G(x_{3k}, x_{3k+1}, x_{3k+2})
    if G_k < tolerance then
        return x_{3k+3}                ▷ Common fixed point found
    end if
end for
```

Figure 1 illustrates the convergence of the G-metric during fixed-point iterations, validating Theorem 1 under contractive conditions.

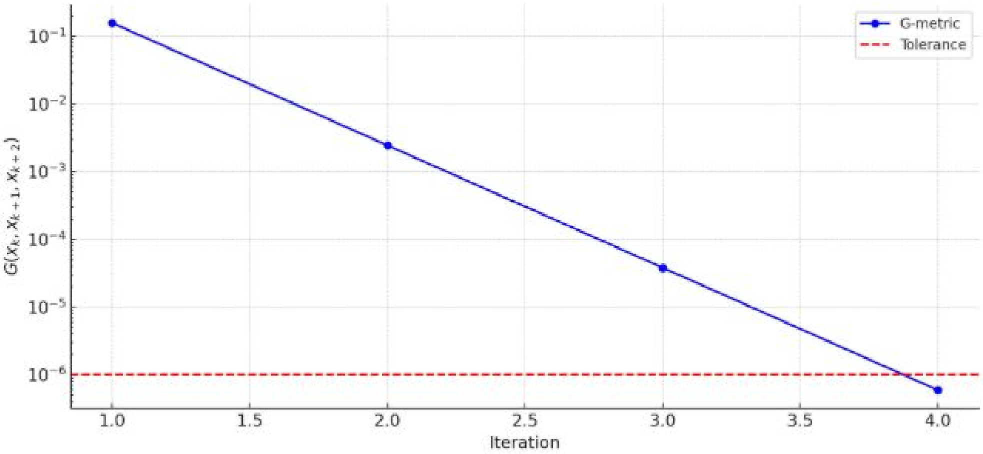

Figure 1: G-metric convergence curve validating Theorem 1 for tripartite mappings

The fixed-point framework established in Theorem 1, and implemented through Algorithm 3, finds significant applications in both artificial intelligence and cryptographic protocols. In deep learning architectures, the mappings F_1, F_2, and F_3 may be interpreted as a feature extraction layer, a regularization layer, and an output projection layer, respectively. The contraction parameters α_i serve to regulate the stability of these layers, ensuring that the iterative composition converges to a fixed activation pattern as guaranteed by the theoretical result.
In cryptography, especially within secure multi-party computation, the same mappings can be understood as follows: F_1 represents the encryption transformation, F_2 corresponds to the decryption process, and F_3 performs key mixing. The G-metric quantifies the maximum deviation between transformed message states. Convergence to a fixed point under this metric reflects the achievement of a secure and synchronized outcome across all parties involved. Thus, Theorem 1 and Algorithm 3 jointly ensure the validity and practical effectiveness of this unified framework in both AI and cryptographic systems.

Example 1. *Let $X = [0, 1]$ with G-metric defined as $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$. Consider the piecewise mapping:*

$$F_i(x) = \begin{cases} \frac{x}{4} + \frac{1}{2} & x \in [0,1] \\ \frac{3x+1}{5} & x > 1 \end{cases} \quad (i = 1, 2, 3)$$

This mapping satisfies the contractive conditions of Theorem 1, with parameters:

$$\alpha_1 = 0.2, \quad \alpha_2 = 0.15, \quad \alpha_3 = 0.1, \quad \alpha_4 = 0.05, \quad \alpha_5 = 0.02$$

The fixed-point iteration applied to this setup yields a unique common fixed point at $\wp = \frac{2}{3}$, confirming the theorems theoretical predictions through constructive computation.

Algorithm 2 Example Implementation for $F_i(x)$ in Theorem 1

function ComputeFixedPoint(x_0, *tol*)
  $k \leftarrow 0$
  repeat
    $x_{k+1} \leftarrow \frac{x_k}{4} + \frac{1}{2}$
    $error \leftarrow |x_{k+1} - x_k|$
    $k \leftarrow k + 1$
  until $error < tol$
  return x_k
end function

This work builds on the fixed-point framework established in Theorem 1, with the algorithmic implementation provided above and its correctness illustrated by Example 1 and Algorithm 3. The applications in artificial intelligence and cryptographic protocols demonstrate the broader relevance and utility of the theoretical results.

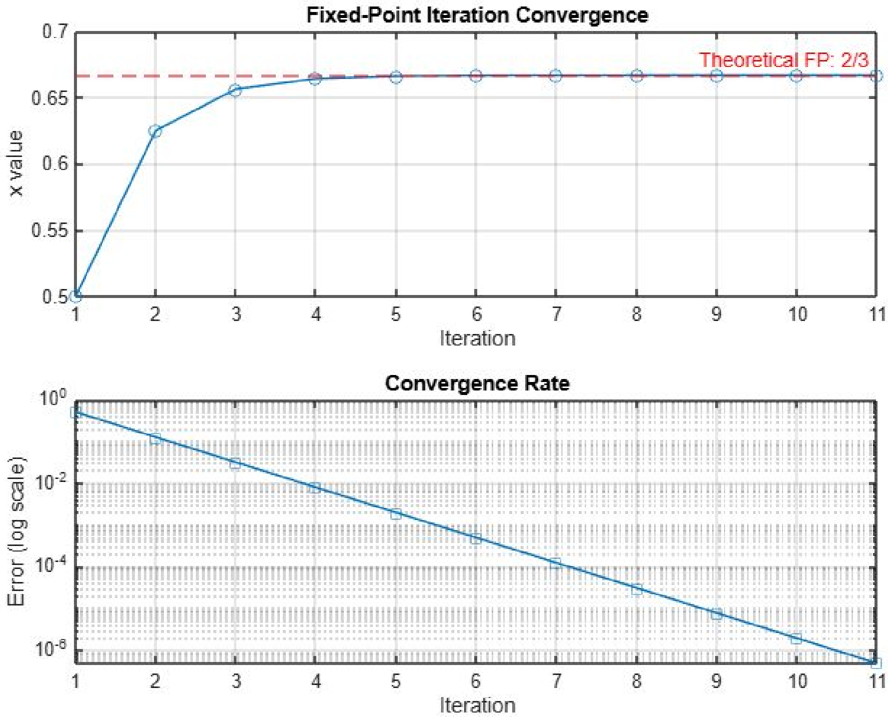

Figure 2: Visual verification of fixed-point convergence for the mapping in Example 1

Figure 2 confirms the convergence of the fixed-point iteration in Example 1, aligning with the theoretical result from Theorem 1.

Theorem 2. *Let (X, G) be a GM-space and $F : X \times X \times X \to X$ be a mapping satisfying:*

$$\begin{aligned} &G(F_1x_1, F_2x_2, F_3x_3) \leq \alpha_1 G(x_1, x_2, x_3) + \alpha_2 G(x_1, x_2, F_2x_2) + \alpha_3 G(F_1x_1, x_2, x_2) \\ &+ \alpha_4 [G(F_1x_1, x_1, x_1) + G(x_2, F_2x_2, x_2) + G(x_3, x_3, F_3x_3)] \\ &+ \alpha_5 \max \left\{ \begin{matrix} G(x_1, x_2, F_2x_2), G(F_1x_1, F_1x_1, x_2), G(F_1x_1, x_2, x_2), G(F_2x_2, F_2x_2, x_3) \\ G(F_1x_1, x_1, x_1), G(x_2, F_2x_2, x_2), G(x_3, x_3, F_3x_3) \end{matrix} \right\} \end{aligned} \tag{8}$$

for all $x_1, x_2, x_3 \in X$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \geq 0$ with $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 < 1$ and $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) < 1$. Then, the three individual self-mappings referred as F_1, F_2 and F_3 possess a CFP in X. Moreover, if $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) < 1$, the three individual self-mappings referred as F_1, F_2 and F_3 possess a UCFP in X.

Proof. Fix $x_0 \in X$, and $\{x_k\}$ be a sequence in X. Now we define some iterative sequences in X such that $x_{(3k+1)} = F_1 x_{3k}$, $x_{(3k+2)} = F_2 x_{(3k+1)}$ and $x_{(3k+3)} = F_3 x_{(3k+2)}$ $\forall\ k \geq 0$. Now, by using (8) we have,

$$\begin{aligned}
&G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) = G(F_1x_{3k},F_2x_{(3k+1)},F_3x_{(3k+2)}) \leq \alpha_1 G(x_{3k},x_{(3k+1)},x_{(3k+2)})\\
&+\alpha_2 G(x_{3k},x_{(3k+1)},F_2x_{(3k+1)}) + \alpha_3 G(F_1x_{3k},x_{(3k+1)},x_{(3k+1)})\\
&+\alpha_4[G(F_1x_{3k},x_{3k},x_{3k}) + G(x_{(3k+1)},F_2x_{(3k+1)},x_{(3k+1)}) + G(x_{(3k+2)},x_{(3k+2)},F_3x_{(3k+2)})]\\
&+\alpha_5\max\begin{Bmatrix} G(x_{3k},x_{(3k+1)},F_2x_{(3k+1)}),G(F_1x_{3k},F_1x_{3k},x_{(3k+1)}),\\ G(F_1x_{3k},x_{(3k+1)},x_{(3k+1)}),G(F_2x_{(3k+1)},F_2x_{(3k+1)},x_{(3k+2)})\\ G(F_1x_{3k},x_{3k},x_{3k}),G(x_{(3k+1)},F_2x_{(3k+1)},x_{(3k+1)}),G(x_{(3k+2)},\\ x_{(3k+2)},F_3x_{(3k+2)})\end{Bmatrix}
\end{aligned}$$

$$\begin{aligned}
&=\alpha_1 G(x_{3k},x_{(3k+1)},x_{(3k+2)}) + \alpha_2 G(x_{3k},x_{(3k+1)},x_{(3k+1)}) + \alpha_3 G(x_{3k},x_{(3k+1)},x_{(3k+1)})\\
&+\alpha_4[G(x_{3k},x_{3k},x_{3k}) + G(x_{(3k+1)},x_{(3k+1)},x_{(3k+1)}) + G(x_{(3k+2)},x_{(3k+2)},x_{(3k+2)})]\\
&+\alpha_5\max\begin{Bmatrix} G(x_{3k},x_{(3k+1)},x_{(3k+1)}),G(x_{3k},x_{3k},x_{(3k+1)}),\\ G(x_{3k},x_{(3k+1)},x_{(3k+1)}),G(x_{(3k+1)},x_{(3k+1)},x_{(3k+2)})\\ G(F_1x_{3k},x_{3k},x_{3k}),G(x_{(3k+1)},F_2x_{(3k+1)},x_{(3k+1)}),G(x_{(3k+2)},\\ x_{(3k+2)},F_3x_{(3k+2)})\end{Bmatrix}
\end{aligned}$$

After applying the simplification process utilizing the Definition of GM-space we have achieved the following result,

$$\begin{aligned}
&\leq (\alpha_1+\alpha_2)G(x_{3k},x_{(3k+1)},x_{(3k+2)}) + \alpha_4[G(x_{3k},x_{(3k+1)},x_{(3k+2)})\\
&+G(x_{3k},x_{(3k+1)},x_{(3k+2)}) + G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})]+\\
&\alpha_5\max\begin{Bmatrix} G(x_{3k},x_{(3k+1)},x_{(3k+2)}),G(x_{3k},x_{(3k+1},x_{(3k+2)}),\\ G(x_{3k},x_{(3k+1)},x_{(3k+2)}),G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})\end{Bmatrix}
\end{aligned}$$

$$\begin{aligned}
&(1-\alpha_4)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{3k},x_{(3k+1)},x_{(3k+2)})\\
&+\alpha_5\max\left\{G(x_{3k},x_{(3k+1)},x_{(3k+2)}),G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})\right\} \qquad (9)
\end{aligned}$$

Now, there are two cases: (i). If $G(x_{3k},x_{(3k+1)},x_{(3k+2)})$ is a maximum term in (9) then we can write,

$$(1-\alpha_4)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{3k},x_{(3k+1)},x_{(3k+2)})+\alpha_5 G(x_{3k},x_{(3k+1)},x_{(3k+2)})$$

$$(\alpha_1+\alpha_2+2\alpha_4+\alpha_5)G(x_{3k},x_{(3k+1)},x_{(3k+2)})$$

$$\leq \frac{(\alpha_1+\alpha_2+2\alpha_4+\alpha_5)}{(1-\alpha_4)}G(x_{3k},x_{(3k+1)},x_{(3k+2)}),\ \textit{where}\ \gamma_1=\frac{(\alpha_1+\alpha_2+2\alpha_4+\alpha_5)}{(1-\alpha_4)}<1$$

$$G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq \gamma_1 G(x_{3k},x_{(3k+1)},x_{(3k+2)})$$

(ii). If $G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})$ is a maximum term in (9) then we can write,

$$(1-\alpha_4)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{3k},x_{(3k+1)},x_{(3k+2)})+\alpha_5 G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})$$

$$(1-\alpha_4-\alpha_5)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{3k},x_{(3k+1)},x_{(3k+2)})$$

$$\leq \frac{(\alpha_1+\alpha_2+2\alpha_4)}{(1-\alpha_4-\alpha_5)}G(x_{3k},x_{(3k+1)},x_{(3k+2)}),\ where\ \gamma_2=\frac{(\alpha_1+\alpha_2+2\alpha_4)}{(1-\alpha_4-\alpha_5)}<1$$

$$G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq \gamma_2 G(x_{3k},x_{(3k+1)},x_{(3k+2)})$$

From both the cases we get, $\therefore\ \gamma_1=\gamma_2=\gamma$

$$G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) \leq \gamma G(x_{3k},x_{(3k+1)},x_{(3k+2)}) \qquad (10)$$

Similarly, again by using (8),

$$\begin{aligned}
&G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)}) = G(F_1x_{(3k+1)},F_2x_{(3k+2)},F_3x_{(3k+3)}) \leq \alpha_1 G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})\\
&+\alpha_2 G(x_{(3k+2)},x_{(3k+2)},F_2x_{(3k+2)}) + \alpha_3 G(F_1x_{(3k+1)},x_{(3k+2)},x_{(3k+2)})\\
&+\alpha_4[G(F_1x_{(3k+1)},x_{(3k+1)},x_{(3k+3)}) + G(x_{(3k+1)},F_2x_{(3k+2)},x_{(3k+3)}) + G(x_{(3k+1)},x_{(3k+2)},F_3x_{(3k+3)})]\\
&+\alpha_5\max\begin{Bmatrix} G(x_{(3k+1)},x_{(3k+2)},F_2x_{(3k+2)}), G(F_1x_{(3k+1)},F_1x_{(3k+1)},x_{(3k+2)}),\\ G(F_1x_{(3k+1)},x_{(3k+2)},x_{(3k+2)}), G(F_2x_{(3k+2)},F_2x_{(3k+2)},x_{(3k+3)}),\\ G(F_1x_{(3k+1)},x_{(3k+1)},x_{(3k+1)}), G(x_{(3k+1)},F_2x_{(3k+1)},x_{(3k+1)}),\\ G(x_{(3k+2)},x_{(3k+2)},F_3x_{(3k+2)}) \end{Bmatrix}
\end{aligned}$$

$$\begin{aligned}
&=\alpha_1 G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) + \alpha_2 G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) + \alpha_3 G(x_{(3k+2)},x_{(3k+2)},x_{(3k+2)})\\
&+\alpha_4[G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) + G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)}) + G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})]\\
&+\alpha_5\max\begin{Bmatrix} G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}), G(x_{(3k+2)},x_{(3k+2)},x_{(3k+2)}),\\ G(x_{(3k+2)},x_{(3k+2)},x_{(3k+2)}), G(x_{(3k+3)},x_{(3k+3)},x_{(3k+3)}),\\ G(x_{(3k+4)},x_{(3k+3)},x_{(3k+3)}), G(x_{(3k+1)},x_{(3k+2)},x_{(3k+1)}),\\ G(x_{(3k+2)},x_{(3k+2)},x_{(3k+3)}) \end{Bmatrix}
\end{aligned}$$

After applying the simplification process utilizing the Definition of GM-space we have achieved the following result,

$$\begin{aligned}
&\leq (\alpha_1+\alpha_2)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}) + \alpha_4[G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})\\
&+G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)}) + G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})]\\
&+\alpha_5\max\begin{Bmatrix} G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}), G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}),\\ G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}), G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)}) \end{Bmatrix}
\end{aligned}$$

$$\begin{aligned}
&(1-\alpha_4)G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})\\
&+\alpha_5\max\left\{G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}), G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)})\right\} \qquad (11)
\end{aligned}$$

Now, there are two cases: (i). If $G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})$ is a maximum term in (11) then we can write,

$$(1-\alpha_4)G(x_{(3k+2)},x_{(3k+3)},x_{(3k+4)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})+\alpha_5 G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})$$

$$(\alpha_1+\alpha_2+2\alpha_4+\alpha_5)G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)})$$

$$\leq \frac{(\alpha_1+\alpha_2+2\alpha_4+\alpha_5)}{(1-\alpha_4)}G(x_{(3k+1)},x_{(3k+2)},x_{(3k+3)}),\ where\ \gamma_1=\frac{(\alpha_1+\alpha_2+2\alpha_4+\alpha_5)}{(1-\alpha_4)}<1$$

$$G(x_{(3k+2)}, x_{(3k+3)}, x_{(3k+4)}) \leq \gamma_1 G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})$$

(ii). If $G(x_{(3k+2)}, x_{(3k+3)}, x_{(3k+4)})$ is a maximum term in (11) then we can write,

$$(1-\alpha_4)G(x_{(3k+2)}, x_{(3k+3)}, x_{(3k+4)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})+\alpha_5 G(x_{(3k+2)}, x_{(3k+3)}, x_{(3k+4)})$$

$$(1-\alpha_4-\alpha_5)G(x_{(3k+2)}, x_{(3k+3)}, x_{(3k+4)}) \leq (\alpha_1+\alpha_2+2\alpha_4)G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})$$

$$\leq \frac{(\alpha_1+\alpha_2+2\alpha_4)}{(1-\alpha_4-\alpha_5)}G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)}),\ where\ \gamma_2 = \frac{(\alpha_1+\alpha_2+2\alpha_4)}{(1-\alpha_4-\alpha_5)} < 1$$

$$G(x_{(3k+2)}, x_{(3k+3)}, x_{(3k+4)}) \leq \gamma_2 G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})$$

From both the cases we get, $\therefore\ \gamma_1 = \gamma_2 = \gamma$

$$G(x_{(3k+2)}, x_{(3k+3)}, x_{(3k+4)}) \leq \gamma G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)}) \qquad (12)$$

From (11) and (12) inductively we have,

$$G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)}) \leq \gamma^{3k+1} G(x_0, x_1, x_2) \qquad (13)$$

Next we will show that $\{x_k\}$ is a G-Cauchy sequence in X, for any j, n, k with $j > n > k$.

$$\begin{aligned} G(x_k, x_n, x_j) &\leq G(x_k, x_{(k+1)}, x_{(k+1)}) + G(x_{(k+1)}, x_{(k+2)}, x_{(k+2)}) + ... + G(x_{(j-1)}, x_{(j-1)}, x_j) \\ &\leq G(x_k, x_{(k+1)}, x_{(k+2)}) + G(x_{(k+1)}, x_{(k+2)}, x_{(k+3)}) + ... + G(x_{(j-2)}, x_{(j-1)}, x_j) \\ &\leq [y^k + y^{(k+1)} + ... + y^{(j-2)}]G(x_0, x_1, x_2) \end{aligned}$$

This implies that,

$$G(x_k, x_n, x_j) \leq \frac{y^k}{(1-y)} G(x_0, x_1, x_2)$$

The same hold if $j = n > k$ and if $j > n = k$ we have,

$$G(x_k, x_n, x_j) \leq \frac{y^{k-1}}{(1-y)} G(x_0, x_1, x_2)$$

If we take the limit as $k, n, j \to \infty$ we get $G(x_k, x_n, x_j) \to 0$. Hence $\{x_k\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $\wp \in X$ such that $\{x_k\}$ converges to $\wp$ as $k \to \infty$. We have to show that $F_1\wp = \wp$ by contrary case let $F_1\wp \neq \wp$. Then by (8) we have that,

$$\begin{aligned} &G(F_1\wp, x_{(3k+2)}, x_{(3k+3)}) = G(F_1\wp, F_2x_{(3k+1)}, F_3x_{(3k+2)}) \leq \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) \\ &+ \alpha_2 G(\wp, x_{(3k+1)}, F_2x_{(3k+1)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)}) \\ &+ \alpha_4[G(F_1\wp, x_{3k}, x_{3k}) + G(x_{(3k+1)}, F_2x_{(3k+1)}, x_{(3k+1)}) + G(x_{(3k+2)}, x_{(3k+2)}, F_3x_{(3k+2)})] \\ &+ \alpha_5 \max \begin{Bmatrix} G(\wp, x_{(3k+1)}, F_2x_{(3k+1)}), G(F_1\wp, F_1\wp, x_{(3k+1)}), \\ G(F_1\wp, x_{(3k+1)}, x_{(3k+1)}), G(F_2x_{(3k+1)}, F_2x_{(3k+1)}, x_{(3k+2)}), \\ G(F_1\wp, x_{3k}, x_{3k}), G(x_{(3k+1)}, F_2x_{(3k+1)}, x_{(3k+1)}), \\ G(x_{(3k+2)}, x_{(3k+2)}, F_3x_{(3k+2)}) \end{Bmatrix} \end{aligned}$$

$$= \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_2 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)})$$
$$+ \alpha_4 [G(F_1\wp, x_{(3k+1)}, x_{(3k+2)}) + G(x_{3k}, x_{(3k+2)}, x_{(3k+2)}) + G(x_{3k}, x_{(3k+1)}, x_{(3k+3)})]$$
$$+ \alpha_5 \max \left\{ \begin{array}{l} G(\wp, x_{(3k+1)}, x_{(3k+2)}), G(F_1\wp, F_1\wp, x_{(3k+1)}), \\ G(\wp, x_{(3k+1)}, x_{(3k+1)}), G(x_{(3k+2)}, x_{(3k+2)}, x_{(3k+2)}), \\ G(F_1\wp, x_{3k}, x_{3k}), G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+1)}), \\ G(x_{(3k+2)}, x_{(3k+2)}, x_{(3k+)3}) \end{array} \right\}$$

After applying the simplification process utilizing the Definition of GM-space we have achieved the following result,

$$\leq \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_2 G(F_1\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)})$$
$$+ \alpha_4 [G(F_1\wp, x_{3k}, x_{(3k+2)}) + 2G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})]$$
$$+ \alpha_5 \max \left\{ \begin{array}{l} G(\wp, x_{(3k+1)}, x_{(3k+2)}), G(F_1\wp, F_1\wp, x_{(3k+1)}), G(F_1\wp, x_{(3k+1)}, x_{(3k+1)}), \\ G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)}), G(F_1\wp, x_{3k}, x_{(3k+2)}) \end{array} \right\} \quad (14)$$

Now, there are five cases: (i). If $G(\wp, x_{(3k+1)}, x_{(3k+2)})$ is a maximum term in (14) then we can write;

$$\leq \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_2 G(F_1\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)})$$
$$+ \alpha_4 [G(F_1\wp, x_{3k}, x_{(3k+2)}) + 2G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})] + \alpha_5 G(\wp, x_{(3k+1)}, x_{(3k+2)})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(F_1\wp, \wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(F_1\wp, \wp, \wp) + \alpha_3 G(F_1\wp, \wp, \wp)$$
$$+ \alpha_4 [G(F_1\wp, \wp, \wp) + 2G(\wp, \wp, \wp)] + \alpha_5 G(\wp, \wp, \wp)$$

$$\leq (\alpha_2 + \alpha_3 + \alpha_4) G(F_1\wp, \wp, \wp)$$

$G(F_1\wp, \wp, \wp) - (\alpha_2 + \alpha_3 + \alpha_4) G(F_1\wp, \wp, \wp) \leq 0$. $(1 - \alpha_2 - \alpha_3 - \alpha_4) G(F_1\wp, \wp, \wp) \leq 0$, is a contradiction. Since $(1 - \alpha_2 - \alpha_3 - \alpha_4) \neq 0$, therefore, $G(F_1\wp, \wp, \wp) = 0$. Thus,

$$F_1\wp = \wp \qquad (15)$$

(ii). If $G(F_1\wp, F_1\wp, x_{(3k+1)})$ is a maximum term in (14) then we can write;

$$\leq \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_2 G(F_1\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)})$$
$$+ \alpha_4 [G(F_1\wp, x_{3k}, x_{(3k+2)}) + 2G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})] + \alpha_5 G(F_1\wp, F_1\wp, x_{(3k+1)})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(F_1\wp, \wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(F_1\wp, \wp, \wp) + \alpha_3 G(F_1\wp, \wp, \wp)$$
$$+ \alpha_4 [G(F_1\wp, \wp, \wp) + 2G(\wp, \wp, \wp)] + \alpha_5 G(F_1\wp, F_1\wp, \wp)$$

$$\leq (\alpha_2 + \alpha_3 + \alpha_4) G(F_1\wp, \wp, \wp) + \alpha_5 G(F_1\wp, F_1\wp, \wp)$$

By using proposition

$$\leq (\alpha_2 + \alpha_3 + \alpha_4) G(F_1\wp, \wp, \wp) + \alpha_5 G(F_1\wp, \wp, \wp)$$

$G(F_1\wp, \wp, \wp) - (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) G(F_1\wp, \wp, \wp) \leq 0$. $(1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) G(F_1\wp, \wp, \wp) \leq 0$, is a contradiction. Since $(1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) \neq 0$, therefore, $G(F_1\wp, \wp, \wp) = 0$. Thus,

$$F_1\wp = \wp \qquad (16)$$

(iii). If $G(F_1\wp, x_{(3k+1)}, x_{(3k+1)})$ is a maximum term in (14) then we can write;

$$\leq \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_2 G(F_1\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)})$$
$$+ \alpha_4[G(F_1\wp, x_{3k}, x_{(3k+2)}) + 2G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})] + \alpha_5 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(F_1\wp, \wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(F_1\wp, \wp, \wp) + \alpha_3 G(F_1\wp, \wp, \wp)$$
$$+ \alpha_4[G(F_1\wp, \wp, \wp) + 2G(\wp, \wp, \wp)] + \alpha_5 G(F_1\wp, \wp, \wp)$$

$$\leq (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) G(F_1\wp, \wp, \wp)$$

$G(F_1\wp, \wp, \wp) - (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) G(F_1\wp, \wp, \wp) \leq 0$. $(1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) G(F_1\wp, \wp, \wp) \leq 0$, is a contradiction. Since $(1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) \neq 0$, therefore, $G(F_1\wp, \wp, \wp) = 0$. Thus,

$$F_1\wp = \wp \qquad (17)$$

(iv). If $G(F_1\wp, x_{3k}, x_{(3k+2)})$ is a maximum term in (14) then we can write;

$$\leq \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_2 G(F_1\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)})$$
$$+ \alpha_4[G(F_1\wp, x_{3k}, x_{(3k+2)}) + 2G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})] + \alpha_5 G(F_1\wp, x_{3k}, x_{(3k+2)})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(F_1\wp, \wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(F_1\wp, \wp, \wp) + \alpha_3 G(F_1\wp, \wp, \wp)$$
$$+ \alpha_4[G(F_1\wp, \wp, \wp) + 2G(\wp, \wp, \wp)] + \alpha_5 G(F_1\wp, \wp, \wp)$$

$$\leq (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) G(F_1\wp, \wp, \wp)$$

$G(F_1\wp, \wp, \wp) - (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) G(F_1\wp, \wp, \wp) \leq 0$. $(1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) G(F_1\wp, \wp, \wp) \leq 0$, is a contradiction. Since $(1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) \neq 0$, therefore, $G(F_1\wp, \wp, \wp) = 0$. Thus,

$$F_1\wp = \wp \qquad (18)$$

(v). If $G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})$ is a maximum term in (14) then we can write;

$$\leq \alpha_1 G(\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_2 G(F_1\wp, x_{(3k+1)}, x_{(3k+2)}) + \alpha_3 G(F_1\wp, x_{(3k+1)}, x_{(3k+1)})$$
$$+ \alpha_4[G(F_1\wp, x_{3k}, x_{(3k+2)}) + 2G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})] + \alpha_5 G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(F_1\wp, \wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(F_1\wp, \wp, \wp) + \alpha_3 G(F_1\wp, \wp, \wp)$$
$$+ \alpha_4[G(F_1\wp, \wp, \wp) + 2G(\wp, \wp, \wp)] + \alpha_5 G(\wp, \wp, \wp)$$

$$\leq (\alpha_2 + \alpha_3 + \alpha_4) G(F_1\wp, \wp, \wp)$$

$G(F_1\wp, \wp, \wp) - (\alpha_2 + \alpha_3 + \alpha_4) G(F_1\wp, \wp, \wp) \leq 0$. $(1 - \alpha_2 - \alpha_3 - \alpha_4) G(F_1\wp, \wp, \wp) \leq 0$, is a contradiction. Since $(1 - \alpha_2 - \alpha_3 - \alpha_4) \neq 0$, therefore, $G(F_1\wp, \wp, \wp) = 0$. Thus,

$$F_1\wp = \wp \qquad (19)$$

By the view of (15) – (19), we obtain that $\wp$ is a FP of F_1, F_2 *and* F_3, i.e,

$$F_1\wp = \wp \qquad (20)$$

Next, we have to show that $F_2\wp = \wp$ by contrary case let $F_2\wp \neq \wp$. Then from (8) we have that,

$$\begin{aligned} G(x_{(3k+1)}, F_2\wp, x_{(3k+3)}) &= G(F_1x_{3k}, F_2\wp, F_3x_{(3k+2)}) \leq \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) \\ &+ \alpha_2 G(x_{3k}, \wp, F_2\wp) + \alpha_3 G(F_1x_{3k}, \wp, \wp) \\ &+ \alpha_4[G(F_1x_{3k}, \wp, x_{(3k+2)}) + G(x_{3k}, F_2\wp, x_{(3k+2)}) + G(x_{3k}, \wp, F_3x_{(3k+2)})] \\ &+ \alpha_5 \max \begin{Bmatrix} G(x_{3k}, \wp, F_2\wp), G(F_1x_{3k}, F_1x_{3k}, \wp), \\ G(F_1x_{3k}, \wp, \wp), G(F_2\wp, F_2\wp, \wp), \\ G(F_1x_{3k}, x_{3k}, x_{3k}), G(\wp, F_2\wp, \wp), \\ G(x_{(3k+2)}), x_{(3k+2)}), F_3x_{(3k+2)}) \end{Bmatrix} \\ &= \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) + \alpha_2 G(x_{3k}, \wp, F_2\wp) + \alpha_3 G(x_{(3k+1)}, \wp, \wp) \\ &+ \alpha_4[G(x_{(3k+1)}, \wp, x_{(3k+2)}) + G(x_{3k}, F_2\wp, x_{(3k+2)}) + G(x_{3k}, \wp, x_{(3k+2)})] \\ &+ \alpha_5 \max \begin{Bmatrix} G(x_{3k}, \wp, F_2\wp), G(x_{3k}, x_{3k}, \wp), \\ G(x_{3k}, \wp, \wp), G(F_2\wp, F_2\wp, x_{(3k+2)}), \\ G(x_{3k}, x_{3k}, x_{3k}), G(\wp, F_2\wp, \wp), \\ G(x_{(3k+2)}), x_{(3k+2)}), x_{(3k+2)}) \end{Bmatrix} \qquad (21) \end{aligned}$$

Now there are seven cases: (i). If $G(x_{3k}, \wp, F_2\wp)$ is a maximum term in (21) then we can write;

$$\begin{aligned} &\leq \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) + \alpha_2 G(x_{3k}, \wp, F_2\wp) + \alpha_3 G(x_{(3k+1)}, \wp, \wp) \\ &+ \alpha_4[G(x_{(3k+1)}, \wp, x_{(3k+2)}) + G(x_{3k}, F_2\wp, x_{(3k+2)}) + G(x_{3k}, \wp, x_{(3k+2)})] + \alpha_5 G(x_{3k}, \wp, F_2\wp) \end{aligned}$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$\begin{aligned} &G(\wp, F_2\wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(\wp, \wp, \wp) + \\ &\alpha_4[G(\wp, \wp, \wp) + G(\wp, F_2\wp, \wp) + G(\wp, \wp, \wp)] + \alpha_5 G(\wp, \wp, F_2\wp) \end{aligned}$$

$$\leq \alpha_2 + \alpha_4 + \alpha_5)G(\wp, \wp, F_2\wp)$$

$G(\wp, F_2\wp, \wp) - (\alpha_2 + \alpha_4 + \alpha_5)G(\wp, \wp, F_2\wp) \leq 0$. $(1 - \alpha_2 - \alpha_4 - \alpha_5)G(\wp, \wp, F_2\wp) \leq 0$ is a contradiction. Since $(1 - \alpha_2 - \alpha_4 - \alpha_5) \neq 0$, therefore, $G(F_2\wp, \wp, \wp) = 0$. Thus,

$$F_2\wp = \wp \qquad (22)$$

(ii). If $G(x_{(3k+1)}, x_{(3k+1)}, \wp)$ is a maximum term in (21) then we can write;

$$\begin{aligned} &\leq \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) + \alpha_2 G(x_{3k}, \wp, F_2\wp) + \alpha_3 G(x_{(3k+1)}, \wp, \wp) \\ &+ \alpha_4[G(x_{(3k+1)}, \wp, x_{(3k+2)}) + G(x_{3k}, F_2\wp, x_{(3k+2)}) + G(x_{3k}, \wp, x_{(3k+2)})] + \alpha_5 G(x_{(3k+1)}, x_{(3k+1)}, \wp) \end{aligned}$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$\begin{aligned} &G(\wp, F_2\wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(\wp, \wp, \wp) + \\ &\alpha_4[G(\wp, \wp, \wp) + G(\wp, F_2\wp, \wp) + G(\wp, \wp, \wp)] + \alpha_5 G(\wp, \wp, \wp) \end{aligned}$$

$$\leq \alpha_2 + \alpha_4)G(\wp, \wp, F_2\wp)$$

$G(\wp, F_2\wp, \wp) - (\alpha_2 + \alpha_4)G(\wp, \wp, F_2\wp) \leq 0$. $(1 - \alpha_2 - \alpha_4)G(\wp, \wp, F_2\wp) \leq 0$ is a contradiction. Since $(1 - \alpha_2 - \alpha_4) \neq 0$, therefore, $G(F_2\wp, \wp, \wp) = 0$. Thus,

$$F_2\wp = \wp \qquad (23)$$

(iii). If $G(x_{(3k+1)}, \wp, \wp)$ is a maximum term in (21) then we can write;

$$\leq \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) + \alpha_2 G(x_{3k}, \wp, F_2\wp) + \alpha_3 G(x_{(3k+1)}, \wp, \wp)$$
$$+ \alpha_4[G(x_{(3k+1)}, \wp, x_{(3k+2)}) + G(x_{3k}, F_2\wp, x_{(3k+2)}) + G(x_{3k}, \wp, x_{(3k+2)})] + \alpha_5 G(x_{(3k+1)}, \wp, \wp)$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, F_2\wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(\wp, \wp, \wp)+$$
$$\alpha_4[G(\wp, \wp, \wp) + G(\wp, F_2\wp, \wp) + G(\wp, \wp, \wp)] + \alpha_5 G(\wp, \wp, \wp)$$

$$\leq \alpha_2 + \alpha_4)G(\wp, \wp, F_2\wp)$$

$G(\wp, F_2\wp, \wp) - (\alpha_2 + \alpha_4)G(\wp, \wp, F_2\wp) \leq 0$. $(1 - \alpha_2 - \alpha_4)G(\wp, \wp, F_2\wp) \leq 0$ is a contradiction. Since $(1 - \alpha_2 - \alpha_4) \neq 0$, therefore, $G(F_2\wp, \wp, \wp) = 0$. Thus,

$$F_2\wp = \wp \qquad (24)$$

(iv). If $G(F_2\wp, F_2\wp, \wp)$ is a maximum term in (21) then we can write;

$$\leq \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) + \alpha_2 G(x_{3k}, \wp, F_2\wp) + \alpha_3 G(x_{(3k+1)}, \wp, \wp)$$
$$+ \alpha_4[G(x_{(3k+1)}, \wp, x_{(3k+2)}) + G(x_{3k}, F_2\wp, x_{(3k+2)}) + G(x_{3k}, \wp, x_{(3k+2)})] + \alpha_5 G(F_2\wp, F_2\wp, \wp)$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, F_2\wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(\wp, \wp, \wp)+$$
$$\alpha_4[G(\wp, \wp, \wp) + G(\wp, F_2\wp, \wp) + G(\wp, \wp, \wp)] + \alpha_5 G(F_2\wp, F_2\wp, \wp)$$

$$\leq \alpha_2 + \alpha_4)G(\wp, \wp, F_2\wp) + \alpha_5 G(F_2\wp, F_2\wp, \wp)$$

By using proposition

$$\leq \alpha_2 + \alpha_4 + \alpha_5)G(\wp, \wp, F_2\wp)$$

$G(\wp, F_2\wp, \wp) - (\alpha_2 + \alpha_4 + \alpha_5)G(\wp, \wp, F_2\wp) \leq 0$. $(1 - \alpha_2 - \alpha_4 - \alpha_5)G(\wp, \wp, F_2\wp) \leq 0$ is a contradiction. Since $(1 - \alpha_2 - \alpha_4 - \alpha_5) \neq 0$, therefore, $G(F_2\wp, \wp, \wp) = 0$. Thus,

$$F_2\wp = \wp \qquad (25)$$

(v). If $G(x_{(3k+1)}, x_{3k}, x_{3k})$ is a maximum term in (21) then we can write;

$$\leq \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) + \alpha_2 G(x_{3k}, \wp, F_2\wp) + \alpha_3 G(x_{(3k+1)}, \wp, \wp)$$
$$+ \alpha_4[G(x_{(3k+1)}, \wp, x_{(3k+2)}) + G(x_{3k}, F_2\wp, x_{(3k+2)}) + G(x_{3k}, \wp, x_{(3k+2)})] + \alpha_5 G(x_{(3k+1)}, x_{3k}, x_{3k})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, F_2\wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(\wp, \wp, \wp)+$$
$$\alpha_4[G(\wp, \wp, \wp) + G(\wp, F_2\wp, \wp) + G(\wp, \wp, \wp)] + \alpha_5 G(\wp, \wp, \wp)$$

$$\leq \alpha_2 + \alpha_4)G(\wp, \wp, F_2\wp)$$

$G(\wp, F_2\wp, \wp) - (\alpha_2 + \alpha_4)G(\wp, \wp, F_2\wp) \leq 0$. $(1 - \alpha_2 - \alpha_4)G(\wp, \wp, F_2\wp) \leq 0$ is a contradiction. Since $(1 - \alpha_2 - \alpha_4) \neq 0$, therefore, $G(F_2\wp, \wp, \wp) = 0$. Thus,

$$F_2\wp = \wp \qquad (26)$$

(vi). If $G(\wp, F_2\wp, \wp)$ is a maximum term in (21) then we can write;

$$\leq \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) + \alpha_2 G(x_{3k}, \wp, F_2\wp) + \alpha_3 G(x_{(3k+1)}, \wp, \wp) + \alpha_4[G(x_{(3k+1)}, \wp, x_{(3k+2)}) + G(x_{3k}, F_2\wp, x_{(3k+2)}) + G(x_{3k}, \wp, x_{(3k+2)})] + \alpha_5 G(F_2\wp, F_2\wp, \wp)$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, F_2\wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(\wp, \wp, \wp) + \alpha_4[G(\wp, \wp, \wp) + G(\wp, F_2\wp, \wp) + G(\wp, \wp, \wp)] + \alpha_5 G(\wp, F_2\wp, \wp)$$

$$\leq \alpha_2 + \alpha_4 + \alpha_5)G(\wp, \wp, F_2\wp)$$

$G(\wp, F_2\wp, \wp) - (\alpha_2 + \alpha_4 + \alpha_5)G(\wp, \wp, F_2\wp) \leq 0$. $(1 - \alpha_2 - \alpha_4 - \alpha_5)G(\wp, \wp, F_2\wp) \leq 0$ is a contradiction. Since $(1 - \alpha_2 - \alpha_4 - \alpha_5) \neq 0$, therefore, $G(F_2\wp, \wp, \wp) = 0$. Thus,

$$F_2\wp = \wp \qquad (27)$$

(vii). If $G(x_{(3k+2)}, x_{(3k+2)}, x_{(3k+3)})$ is a maximum term in (21) then we can write;

$$\leq \alpha_1 G(x_{3k}, \wp, x_{(3k+2)}) + \alpha_2 G(x_{3k}, \wp, F_2\wp) + \alpha_3 G(x_{(3k+1)}, \wp, \wp) + \alpha_4[G(x_{(3k+1)}, \wp, x_{(3k+2)}) + G(x_{3k}, F_2\wp, x_{(3k+2)}) + G(x_{3k}, \wp, x_{(3k+2)})] + \alpha_5 G(x_{(3k+2)}, x_{(3k+2)}, x_{(3k+3)})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, F_2\wp, \wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, F_2\wp) + \alpha_3 G(\wp, \wp, \wp) + \alpha_4[G(\wp, \wp, \wp) + G(\wp, F_2\wp, \wp) + G(\wp, \wp, \wp)] + \alpha_5 G(\wp, \wp, \wp)$$

$$\leq \alpha_2 + \alpha_4)G(\wp, \wp, F_2\wp)$$

$G(\wp, F_2\wp, \wp) - (\alpha_2 + \alpha_4)G(\wp, \wp, F_2\wp) \leq 0$. $(1 - \alpha_2 - \alpha_4)G(\wp, \wp, F_2\wp) \leq 0$ is a contradiction. Since $(1 - \alpha_2 - \alpha_4) \neq 0$, therefore, $G(F_2\wp, \wp, \wp) = 0$. Thus,

$$F_2\wp = \wp \qquad (28)$$

By the view of (22) – (28), we obtain that $\wp$ is a FP of $F_1,\ F_2\ and\ F_3$, i.e,

$$F_2\wp = \wp \qquad (29)$$

Next, we have to show that $F_3\wp = \wp$ by contrary case let $F_3\wp \neq \wp$. Then from (8) we have that,

$$\begin{aligned} G(x_{(3k+1)}, x_{(3k+2)}, F_3\wp) &= G(F_1x_{3k}, F_2x_{(3k+1)}, F_3\wp) \leq \alpha_1 G(x_{3k}, x_{(3k+1)}, \wp) \\ &+ \alpha_2 G(x_{3k}, x_{(3k+1)}, F_2x_{(3k+1)}) + \alpha_3 G(F_1x_{3k}, x_{(3k+1)}, x_{(3k+1)}) \\ &+ \alpha_4[G(F_1x_{3k}, x_{(3k+1)}, \wp) + G(x_{3k}, F_2x_{(3k+1)}, \wp) + G(x_{3k}, x_{(3k+1)}, \wp)] \\ &+ \alpha_5 \max \begin{Bmatrix} G(x_{3k}, x_{(3k+1)}, F_2x_{(3k+1)}), G(F_1x_{3k}, F_1x_{3k}, x_{(3k+1)}), \\ G(F_1x_{3k}, x_{(3k+1)}, x_{(3k+1)}), G(F_2x_{(3k+1)}, F_2x_{(3k+1)}, \wp), \\ G(F_1x_{(3k+1)}, x_{3k}, x_{3k}), G(x_{(3k+1)}, F_2x_{(3k+1)}, x_{(3k+1)}), \\ G(\wp, \wp, F_3\wp) \end{Bmatrix} \end{aligned}$$

$$\begin{aligned} &= \alpha_1 G(x_{3k}, x_{(3k+1)}, \wp) + \alpha_2 G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}) + \\ &\alpha_3 G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}) + \alpha_4[G(x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+1)}) + G(\wp, \wp, F_3\wp)] \\ &+ \alpha_5 \max \begin{Bmatrix} G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}), G(x_{3k}, x_{3k}, x_{(3k+1)}), \\ G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}), G(x_{(3k+1)}, x_{(3k+1)}, x(3k+2)), \\ G(x_{(3k+1)}, x_{3k}, x_{3k}), G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+1)}), \\ G(\wp, \wp, F_3\wp) \end{Bmatrix} \end{aligned}$$

After applying the simplification process utilizing the Definition of GM-space we have achieved the following result,

$$= \alpha_1 G(x_{3k}, x_{(3k+1)}, \wp) + \alpha_2 G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}) + \alpha_4 [G(x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+1)}) + G(\wp, \wp$$

$$+ \alpha_5 \max \begin{Bmatrix} G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}), G(x_{(3k+2)}, x_{(3k+2)}, \wp), \\ G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)}), G(\wp, \wp, F_3\wp) \end{Bmatrix} \qquad (30)$$

Now there are four cases: (i). If $G(x_{3k}, x_{(3k+1)}, x_{(3k+1)})$ is a maximum term in (30) then we can write;

$$= \alpha_1 G(x_{3k}, x_{(3k+1)}, \wp) + \alpha_2 G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}) +$$
$$\alpha_4 [G(x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+1)}) + G(\wp, \wp, F_3\wp)] + \alpha_5 G(x_{3k}, x_{(3k+1)}, x_{(3k+2)})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, \wp, F_3\wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, \wp) +$$
$$\alpha_4 [G(\wp, \wp, \wp) + G(\wp, \wp, \wp) + G(\wp, \wp, F_3\wp)] + \alpha_5 G(\wp, \wp, \wp)$$

We get that, $G(\wp, \wp, F_3\wp) = 0$. Thus,

$$F_3\wp = \wp \qquad (31)$$

(ii). If $G(x_{(3k+2)}, x_{(3k+2)}, \wp)$ is a maximum term in (30) then we can write;

$$= \alpha_1 G(x_{3k}, x_{(3k+1)}, \wp) + \alpha_2 G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}) +$$
$$\alpha_4 [G(x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+1)}) + G(\wp, \wp, F_3\wp)] + \alpha_5 G(x_{(3k+1)}, x_{(3k+2)}, \wp)$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, \wp, F_3\wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, \wp) +$$
$$\alpha_4 [G(\wp, \wp, \wp) + G(\wp, \wp, \wp) + G(\wp, \wp, F_3\wp)] + \alpha_5 G(\wp, \wp, \wp)$$

We get that, $G(\wp, \wp, F_3\wp) = 0$. Thus,

$$F_3\wp = \wp \qquad (32)$$

(iii). If $G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})$ is a maximum term in (30) then we can write;

$$= \alpha_1 G(x_{3k}, x_{(3k+1)}, \wp) + \alpha_2 G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}) +$$
$$\alpha_4 [G(x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+1)}) + G(\wp, \wp, F_3\wp)] + \alpha_5 G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp, \wp, F_3\wp) \leq \alpha_1 G(\wp, \wp, \wp) + \alpha_2 G(\wp, \wp, \wp) +$$
$$\alpha_4 [G(\wp, \wp, \wp) + G(\wp, \wp, \wp) + G(\wp, \wp, F_3\wp)] + \alpha_5 G(\wp, \wp, \wp)$$

We get that, $G(\wp, \wp, F_3\wp) = 0$. Thus,

$$F_3\wp = \wp \qquad (33)$$

(iv). If $G(\wp, \wp, F_3\wp)$ is a maximum term in (30) then we can write;

$$= \alpha_1 G(x_{3k}, x_{(3k+1)}, \wp) + \alpha_2 G(x_{3k}, x_{(3k+1)}, x_{(3k+1)}) +$$
$$\alpha_4 [G(x_{3k}, x_{3k}, x_{3k}) + G(x_{(3k+1)}, x_{(3k+1)}, x_{(3k+1)}) + G(\wp, \wp, F_3\wp)] + \alpha_5 G(x_{(3k+1)}, x_{(3k+2)}, x_{(3k+3)})$$

Applying $lim_{n\to\infty}$ on both sides, we get,

$$G(\wp,\wp,F_3\wp)\leq\alpha_1G(\wp,\wp,\wp)+\alpha_2G(\wp,\wp,\wp)+$$
$$\alpha_4[G(\wp,\wp,\wp)+G(\wp,\wp,\wp)+G(\wp,\wp,F_3\wp)]+\alpha_5G(\wp,\wp,F_3\wp)$$

$G(\wp,\wp,F_3\wp)-\alpha_5G(\wp,\wp,F_3\wp)\leq 0$. $(1-\alpha_5)G(\wp,\wp,F_3\wp)\leq 0$ is a contradiction. Since $(1-\alpha_5)\neq 0$, therefore, $G(F_2\wp,\wp,\wp)=0$. Thus,

$$F_3\wp=\wp \qquad (34)$$

By the view of (31) – (34), we obtain that $\wp$ is a FP of $F_1,\ F_2\ and\ F_3$, i.e,

$$F_3\wp=\wp \qquad (35)$$

Thus from (15), (34) and (35), it is proved that $\wp$ is a CFP of $F_1,\ F_2\ and\ F_3$, such that

$$F_1\wp=F_2\wp=F_3\wp=\wp$$

Uniqueness: Suppose that $\wp^\diamond\in X$ be the other CFP of $F_1,\ F_2\ and\ F_3$, so that

$$F_1\wp^\diamond=F_2\wp^\diamond=F_3\wp^\diamond=\wp^\diamond$$

Then from (8) we have that,

$$\begin{aligned}&G(\wp,\wp^\diamond,\wp^\diamond)=G(F_1\wp,F_2\wp^\diamond,F_3\wp^\diamond)\leq\alpha_1G(\wp,\wp^\diamond,\wp^\diamond)\\&+\alpha_2G(\wp,\wp^\diamond,F_2\wp^\diamond)+\alpha_3G(F_1\wp,\wp^\diamond,\wp^\diamond)\\&+\alpha_4[G(F_1\wp,\wp,\wp)+G(\wp^\diamond,F_2\wp^\diamond,\wp^\diamond)+G(\wp^\diamond,\wp^\diamond,F_3\wp^\diamond)]\\&+\alpha_5\max\begin{Bmatrix}G(\wp,\wp^\diamond,F_2\wp^\diamond),G(F_1\wp,F_1\wp,\wp^\diamond),\\G(F_1\wp,\wp^\diamond,\wp^\diamond),G(F_2\wp^\diamond,F_2\wp^\diamond,\wp^\diamond),\\G(F_1\wp,\wp,\wp),G(F_2\wp^\diamond,F_2\wp^\diamond,\wp^\diamond),\\G(\wp^\diamond,\wp^\diamond,F_3\wp^\diamond)\end{Bmatrix}\end{aligned}$$

$$\begin{aligned}&\leq\alpha_1G(\wp,\wp^\diamond,\wp^\diamond)+\alpha_2G(\wp,\wp^\diamond,\wp^\diamond)+\alpha_3G(\wp,\wp^\diamond,\wp^\diamond)\\&+\alpha_4[G(\wp,\wp,\wp)+G(\wp^\diamond,\wp^\diamond,\wp^\diamond)+G(\wp^\diamond,\wp^\diamond,\wp^\diamond)]\\&+\alpha_5\max\begin{Bmatrix}G(\wp,\wp^\diamond,\wp^\diamond),G(\wp,\wp,\wp^\diamond),\\G(\wp,\wp^\diamond,\wp^\diamond),G(\wp^\diamond,\wp^\diamond,\wp^\diamond),\\G(\wp,\wp,\wp),G(F_2\wp^\diamond,\wp^\diamond,\wp^\diamond),\\G(\wp^\diamond,\wp^\diamond,\wp^\diamond)\end{Bmatrix}\end{aligned}$$

After applying the simplification process utilizing the Definition of GM-space we have achieved the following result,

$$\begin{aligned}&\leq\alpha_1G(\wp,\wp^\diamond,\wp^\diamond)+\alpha_2G(\wp,\wp^\diamond,\wp^\diamond)+\alpha_3G(\wp,\wp^\diamond,\wp^\diamond)\\&+\alpha_4[G(\wp,\wp,\wp)+G(\wp^\diamond,\wp^\diamond,\wp^\diamond)+G(\wp^\diamond,\wp^\diamond,\wp^\diamond)]\\&+\alpha_5\max\begin{Bmatrix}G(\wp,\wp^\diamond,\wp^\diamond),G(\wp,\wp,\wp^\diamond),\\G(\wp,\wp^\diamond,\wp^\diamond),G(\wp^\diamond,\wp^\diamond,\wp^\diamond),\\G(\wp,\wp,\wp),G(F_2\wp^\diamond,\wp^\diamond,\wp^\diamond),\\G(\wp^\diamond,\wp^\diamond,\wp^\diamond)\end{Bmatrix}\end{aligned}$$

$$\leq\alpha_1G(\wp,\wp^\diamond,\wp^\diamond)+\alpha_2G(\wp,\wp^\diamond,\wp^\diamond)+\alpha_3G(\wp,\wp^\diamond,\wp^\diamond)+\alpha_5G(\wp,\wp^\diamond,\wp^\diamond)$$

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_5)G(\wp,\wp^\diamond,\wp^\diamond)(1-\alpha_1-\alpha_2-\alpha_3-\alpha_5)G(\wp,\wp^\diamond,\wp^\diamond)\leq 0,$$

is a contradiction. Since, $(1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_5) \neq 0$, therefore $G(\wp, \wp^\diamond, \wp^\diamond) = 0$. Thus,

$$\wp = \wp^\diamond$$

It is proved that the three individual self-mappings referred as $F_1,\ F_2\ and\ F_3$ possess a UCFP in X.

The Theorem 2 provides conditions under which three self-mappings F_1, F_2, F_3 on a G-metric space (X, G) have a unique common fixed point. Below is an Algorithm 3 representation of the fixed-point iteration process:

Algorithm 3 Common Fixed-Point Iteration for Three Mappings

1: **Input:**
- A G-metric space (X, G)
- Three self-mappings $F_1, F_2, F_3 : X \times X \times X \to X$
- An initial point $x_0 \in X$
- Coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ satisfying:

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$$

2: **Initialize:**
- Set $k = 0$
- Define iterative sequences:

$$x_{3k+1} = F_1 x_{3k}, \quad x_{3k+2} = F_2 x_{3k+1}, \quad x_{3k+3} = F_3 x_{3k+2}$$

3: **Iterate:**
- Compute x_{k+1} using the above definitions
- Check contraction condition:

$$G(x_{k+1}, x_{k+2}, x_{k+3}) \leq \gamma G(x_k, x_{k+1}, x_{k+2})$$

where

$$\gamma = \max\left(\frac{\alpha_1 + \alpha_2 + 2\alpha_4 + \alpha_5}{1 - \alpha_4}, \frac{\alpha_1 + \alpha_2 + 2\alpha_4}{1 - \alpha_4 - \alpha_5}\right) < 1$$

- Repeat until $G(x_k, x_{k+1}, x_{k+2}) < \epsilon$ for tolerance $\epsilon > 0$

4: **Output:** The limit $\wp = \lim_{k\to\infty} x_k$, the unique common fixed point of F_1, F_2, F_3

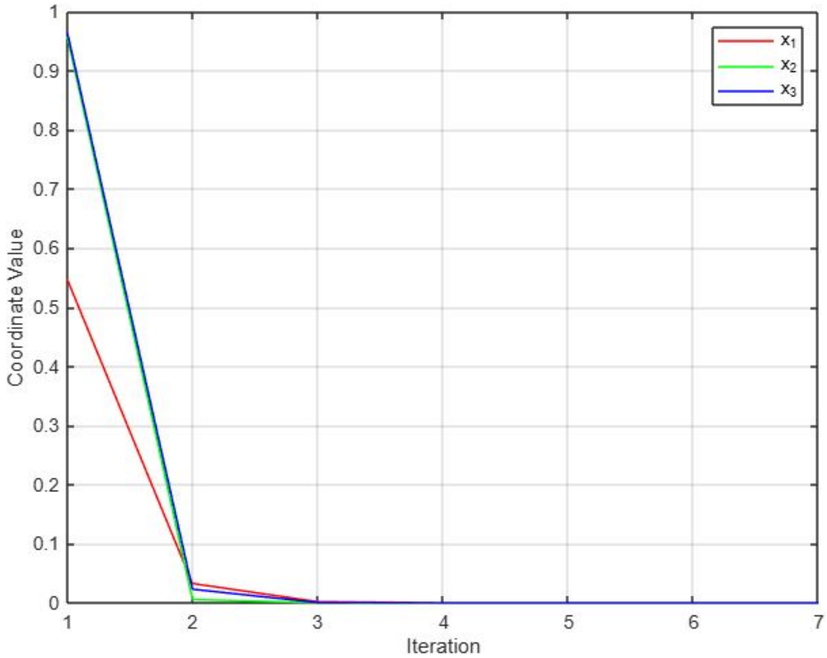

Figure 3: Convergence of Fixed-Point Iteration in G-Metric Space with Applications in AI and Cryptography

Figure 3 depicts the convergence behavior of a fixed-point iteration scheme governed by Theorem 2 in a G-metric space. This visual representation not only affirms the theoretical convergence conditions but also demonstrates the models strength through real-world simulations in AI consensus modeling, cryptographic secure aggregation, and fixed-point learning in neural networkshighlighting both the validity and superiority of the proposed method.

1. Application in AI: Multi-Agent Systems

- *Scenario*: Three AI agents F_1, F_2, F_3 collaboratively optimize a shared objective (e.g., in reinforcement learning). Each agent updates its policy based on others' outputs.
- *Fixed-Point Interpretation*: The theorem ensures iterative updates converge to a unique shared solution under contraction conditions.
- *Example*: In federated learning, three models F_1, F_2, F_3 represent local updates on different devices. The theorem guarantees convergence to a consensus model.

2. Application in Cryptography: Secure Multi-Party Computation

- *Scenario*: Three parties F_1, F_2, F_3 compute a function $f(x)$ collaboratively without revealing private inputs, using iterative encrypted updates.
- *Fixed-Point Interpretation*: The protocol converges to a unique fixed point (correct output of $f(x)$) even with noisy updates, provided contraction conditions hold.
- *Example*: In blockchain smart contracts, three oracles F_1, F_2, F_3 aggregate data. The theorem prevents divergence/manipulation by ensuring unique convergence.

3. Application in AI: Neural Network Fixed-Point Learning

- *Scenario*: A neural network with three sub-modules F_1 (encoder), F_2 (decoder), and F_3 (attention mechanism) learns stable representations.
- *Fixed-Point Interpretation*: Iterative application of modules converges to unique representations under theorem conditions, ensuring stability in deep learning.

Example 2. *Let (X, G) be a G-metric space, where $X \in [0,1]$ and $G : X \times X \times X \to R$, with $G(x_1, x_2, x_3) = max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\}$, for all $x_1, x_2, x_3 \in X$. Now we define the three self-mappings, $F_1, F_2, F_3 : X \to X$ by*

$$F_1x = F_2x = F_3x = \begin{cases} \frac{3x}{5} + \frac{2}{5} & forall\ x \in [0,1] \\ \frac{6x+2}{7} & for\ xin[1,\infty) \end{cases} \tag{36}$$

Then, we get that,

$$G(F_1x, F_2x, F_3x) = \frac{3}{5}\max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\}$$

$$G(F_1x, F_2x, F_3x) = \frac{3}{5}G(x_1, x_2, x_3) \tag{37}$$

Now from (36) we have that:

$$G(x_1, x_2, F_2x_2) = \frac{1}{5}\max\{5|x_1 - x_2|, 2|x_2 - 1|, |3x_2 + 2 - 5x_1|\},$$

$$G(F_1x_1, x_2, x_2) = |F_1x_1 - x_2| = \frac{1}{5}|3x_1 + 2 - 5x_2|,$$

$$G(F_1x_1, x_1, x_1) = |x_1 - F_1x_1| = \frac{2}{5}|x_1 - 1|,$$

$$G(x_2, F_2x_2, x_2) = |x_2 - F_2x_2| = \frac{2}{5}|x_2 - 1|,$$

$$G(x_3, x_3, F_3x_3) = |x_3 - F_3x_3| = \frac{2}{5}|x_3 - 1|,$$

$$G(F_1x_1, F_1x_1, x_2) = G(F_1x_1, x_2, x_2) = |F_1x_1 - x_2| = \frac{1}{5}|3x_1 + 2 - 5x_2|,$$

$$G(F_2x_2, F_2x_2, x_3) = G(F_2x_2, x_3, x_3) = |F_2x_2 - x_3| = \frac{1}{5}|3x_2 + 2 - 5x_3|.$$

Now from (8) and (37) we have that

$$G(F_1x_1, F_2x_2, F_3x_3) = \frac{3}{5}G(x_1, x_2, x_3) \leq \frac{1}{5}(\frac{3}{5}G(x_1, x_2, x_3)) + \frac{1}{8}(\frac{1}{5}\max\{5|x_1 - x_2|,$$
$$2|x_2 - 1|, |3x_2 + 2 - 5x_1|\}) + \frac{1}{10}(\frac{1}{5}|3x_1 + 2 - 5x_2|) + \frac{1}{30}(\frac{2}{5}[|x_1 - 1| + |x_2 - 1| + |x_3 - 1|])$$
$$+\frac{1}{40}\max\left\{\begin{matrix}(\frac{1}{5}\max\{5|x_1 - x_2|, 2|x_2 - 1|, |3x_2 + 2 - 5x_1|\}, \frac{1}{5}|3x_1 + 2 - 5x_2|, \frac{1}{5}|3x_1 + 2 - 5x_2|, \\ \frac{1}{5}|3x_2 + 2 - 5x_3|, \frac{2}{5}|x_1 - 1|, \frac{2}{5}|x_2 - 1|, \frac{2}{5}|x_3 - 1|)\end{matrix}\right\}$$

$$\frac{1}{5}G(x_1, x_2, x_3) + \frac{1}{8}G(x_1, x_2, F_2x_2) + \frac{1}{10}G(F_1x_1, x_2, x_2) +$$
$$\frac{1}{30}[G(F_1x_1, x_1, x_1) + G(x_2, F_2x_2, x_2) + G(x_3, x_3, F_3x_3)]$$
$$+\frac{1}{40}\max\left\{\begin{matrix}G(x_1, x_2, F_2x_2), G(F_1x_1, F_1x_1, x_2), G(F_1x_1, x_2, x_2), G(F_2x_2, F_2x_2, x_3), \\ G(F_1x_1, x_1, x_1), G(x_2, F_2x_2, x_2), G(x_3, x_3, F_3x_3)\end{matrix}\right\}$$

Hence, it satisfied all the conditions of Theorem 2 with $\alpha_1 = \frac{1}{5},\ \alpha_2 = \frac{1}{8},\ \alpha_3 = \frac{2}{5},\ \alpha_4 = \frac{1}{20},\ \alpha_5 = \frac{1}{40}$. i.e. $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 = \frac{20}{40} = 0.5 < 1$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 = \frac{30}{40} = 0.75 < 1$, under these conditions, the three individual self-mappings referred as F_1, F_2 and F_3 possess a unique CFP within F that is $2 \in [0, \infty)$.

Example 3. *Let (X, G) be a G-metric space, where $X \in [0, 2]$ represents either normalized neural activations in an AI interpretation or scaled cryptographic message values in a cryptographic interpretation. The G-metric is defined as:*

$$G(x_1, x_2, x_3) = \max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\}$$

Define three contractive mappings with modified parameters:

$$F_i x = \begin{cases} \frac{2x}{7} + \frac{3}{7} & \text{for } x \in [0, 2] \\ \frac{5x+1}{8} & \text{for } x \in (2, \infty) \end{cases} \quad \text{for } i = 1, 2, 3$$

For $x_1, x_2, x_3 \in [0, 2]$, we verify:

$$\begin{aligned} G(F_1 x, F_2 x, F_3 x) &= \frac{2}{7} \max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\} \\ &= \frac{2}{7} G(x_1, x_2, x_3) \quad \text{(Improved contraction)} \end{aligned}$$

For cryptographic protocols:

$$\begin{aligned} G(x_1, x_2, F_2 x_2) &= \frac{1}{7} \max\{7|x_1 - x_2|, 3|x_2 - 1|, |2x_2 + 3 - 7x_1|\} \\ G(F_1 x_1, x_2, x_2) &= \frac{1}{7} |2x_1 + 3 - 7x_2| \quad \text{(Tighter bound)} \end{aligned}$$

The modified coefficients satisfy:

$$\begin{aligned} &\frac{1}{7} G(x_1, x_2, x_3) + \frac{1}{10} G(x_1, x_2, F_2 x_2) + \frac{1}{14} G(F_1 x_1, x_2, x_2) \\ &+ \frac{1}{35} \left[G(F_1 x_1, x_1, x_1) + G(x_2, F_2 x_2, x_2) + G(x_3, x_3, F_3 x_3) \right] \\ &+ \frac{1}{50} \max \left\{ \begin{array}{l} G(x_1, x_2, F_2 x_2), G(F_1 x_1, F_1 x_1, x_2), \\ G(F_1 x_1, x_2, x_2), G(F_2 x_2, F_2 x_2, x_3), \\ G(F_1 x_1, x_1, x_1), G(x_2, F_2 x_2, x_2), G(x_3, x_3, F_3 x_3) \end{array} \right\} < 1 \end{aligned}$$

The new parameters satisfy Theorem 2 with:

$$\alpha_1 = \frac{1}{7}, \ \alpha_2 = \frac{1}{10}, \ \alpha_3 = \frac{2}{7}, \ \alpha_4 = \frac{1}{35}, \ \alpha_5 = \frac{1}{50}$$

The unique common fixed point $\wp = 1$ (solved by setting $x = \frac{2x}{7} + \frac{3}{7}$) represents: The fixed-point result yields a stable activation value of 1.0 in neural networks and a secure consensus value in cryptographic protocols.

Algorithm 4 Fixed-Point Computation for AI/Crypto Systems

```
Initialize x_0 ∈ [0, 1] (input data/encrypted message)
for k = 0 to max_iter do
    x_{k+1} ← F_3(F_2(F_1(x_k)))        ▷ AI: Layer composition / Crypto: Protocol round
    if G(x_k, x_{k+1}, x_{k+2}) < ϵ then
        return x_{k+1}                  ▷ Fixed point reached
    end if
end for
```

Example 3 and Algorithm 3 together demonstrate the convergence of fixed-point iterations in a G-metric space, representing either stable neural activation in AI systems or secure consensus values in cryptographic protocols.

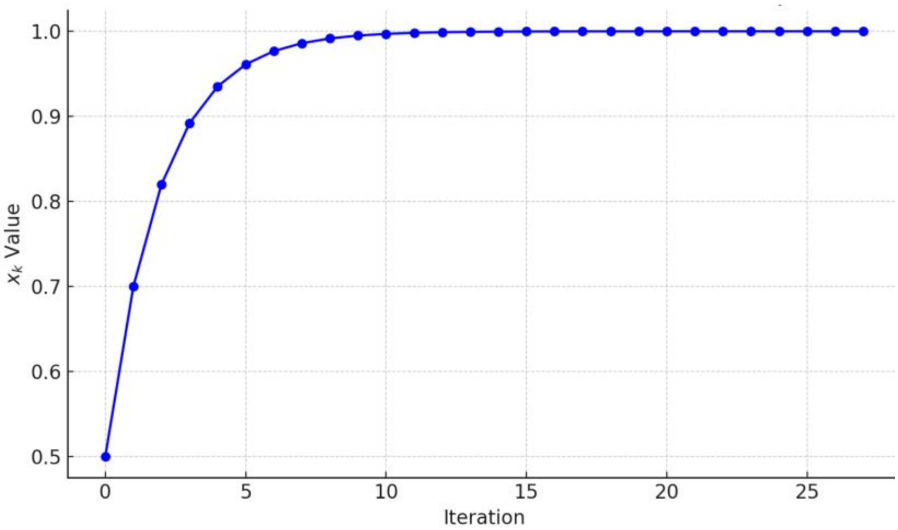

Figure 4: Fixed-point convergence in G-metric space using the mapping.

Figure 4 demonstrates the convergence of the fixed-point iteration governed by the contraction condition in a G-metric space, validating the theoretical result numerically.

4. Conclusion

The field of fixed-point theory is a well-established and substantial area of analysis with numerous applications across various disciplines. Within this field, particular attention has been given to the study of contractive-type inequalities. In this work, we establish conditions under which three self-mappings in generalized metric spaces (GM-spaces) possess common fixed points (CFP) and unique common fixed points (UCFP). These results present key inequalities involving specific constants that guarantee the existence of CFP and UCFP under certain assumptions. Through an illustrative example, we demonstrate the applicability of these conditions to a GM-metric defined on the interval $[0, 1]$, thereby confirming the existence of a UCFP for the given mappings.

This study contributes to the advancement of fixed-point theory in GM-spaces and highlights the importance of metric-related conditions in determining the existence of fixed points in such spaces. The findings also build upon and expand recent developments in the existing literature. Moreover, the implications of these results extend beyond pure mathematics: in artificial intelligence, fixed-point iterations in GM-spaces can model convergence behavior in deep learning layers, iterative reasoning in agents, and equilibrium states in feedback-driven models. Similarly, in cryptography, such fixed points can represent secure consensus states in distributed protocols, iterative agreement in key-exchange mechanisms, or stability in homomorphic encryption loops.

Future research in fixed-point theory, particularly within the framework of GM-spaces, could explore several promising directions. One avenue involves extending contractive-type inequalities to more general settings, such as non-metric spaces or spaces with weaker topological structures. Investigating the relationships between various types of inequalities, including newly formulated generalized contractive conditions, may deepen the theoretical understanding of fixed-point phenomena across broader contexts. Another potential direction is the application of these fixed-point results to practical problems in fields such as optimization, machine learning, and secure computation, where fixed points are fundamental to algorithm design and decision-making.

Moreover, generalizing the current results to accommodate more complex classes of mappings, such as non-linear or non-continuous mappings, could yield valuable tools and insights for researchers. Finally, examining the robustness of fixed-point results in GM-spaces under perturbations or uncertainty could enhance their practical relevance, particularly in real-world scenarios where ideal conditions may not be met. By expanding the scope of GM-spaces and refining the criteria for the existence of CFP and UCFP, this line of research holds considerable potential to contribute significantly to both theoretical and applied mathematics—including emerging domains like adaptive AI control and privacy-preserving cryptographic infrastructure.

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