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Exact Solutions of the Damped Telegrapher's Equation with Harmonic Potential via the Generalized First Integral Method

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Abstract. This paper aims to develop exact analytical solutions for the telegrapher's equation incorporating both damping and harmonic potential by employing the generalized first integral method. This approach extends the classical first integral technique through the use of Laurent polynomials, enhancing its ability to address complex nonlinear structures. The telegrapher's equation, a fundamental model in applied mathematics and physics, describes wave propagation influenced by both dispersive and damping effects, with applications across various engineering and physical systems. By applying suitable transformations, the original nonlinear partial differential equation is reduced to an ordinary differential form. The generalized method is then utilized to derive exact solutions under different parametric conditions. These solutions offer valuable analytical insight into how damping influences wave amplitude, speed, and qualitative behavior. In particular, the method effectively captures the modulation in wave attenuation and propagation characteristics caused by dissipative effects. The key outcome of this study is the demonstration that the generalized first integral method serves as a robust and versatile analytical tool for solving nonlinear damped wave models, where conventional methods often encounter limitations. Its strength lies in simplifying complex nonlinear systems while preserving essential physical effects, providing precise analytical descriptions of wave behavior. Additionally, three-dimensional graphical visualizations of the obtained solutions offer a detailed understanding of the system's spatial and temporal dynamics. This work contributes to the ongoing advancement of analytical techniques for nonlinear evolution equations and establishes a foundation for extending this method to other complex dynamical systems involving damping, dissipation, and external potentials.

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1. Introduction

Nonlinear partial differential equations play a crucial role in accurately modeling complex physical, biological, and engineering systems where linear models often fail to capture intricate dynamics. These equations are fundamental in describing a wide range of phenomena, including fluid dynamics, heat transfer, nonlinear optics, population dynamics, and financial markets, where interactions among variables are inherently nonlinear. The importance of nonlinear PDEs is underscored by their ability to describe real-world processes with greater fidelity, accounting for effects such as turbulence, shock waves, pattern formation, and chaotic behavior. In recent years, fractional calculus has emerged as a powerful tool in modeling complex dynamical systems, offering more accurate descriptions of memory and hereditary properties inherent in various physical and engineering processes [1], [2], [3]. Coupled with advances in numerical methods, including finite difference schemes and spectral techniques, these developments have significantly enhanced the analysis and solution of nonlinear partial differential equations [4], [5]. The incorporation of fractional derivatives and novel computational approaches has expanded the capability to capture anomalous diffusion, viscoelasticity, and chaotic behaviors across multiple scientific domains. Recent studies highlight the growing interdisciplinary interest and provide a strong foundation for further exploration of nonlinear wave equations with damping and complex potentials. Despite their significance, solving nonlinear PDEs poses a substantial challenge due to the absence of universal solution methods. As a result, various analytical techniques such as perturbation methods, similarity transformations, and variational approaches are often employed. These are complemented by numerical techniques like the finite element method, finite difference schemes, and spectral methods. However, many traditional analytical methods suffer from limitations such as the requirement for small parameters, approximate series expansions, or difficulties in handling damping and potential terms in complex equations.

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta(u) = c^2 u_{xx}, \tag{1}$$

commonly referred to as the telegrapher's equation with damping and harmonic potential, describes a system where wave propagation is influenced by both temporal damping and spatial diffusion. Here, u_{tt} represents the time acceleration, $(\alpha + \beta)u_t$ is a damping term reflecting system resistance or friction, and $\alpha\beta u$ introduces a harmonic potential. The term $c^2 u_{xx}$ corresponds to wave propagation, where c denotes the wave speed. This equation serves as a core model in describing physical processes such as signal transmission, mechanical vibrations, and heat conduction, particularly in contexts where both wave-like behavior and energy dissipation are significant [6], [7]. The parameters α, β, and c have distinct physical interpretations that govern the dynamics of the system:

- **Damping coefficients α and β:** These parameters characterize the dissipative effects within the system. The term $(\alpha+\beta)u_t$ models damping forces proportional to velocity, representing resistance or frictional effects that gradually reduce the wave amplitude over time. Meanwhile, $\alpha\beta u$ acts as a harmonic potential, providing a restoring force proportional to the displacement, which can model elastic or reactive effects in the medium. Physically, α and β may correspond to different mechanisms of energy loss or gain, such as electrical resistance in transmission lines, viscous damping in mechanical vibrations, or absorption effects in wave propagation through lossy media.

- **Wave speed c:under** The parameter c represents the characteristic speed of wave propagation through the medium in the absence of damping and potential effects. It governs the dispersive term $c^2 u_{xx}$, modeling spatial diffusion or wave transmission. In physical contexts, c may represent the speed of signal transmission in telegraph lines, the speed of sound in vibrating strings, or diffusion rates in thermal and fluid systems.

Together, these parameters control the interplay between wave propagation, dissipation, and restoring forces, shaping solution characteristics such as amplitude attenuation, wave speed modulation, and oscillatory behavior. In recent years, there has been considerable interest in developing both analytical and numerical methods for solving damped wave equations and related nonlinear models. Techniques such as the exp-function method, tanh-coth method, sine-Gordon expansion method, and homotopy analysis method have been applied to obtain approximate and exact traveling wave solutions in various contexts [8], [9], [10], [11]. Additionally, advanced numerical algorithms and hybrid analytical-numerical techniques have been explored to tackle increasingly complex models incorporating variable coefficients, memory effects, and fractional derivatives. These developments have significantly broadened the range of solvable models, although challenges remain in deriving exact solutions for systems involving both nonlinearities and damping. Traditional methods for solving the telegrapher's equation, especially in the presence of damping, often face difficulties due to their reliance on complex or approximate procedures [12], [13], [14], [15], [16], [17], [18], [19]. Many of these techniques are limited in their ability to derive exact, closed-form solutions, especially when the equation involves nonlinear terms and damping effects. The Generalized First Integral Method provides a more effective approach to overcoming these limitations. This method extends the classical first integral technique by introducing Laurent polynomials instead of traditional polynomials, allowing for a broader class of solutions. Additionally, it facilitates a wave transformation that reduces the PDE to an ordinary differential equation, significantly simplifying the solution process. Unlike perturbative or numerical techniques, Generalized First Integral Method allows for the direct construction of exact traveling wave solutions.

In this study, we conduct an analytical investigation of the telegrapher's equation with damping and harmonic potential using the Generalized First Integral Method. Through an appropriate wave transformation, we reduce the equation to an form and systematically derive exact solutions under varying conditions. These solutions offer insight into how damping influences wave behavior, including attenuation, distortion, and propagation speed variation. The Generalized First Integral Method not only simplifies the analytical process but also deepens our understanding of dissipative wave phenomena. Given its applicability to problems in signal transmission, mechanical systems, and thermal con-

duction, this method serves as a valuable tool in both theoretical research and practical modeling within applied mathematics.
The structure of the paper is as follows. In Section 2, we provide a concise overview of the Generalized First Integral Method. Section 3 applies this method to derive exact solutions of the telegrapher's equation with damping and harmonic potential. Section 4 presents a detailed discussion of the results, including physical interpretations and stability considerations. Finally, Section 5 offers graphical representations of the obtained solutions to illustrate their spatial and temporal behaviors.

2. Structural Layout of the Generalized First Integral Method

Step 1. Consider a general nonlinear PDE in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0. \tag{2}$$

To find the travelling wave solutions to Equation (2), we introduce the wave variable

$$\xi = x - ct, \tag{3}$$

$$u(x,t) = u(\xi), \tag{4}$$

Differentiating equation (3) and (4), according to (2) we have

$$u_x = u^{'}(\xi), \tag{5}$$

$$u_t = -cu^{'}(\xi), \tag{6}$$

$$u_{xx} = u^{''}(\xi), \tag{7}$$

$$u_{xt} = -cu^{''}(\xi), \tag{8}$$

and so on for the other derivatives. Using equation (5-8) the PDE (2) is converted to an Ordinary differential equation of the form

$$F(u, u^{'}, -cu^{'}, u^{''}, -cu^{''}, ...) = 0. \tag{9}$$

where $u = u(\xi)$ is an unknown function, F is a polynomial in the variable u and its derivatives.
Step 2. Suppose the solution of ODE (9) can be written as follows:

$$U(x,t) = f(\xi), \tag{10}$$

and furthermore, we introduce a new independent variable

$$\begin{cases} X(\xi) = f(\xi) \\ Y(\xi) = f'(\xi) \end{cases} \tag{11}$$

Step 3. Under the conditions of Step 2, Equation (9) can be converted to a system of nonlinear ODEs as follows

$$\begin{cases} X'(\xi) = Y(\xi) \\ Y'(\xi) = F(X(\xi), Y(\xi)) \end{cases} \tag{12}$$

If we are able to find the integrals for Equation (12), the general solutions to these equations can be determined directly. However, in most cases, even finding a single first integral is challenging. This difficulty arises because, for a given autonomous system, there is no systematic theory that explicitly guides us on how to find its first integrals, nor is there a clear method for identifying what these first integrals are. To address this, we will apply the Division Theorem for Laurent polynomials to obtain one first integral for Equation (13) and (14) [20], which will reduce Equation (9) to a first or second order, integrable ordinary differential equation (ODE). The exact solution to Equation (9) can then be obtained by solving this reduced equation.

Division theorem for Laurent Polynomials

Suppose $P(\omega, z)$ and $Q(\omega, z)$ are two Laurent polynomials in $C(\omega, z)$ and $P(\omega, z)$ is irreducible in $C[\omega, z]$. If $Q(\omega, z)$ vanishes at any zero point of $P(\omega, z)$ then there exist a Laurent polynomial $G(\omega, z)$ in $C[\omega, z]$ such that $Q(\omega, z)= P(\omega, z) \cdot G(\omega, z)$.

3. Employing a Generalized First Integral Method for Solving the Telegrapher's equation with damping and harmonic potential

Consider the wave variable,

$$\xi = x + t, \tag{13}$$

and the function transformation as

$$U(x, t) = f(\xi), \tag{14}$$

Differentiating equations (13) and (14) according to (1), we have

$$\begin{gathered} U_t = f^{'}(\xi), \\ U_{tt} = f^{''}(\xi), \\ U_x = f^{'}(\xi), \\ U_{xx} = f^{''}(\xi), \end{gathered}$$

Now, equation(1) becomes

$$f''(\xi) = \left(\frac{-(\alpha + \beta)}{1 - c^2}\right) f'(\xi) - \left(\frac{\alpha\beta}{1 - c^2}\right) f(\xi). \tag{15}$$

Equation (15) is the required ODE of order two,
let

$$\begin{cases} X(\xi) = f(\xi) \\ Y(\xi) = f'(\xi) \end{cases} \tag{16}$$

$$\begin{cases} X'(\xi) & = f'(\xi) \\ Y'(\xi) & = f''(\xi) \end{cases}$$

$$\begin{cases} X'(\xi) &= Y(\xi) \\ Y'(\xi) &= \left(\dfrac{-(\alpha+\beta)}{1-c^2}\right) Y - \left(\dfrac{\alpha\beta}{1-c^2}\right) X \end{cases}$$

Thus, equation (15) is equivalent to the two dimensional autonomous system

$$\begin{cases} X'(\xi) &= Y(\xi) \\ Y'(\xi) &= \left(\dfrac{-(\alpha+\beta)}{1-c^2}\right) Y - \left(\dfrac{\alpha\beta}{1-c^2}\right) X \end{cases} \tag{17}$$

let $X(\xi)$ and $Y(\xi)$ be non-trivial solutions of equation (16) and let

$$Q(X,Y) = \sum_{k=\alpha}^{\beta} a_k(X) Y^k$$

is an irreducible laurent polynomial in $\mathbb{C}[X(\xi), Y(\xi)]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{k=\alpha}^{\beta} a_k(X) Y^k = 0 \tag{18}$$

where $a_k(X)(k = 0, 1, 2, ...,)$ are Laurent polynomials of X. Above equation is said to be the first integral of Eq (16). By applying division theorem, a laurent polynomial $[g(X) + h(X)Y]$ exist in the complex domain $\mathbb{C}[X, Y]$ such that, by chain rule

$$\begin{aligned} \frac{dq}{d\xi} = \sum_{k=\alpha}^{\beta} a'_k(X) Y^{k+1} + \sum_{k=\alpha}^{\beta} k a_k(X) Y^k \cdot \left(\frac{-(\alpha+\beta)}{1-c^2}\right) \\ + \sum_{k=\alpha}^{\beta} k a_k(X) Y^{k-1} \cdot \left(\frac{-\alpha\beta}{1-c^2} X\right) \end{aligned} \tag{19}$$

$$\begin{aligned} [g(X) + h(X)Y] \left[\sum_{k=\alpha}^{\beta} a_k(X) Y^k\right] = \left[\sum_{k=\alpha}^{\beta} a'_k(X) Y^{k+1}\right] \\ + \left[\sum_{k=\alpha}^{\beta} k a_k(X) Y^k\right] \cdot \left(\frac{-(\alpha+\beta)}{1-c^2}\right) \\ + \left[\sum_{k=\alpha}^{\beta} k a_k(X) Y^{k-1}\right] \cdot \left(\frac{-\alpha\beta}{1-c^2} X\right) \end{aligned} \tag{20}$$

now for $\alpha = 0$ and $\beta = 1$ equation (17) becomes

$$\begin{aligned} [g(X) + h(X)Y] \left\{a_0(X) Y^0 + a_1(X) Y^1\right\} = \left\{a'_0(X) Y^1 + a'_1(X) Y^2\right\} \\ + \left\{a_1(X) Y^1\right\} \cdot \left(\frac{-(\alpha+\beta)}{1-c^2}\right) \\ + \left\{a_1(X) Y^0\right\} \cdot \left(\frac{-\alpha\beta}{1-c^2} X\right) \end{aligned} \tag{21}$$

comparing coefficients of Y^2, Y^1, Y^0
Y^2;

$$h(X)a_1(X) = a_1^{'}(X). \tag{22}$$

Y^1;

$$h(X)a_o(X) + g(X)a_1(X) = a_(^{'}0)(X) - a_1(X)\frac{\alpha+\beta}{1-c^2}. \tag{23}$$

Y^0;

$$g(X)a_o(X) = -a_1(X)(\frac{\alpha\beta}{1-c^2}X). \tag{24}$$

from equation (21) we suppose a Laurent polynomial

$$h(X) = \frac{1}{X},$$

so equation (21) $\Longrightarrow$

$$\frac{1}{X} = \frac{a_o^{'}(X)}{a_o(X)},$$

Integration gives

$$a_o(X) = C_1(X). \tag{25}$$

equation (22) $\Longrightarrow$

$$\frac{a_0(X)}{X} + C_1 X g(X) = a_0'(X) - C_1 X \cdot \left[\frac{\alpha+\beta}{1-c^2}\right]. \tag{26}$$

$$C_1 X g(X) + C_1 X \cdot \left[\frac{\alpha+\beta}{1-c^2}\right] = a_0'(X) - \frac{a_0(X)}{X}. \tag{27}$$

$$C_1 g(X) + C_1 \cdot \left[\frac{\alpha+\beta}{1-c^2}\right] = \frac{d}{dX}\left(\frac{a_0(X)}{X}\right). \tag{28}$$

Integrating (27)

$$C_1 \int g(X)dX + C_1 X \cdot \frac{\alpha+\beta}{1-c^2} + C_2 = \frac{a_0(X)}{X},$$

$$C_1 X \int g(X)dX + C_1 X^2 \cdot \frac{\alpha+\beta}{1-c^2} + C_2 X = a_0(X). \tag{29}$$

For equation (21) and (22), equation (23) $\Longrightarrow$

$$\left[C_1 X \int g(X)\,dX + C_1 X^2 \cdot \frac{\alpha+\beta}{1-c^2} + C_2 X\right] g(X) = -C_1 X^2 \left(\frac{\alpha\beta}{1-c^2}\right)$$

$$\left[g(X) \int g(X)\,dX + X g(X) \cdot \frac{\alpha+\beta}{1-c^2} + A g(X)\right] = -X \cdot \frac{\alpha\beta}{1-c^2} \tag{30}$$

where $A = \frac{C_2}{C_1}$
In applying the Generalized First Integral Method, the function $g(X)$ is assumed as a constant B to simplify the integration process, with its value subsequently determined from the consistency condition of the reduced equation, expressed in terms of the system parameters α, β, and c, so equation(29) $\Longrightarrow$

$$B^2X + BX \cdot [\frac{\alpha+\beta}{1-c^2}] + BA = -X[\frac{\alpha\beta}{1-c^2}]. \tag{31}$$

compairing Power of X on both sides of (30)
X^1;

$$B^2 + B \cdot \left[\frac{\alpha+\beta}{1-c^2}\right] = \frac{-\alpha\beta}{1-c^2}.$$

X^0;

$$AB = 0.$$

Here $A = 0$ because $B \neq 0$ Now, $\Longrightarrow$

$$B^2 + B \cdot \left[\frac{\alpha+\beta}{1-c^2}\right] = \frac{-\alpha\beta}{1-c^2},$$

Quadratic in B

$$B = \frac{\frac{-(\alpha+\beta)}{1-c^2} \pm \sqrt{\frac{(\alpha+\beta)^2-4(\alpha\beta)(1-c^2)}{(1-c^2)^2}}}{2}. \tag{32}$$

$g(X) = B$ where B is the solution of above quadratic equation.
Now equation (23) $\Longrightarrow$

$$a_0(X)g(X) = -C_1X\left(\frac{\alpha\beta}{1-c^2}\right)X$$

$$a_0(X) = \frac{-C_1X^2\left(\frac{\alpha\beta}{1-c^2}\right)}{B},$$

Now, for $\alpha = 0$, $\beta = 1$ equation (17) $\Longrightarrow$

$$a_0(X) + a_1(X)Y = 0,$$

$$Y = \frac{-a_0(X)}{a_1(X)},$$

$$Y = \frac{(\alpha\beta)X}{B(1-c^2)},$$

$$\frac{d\xi}{dX} = \frac{1}{\frac{(\alpha\beta)X}{B(1-c^2)}},$$

Integration gives

$$\xi = \frac{B(1-c^2)}{\alpha\beta} \ln X + K,$$

$$\xi - K = \frac{B(1-c^2)}{\alpha\beta} \ln X,$$

$$\frac{\alpha\beta(\xi - K)}{B(1-c^2)} = \ln X$$

$\implies$

$$X = f(\xi) = u = \exp\left(\frac{\alpha\beta(\xi - K)}{B(1-c^2)}\right). \tag{33}$$

Where

$$\xi = x + t$$

and

$$B = \frac{\frac{-(\alpha+\beta)}{1-c^2} \pm \sqrt{\frac{(\alpha+\beta)^2 - 4\alpha\beta(1-c^2)}{(1-c^2)^2}}}{2}.$$

Equation (32) is the required solution of Telegrapher's equation with damping and harmonic potential. This solution best describes *underdamped* or *overdamped* wave propagation depending on the discriminant $(\alpha+\beta)^2 - 4\alpha\beta(1-c^2)$:

- If the discriminant is positive, the system is **overdamped**, leading to non-oscillatory exponential decay.
- If the discriminant is zero, the system is **critically damped**.
- If the discriminant is negative, the system exhibits **underdamped oscillations**, where waves persist but are modulated by exponential decay.

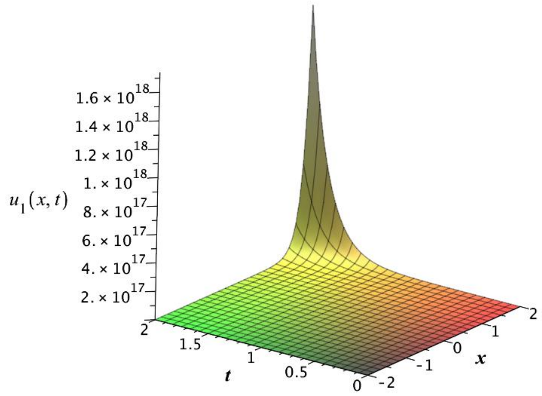

Figure 1: Three-dimensional plot of the exact solution $u_1(x,t)$ showing rapid exponential growth as $t \to 0^+$ and $x \to 0$. The solution represents a highly localized pulse that sharply increases in amplitude. Physically, this may correspond to a resonance or instability triggered by specific combinations of damping parameters α and β, where energy is not dissipated fast enough to suppress the growth.

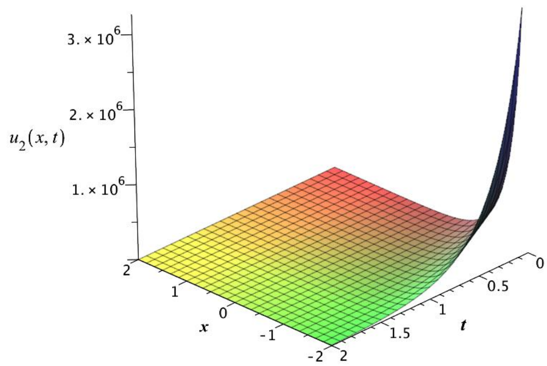

Figure 2: Plot of the exact solution $u_2(x,t)$ demonstrating smooth exponential growth as $t \to 0$, with wave attenuation visible along the spatial axis. This solution reflects a scenario where the damping cannot fully prevent energy concentration near the origin. The result can model physical systems with weak dissipation or external energy input, such as electrical pulses or signals in low-resistance media.

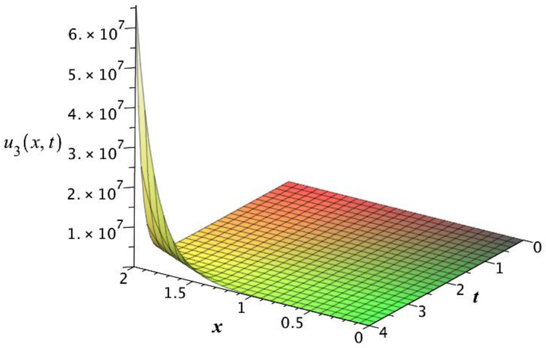

Figure 3: Plot of $u_3(x,t)$, exhibiting rapid spatial decay with increasing x and temporal flattening as t increases. The solution characterizes a strongly damped wave where energy dissipates quickly in both time and space. Such behavior is typical in overdamped electrical or thermal systems, where signals are quickly attenuated.

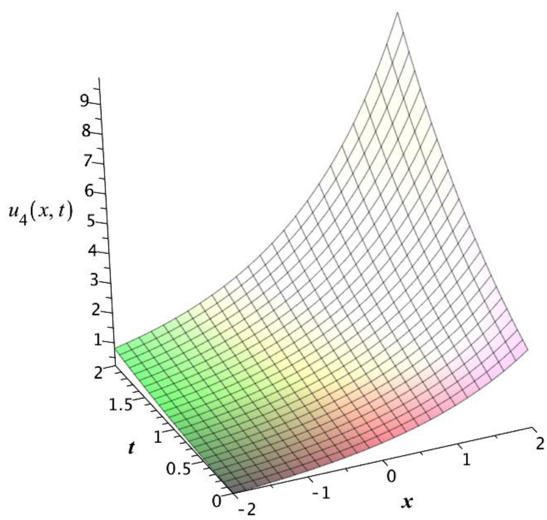

Figure 4: Three-dimensional surface plot of the approximate solution $u_4(x,t)$ to the damped wave equation $u_{tt} + (\alpha+\beta)u_t + \alpha\beta u = c^2 u_{xx}$. The plot illustrates the temporal and spatial evolution of the solution. The exponential growth in $u_4(x,t)$ over time, visible from the surface, indicates a scenario where the damping effect is insufficient to counteract the restoring and wave propagation forces. The parameters α, β, and c significantly influence the system's dynamic response.

4. Results and Discussion

The provided graphs represent the explicit solutions to the telegrapher's equation

$$u_{tt} + (\alpha + \beta)u_t + (\alpha\beta)u = c^2 u_{xx},$$

with the solution expressed as

$$X = f(\xi) = u = \exp\left(\frac{\alpha\beta(\xi - K)}{B(1-c^2)}\right),$$

and B is the perimeter depending on α, β and c defined by

$$B = \frac{\frac{-(\alpha+\beta)}{1-c^2} \pm \sqrt{\frac{(\alpha+\beta)^2 - 4\alpha\beta(1-c^2)}{(1-c^2)^2}}}{2},$$

Each of the four 3D surface plots corresponds to different choices of the parameters α, β, c and K highlighting the diverse behaviors this solution can capture.

Figure 1(Graph of $u_1(x,t)$ displays a highly singular behavior centered near $x = 0$ $t = 0$. The solution grows extremely large in a very narrow region, suggesting that for specific parameter values, the system experiences rapid and significant amplification. This behavior models real-world phenomena where initial conditions or parameter choices lead to instability or catastrophic failure, such as resonance in mechanical systems or blow-up in nonlinear media.

Figure 2(Graph of $u_2(x,t)$ shows rapid growth along the time axis while remaining relatively flat along the spatial direction. This indicates that the temporal damping is not sufficient to counteract the intrinsic growth, making this solution appropriate to model systems where energy input over time leads to an unstable build-up, such as certain electrical circuits or unstable feedback systems.

Figure 3(Graph of $u_3(x,t)$ portrays a localized sharp peak at an early time that dissipates as time progresses. This suggests a pulse-like or shock-like solution that decays naturally due to the damping effects. It captures scenarios like dissipative wave propagation, where an initial disturbance gradually loses energy as it travels, which is crucial in fields like seismology, material sciences, and acoustic wave modeling.

Figure 4(Graph of $u_4(x,t)$ differs significantly by exhibiting a smooth, controlled increase in amplitude without any signs of instability. Here, the solution remains bounded and grows predictably, which corresponds to stable regimes where the damping dominates and prevents any explosive behavior. Such behavior is desirable in engineering applications where stability under oscillations is critical, such as suspension bridges, skyscraper design, and vibration control in aerospace structures.

The advantage of this explicit solution structure is profound. It provides a direct and exact way to understand how damping parameters α, β and the wave propagation speed c interact to govern the overall behavior of the system. Analytical solutions like these are extremely valuable because they offer insight into the qualitative behavior of the system without requiring extensive numerical simulations. They allow for a clear understanding of stability conditions, energy dissipation, and the critical thresholds for transition from stability to instability. Furthermore, in practical applications, having an explicit solution enables engineers and scientists to design systems with desired dynamic properties. By adjusting the parameters based on the explicit formula, one can tailor the system's response to prevent dangerous instabilities or to harness controlled amplification where needed (such as signal boosting). The importance of such solutions extends to various fields, including mechanical and civil engineering, physics, materials science, and control theory.

In conclusion, the diverse behaviors captured by these graphs ranging from explosive growth to smooth decay highlight the richness of the dynamics described by the damped wave equation. The explicit solutions serve as powerful tools both for theoretical analysis and for practical design and optimization in complex systems.

5. Stability Analysis of the Exact Solution

To assess the stability of the exact traveling wave solution

$$f(\xi) = \exp\left(\frac{\alpha\beta(\xi - K)}{B(1 - c^2)}\right),$$

we consider a small perturbation of the form,

$$u(\xi, t) = f(\xi) + \epsilon\phi(\xi, t),$$

where $0 < \epsilon \ll 1$ and $\phi(\xi, t)$ denotes the perturbation function.

Substituting this perturbed solution into the original equation and linearizing by neglecting terms of order ϵ^2 and higher, we obtain the linearized equation governing the perturbation dynamics:

$$\phi_{tt} + (\alpha + \beta)\phi_t + \alpha\beta\phi = c^2\phi_{xx}.$$

Assuming a solution of the form $\phi(\xi, t) = e^{\lambda t}\psi(\xi)$, where λ is the growth rate parameter, leads to the characteristic equation

$$\lambda^2 + (\alpha + \beta)\lambda + \alpha\beta = 0.$$

The roots of this quadratic equation are

$$\lambda = \frac{-(\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}}{2}.$$

For positive damping coefficients $\alpha > 0$ and $\beta > 0$, both roots have negative real parts. This implies that any small perturbation will decay exponentially over time, confirming the linear stability of the exact traveling wave solution.

Hence, the solution obtained through the generalized first integral method is stable and physically relevant for modeling wave propagation under the influence of damping and harmonic potential.

6. Conclusions

This paper introduces a novel approach to solving partial differential equations exactly, by replacing conventional polynomials with Laurent polynomials within the first integral method. This innovation is grounded in the Division Theorem for Laurent polynomials in two complex variables, which provides a more flexible and powerful algebraic framework for deriving first integrals. To validate the proposed technique, we revisit and apply the Euclidean Algorithm for Laurent polynomials, demonstrating its effectiveness in simplifying the solution process for nonlinear differential equations. The core contribution of this work is the development of the generalized first integral method, a new tool for obtaining exact solutions to complex dynamical systems, particularly those involving damping effects and external potentials. This method not only facilitates the derivation of precise analytical solutions but also highlights the critical role of algebraic methods, particularly the properties of Laurent polynomials, in advancing the theory of nonlinear evolution equations. By transforming complex systems into integrable first-order ordinary differential equations, we were able to uncover exact solutions that provide valuable insights into the dynamic behavior of such systems. The importance of this approach extends beyond the specific case studied here. The generalized first integral method represents a significant advancement in the analytical study of nonlinear systems, offering a systematic and efficient tool for obtaining exact solutions to a wide range of partial differential equations that involve damping or dissipation. The technique is applicable to various fields of applied mathematics, such as fluid dynamics, signal processing, and heat diffusion, where the need for exact solutions to nonlinear models is critical. In conclusion, this paper not only introduces an effective new method for solving nonlinear partial differential equations but also contributes to the broader analytical framework available for the study of complex dynamical systems. Future research could explore the application of the generalized first integral method to other types of nonlinear differential equations and more complicated systems involving variable coefficients or coupled components. Furthermore, extending this method to higher-dimensional problems and systems with spatially or temporally varying damping and potential terms could significantly broaden its applicability. Another valuable direction would be integrating this technique with numerical or perturbation-based approaches to develop hybrid analytical-numerical frameworks for problems where exact solutions are difficult to obtain. These prospective extensions would enhance the versatility of the method and promote its adoption in various scientific and engineering disciplines.

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