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Friendly Domination in Graph

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Abstract. Friendly domination combines two ideas in graph theory: domination, which captures influence, and friendly sets, which balance that influence. We study the friendly dominating set and the friendly domination number, the smallest size of a set that is both dominating and friendly. First, we prove that for every graph the usual domination number is never larger than its friendly domination number, and we identify all graphs whose friendly domination number equals one or two. We then show that the gap between these two parameters can be made arbitrarily large. Exact formulas are derived for paths and cycles. Sharp Nordhaus–Gaddum bounds are obtained for the sum and product of the friendly domination number of a graph and its complement. Finally, we give complete structural characterizations of friendly dominating sets in the join and corona of two graphs.

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1. Introduction

Graph theory has been around since the eighteenth century and has grown into an important part of mathematics. It helps us understand communication networks, chemical structures, disease spread, and social connections [1, 2]. One of the key ideas studied is domination, where a small group of vertices can control or influence the entire graph. Many researchers, like Henning and Yeo [3], have expanded these ideas and connected them to other topics like hypergraphs, graphs with special distance properties, and computer algorithms.

Recently, the idea of a friendly set, where each vertex outside the set has at least as many neighbors outside as inside, was introduced by Haynes, Hedetniemi, and Henning [4]. Building on this concept, we define friendly domination as a dominating set that is also a friendly set. This new concept reflects situations where influence needs to be applied carefully without overwhelming any single vertex. Friendly domination is closely related to other ideas like offensive alliance [5–8] and defensive alliances [9–12].

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At the same time, there has been growing interest in other fairness conditions on dominating sets. One example is fair domination, where each vertex outside the set must have the same number of connections to the set [13]. These ideas show a trend toward studying domination with more detailed and realistic rules compared to traditional domination.

Even with these developments, there are still many open questions about the friendly domination. Basic properties that were solved long ago for ordinary domination, such as Nordhaus–Gaddum-type inequalities [14], are still open for friendly domination. Also, exact values and sharp bounds are only known for a few simple types of graphs, and there is little known about how friendly domination behaves under graph operations like graph products.

In this paper, we prove that the domination number is always less than or equal to the friendly domination number, and we describe graphs where the friendly domination number equals one or two. We also show that the difference between the friendly domination number and the domination number can be made as large as desired by constructing appropriate graphs. Furthermore, we find exact formulas for the friendly domination number in paths, cycles, and graphs with maximum degree two. In addition, we establish sharp Nordhaus–Gaddum bounds for the sum and product of the friendly domination number of a graph and its complement. Finally, we characterize friendly dominating sets in the join and corona of graphs.

2. Terminology and Notation

Given a graph $G = (V(G), E(G))$, the *open neighborhood* of a vertex $v \in V$, denoted $N(v)$, is the set of all vertices adjacent to v, that is,$N(v) = \{u \in V \mid \{u, v\} \in E(G)\}$. Its *closed neighborhood*, denoted $N[v]$, is obtained by adding the vertex itself: $N[v] = N(v) \cup \{v\}$. More generally, for any subset $S \subseteq V$, the *open neighborhood* of S is the union of the open neighborhoods of its elements, $N(S) = \bigcup_{v\in S} N(v)$, and the *closed neighborhood* of S is the union of their closed neighborhoods (equivalently $N(S) \cup S$), $N[S] = \bigcup_{v\in S} N[v] = N(S) \cup S$. The *complement* of G, denoted $\overline{G}$, is the simple graph on the same vertex set V whose edge set is $E(\overline{G}) = \big\{\{u, v\} \subseteq V :\ u \neq v \text{ and } \{u, v\} \notin E(G)\big\}$.

The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is defined to be the number of vertices adjacent to v. Equivalently, $\deg(v) = \big|N(v)\big|, N(v) = \{u \in V \mid \{u, v\} \in E(G)\}$. A vertex v for which $\deg(v) = 1$ is called a *leaf.* The *maximum degree* of G is $\Delta(G) = \max_{v\in V} \deg(v)$, and the *minimum degree* of G is $\delta(G) = \min_{v\in V} \deg(v)$. For any subset $S \subseteq V$, the *degree of v restricted to S* is $\deg_S(v) = \big|N(v) \cap S\big|$.

A subset $D \subseteq V$ is called a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. Equivalently, the closed neighborhood of D, defined by $N[D] = \bigcup_{v\in D}\big(\{v\} \cup N(v)\big)$, satisfies $N[D] = V$, so that every vertex of G lies either in D or has a neighbor in D. The *domination number* of G, denoted $\gamma(G)$, is the smallest size of any dominating set in G: $\gamma(G) = \min\big\{|D| : D \subseteq V,\ N[D] = V\big\}$. [1, 15]

A subset $F \subseteq V$ is called a *friendly set* if each vertex outside F has at least as many neighbors outside F as inside F. Equivalently, for every $v \in V(G) \setminus F$, $\deg_F(v) \leq \deg_{V(G)\setminus F}(v)$. This notion was introduced by Haynes, Hedetniemi, and Henning in their study of structures of domination in graphs. [4].

Standard graph-theoretic terminology not defined herein may be found in classic texts such as [1, 2, 15]. For more advanced notions of domination and friendly sets, the reader is referred to Haynes et al. [4].

The *join* of two graphs G and H, written $G \vee H$, is formed by first taking disjoint copies of G and H (so no vertex is shared and no edge connects the two graphs), and then adding every possible edge between each vertex of G and each vertex of H. Thus the vertex set of $G \vee H$ is $V(G) \cup V(H)$, its edge set contains $E(G)$ and $E(H)$, and in addition it contains the set of edges $\{\, \{u, v\} \mid u \in V(G),\, v \in V(H)\, \}$. Intuitively, the join "glues" the two graphs together by making every vertex of one graph adjacent to every vertex of the other. [1, 15]

The *corona* of two graphs G and H, written $G \circ H$, is obtained by taking one copy of G and, for every vertex x of G, adjoining an entire disjoint copy H_x of H and then adding edges from x to every vertex of that copy. Thus each vertex of G becomes the centre of a "star" whose leaves induce a copy of H, while all internal adjacencies of G and of each H_x are retained. [16].

Given any statement or property $\mathcal{A}$, the *indicator* (or *characteristic*) function $\mathbf{1}_{\{\mathcal{A}\}}$ is defined to be 1 if the property $\mathcal{A}$ is true, and 0 if the property $\mathcal{A}$ is false. In particular, for a subset S of vertices, the indicator $\mathbf{1}_{\{x \in S\}}$ equals 1 when the vertex x belongs to S and 0 otherwise, while $\mathbf{1}_{\{x \notin S\}}$ equals 1 when x does not belong to S and 0 otherwise. For any fixed vertex x, exactly one of these two indicators equals 1 and the other equals 0. This convenient notation allows case distinctions to be written compactly inside a single unified formula, simplifying many expressions and arguments.

For non–negative real numbers $a_1, a_2, \ldots, a_n$ the *Arithmetic–Geometric Mean Inequality* asserts that the arithmetic mean $\frac{a_1 + \cdots + a_n}{n}$ is never smaller than the geometric mean $(a_1 \cdots a_n)^{1/n}$; in symbols,

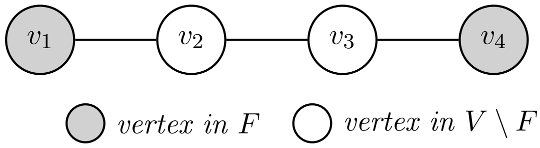
$$\frac{a_1 + a_2 + \cdots + a_n}{n} \;\geq\; (a_1 a_2 \cdots a_n)^{1/n},$$

with equality precisely when all the a_i are equal [17].

3. Results

Definition 1. *Let G be a nontrivial connected graph. A nonempty set $F \subseteq V(G)$ is a friendly dominating set if $N[F] = V(G)$ and for every $v \in V(G) \setminus F$, $deg_F(v) \leq deg_{V(G)\setminus F}(v)$. The minimum cardinality of a friendly dominating set in G is called the friendly domination number of G, denoted by $\gamma_f(G)$. A friendly dominating set whose cardinality is the $\gamma_f(G)$, is called the γ_f-set of G.*

Example 1. *Consider the path graph P_4 with vertices v_1, v_2, v_3, v_4 in order, and let $F = \{v_1, v_4\}$.*



Clearly $N[F] = V(P_4)$, since v_1 dominates v_2 and v_4 dominates v_3. For every vertex outside F we have

$$\deg_F(v_2) = 1 \le \deg_{V\setminus F}(v_2) = 1, \qquad \deg_F(v_3) = 1 \le \deg_{V\setminus F}(v_3) = 1.$$

Thus F is a friendly dominating set of P_4. In particular, $\gamma_f(P_4) = 2$.

3.1. Some Realizations and the Friendly Dominating Sets of Some Known Graphs

Given nontrivial connected graph $G = (V(G), E(G))$. Every friendly dominating set of G is a dominating set of G. Thus we have the following remark:

Remark 1. *Let G be a graph. Then, $\gamma(G) \le \gamma_f(G)$.*

Let F be a nonempty subset of $V(G)$. If $F = V(G)$, then F is trivially a dominating set because it contains every vertex of G. Moreover, the friendly condition only needs to be verified for vertices in $V(G) \setminus F$. Since $V(G) \setminus F = \varnothing$, there are no vertices to check the inequality, and hence the condition is satisfied vacuously.

Remark 2. *Let G be a graph. Then the vertex set, $V(G)$, is a friendly dominating set of G.*

Theorem 1. *Let G be a nontrivial connected graph and let $F \subseteq V(G)$ be a friendly dominating set of G. If G contains any leaf vertices, then every leaf vertex of G must belong to F.*

Proof. Let G be a nontrivial connected graph and let $F \subseteq V(G)$ be a friendly dominating set of G. Suppose that there exists a leaf vertex $u \in V(G)$ such that $u \notin F$. Since F is a dominating set, u must be adjacent to some vertex in F. However, as u is a leaf, it has exactly one neighbor in G, say w. Thus, $w \in F$ and consequently $\deg_F(u) = 1$. On the other hand, u has no other neighbors, so $\deg_{V(G)\setminus F}(u) = 0$. This yields to

$$1 = \deg_F(u) > \deg_{V(G)\setminus F}(u) = 0,$$

which is a contradiction. Hence, it follows that every leaf vertex of G belongs to F. □

Theorem 2. *Let G be a nontrivial connected graph and $F \subseteq V(G)$ be a friendly dominating set of G. Then the induced subgraph of $V(G) \setminus F$ in G, has no isolated vertices.*

Proof. Suppose that the induced subgraph $G[V(G) \setminus F]$ contains an isolated vertex w. Then $w \in V(G) \setminus F$ and $\deg_{V(G)\setminus F}(w) = 0$. Since F is a dominating set, w must be adjacent to some vertex in F, implying that $\deg_F(w) \geq 1$. This yields the inequality

$$1 \leq \deg_F(w) \leq \deg_{V(G)\setminus F}(w) = 0,$$

which is a contradiction. Therefore, the induced subgraph $G[V(G) \setminus F]$ cannot contain an isolated vertex. □

Theorem 3. *Let G be a nontrivial connected graph of order $n > 2$. Then, $\gamma_f(G) = 1$ if and only if there exist a vertex $v \in V(G)$ such that deg$(v) = n - 1$ and for every other vertex $u \neq v$ has at least one neighbor besides v.*

Proof. Suppose $\gamma_f(G) = 1$. Then there is a friendly dominating set $F \subseteq V(G)$ with $|F| = 1$. Let $F = \{v\}$. Since F is a dominating set, v must be adjacent to every other vertex. Hence, $\deg(v) = n-1$. Now, for each $u \in V(G) \setminus \{v\}$, we have $\deg_F(u) = 1$. Since F is a friendly set, $\deg_F(u) \leq \deg_{V(G)\setminus\{v\}}(u)$ for all $u \notin F$. Thus, $\deg_{V(G)\setminus\{v\}}(u) \geq 1$, meaning each $u \neq v$ must have at least one neighbor other than v.

Conversely, suppose there exists a vertex $v \in V(G)$ with $\deg(v) = n - 1$ and for every $u \neq v$, $\deg_{V(G)\setminus F}(u) \geq 1$. Consider $F = \{v\}$. Then, F is a dominating set of G. Also, for each $u \neq v$, $\deg_F(u) = 1$. Thus, $\deg_F(u) = 1 \leq \deg_{V(G)\setminus F}(u)$. Hence, F is a friendly set in G. Therefore, $F = \{v\}$ is a friendly dominating set of G and that $\gamma_f(G) \leq 1$. Clearly, $\gamma_f(G) \geq 1$. Thus, $\gamma_f(G) = 1$. □

Corollary 1. *For the complete graph K_n, fan graph $F_{1,n}$, and wheel graph $W_{1,m}$ with $n > 1$ and $m > 2$, $\gamma_f(K_m) = \gamma_f(F_{1,n}) = \gamma_f(W_{1,m}) = 1$.*

Proof. This immediately follows from Theorem 3. □

Theorem 4. *Let G be a non-trivial connected graph of order $n \geq 3$. Then $\gamma_f(G) = 2$ if and only if all three conditions hold.*

(i) There is no vertex $u \in V(G)$ such that $N[u] = V(G)$ and the induced subgraph of $G - \{u\}$ has minimum degree $\delta\big(G - \{u\}\big) \geq 1$.

(ii) There exist two distinct vertices $x, y \in V(G)$ with the property that every other vertex $v \in V(G) \setminus \{x, y\}$ is adjacent to at least one of x or y.

(iii) For every vertex $v \in V(G) \setminus \{x, y\}$ we have $\deg_{\{x,y\}}(v) \ \leq \ \deg_{V(G)\setminus\{x,y\}}(v)$.

Proof. Assume that the friendly domination number of G is $\gamma_f(G) = 2$. Then there exists a friendly dominating set $F = \{x, y\} \subseteq V(G)$ of cardinality two. Since F is dominating, every vertex outside the pair is adjacent to at least one of x or y; this is exactly the content of condition (ii). Moreover, since F is friendly, $\deg_F(v) \leq \deg_{V(G)\setminus F}(v)$ for every $v \in V(G) \setminus F$, is identical to condition (iii) when written for the set $\{x, y\}$. Now, suppose (i) is not true, then exist a single vertex u that is itself a friendly dominating set, which would force $\gamma_f(G) = 1$, contradicting our assumption that $\gamma_f(G) = 2$. Hence all three conditions $(i) - (iii)$ must hold.

Conversely, suppose that conditions $(i), (ii)$ and (iii) are satisfied. By condition (ii) the pair $\{x, y\}$ dominates the graph, while condition (iii) shows that the set $\{x, y\}$ is friendly . Thus $\{x, y\}$ is a friendly dominating set, so $\gamma_f(G) \leq 2$. Condition (i) rules out the possibility that a single vertex can act as a friendly dominating set. Thus, $\gamma_f(G) > 1$, hence, $\gamma_f(G) \geq 2$. Therefore, $\gamma_f(G) = 2$. □

Corollary 2. *Let G and H be graphs with $|V(G)|, |V(H)| \geq 2$ such that $\Delta(G) < |V(G)|-1$ and $\Delta(H) < |V(H)| - 1$. Then, $\gamma_f(G \vee H) = 2$. In particular, for every complete bipartite graph $K_{m,n}$ with $m, n \geq 2$ we have $\gamma_f(K_{m,n}) = 2$.*

Proof. This immediately follows from Theorem 4. □

Theorem 5. *Let a and b be positive integers such that $a \leq b$. Then there exist a nontrivial connected graph G such that $\gamma(G) = a$ and $\gamma_f(G) = b$.*

Proof. Let a and b be positive integers with $a \leq b$. Consider the following cases:

Case 1: $a = b$

Consider the graph G shown in Figure 1. Clearly, the set $F = \{x_i | i = 1, 2, 3, \ldots, a - 1, a\}$ is both a γ-set and a γ_f-set of G. Thus, $\gamma(G) = \gamma_f(G) = |F| = a = b$. This completes the proof for the case $a = b$.

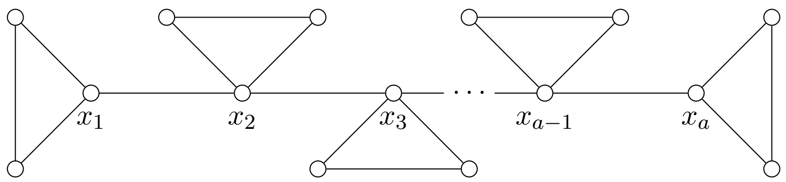


Figure 1: A graph with $\gamma(G) = \gamma_f(G)$

Case 2: $a < b$

Let G be the graph shown in Figure 2. Observe that the set $D = \{x_1, x_2, x_3, \ldots, x_{a-1}, x_a\}$ is a γ-set and the set $F = D \cup \{y_1, y_2, y_3, \ldots, y_{b-a-1}, y_{b-a}\}$ is a γ_f-set. Thus, $\gamma(G) = |D| = a$ and $\gamma_f(G) = |F| = |D| + |\{y_1, y_2, y_3, \ldots, y_{b-a-1}, y_{b-a}\}| = a + b - a = b$. This proves the assertion. □

Corollary 3. *For each positive integer n, there exist a connected graph G such that $\gamma_f(G) - \gamma(G) = n$, that is, the difference between $\gamma_f(G)$ and $\gamma(G)$ can be made arbitrarily large.*

Proof. By Theorem 5, for any pair of positive integers a and b with $a < b$, there exists a connected nontrivial graph H such that $\gamma(H) = a$ and $\gamma_f(H) = b$. Taking $a = 1$ and

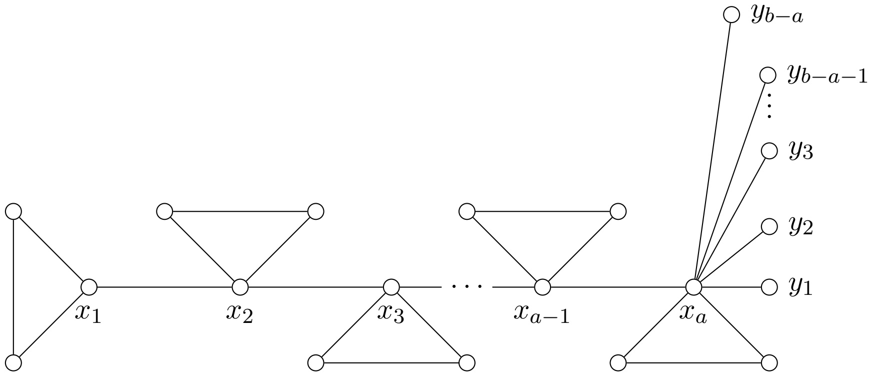


Figure 2: A graph with $\gamma(G) < \gamma_f(G)$

$b = n + 1$. Thus, we have $\gamma_f(G) - \gamma(G) = (n + 1) - 1 = n$. Since n is arbitrary, the difference $\gamma_f(G) - \gamma(G)$ can be made as large as desired. □

Theorem 6. *Let G be a connected nontrivial graph with maximum degree at most 2, and let $F \subseteq V(G)$ be a nonempty subset. Then F is a friendly dominating set of G if and only if the following conditions hold:*

(i) Every vertex $v \in V(G) \setminus F$ satisfies $\deg_F(v) = 1$; that is, every vertex outside F is adjacent to exactly one vertex in F.

(ii) Every vertex $v \in V(G)$ with $\deg(v) = 1$ belongs to F.

Proof. Let $F \subseteq V(G)$ be a friendly dominating set of G. By Theorem 1, every vertex $u \in V(G)$ with $\deg(u) = 1$ belongs to F. Since the maximum degree in G is 2, each vertex v satisfies $\deg(v) \leq 2$. If a vertex $v \notin F$ were adjacent to two vertices in F, then $\deg_F(v) = 2$ and consequently $\deg_{V(G)\setminus F}(v) = 0$, which is a contradiction to the assumption of F. Therefore, every vertex $v \notin F$ must satisfy $\deg_F(v) = 1$.

Conversely, assume that for every vertex $v \notin F$ we have $\deg_F(v) = 1$ and that every leaf of G is contained in F. Then F is clearly a dominating set since every vertex not in F has exactly one neighbor in F. Moreover, for every vertex $v \notin F$ with $\deg(v) \leq 2$, if v is not a leaf it must have degree 2, and the only possibility is that it has one neighbor in F and one neighbor in $V(G) \setminus F$; hence $\deg_{V(G)\setminus F}(v) = 1$ and the inequality

$$\deg_F(v) = 1 \leq 1 = \deg_{V(G)\setminus F}(v)$$

holds. Thus, F is a friendly dominating set of G. □

Corollary 4. *Let P_n be the path graph on $n \geq 2$ vertices. Then the friendly domination* *number of P_n, is given by*

$$\gamma_f(P_n) = \begin{cases} \dfrac{n+6}{3}, & \text{if } n \equiv 0 \pmod 3, \\ \dfrac{n+2}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \dfrac{n+4}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \ldots, v_{n-1}, v_n]$. Consider the following cases for n:

Case 1: $n \equiv 0 \pmod 3$

If $n = 3$, then the γ_f-set of P_3 is $V(P_3)$. Thus, $\gamma_f(P_3) = \frac{n}{3} + 2 = \frac{n+6}{3} = \frac{3+6}{3} = 3 = V(P_3)$. Now, for $n > 3$. Choose $F = \{v_1, v_n\} \cup \{v_4, v_7, \ldots, v_{n-2},, v_{n-1}\}$, then $|F| = 2 + \frac{n}{3} = \frac{n+6}{3}$. By Theorem 6, F is a friendly dominating set in P_n. Thus, $\gamma_f(G) \leq \frac{n+6}{3}$. Since there can be no another friendly dominating sets whose cardinality is strictly less than $\frac{n+6}{3}$, we can have $\gamma_f(P_n) \geq \frac{n+6}{3}$. Thus, $\gamma_f(P_n) = \frac{n+6}{3}$.

Case 2: $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$

An entirely analogous argument as Case 1. Choosing $F = \{v_1, v_n\} \cup \{v_4, v_7, \ldots, v_{n-3}\}$ with $|F| = 2 + \frac{n-4}{3} = \frac{n+2}{3}$ for $n \equiv 1 \pmod 3$ and $F = \{v_1, v_n\} \cup \{v_4, v_7, \ldots, v_{n-2}\}$ with $|F| = 2 + \frac{n-2}{3} = \frac{n+4}{3}$ for $n \equiv 2 \pmod 3$. □

Corollary 5. *Let C_n be the cycle graph on $n \geq 3$ vertices. Then the friendly domination number of C_n is*

$$\gamma_f(C_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Proof. By Theorem 6 (i), finding the friendly dominating set in C_n is equivalent to finding a set F such that every vertex is either in F or adjacent to exactly one vertex of F. Thus, in C_n, the minimum number of vertices needed so that every vertex in C_n is dominated by exactly one member of the set is $\left\lceil \frac{n}{3} \right\rceil$ and no smaller set can satisfy this condition in C_n. Thus, $\gamma_f(C_n) = \left\lceil \frac{n}{3} \right\rceil$. □

3.2. Nordhaus-Gaddum Type Inequalities for γ_f-set.

Theorem 7. *Let G be a connected nontrivial graph on n vertices. Then,*

$$0 \leq \left|\gamma_f(G) - \gamma_f(\overline{G})\right| \leq n - 1.$$

Proof. Let G be a connected nontrivial graph on n vertices, and let $\gamma_f(G)$ and $\gamma_f\left(\overline{G}\right)$ be the friendly domination numbers of G and its complement $\overline{G}$, respectively. Now, $\gamma_f(G)$

and $\gamma_f(\overline{G})$ are each at least 1 since a friendly dominating set must be nonempty, and they are each at most n since taking the entire vertex set $V(G)$ and $V(\overline{G})$ always yields a friendly dominating set. It follows that $\gamma_f(G)$ and $\gamma_f(\overline{G})$ both lie in the integer interval $[1, n]$, so their absolute difference satisfies

$$0 \leq \left|\gamma_f(G) - \gamma_f(\overline{G})\right| \leq n-1.$$

To see that these bounds are sharp, consider the complete graph K_n. By Corollary 1, $\gamma_f(K_n) = 1$. However, the complement of K_n is the empty graph on n vertices, $\overline{K_n}$, in which no vertices are adjacent. In $\overline{K_n}$, every vertex must be in the friendly dominating set to dominate itself, so $\gamma_f(\overline{K_n}) = n$. Hence $\left|\gamma_f(K_n) - \gamma_f(\overline{K_n})\right| = \left|1 - n\right| = n-1$, matching the upper bound exactly. On the other hand, when G is self-complementary, $\gamma_f(G) = \gamma_f(\overline{G})$, implying that, $\left|\gamma_f(G) - \gamma_f(\overline{G})\right| = 0$. Consequently, for general connected nontrivial graphs G of order n, the difference $\left|\gamma_f(G) - \gamma_f(\overline{G})\right|$ is always between 0 and $n-1$, and both extremes can be attained. □

Let G be a graph of order n. For the empty graph we clearly have both the domination number and the friendly-domination number equal to n. When G is connected and nontrivial one might expect these parameters to drop, and indeed the ordinary domination number always does. In contrast, the friendly-domination number can remain as large as n, even in connected graphs, as the next lemma shows.

Lemma 1. *Let G be a connected graph on at least two vertices. Then*

$$\gamma_f(G) \leq n,$$

and the bound is best possible.

Proof. Since G is connected, By Remark 2, the whole vertex set $V(G)$ is a non-empty friendly dominating set. Hence $\gamma_f(G) \leq |V(G)| = n$.

To show that the bound is tight. Consider the star $K_{1,n-1}$ with center c and leaves $L = \{\ell_1, \ldots, \ell_{n-1}\}$. Let $F \subseteq V(K_{1,n-1})$ be a friendly dominating set in $K_{1,n-1}$. If $c \notin F$, then c must be dominated by some leaf $\ell_i \in F$. Vertex c has no neighbor outside F while $\deg_F(c) \geq 1$, contradicting $\deg_F(c) \leq \deg_{V(K_{1,n-1})\setminus F}(c)$. If $c \in F$ but some leaf $\ell \notin F$, then $\deg_F(\ell) = 1$ (its neighbour c) and $\deg_{V(K_{1,n-1})\setminus F}(\ell) = 0$, again violating the friendly condition. Therefore every friendly dominating set of the star must be the entire vertex set, giving $\gamma_f(K_{1,n-1}) = n$. Thus, the upper bound n is tight. □

Theorem 8. *Let G be a graph on $n \geq 4$ vertices such that both G and its complement $\overline{G}$ are connected. Then*

$$\gamma_f(G) + \gamma_f(\overline{G}) \leq 2n.$$

Proof. By Lemma 1, $\gamma_f(G) \leq n$. Thus, $\gamma f(G) + \gamma_f(\overline{G}) \leq n + n = 2n$.

Corollary 6. *For nontrivial connected graph G on $n \geq 4$ vertices, we have*

$$\gamma_f(G)\,\gamma_f(\overline{G}) \leq n^2.$$

Proof. By Theorem 8, $\gamma_f(G) + \gamma_f(\overline{G}) \leq 2n$. Since both $\gamma_f(G)$ and $\gamma_f(\overline{G})$ are nonnegative numbers, the Arithmetic Mean–Geometric Mean (AM–GM) Inequality yields

$$\gamma_f(G)\,\gamma_f(\overline{G}) \leq \left(\frac{\gamma_f(G) + \gamma_f(\overline{G})}{2}\right)^2.$$

Substituting the upper bound on the sum, we obtain, $\gamma_f(G)\,\gamma_f(\overline{G}) \leq \left(\frac{2n}{2}\right)^2 = n^2$. □

3.3. Structure of Friendly Dominating Sets in the Join of Graphs

Remark 3. *Let G and H be empty graphs and let $F \subseteq V(G \vee H)$ be a nonempty set. If $F \subseteq V(G)$ or $F \subseteq V(H)$, then F cannot be a friendly dominating set in $G \vee H$.*

Theorem 9. *Let G and H be empty graphs with $|V(G)| = n_1$ and $|V(H)| = n_2$. Let $F = F_1 \cup F_2$, where $F_1 \subseteq V(G)$ and $F_2 \subseteq V(H)$ are both nonempty and assume $F \neq V(G \vee H)$ (equivalently, $V(G \vee H) \setminus F \neq \varnothing$). Then F is a friendly dominating set in the join $G \vee H$ if and only if $|F_1| \leq \left\lfloor \frac{n_1}{2} \right\rfloor$ and $|F_2| \leq \left\lfloor \frac{n_2}{2} \right\rfloor$.*

Proof. Assume that F is a friendly dominating set with $F \neq V(G \vee H)$. Because F_1 and F_2 are both nonempty, every vertex is either in F or adjacent to some vertex of F. Hence, F is a dominating set. Now, pick any vertex $y \in V(H) \setminus F_2$. All of y's neighbors lie in $V(G)$; specifically, $\deg_F(y) = |F_1|$ and $\deg_{V(G\vee H)\setminus F}(y) = n_1 - |F_1|$. Since F is friendly, we have, $|F_1| \leq n_1 - |F_1|$. Thus, $|F_1| \leq \left\lfloor \frac{n_1}{2} \right\rfloor$. Repeating the same argument with a vertex $x \in V(G) \setminus F_1$ gives $|F_2| \leq \left\lfloor \frac{n_2}{2} \right\rfloor$.

Conversely, suppose that $|F_1| \leq \left\lfloor \frac{n_1}{2} \right\rfloor$ and $|F_2| \leq \left\lfloor \frac{n_2}{2} \right\rfloor$ with F_1 and F_2 both nonempty. Because $F_1 \neq \varnothing$, every vertex in $V(H)$ is adjacent to some vertex of F; similarly, $F_2 \neq \varnothing$ guarantees that every vertex in $V(G)$ is adjacent to some vertex of F. Hence, $N[F] = V(G \vee H)$. Thus, F is a dominating set in $G \vee H$. Now, let $v \in V(G) \setminus F_1$. Its neighbors are precisely the n_2 vertices of $V(H)$, of which $|F_2|$ lie in F and $n_2 - |F_2|$ lie outside F. Hence, $\deg_F(v) = |F_2|$ and $\deg_{V(G\vee H)\setminus F}(v) = n_2 - |F_2|$. By assumption, $|F_2| \leq \left\lfloor \frac{n_2}{2} \right\rfloor$, implying that,$|F_2| \leq n_2 - |F_2|$. Thus, $\deg_F(v) \leq \deg_{V(G\vee H)\setminus F}(v)$. An analogous calculation applies for $w \in V(H) \setminus F_2$, using $|F_1| \leq n_1 - |F_1|$ gives $\deg_F(w) \leq \deg_{V(G\vee H)\setminus F}(w)$. Thus, F is friendly. Therefore, F is a friendly dominating set in $G \vee H$. □

Theorem 10. *Let G and H be non-trivial connected graphs and let $F \subseteq V(G)$. Then F is a friendly dominating set of $G \vee H$ if and only if the following three conditions hold:*

(i) F *dominates* G*;*

(ii) $|F| \leq |V(G) \setminus F| + \delta(H)$*;*

(iii) $\deg_F^G(x) \leq |V(H)| + \deg_{V(G)\setminus F}^G(x)$ *for every* $x \in V(G) \setminus F$

Proof. Assume F is a friendly dominating set of $G \vee H$. Since every vertex of H is adjacent to all vertices of G in the join, F already dominates $G \vee H$, hence in particular G; this is (i). Choose $y \in V(H)$. Then $\deg_F(y) = |F|$ and $\deg_{V(G\vee H)\setminus F}(y) = |V(G) \setminus F| + \deg_H(y)$. Since F is friendly and $\deg_H(y) \geq \delta(H)$, we have, $|F| \leq |V(G) \setminus F| + \delta(H)$. Thus (ii) holds. Now take $x \in V(G)\setminus F$. Then, $\deg_F(x) = \deg_F^G(x)$ and $\deg_{V(G\vee H)\setminus F}(x) = |V(H)| + \deg_{V(G)\setminus F}^G(x)$. Again, since F is friendly, $\deg_F^G(x) \leq |V(H)| + \deg_{V(G)\setminus F}^G(x)$. Thus, (iii) holds.

Conversely, suppose (i)–(iii) hold. Because F dominates G and every vertex of H is adjacent to all vertices of G in the join, F also dominates H; hence it dominates $G \vee H$. To show that F is friendly, we consider two cases.

Case 1: Let $y \in V(H)$. We have $\deg_F(y) = |F|$ and $\deg_{V(G\vee H)\setminus F}(y) = |V(G) \setminus F| + \deg_H(y)$. With $\deg_H(y) \geq \delta(H)$ and (ii) we get $\deg_F(y) \leq \deg_{V(G\vee H)\setminus F}(y)$.

Case 2: Let $x \in V(G) \setminus F$. Here $\deg_F(x) = \deg_F^G(x)$ and $\deg_{V(G\vee H)\setminus F}(x) = |V(H)| + \deg_{V(G)\setminus F}^G(x)$. By condition (iii), again, $\deg_F(y) \leq \deg_{V(G\vee H)\setminus F}(y)$.

Thus F is friendly. Hence, F is a friendly dominating set in $G \vee H$. □

Theorem 11. *Let G and H be non-trivial connected graphs and let $F \subseteq V(H)$. Then F is a friendly dominating set of $G \vee H$ if and only if the following three conditions hold:*

(i) *F dominates H;*

(ii) $|F| \leq |V(H) \setminus F| + \delta(G)$;

(iii) $\deg_F^H(x) \leq |V(G)| + \deg_{V(H)\setminus F}^H(x)$ *for every* $x \in V(H) \setminus F$

Proof. The same analogous proof as Theorem 10. □

Theorem 12. *Let G and H be non-trivial connected graphs on $|V(G)| = n_1$ and $|V(H)| = n_2$ vertices, respectively. Let $F_1 \subseteq V(G)$ and $F_2 \subseteq V(H)$ and put $F = F_1 \cup F_2$. Then F is a friendly dominating set of $G \vee H$ if and only if the following two inequalities are satisfied:*

(i) $\deg_{F_1}^G(x) \leq \deg_{V(G)\setminus F_1}^G(x) + n_2 - 2|F_2|$, *for every* $x \in V(G) \setminus F_1$,

(ii) $\deg_{F_2}^H(y) \leq \deg_{V(H)\setminus F_2}^H(y) + n_1 - 2|F_1|$, *for every* $y \in V(H) \setminus F_2$.

Proof. Assume F is friendly dominating set in $G \vee H$. Pick $x \in V(G) \setminus F_1$ and count separately the neighbors of x inside and outside F. Then, $\deg_F(x) = \deg_{F_1}^G(x) + |F_2|$ and $\deg_{V(G\vee H)\setminus F}(x) = \deg_{V(G)\setminus F_1}^G(x) + |V(H) \setminus F_2| = \deg_{V(G)\setminus F_1}^G(x) + n_2 - |F_2|$. Since F is friendly, we have, $\deg_{F_1}^G(x) \leq \deg_{V(G)\setminus F_1}^G(x) + n_2 - 2|F_2|$. An analogous statement for $y \in V(G) \setminus F_2$ gives $\deg_{F_2}^H(y) \leq \deg_{V(H)\setminus F_2}^H(y) + n_1 - 2|F_1|$.

Conversely, suppose (i) and (ii) hold. Since F_1 (resp. F_2) contains at least one vertex of G (resp. H), every vertex of H (resp. G) is adjacent (through the join) to a vertex of F;

hence F is a dominating set in $G \vee H$. To show that F is friendly. Let $x \in V(G) \setminus F_1$. By (i), $\deg^G_{F_1}(x) \leq \deg^G_{V(G)\setminus F_1}(x) + n_2 - 2|F_2|,$. This implies that, $\deg^G_{F_1}(x) + |F_2| \leq \deg^G_{V(G)\setminus F_1}(x) + n_2 - |F_2|$. Now, $\deg_F(x) = \deg^G_{F_1}(x) + |F_2|$ and $\deg_{V(G\vee H)\setminus F}(x) = \deg^G_{V(G)\setminus F_1}(x) + n_2 - |F_2|$. Thus, $\deg_F(x) \leq \deg_{V(G\vee H)\setminus F}(x)$. Similarly for $y \in V(H) \setminus F_2$ and by (ii) gives, $\deg_F(y) \leq \deg_{V(G\vee H)\setminus F}(y)$. Hence, F is friendly. Therefore, F is a friendly dominating set in $G \vee H$. □

3.4. Friendly Dominating Sets in the Corona of Graphs

Theorem 13. *Let G and H be connected non-trivial graphs. Suppose $F \subseteq V(G \circ H)$ for which $F \subseteq V(G)$. Then F is a friendly dominating set of $G \circ H$ if and only if $F = V(G)$.*

Proof. Assume first that F is a friendly dominating set of $G \circ H$. If $F \neq V(G)$ choose a vertex $y \in V(G) \setminus F$. Since F contains no vertices of the copy H_y, every vertex $h \in V(H_y)$ is adjacent only to its root y and to vertices of H_y itself. Since neither y nor any vertex of H_y lies in F, the vertex h has no neighbor in F; hence $h \notin N[F]$, contradicting the assumption that F is dominating. Therefore domination forces $F = V(G)$.

Conversely, suppose $F = V(G)$. Clearly, F is a dominating set in $G \circ H$. Fix $x \in V(G)$ and let $h \in H_x \setminus F$. Exactly one edge, namely $\{h, x\}$, joins h to F, so $\deg_F(h) = 1$. On the other hand, h is adjacent inside H_x to $\deg_H(h)$ vertices, all of which lie in $V(G \circ H) \setminus F$. Since H is connected and non–trivial, $\deg_H(h) \geq 1$, whence, $\deg_F(h) = 1 \leq \deg_{V(G\circ H)\setminus F}(h) = \deg_H(h)$. Thus the inequality holds for every vertex outside F, so F is friendly. Therefore $F = V(G)$ is indeed a friendly dominating set. □

Theorem 14. *Let G and H be connected non–trivial graphs and let $F \subseteq V(G \circ H)$. Suppose $F \subseteq \bigcup_{x\in V(G)} V(H_x)$ and $P_x = F \cap V(H_x) \neq \varnothing$ for every $x \in V(G)$. Then F is a friendly dominating set of $G \circ H$ if and only if the following hold for every $x \in V(G)$:*

(i) *P_x is a dominating set of H_x;*

(ii) $|P_x| \leq \left\lfloor \dfrac{|V(H)| + \deg_G(x)}{2} \right\rfloor$.

Proof. Assume F is a friendly dominating set. Since $x \notin F$ for each $x \in V(G)$, domination of x must occur via an edge xp with $p \in P_x$, so $P_x \neq \varnothing$. If some $h \in H_x \setminus P_x$ had no neighbour in P_x, it would not be dominated; hence P_x dominates H_x, establishing (i). For (ii), observe that, $\deg_F(x) = |P_x|$ and $\deg_{V(G\circ H)\setminus F}(x) = \deg_G(x) + |V(H)| - |P_x|$, so the friendly inequality $\deg_F(x) \leq \deg_{V(G\circ H)\setminus F}(x)$ is equivalent to $|P_x| \leq \dfrac{|V(H)| + \deg_G(x)}{2}$.

Conversely, suppose that (i) and (ii) hold. Every $x \in V(G)$ is adjacent to $P_x \neq \varnothing$, and (i) ensures P_x dominates H_x; hence $N_{G\circ H}[F] = V(G \circ H)$. To show that F is friendly. Let $v \in V(G \circ H) \setminus F$. If $v = x \in V(G)$, then $\deg_F(v) = |P_x| \leq \frac{|V(H)|+\deg_G(x)}{2} = \deg_{V(G\circ H)\setminus F}(v)$ by (ii). Now, if $v = h \in H_x \setminus P_x$, put $a = \deg_{P_x}(h)$. Then $\deg_F(h) = a$ and $\deg_{V(G\circ H)\setminus F}(h) = \deg_H(h) - a + 1$. Because $a \leq |P_x|$ and $2|P_x| \leq |V(H)| + \deg_G(x)$ by (ii), we have $2a \leq |V(H)| + \deg_G(x)$. Since $\deg_H(h) + 1 \leq |V(H)| + \deg_G(x)$, it follows that $2\deg_F(h) \leq \deg_H(h) + 1 = \deg_{G\circ H}(h)$, i.e. $\deg_F(h) \leq \deg_{V(G\circ H)\setminus F}(h)$. Hence, F is friendly. Therefore, F is a friendly dominating set of $G \circ H$. □

Theorem 15. *Let G and H be connected non-trivial graphs and let $F \subseteq V(G\circ H)$. Choose a non-empty subset $F_0 \subseteq V(G)$ and, for every $x \in V(G)$, a subset $P_x \subseteq V(H_x)$ such that $P_x \neq \varnothing$ for at least one vertex x. Suppose $F = F_0 \cup \left(\bigcup_{x\in V(G)} P_x\right)$. Then the set F is a friendly dominating set of $G \circ H$ if and only if the following conditions are satisfied.*

(i) For every $x \in V(G)$:

- *(a) $x \in F_0$, or $N_G(x) \cap F_0 \neq \varnothing$, or $P_x \neq \varnothing$;*
- *(b) if $x \notin F_0$ then every vertex of H_x is adjacent to a vertex of P_x; equivalently $N_{H_x}[P_x] = V(H_x)$.*

(ii) For every vertex $x \in V(G) \setminus F_0$,

$$|P_x| + \deg_{F_0}(x) \;\leq\; \big(|V(H)| - |P_x|\big) + \big(\deg_G(x) - \deg_{F_0}(x)\big).$$

(iii) For every vertex $h \in V(H_x) \setminus P_x$,

$$\deg_{P_x}(h) \;+\; \mathbf{1}_{\{x\in F_0\}} \;\leq\; \big(\deg_H(h) - \deg_{P_x}(h)\big) + \mathbf{1}_{\{x\notin F_0\}}.$$

Proof. Assume that F is a friendly dominating set of $G \circ H$. Let $x \in V(G)$. If $x \in F_0$, then we are done. Otherwise $x \notin F$, so x must be dominated by a neighbor belonging to F. Its neighbors are exactly the vertices of $N_G(x) \cup H_x$; hence either $N_G(x) \cap F_0 \neq \varnothing$ or $P_x \neq \varnothing$, which is condition $(i.a)$. For condition $(i.b)$ keep $x \notin F_0$ and choose an arbitrary $h \in H_x \setminus P_x$. Since $x \notin F$, the vertex h can be dominated only through a neighbor in P_x, so $N_{H_x}(h) \cap P_x \neq \varnothing$. Thus, P_x dominates H_x. Now take $x \in V(G) \setminus F_0$. The number of neighbors of x that lie in F equals $\deg_F(x) = \deg_{F_0}(x) + |P_x|$, while the number that lie outside F is $\deg_{V(G\circ H)\setminus F}(x) = \big(|V(H)| - |P_x|\big) + \big(\deg_G(x) - \deg_{F_0}(x)\big)$. Hence the inequality $\deg_F(x) \leq \deg_{V(G\circ H)\setminus F}(x)$ reproduces condition (ii). Finally, fix $h \in H_x \setminus P_x$ and write $a = \deg_{P_x}(h)$. Then, $\deg_F(h) = a + \mathbf{1}_{\{x\in F_0\}}$, $\deg_{V(G\circ H)\setminus F}(h) = \big(\deg_H(h) - a\big) + \mathbf{1}_{\{x\notin F_0\}}$, so the friendly inequality for h is exactly condition (iii).

Conversely, suppose that conditions $(i) - (iii)$ hold. For every $x \in V(G)$, by $(i.a)$, guarantees a neighbor of x that lies in F; hence x is dominated. Now fix $h \in H_x$. If $x \in F_0$, the edge xh dominates h. If $x \notin F_0$, by $(i.b)$, guarantees a neighbor of h in $P_x \subseteq F$. Thus every vertex of H_x is dominated, and we obtain $N_{G\circ H}[F] = V(G \circ H)$. Thus, F is a dominating set. To show that F is friendly. Let $v \notin F$. If $v = x \in V(G) \setminus F_0$, by (ii), we have $\deg_F(v) \leq \deg_{V(G\circ H)\setminus F}(v)$. If $v = h \in H_x \setminus P_x$, by (iii), again, $\deg_F(v) \leq \deg_{V(G\circ H)\setminus F}(v)$. Hence, F is friendly. Therefore, F is a friendly dominating set of $G \circ H$. □

Corollary 7. *Let G and H be connected, non-trivial graphs and let $G \circ H$ denote their corona. Then*

$$\gamma_f(G \circ H) \;=\; |V(G)|.$$

Proof. Theorem 13 shows that the set $F = V(G)$ is a friendly dominating set of $G \circ H$, so $\gamma_f(G \circ H) \leq |V(G)|$. Now, Theorem 14 and Theorem 15 together implies that every friendly dominating set which omits at least one vertex of $V(G)$ must contain strictly more than $|V(G)|$ vertices. Hence no friendly dominating set of smaller size than $|V(G)|$ exists, so $\gamma_f(G \circ H) \geq |V(G)|$. Therefore, $\gamma_f(G \circ H) = |V(G)|$.

4. Conclusion

This paper demonstrated that the *friendly domination number* γ_f, the smallest size of a dominating set that is also friendly—always lies above or on the ordinary domination number γ. We gave complete characterizations of graphs with $\gamma_f = 1$ or 2, construct families in which the gap $\gamma_f - \gamma$ is arbitrarily large, and derive exact expression for paths and cycles. Moreover, we established sharp Nordhaus–Gaddum bounds for the difference, the sum, and the product of γ_f of a graph and its complement, and we identified all friendly dominating sets in the join and the corona of two graphs, proving that in every corona γ_f equals the order of the host graph.

The present work is restricted to simple, undirected graphs and to two basic graph operations. It does not treat weighted or directed variants, computational complexity, or behaviour under richer products such as Cartesian or strong products.

Future research should therefore address several open directions: determining γ_f for additional graph families and products; analysing the complexity of computing γ_f and designing efficient approximation schemes; extending the concept to weighted, directed, or temporal graphs; and studying probabilistic thresholds for friendly domination in random graph models. Progress on these questions will deepen our understanding of how equitable influence can be maintained in increasingly complex networks.

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