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On Quotient KS-semigroups Induced by Fuzzy KS-ideals

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Abstract. In this paper, we introduce new types of fuzzy ideals on KS-semigroups, compare them with existing fuzzy KS-ideals and investigate their properties. We discuss the construction of quotient KS-semigroups induced by fuzzy KS-ideals, show that this quotient structure is a generalization of the existing quotient structure and prove properties of this generalized structure. Moreover, we investigate the quotient structure of product KS-semigroups induced by fuzzy KS-ideals.

2020 Mathematics Subject Classifications: 08A30, 08A72

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1. Introduction

The class of BCK-algebras was introduced by Imai and Iseki [1] in 1966 that describes fragments of propositional calculus involving implication, known as BCK logic. It was also developed as a generalization of set difference in set theory. Since then, a great deal of literature has been produced in the theory of BCK-algebras. The initial paper on the class of semigroups emerged in 1905 as a concise work by Dickson. However, the true inception of the theory occurred in 1928 when Suschkewitsch [2] published a paper of paramount significance. In contemporary language, he demonstrated that within any finite semigroup, there exists a "kernel" (referred to as a simple ideal), and he comprehensively characterized the structure of finite simple semigroups. Semigroups provide a foundational framework for understanding how elements combine under certain operations, and their applications span across multiple branches of mathematics and various interdisciplinary fields such as Coding theory, Automata, etc. For more notions about semigroups and history, we refer to [3].

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The notion of fuzzy sets was introduced by Zadeh [4] in 1965 to provide a mathematical framework for dealing with vagueness and imprecision. It departs from classical set theory by allowing elements to have degrees of membership in a set, rather than requiring absolute membership or non-membership. Fuzzy sets quickly garnered attention across various disciplines, including algebraic structures. The combination of the two concepts gave rise to fuzzy algebraic structures. The latter was established by Rosenfeld in 1971 when he applied the concept of fuzzy sets to groups.

In 2006, Kim [5] introduced the class of KS-semigroups which is both a BCK-algebra and a semigroup. The quotient KS-semigroups via ideals was established and the isomorphism theorems were proved in 2009 by Cawi and Vilela [6]. In 2007, Prince Williams and Husain [7] applied the concept of fuzzy sets to KS-semigroup, and referred to it as a fuzzy KS-semigroup. The notions of fuzzy KS-ideals and fuzzy KS-p-ideals were introduced and some of their properties were investigated. In 2011, Bautista and Vilela [8] introduced and investigated fuzzy topology on a KS-semigroup and fuzzy topological KS-semigroup.

There are many research items in the literature characterizing the class of semigroups through its (fuzzy) sets, particularly, through generalization and fuzzification of ideals. For example, generalization of bi-antiideals in semigroups and their fuzzification were studied in [9] and (fuzzy) KS-H-ideals in KS-semigroups and their properties were studied in [10]. Moreover, the construction of quotient semigroups induced by fuzzy ideals and their properties were studied in [11]. In this paper, we introduce new types of fuzzy ideals on KS-semigroups, compare them with existing fuzzy KS-ideals and investigate their properties. We construct quotient KS-semigroups induced by fuzzy KS-ideals and investigate their properties. Furthermore, we investigate the quotient structure of product KS-semigroups induced by fuzzy KS-ideals.

2. Preliminaries

In this section, we present some basic concepts and known results that are useful in this study.

Definition 1. [5] *An algebraic system* $(X, *, 0)$ *is called a BCK-algebra if it satisfies the following conditions: for all* $x, y, z \in X$,

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $0 * x = 0$,

(v) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

Definition 2. [5] *If* X *is a BCK-algebra, then the relation* $x \leq y$ *if and only if* $x * y = 0$ *is a partial order on* X, *which will be called the natural ordering on* X.

Remark 1. [5] *A BCK-algebra* X *has the following properties for any* $x, y, z \in X$:

(i) $x * 0 = x$,

(iv) $x \leq y$ *implies that* $x * z \leq y * z$ *and* $z * y \leq z * x$,

(ii) $x * y \leq x$,

(iii) $(x * y) * z = (x * z) * y$,

(v) $(x * z) * (y * z) \leq x * y$.

Proposition 1. [12] *In a BCK-algebra X, the following hold: for all $x, y, z \in X$,*

(i) $((x * z) * z) * (y * z) \leq (x * y) * z$,

(ii) $(x * z) * (x * (x * z)) = (x * z) * z$,

(iii) $(x * (y * (y * x))) * (y * (x * (y * (y * x)))) \leq x * y$.

Definition 3. [13] *Let X be a non-empty set. The system $(X, \cdot)$ is called a semigroup if " $\cdot$ " is an associative binary operation. For convenience, we write $x \cdot y$ by xy.*

Definition 4. [5] *An algebraic system $(X, *, \cdot, 0)$ is called a KS-semigroup if it satisfies the following conditions:*

(i) *$(X, *, 0)$ is a BCK-algebra*

(ii) *$(X, \cdot)$ is a semigroup*

(iii) *The operation $\cdot$ is distributive over the operation $*$ on both sides, that is, for all $x, y, z \in X$, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$.*

Example 1. [5] *Let $X = \{0, a, b, c\}$. Define $*$ and $\cdot$ by the following tables:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	c	0	0

Then X is a KS-semigroup.

Definition 5. [5] *A KS-semigroup X is said to be*

(i) *commutative if $x * (x * y) = y * (y * x)$ for all $x, y \in X$,*

(ii) *positive implicative if $(x * y) * z = (x * z) * (y * z)$, or equivalently, $x * y = (x * y) * y$ for all $x, y, z \in X$,*

(iii) *implicative if $x * (y * x) = x$ for all $x, y \in X$, or if it is both commutative and positive implicative.*

Proposition 2. [5] *A KS-semigroup X is commutative if and only if for all $x, y \in X$, $x \leq y$ implies $x = y * (y * x)$.*

Definition 6. [5] *A non-empty subset A of a semigroup $(X, \cdot)$ is said to be left (resp. right) stable if $xa \in A$ (resp. $ax \in A$) whenever $x \in X$ and $a \in A$. A non-empty subset of X which is both left and right stable is called two-sided stable or simply stable.*

Definition 7. [5] *A non-empty subset A of a KS-semigroup X is called a left (resp. right) ideal of X if*

(i) *A is a left (resp. right) stable subset of $(X, \cdot)$,*

(ii) *for any $x, y \in X$, $x * y \in A$ and $y \in A$ imply that $x \in A$.*

If A is both a left and a right ideal, then it is called a two-sided ideal or simply an ideal.

Remark 2. [5] *Let X be a KS-semigroup. Then*

(i) *$\{0\}$ and X are ideals of X.*

(ii) *If A is a left (resp. right) ideal of X, then $0 \in A$.*

Definition 8. [5] *Let X be a KS-semigroup and let $\sim$ be a binary relation on X. Then*

(i) *$\sim$ is said to be right (resp. left) compatible if whenever $x \sim y$ then $x * z \sim y * z$ (resp. $z * x \sim z * y$) and $xz \sim yz$ (resp. $zx \sim zy$) for all $x, y, z \in X$;*

(ii) *$\sim$ is said to be compatible if $x \sim y$ and $u \sim v$ imply $x * u \sim y * v$ and $xu \sim yv$ for all $x, y, u, v \in X$;*

(iii) *A compatible equivalence relation is called a congruence relation.*

Theorem 1. [5] *Let X be a KS-semigroup. An equivalence relation $\sim$ on X is a congruence relation if and only if it is both left and right compatible.*

Let X be a non-empty set. A *fuzzy set* on X is a function $\mu : X \to [0, 1]$. The characteristic function of a set I is a fuzzy set denoted by χ_I.

Definition 9. [7] *A fuzzy set μ on a KS-semigroup X is called a left (resp. right) fuzzy KS-ideal of X if for all $x, y, a \in X$,*

(F1) $\mu(0) \geq \mu(x)$,

(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$,

(F3) $\mu(xa) \geq \min\{\mu(x), \mu(a)\}$ (*resp.* $\mu(ax) \geq \min\{\mu(x), \mu(a)\}$).

A fuzzy set μ on a KS-semigroup X is called a fuzzy KS-ideal of X if it is both a left and a right fuzzy KS-ideal of X.

Definition 10. [7] *Let X be a KS-semigroup. A fuzzy set μ on X is called a left (resp. right) fuzzy KS-p-ideal of X if it satisfies* (F1), (F3) *and*

$$(\text{F4}) : \mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\} \text{ for all } x, y, z \in X.$$

A fuzzy set μ on a KS-semigroup X is called a fuzzy KS-p-ideal of X if it is both a left and a right fuzzy KS-p-ideal of X.

Theorem 2. [7] *Let X be a KS-semigroup. Then any fuzzy KS-p-ideal of X is a fuzzy KS-ideal of X. However, the converse is not true.*

3. Some Fuzzy Ideals on KS-semigroup

In this section, we investigate useful properties of fuzzy KS-ideals and give criterion for a fuzzy KS-ideal to be a fuzzy KS-p-ideal. Moreover, we introduce new types of fuzzy ideals on KS-semigroups, compare them with existing fuzzy KS-ideals and investigate their properties.

The following lemma gives some useful properties of fuzzy KS-ideals.

Lemma 1. *Let μ be a fuzzy KS-ideal of a KS-semigroup X.*

(i) *If $x, y \in X$ such that $x \leq y$, then $\mu(y) \leq \mu(x)$.*

(ii) *If $x, y, z \in X$ such that $x * y \leq z$, then $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.*

Proof. Let μ be a fuzzy KS-ideal of a KS-semigroup X.

(i) Suppose $x, y \in X$ such that $x \leq y$. Then by (F2), $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ and $x * y = 0$. Thus, $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} = \min\{\mu(0), \mu(y)\} = \mu(y)$.

(ii) Suppose $x, y, z \in X$ such that $x * y \leq z$. Then by (i), $\mu(z) \leq \mu(x * y)$. By (F2), $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(z), \mu(y)\}$. □

In Theorem 2, a fuzzy KS-ideal may not be a fuzzy KS-p-ideal. The following theorem gives a criterion for a fuzzy KS-ideal to be a fuzzy KS-p-ideal.

Theorem 3. *A fuzzy set μ on a KS-semigroup X is a fuzzy KS-p-ideal if and only if it is a fuzzy KS-ideal satisfying $\mu(x * y) = \mu((x * y) * y)$ for all $x, y \in X$.*

Proof. Let X be a KS-semigroup and μ be a fuzzy set on X. Suppose μ is a fuzzy KS-p-ideal of X. Then by Theorem 2, μ is a fuzzy KS-ideal. By (F4), (F1) and Definition 1, for all $x, y \in X$,

$$\begin{aligned} \mu(x * y) &\geq \min\{\mu((x * y) * y), \mu(y * y)\} \\ &= \min\{\mu((x * y) * y), \mu(0)\} \\ &= \mu((x * y) * y). \end{aligned}$$

Moreover, observe that by Remark 1(ii), $(x * y) * y \leq x * y$. Thus, by Lemma 1(i), $\mu((x * y) * y) \geq \mu(x * y)$. Hence, $\mu(x * y) = \mu((x * y) * y)$ for all $x, y \in X$.

Conversely, suppose μ is a fuzzy KS-ideal of X satisfying $\mu(x * y) = \mu((x * y) * y)$ for all $x, y \in X$. Note that by Proposition 1(i), $((x * z) * z) * (y * z) \leq (x * y) * z$. Thus, by Lemma 1(ii), we have $\mu(x * z) = \mu((x * z) * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}$. Therefore, μ is a fuzzy KS-p-ideal of X. □

Now, we introduce some fuzzy ideals on KS-semigroups, investigate their relationships with fuzzy KS-ideals and discuss some of their properties.

Definition 11. *Let X be a KS-semigroup. A fuzzy set μ on X is called a left (resp. right) fuzzy commutative KS-ideal of X if it satisfies* (F1), (F3) *and*

$$(\mathrm{F5}) : \mu(x * (y * (y * x))) \geq \min\{\mu((x * y) * z), \mu(z)\} \text{ for all } x, y, z \in X.$$

A fuzzy set μ on a KS-semigroup X is called a fuzzy commutative KS-ideal of X if it is both a left and a right fuzzy commutative KS-ideal of X.

Example 2. *Let $X = \{0, a, b, c\}$. Define the operations $*$ and $\cdot$ by the following tables.*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	a	b	c

By routine calculations, we can see that X is a KS-semigroup. Let $t_0, t_1, t_2 \in [0, 1]$ such that $t_0 > t_1 > t_2$. Define a fuzzy set μ on X by $\mu(0) = t_0$, $\mu(a) = t_1$ and $\mu(b) = \mu(c) = t_2$. Then μ is a fuzzy commutative KS-ideal of X.

The following theorem tells us that every fuzzy commutative KS-ideal is a fuzzy KS-ideal.

Theorem 4. *Let X be a KS-semigroup. Then any fuzzy commutative KS-ideal of X is a fuzzy KS-ideal of X.*

Proof. Let μ be a fuzzy commutative KS-ideal of X. Then by Remark 1(i) and Definition 1, for any $x, y \in X$,

$$\begin{aligned} \mu(x) &= \mu(x * 0) \\ &= \mu(x * (0 * 0)) \\ &= \mu(x * (0 * (0 * x))) \\ &\geq \min\{\mu((x * 0) * y), \mu(y)\} \\ &= \min\{\mu(x * y), \mu(y)\}. \end{aligned}$$

Therefore, μ is a fuzzy KS-ideal of X. □

A fuzzy KS-ideal of a KS-semigroup X may not be a fuzzy commutative KS-ideal of X as shown in the following example.

Example 3. *Let $X = \{0, 1, 2, 3, 4\}$. Define the operations $*$ and $\cdot$ by the following tables.*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	3
4	0	1	2	3	4

Then X is a KS-semigroup. Let $t_0, t_1, t_2 \in [0,1]$ such that $t_0 > t_1 > t_2$. Define a fuzzy set μ on X by $\mu(0) = t_0$, $\mu(1) = t_1$ and $\mu(2) = \mu(3) = \mu(4) = t_2$. By routine calculations, we can see that μ is a fuzzy KS-ideal of X. However, μ is not a fuzzy commutative KS-ideal of X since (F5) *is not satisfied:*

$$\mu(2 * (3 * (3 * 2))) = \mu(2) = t_2 < t_0 = \mu(0) = \min\{\mu((2 * 3) * 0), \mu(0)\}.$$

The following theorem gives a criterion for a fuzzy KS-ideal to be a fuzzy commutative KS-ideal.

Theorem 5. *A fuzzy set μ on a KS-semigroup X is a fuzzy commutative KS-ideal if and only if it is a fuzzy KS-ideal satisfying $\mu(x * (y * (y * x))) = \mu(x * y)$ for all $x, y \in X$.*

Proof. Let X be a KS-semigroup and μ be a fuzzy set on X. Suppose μ is a fuzzy commutative KS-ideal of X. Then by Theorem 4, μ is a fuzzy KS-ideal. By (F5), for all $x, y \in X$,

$$\begin{aligned} \mu(x * (y * (y * x))) &\geq \min\{\mu((x * y) * 0), \mu(0)\} \\ &= \min\{\mu((x * y)), \mu(0)\} \\ &= \mu(x * y). \end{aligned}$$

Moreover, note that by Remark 1(ii), $y * (y * x) \leq y$. Thus, by Remark 1(iv), $x * y \leq x * (y * (y * x))$. Hence, by Lemma 1(i), $\mu(x * y) \geq \mu(x * (y * (y * x)))$. Therefore, $\mu(x * (y * (y * x))) = \mu(x * y)$.

Conversely, suppose μ is a fuzzy KS-ideal of X satisfying $\mu(x * (y * (y * x))) = \mu(x * y)$ for all $x, y \in X$. By (F2), $\mu(x * (y * (y * x))) = \mu(x * y) \geq \min\{\mu((x * y) * z), \mu(z)\}$ for all $x, y, z \in X$. Therefore, μ is a fuzzy commutative KS-ideal of X. □

The following theorem provides a necessary condition for a fuzzy KS-ideal to be a fuzzy commutative KS-ideal.

Theorem 6. *Let X be a commutative KS-semigroup. Then every fuzzy KS-ideal is a fuzzy commutative KS-ideal.*

Proof. Let μ be a fuzzy KS-ideal of a commutative KS-semigroup X. It is sufficient to show that μ satisfies (F5). Let $x, y, z \in X$. Then by Remark 1 and since X is commutative,

$$\begin{aligned} ((x * (y * (y * x))) * ((x * y) * z)) * z &= ((x * (y * (y * x))) * z) * ((x * y) * z) \\ &\leq (x * (y * (y * x))) * (x * y) \\ &= (x * (x * y)) * (y * (y * x)) \\ &= 0. \end{aligned}$$

By Definition 2, $(x * (y * (y * x))) * ((x * y) * z) \leq z$. Thus, by Lemma 1(ii),

$$\mu(x * (y * (y * x))) \geq \min\{\mu((x * y) * z), \mu(z)\}.$$

Hence, (F5) holds. Therefore, μ is a fuzzy commutative KS-ideal of X. □

Definition 12. *Let X be a KS-semigroup. A fuzzy set μ on X is called a left (resp. right) fuzzy implicative KS-ideal of X if it satisfies* (F1), (F3) *and*

$$\text{(F6)}: \mu(x) \geq \min\{\mu((x*(y*x))*z), \mu(z)\} \text{ for all } x, y, z \in X.$$

A fuzzy set μ on a KS-semigroup X is called a fuzzy implicative KS-ideal of X if it is both a left and a right fuzzy implicative KS-ideal of X.

Example 4. *Let $X = \{0, 1, 2, 3, 4\}$. Define the operations $*$ and $\cdot$ by the following tables.*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	4

Then X is a KS-semigroup. Let $s, t \in [0, 1]$ such that $s < t$. Define a fuzzy set μ on X by $\mu(0) = \mu(1) = \mu(2) = t$ and $\mu(3) = \mu(4) = s$. By routine calculations, μ is a fuzzy implicative KS-ideal of X.

We now give a relationship between a fuzzy KS-ideal and a fuzzy implicative KS-ideal.

Theorem 7. Let X be a KS-semigroup. Then any fuzzy implicative KS-ideal of X is a fuzzy KS-ideal of X.

Proof. Let μ be a fuzzy implicative KS-ideal of a KS-semigroup X. Then for all $x, y \in X$,

$$\begin{aligned} \mu(x) &\geq \min\{\mu((x*(x*x))*y), \mu(y)\} \\ &= \min\{\mu((x*0)*y), \mu(y)\} \\ &= \min\{\mu(x*y), \mu(y)\}. \end{aligned}$$

Therefore, μ is a fuzzy KS-ideal of X. □

The converse of Theorem 7 may not be true as shown in the following example.

Example 5. *Consider the KS-semigroup in Example* 4*. Define a fuzzy set μ on X by $\mu(0) = \mu(2) = 0.8$ and $\mu(1) = \mu(3) = \mu(4) = 0.5$. By routine calculations, μ is a fuzzy KS-ideal of X. However, μ is not a fuzzy implicative KS-ideal of X since*

$$\mu(1) = 0.5 < 0.8 = \min\{\mu((1*(3*1))*2), \mu(2)\}.$$

The following result provides a criterion for a fuzzy KS-ideal to be a fuzzy implicative KS-ideal.

Theorem 8. *A fuzzy set μ on a KS-semigroup X is a fuzzy implicative KS-ideal if and only if it is a fuzzy KS-ideal satisfying $\mu(x) = \mu(x * (y * x))$ for all $x, y \in X$.*

Proof. Let X be a KS-semigroup and μ be a fuzzy set on X. Suppose μ is a fuzzy implicative KS-ideal of X. Then by Theorem 7, μ is a fuzzy KS-ideal of X. By (F6), for all $x, y \in X$,

$$\begin{aligned} \mu(x) &\geq \min\{\mu(x * (y * x) * 0), \mu(0)\} \\ &= \min\{\mu(x * (y * x)), \mu(0)\} \\ &= \mu(x * (y * x)). \end{aligned}$$

Moreover, observe that $x*(y*x) \leq x$ by Remark 1(ii). Thus, by Lemma 1(i), $\mu(x*(y*x)) \geq \mu(x)$. Hence, $\mu(x) = \mu(x * (y * x))$.

Conversely, suppose μ is a fuzzy KS-ideal of X satisfying $\mu(x) = \mu(x * (y * x))$ for all $x, y \in X$. By (F2), $\mu(x) = \mu(x*(y*x)) \geq \min\{\mu((x*(y*x))*z), \mu(z)\}$ for all $x, y, z \in X$. Thus, μ is a fuzzy implicative KS-ideal of X. □

The following theorem gives a necessary condition for a fuzzy KS-ideal to be a fuzzy implicative KS-ideal.

Theorem 9. *If X is an implicative KS-semigroup, then every fuzzy KS-ideal of X is fuzzy implicative KS-ideal.*

Proof. Let X be an implicative KS-semigroup and μ a fuzzy KS-ideal of X. Then by Definition 5(ii), $x = x * (y * x)$ for all $x, y \in X$. Since μ is a fuzzy KS-ideal of X, by (F2),

$$\mu(x) \geq \min\{\mu(x * z), \mu(z)\} = \min\{\mu((x * (y * x)) * z), \mu(z)\}$$

for all $z \in X$. Thus, (F6) holds. Hence, μ is a fuzzy implicative KS-ideal of X. □

Theorem 10. *A fuzzy set μ on a KS-semigroup X is a fuzzy implicative KS-ideal if and only if it is both a fuzzy KS-p-ideal and fuzzy commutative KS-ideal.*

Proof. Let X be a KS-semigroup and μ be a fuzzy set on X. Suppose μ is a fuzzy implicative KS-ideal of X. By Theorem 8, Proposition 1 and Lemma 1(ii), for all $x, y, z \in X$, $\mu(x * z) = \mu((x * z) * (x * (x * z))) = \mu((x * z) * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}$. Thus, (F4) holds. Hence, μ is a fuzzy KS-p-ideal. Moreover, by Proposition 1, Lemma 1(i) and Theorem 8, $\mu(x * y) \leq \mu((x * (y * (y * x))) * (y * (x * (y * (y * x))))) = \mu(x * (y * (y * x)))$ for all $x, y \in X$. By Remark 1, $y * (y * x) \leq y$ implies that $x * y \leq x * (y * (y * x))$. Thus, by Lemma 1(i), $\mu(x * y) \leq \mu(x * (y * (y * x)))$. Hence, $\mu(x * y) = \mu(x * (y * (y * x)))$. Therefore, by Theorem 5, μ is a fuzzy commutative KS-ideal of X.

Conversely, suppose μ is both fuzzy KS-p-ideal and fuzzy commutative KS-ideal of X. By Definition 1(i), $(y * (y * x)) * (y * x) \leq x * (y * x)$ for all $x, y \in X$. Thus, by Lemma 1(i), $\mu(x*(y*x)) \leq \mu((y*(y*x))*(y*x))$. By Theorem 3, $\mu((y*(y*x))*(y*x)) = \mu(y*(y*x))$. Thus, we have

$$\mu(x * (y * x)) \leq \mu(y * (y * x)). \tag{1}$$

Now, by Remark 1, $y * x \leq y$ implies that $x * y \leq x * (y * x)$ for all $x, y \in X$. Thus, by Lemma 1(i), $\mu(x * (y * x)) \leq \mu(x * y)$. Since μ is a fuzzy commutative KS-ideal of X, by Theorem 5, $\mu(x * y) = \mu(x * (y * (y * x)))$. Hence, for all $x, y \in X$,

$$\mu(x * (y * x)) \leq \mu(x * (y * (y * x))). \tag{2}$$

Note that by Definition 1(ii), $x * (x * (y * (y * x))) \leq y * (y * x)$. Thus, by Lemma 1(ii), $\mu(x) \geq \min\{\mu(x * (y * (y * x))), \mu(y * (y * x))\}$. Combining (1) and (2), we get $\mu(x) \geq \min\{\mu(x * (y * (y * x))), \mu(y * (y * x))\} \geq \mu(x * (y * x))$. Since $x * (y * x) \leq x$, it follows that by Lemma 1(i), $\mu(x) \leq \mu(x * (y * x))$. Thus, $\mu(x) = \mu(x * (y * x))$ for all $x, y \in X$. Therefore, by Theorem 8, μ is a fuzzy implicative KS-ideal of X. □

4. Quotient KS-semigroups induced by Fuzzy KS-ideals

Recall that in [5], if A is an ideal of a KS-semigroup X, then the relation " $\sim_A$ " on X defined by $x \sim_A y$ if and only if $x * y \in A$ and $y * x \in A$ is a congruence relation on X. Denote A_x as the equivalence class containing $x \in X$ and X/A as the set of all equivalence classes of X with respect to " $\sim_A$ ", that is, $A_x = \{y \in X : x \sim_A y\}$ and $X/A = \{A_x : x \in X\}$. Furthermore, $(X/A, \circledast, \odot, A_0)$ is a KS-semigroup under the binary operation $A_x \circledast A_y = A_{x*y}$ and $A_x \odot A_y = A_{xy}$ for all $A_x, A_y \in X/A$.

In this section, we present the construction of quotient KS-semigroups via fuzzy KS-ideals.

Definition 13. *Let μ be a non-zero fuzzy KS-ideal of a KS-semigroup X. Define a binary relation $\sim_\mu$ on X by $x \sim_\mu y$ if and only if $\mu(x * y) > 0$ and $\mu(y * x) > 0$.*

We will prove that $\sim_\mu$ is a congruence relation on a KS-semigroup X.

Proposition 3. *$\sim_\mu$ is an equivalence relation on a KS-semigroup X.*

Proof. Let μ be a non-zero fuzzy KS-ideal of a KS-semigroup X. Let $x \in X$. Since μ is non-zero, there exists $z \in X$ such that $\mu(z) \neq 0$, that is, $\mu(z) > 0$. Then by (F1), $\mu(0) \geq \mu(z) > 0$. Thus by Definition 1, $\mu(x * x) = \mu(0) > 0$. Hence, $x \sim_\mu x$, which means $\sim_\mu$ is reflexive. Suppose $x \sim_\mu y$. Then $\mu(x * y) > 0$ and $\mu(y * x) > 0$, or equivalently, $\mu(y * x) > 0$ and $\mu(x * y) > 0$. Thus, $y \sim_\mu x$. Hence, $\sim_\mu$ is symmetric. Now, suppose $x \sim_\mu y$ and $y \sim_\mu z$. Then $\mu(x * y) > 0$, $\mu(y * x) > 0$, $\mu(y * z) > 0$ and $\mu(z * y) > 0$. Since X is a KS-semigroup, $(x * z) * (x * y) \leq y * z$ and $(z * x) * (z * y) \leq y * x$ for all $x, y, z \in X$. Thus, by Lemma 1(ii), we have

$$\mu(x * z) \geq \min\{\mu(x * y), \mu(y * z)\} > 0 \text{ and } \mu(z * x) \geq \min\{\mu(y * x), \mu(z * y)\} > 0.$$

Thus, $x \sim_\mu z$. Hence, $\sim_\mu$ is transitive. Therefore, $\sim_\mu$ is an equivalence relation on X. □

The following example shows that if a non-zero fuzzy KS-ideal μ of a KS-semigroup X does not satisfy condition (C): $\mu(xy) \geq \mu(x)$ and $\mu(xy) \geq \mu(y)$ for all $x, y \in X$, then $\sim_\mu$ is not compatible.

Example 6. *Consider the KS-semigroup $X = \{0, a, b, c\}$ in Example 1. Define a fuzzy set μ in X by $\mu(0) = \mu(a) = \mu(b) = 0.8$ and $\mu(c) = 0$. Then μ is a fuzzy KS-ideal of X. Note that $\mu(ca) = \mu(c) = \min\{\mu(c), \mu(a)\} < \mu(a)$. Thus, μ does not satisfy condition (C). Observe that $\mu(a * b) = \mu(a) = 0.8 > 0$ and $\mu(b * a) = \mu(b) = 0.8 > 0$. Thus, $a \sim_\mu b$. However, $\sim_\mu$ is not compatible since $\mu(ca * cb) = \mu(c * 0) = \mu(c) = 0$, that is, $ca \nsim_\mu cb$.*

Proposition 4. *Let X be a KS-semigroup and μ a non-zero fuzzy KS-ideal of X satisfying condition (C). Then $\sim_\mu$ is both left and right compatible.*

Proof. Let X be a KS-semigroup and μ a non-zero fuzzy KS-ideal of X satisfying condition (C). Let $x, y, z \in X$ such that $x \sim_\mu y$. Then $\mu(x*y) > 0$ and $\mu(y*x) > 0$. Since $(z * x) * (z * y) \leq y * x$ and $(z * y) * (z * x) \leq x * y$, we have

$$\mu((z * x) * (z * y)) \geq \mu(y * x) > 0 \text{ and } \mu((z * y) * (z * x)) \geq \mu(x * y) > 0$$

by Lemma 1(i). Thus, $z * x \sim_\mu z * y$. Now, since $zx * zy = z(x * y)$, we have $\mu(zx * zy) = \mu(z(x * y))$. Thus by condition (C), $\mu(zx * zy) = \mu(z(x * y)) \geq \mu(x * y) > 0$. Similarly, $\mu(zy * zx) > 0$. Hence, $zx \sim_\mu zy$. By Definition 8(i), $\sim_\mu$ is left compatible.

Now, since $(x * z) * (y * z) \leq x * y$ and $(y * z) * (x * z) \leq y * x$, we have

$$\mu((x * z) * (y * z)) \geq \mu(x * y) > 0 \text{ and } \mu((y * z) * (x * z)) > \mu(y * x) > 0$$

by Lemma 1(i). Thus, $x * z \sim_\mu y * z$. Moreover, since $xz * yz = (x * y)z$, we have $\mu(xz * yz) = \mu((x * y)z)$. Thus by condition (C), $\mu(xz * yz) = \mu((x * y)z) \geq \mu(x * y) > 0$. Similarly, $\mu(yz*xz) > 0$. Hence, $xz \sim_\mu yz$. By Definition 8(i), $\sim_\mu$ is right compatible. □

The following corollary follows from Propositions 3 and 4 and Theorem 1.

Corollary 1. *Let X be a KS-semigroup and μ a non-zero fuzzy KS-ideal of X satisfying condition (C). Then $\sim_\mu$ is a congruence relation on X.*

Let X be a KS-semigroup and μ a non-zero fuzzy KS-ideal of X satisfying condition (C). Denote μ_x as the equivalence class containing $x \in X$ and X/μ as the set of all equivalence classes of X with respect to " $\sim_\mu$ ", that is, $\mu_x = \{y \in X : y \sim_\mu x\}$ and $X/\mu = \{\mu_x : x \in X\}$.

Remark 3. *$\mu_x = \mu_y$ if and only if $x \sim_\mu y$.*

Theorem 11. *Let X be a KS-semigroup and μ a non-zero fuzzy KS-ideal of X satisfying condition (C). Then $(X/\mu, \circledast, \odot, \mu_0)$ is a KS-semigroup under the binary operations*

$$\mu_x \circledast \mu_y = \mu_{x*y} \text{ and } \mu_x \odot \mu_y = \mu_{xy}$$

for all $\mu_x, \mu_y \in X/\mu$.

Proof. Let X be a KS-semigroup and μ a non-zero fuzzy KS-ideal of X satisfying condition (C). By Corollary 1, the operations $\circledast$ and $\odot$ are well-defined. For every $\mu_x, \mu_y, \mu_z \in X/\mu$,

(i) $((\mu_x \circledast \mu_y) \circledast (\mu_x \circledast \mu_z)) \circledast (\mu_z \circledast \mu_y) = \mu_{((x*y)*(x*z))*(z*y)} = \mu_0$;

(ii) $(\mu_x \circledast (\mu_x \circledast \mu_y)) \circledast \mu_y = \mu_{(x*(x*y))*y} = \mu_0$;

(iii) $\mu_x \circledast \mu_x = \mu_{x*x} = \mu_0$;

(iv) $\mu_0 \circledast \mu_x = \mu_{0*x} = \mu_0$.

(v) Suppose $\mu_x \circledast \mu_y = \mu_0$ and $\mu_y \circledast \mu_x = \mu_0$. Then $\mu_{x*y} = \mu_{y*x} = \mu_0$. This implies that $x * y \sim_\mu 0$ and $y * x \sim_\mu 0$. Thus, $\mu(x * y) = \mu((x * y) * 0) > 0$ and $\mu(y * x) = \mu((y * x) * 0) > 0$, that is, $x \sim_\mu y$. Hence, $\mu_x = \mu_y$.

Therefore, $(X/\mu, \circledast, \mu_0)$ is a BCK-algebra. Moreover,

$$(\mu_x \odot \mu_y) \odot \mu_z = \mu_{xy} \odot \mu_z = \mu_{(xy)z} = \mu_{x(yz)} = \mu_x \odot \mu_{yz} = \mu_x \odot (\mu_y \odot \mu_z).$$

Thus, $(X/\mu, \odot)$ is a semigroup. Now, observe that

$$\mu_x \odot (\mu_y \circledast \mu_z) = \mu_x \odot \mu_{y*z} = \mu_{x(y*z)} = \mu_{xy*xz} = \mu_{xy} \circledast \mu_{xz} = (\mu_x \odot \mu_y) \circledast (\mu_x \odot \mu_z)$$

and

$$(\mu_x \circledast \mu_y) \odot \mu_z = \mu_{x*y} \odot \mu_z = \mu_{(x*y)z} = \mu_{xz*yz} = \mu_{xz} \circledast \mu_{yz} = (\mu_x \odot \mu_z) \circledast (\mu_y \odot \mu_z).$$

Therefore, $(X/\mu, \circledast, \odot, \mu_0)$ is a KS-semigroup. □

The algebraic system $(X/\mu, \circledast, \odot, \mu_0)$ is called the ***quotient KS-semigroup induced by a fuzzy KS-ideal*** μ.

The following theorem tells us that the condition (C) is necessary to extend the notion of ideal of KS-semigroup.

Theorem 12. *Let I be a nonempty subset of a KS-semigroup X. Then χ_I is a fuzzy KS-ideal of X satisfying condition* (C) *if and only if I is an ideal of X.*

Proof. Let I be a nonempty subset of a KS-semigroup X. Suppose χ_I is a fuzzy KS-ideal of X satisfying condition (C). Let $x \in X$ and $a \in I$. Then by condition (C), $\chi_I(xa) \geq \chi_I(a) = 1$. Thus, $xa \in I$. Similarly, $ax \in I$. Hence, I is stable. Now, let $x, y \in X$ such that $x * y \in I$ and $y \in I$. Then $\chi_I(x * y) = \chi_I(y) = 1$. Since χ_I is a fuzzy KS-ideal in X, $\chi_I(x) \geq \min\{\chi_I(x * y), \chi_I(y)\} = 1$. Thus, $\chi_I(x) = 1$, that is, $x \in I$. Hence, I is an ideal of X.

Conversely, suppose I is an ideal of X. Then by Remark 2(ii), $0 \in I$. Thus, $\chi_I(0) = 1 \geq \chi_I(x)$ for all $x \in X$. Now, let $x, y \in X$. We consider the following cases.

Case 1: $x * y \in I$ and $y \in I$. Since I is an ideal of X, $x \in I$. Thus,

$$\chi_I(x) = 1 \geq \min\{\chi_I(x * y), \chi_I(y)\}.$$

Case 2: $x * y \in I$ and $y \notin I$. Then $\chi_I(y) = 0$. Thus,

$$\chi_I(x) \geq 0 = \min\{\chi_I(x * y), \chi_I(y)\}.$$

Case 3: $x * y \notin I$ and $y \in I$. Then $\chi_I(x * y) = 0$. Thus,

$$\chi_I(x) \geq 0 = \min\{\chi_I(x * y), \chi_I(y)\}.$$

Case 4: $x * y \notin I$ and $y \notin I$. Then $\chi_I(x * y) = \chi_I(y) = 0$. Thus,

$$\chi_I(x) \geq 0 = \min\{\chi_I(x * y), \chi_I(y)\}.$$

In either case, $\chi_I(x) \geq \min\{\chi_I(x*y), \chi_I(y)\}$. Now, let $x, a \in X$. We consider the following cases.

Case 1: $x, a \in I$. Since I is an ideal of X, $xa, ax \in I$. Thus,

$$\chi_I(xa) = 1 \geq \min\{\chi_I(x), \chi_I(a)\} \text{ and } \chi_I(ax) = 1 \geq \min\{\chi_I(x), \chi_I(a)\}.$$

Case 2: $x \notin I$ or $a \notin I$. If $xa \in I$, then $\chi_I(xa) = 1$. Thus, $\chi_I(xa) = 1 \geq \min\{\chi_I(x), \chi_I(a)\}$. If $xa \notin I$, then $x, a \notin I$. Thus, $\chi_I(xa) = 0$ and $\chi_I(x) = 0 = \chi_I(a)$. Hence,

$$\chi_I(xa) = 0 \geq 0 = \min\{\chi_I(x), \chi_I(a)\}.$$

Similarly, $\chi_I(ax) \geq \min\{\chi_I(x), \chi_I(a)\}$.

Thus, χ_I is a fuzzy KS-ideal of X . Moreover, by observing Case 1 and Case 2 above, χ_I satisfies condition (C). □

The following proposition tells us that the equivalence class containing any element of a KS-semigroup X are equal with respect to $\sim_I$ and $\sim_{\chi_I}$ for any ideal I of X.

Proposition 5. *Let I be an ideal of a KS-semigroup X. Then $x \sim_I y$ if and only if $x \sim_{\chi_I} y$.*

Proof. Let I be an ideal of a KS-semigroup X. Then for all $x, y \in X$,

$$\begin{aligned} x \sim_I y &\Leftrightarrow x * y \in I \text{ and } y * x \in I \\ &\Leftrightarrow \chi_I(x * y) = 1 \text{ and } \chi_I(y * x) = 1 \\ &\Leftrightarrow \chi_I(x * y) > 0 \text{ and } \chi_I(y * x) > 0 \\ &\Leftrightarrow x \sim_{\chi_I} y. \quad \square \end{aligned}$$

Let I be an ideal of a KS-semigroup X. Then for all $x \in X$, $I_x = (\chi_I)_x$. Hence, $X/I = X/\chi_I$. Therefore, if μ is a non-zero fuzzy KS-ideal of X satisfying condition (C), the quotient KS-semigroup X/μ is a **generalization** of the quotient KS-semigroup X/I.

Theorem 13. *Let X be a KS-semigroup and μ a non-zero fuzzy KS-ideal of X satisfying condition* (C)*. If J is an ideal of X, then $J/\mu = \{\mu_j : j \in J\}$ is an ideal of X/μ.*

Proof. Let μ be a non-zero fuzzy KS-ideal in a KS-semigroup X satisfying condition (C) and J an ideal of X. Since $J \subseteq X$, it follows that $J/\mu \subseteq X/\mu$. Now, let $\mu_a \in J/\mu$ and $\mu_x \in X/\mu$. Then $a \in J$ and $x \in X$. Since J is an ideal of X, $ax \in J$. Thus, $\mu_a\mu_x = \mu_{ax} \in J/\mu$. Similarly, $\mu_x\mu_a \in J/\mu$. For any $\mu_x, \mu_y \in X/\mu$, suppose $\mu_{x*y} = \mu_x * \mu_y \in J/\mu$ and $\mu_y \in J/\mu$. Then $x * y \in J$ and $y \in J$. Since J is an ideal of X, $x \in J$. Thus, $\mu_x \in J/\mu$. Therefore, J/μ is an ideal of X/μ. □

Theorem 14. *Let X be a KS-semigroup and μ a non-zero fuzzy KS-ideal of X satisfying condition* (C)*. If J^* is an ideal of X/μ, then there exists an ideal $J = \bigcup\{x \in X : \mu_x \in J^*\}$ of X such that $J/\mu = J^*$.*

Definition 14. [5] *Let X and Y be KS-semigroups and $f : X \to Y$ be a mapping. Then f is called a KS-semigroup homomorphism* (*briefly, homomorphism*) *if for all $x, y \in X$, $f(x * y) = f(x) * f(y)$ and $f(xy) = f(x)f(y)$. If f is a one-to-one homomorphism, f is said to be a monomorphism. If f is an onto homomorphism, f is called an epimorphism. If f is a bijective homomorphism, f is called an isomorphism. In this case, X and Y are said to be isomorphic* (*written $X \cong Y$*)*.*

Theorem 15. *Let μ be a non-zero fuzzy KS-ideal of a KS-semigroup X satisfying condition* (C)*. If J is an ideal of X, then $\dfrac{X/\mu}{J/\mu} \cong X/J$.*

Proof. Let μ be a non-zero fuzzy KS-ideal of a KS-semigroup X satisfying condition (C) and J an ideal of X. By Theorem 13, J/μ is an ideal of X/μ, $\dfrac{X/\mu}{J/\mu} = \{(J/\mu)_{\mu_x} : \mu_x \in X/\mu\}$ and $X/J = \{J_x : x \in X\}$.

Consider a map $\varphi : \dfrac{X/\mu}{J/\mu} \to X/J$ defined by $\varphi((J/\mu)_{\mu_x}) = J_x$. Let $(J/\mu)_{\mu_x}, (J/\mu)_{\mu_y} \in \dfrac{X/\mu}{J/\mu}$ such that $(J/\mu)_{\mu_x} = (J/\mu)_{\mu_y}$. Then $\mu_x \sim_{J/\mu} \mu_y$. Thus, $\mu_{x*y} = \mu_x * \mu_y \in J/\mu$, $\mu_{y*x} = \mu_y * \mu_x \in J/\mu$. This means that $x * y \in J$, $y * x \in J$, that is, $x \sim_J y$ and $J_x = J_y$. Thus, $\varphi((J/\mu)_{\mu_x}) = J_x = J_y = \varphi((J/\mu)_{\mu_y})$. Hence, φ is well-defined.

Now, let $(J/\mu)_{\mu_x}, (J/\mu)_{\mu_y} \in \dfrac{X/\mu}{J/\mu}$. Then

$$\begin{aligned}
\varphi((J/\mu)_{\mu_x} * (J/\mu)_{\mu_y}) &= \varphi((J/\mu)_{\mu_x * \mu_y}) \\
&= \varphi((J/\mu)_{\mu_{x*y}}) \\
&= J_{x*y} \\
&= J_x * J_y \\
&= \varphi((J/\mu)_{\mu_x}) * \varphi((J/\mu)_{\mu_y})
\end{aligned}$$

and

$$\varphi((J/\mu)_{\mu_x}(J/\mu)_{\mu_y}) = \varphi((J/\mu)_{\mu_x\mu_y})$$

$$
\begin{aligned}
&= \varphi((J/\mu)_{\mu_{xy}}) \\
&= J_{xy} \\
&= J_x J_y \\
&= \varphi((J/\mu)_{\mu_x})\varphi((J/\mu)_{\mu_y}).
\end{aligned}
$$

Thus, φ is a KS-semigroup homomorphism. For each $J_x \in X/J$, there exists $(J/\mu)_{\mu_x}$ such that $\varphi((J/\mu)_{\mu_x}) = J_x$. Thus, φ is onto.

Now, suppose $\varphi((J/\mu)_{\mu_x}) = \varphi((J/\mu)_{\mu_y})$. Then $J_x = J_y$. Thus, $x \sim_J y$. Let $\mu_\alpha \in (J/\mu)_{\mu_x}$. Then $\mu_\alpha \sim_{J/\mu} \mu_x$. It follows that $\mu_{\alpha * x} = \mu_\alpha * \mu_x \in J/\mu$ and $\mu_{x*\alpha} = \mu_x * \mu_\alpha \in J/\mu$. Thus, $\alpha * x \in J$ and $x * \alpha \in J$, that is, $\alpha \sim_J x$. Since $\sim_J$ is an equivalence relation, $\alpha \sim_J y$. Thus, $\mu_\alpha \in (J/\mu)_{\mu_y}$. Hence, $(J/\mu)_{\mu_x} \subseteq (J/\mu)_{\mu_y}$. Similarly, $(J/\mu)_{\mu_y} \subseteq (J/\mu)_{\mu_x}$. Thus, $(J/\mu)_{\mu_x} = (J/\mu)_{\mu_y}$. Hence, φ is one-to-one. Therefore, φ is an isomorphism, that is, $\dfrac{X/\mu}{J/\mu} \cong X/J$. □

The following are the properties of the generalized quotient via fuzzy KS-ideals with respect to fuzzy KS-p-ideal, fuzzy commutative KS-ideal and fuzzy implicative KS-ideal.

Theorem 16. *Let μ be a non-zero fuzzy commutative KS-ideal of a KS-semigroup X satisfying condition* (C). *Then X/μ is a commutative KS-semigroup.*

Proof. Let μ be a non-zero fuzzy commutative KS-ideal of a KS-semigroup X satisfying condition (C). For all $x, y \in X$, denote $u = y * (y * x)$. Then $u * x = (y * (y * x)) * x = (y*x)*(y*x) = 0$. Thus, by Theorem 5, $\mu(u*(x*(x*u))) = \mu(u*x) = \mu(0) > 0$. Moreover, since $(x * (x * u)) * u = (x * u) * (x * u) = 0$, it follows that $\mu((x * (x * u)) * u) = \mu(0) > 0$. Thus, $y * (y * x) \sim_\mu x * (x * (y * (y * x)))$. Hence,

$$\mu_y \circledast (\mu_y \circledast \mu_x) = \mu_x \circledast (\mu_x \circledast (\mu_y \circledast (\mu_y \circledast \mu_x))).$$

Therefore, by Proposition 2, X/μ is a commutative KS-semigroup. □

Theorem 17. *Let μ be a non-zero fuzzy KS-p-ideal of a KS-semigroup X satisfying condition* (C). *Then X/μ is a positive implicative KS-semigroup.*

Proof. Let μ be a non-zero fuzzy KS-p-ideal of a KS-semigroup X satisfying condition (C). Let $x, y \in X$ and denote $u = x * ((x * y) * y)$. Then $(u * y) * y = 0$. Thus, $\mu((u*y)*y) = \mu(0) > 0$ since μ is non-constant. By Theorem 3, $\mu(u*y) = \mu((u*y)*y) > 0$. Thus, $\mu((x * y) * ((x * y) * y)) = \mu(u * y) > 0$. Moreover, since $((x * y) * y) * (x * y) = 0$, $\mu(((x * y) * y) * (x * y)) = \mu(0) > 0$. This implies that $x * y \sim_\mu (x * y) * y$. Hence,

$$\mu_x \circledast \mu_y = \mu_{x*y} = \mu_{(x*y)*y} = (\mu_x \circledast \mu_y) \circledast \mu_y.$$

Therefore, X/μ is a positive implicative KS-semigroup. □

Theorem 18. *Let μ be a non-zero fuzzy implicative KS-ideal of a KS-semigroup X satisfying condition* (C)*. Then X/μ is a implicative KS-semigroup.*

Proof. Let μ be a non-zero fuzzy implicative KS-ideal of a KS-semigroup X satisfying condition (C). Then by Theorem 10, μ is both a non-zero fuzzy commutative KS-ideal and fuzzy KS-p-ideal satisfying condition (C). Thus, by Theorems 16 and 17, X/μ is both commutative and positive implicative KS-semigroup. Therefore, by Definition 5(iii), X/μ is implicative. □

5. Homomorphic Properties of the Generalized Quotient via fuzzy KS-ideals

In this section, we investigate some results on homomorphism of KS-semigroup.

Lemma 2. *If μ is a non-zero fuzzy KS-ideal of a KS-semigroup X satisfying condition* (C)*, then the set $X_\mu = \{x \in X : \mu(x) > 0\}$ is an ideal of X.*

Proof. Let μ be a non-zero fuzzy KS-ideal of a KS-semigroup X satisfying condition (C) and let $X_\mu = \{x \in X : \mu(x) > 0\}$. Let $x \in X$ and $a \in X_\mu$. Then $\mu(a) > 0$. Thus, $\mu(xa) \geq \mu(a) > 0$ and $\mu(ax) \geq \mu(a) > 0$. This implies that $xa, ax \in X_\mu$. Thus, X_μ is stable.

Now, suppose $x * y \in X_\mu$ and $y \in X_\mu$. Then $\mu(x * y) = \mu(y) > 0$. Since μ is a fuzzy KS-ideal of X, $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} > 0$. Thus, $x \in X_\mu$. Therefore, X_μ is an ideal of X. □

Definition 15. [5] *Let $f : X \to Y$ be a homomorphism of KS-semigroup. Then the kernel of f is denoted and defined as* $\ker f = \{x \in X : f(x) = 0\}$.

Theorem 19. *Let μ be a non-zero fuzzy KS-ideal of a KS-semigroup X satisfying condition* (C)*. Then the mapping $\gamma : X \to X/\mu$ given by $\gamma(x) = \mu_x$ is an epimorphism with* $\ker \gamma = X_\mu$.

Recall that in [7], if $f : X \to Y$ is a mapping of KS-semigroup and ν is a fuzzy set on Y, then the fuzzy set $\nu^f = \nu \circ f$ on X is said to be the *preimage of ν under f*. If f is a homomorphism and ν is a fuzzy KS-ideal of Y, then ν^f is a fuzzy KS-ideal of X. Furthermore, if f is an epimorphism and ν^f is a fuzzy KS-ideal of X, then ν is a fuzzy KS-ideal of Y.

Lemma 3. *Let $f : X \to Y$ be a KS-semigroup homomorphism. If μ is a non-zero fuzzy KS-ideal of Y satisfying condition* (C)*, then μ^f is a non-zero fuzzy KS-ideal of X satisfying condition* (C)*.*

Proof. Let $f : X \to Y$ be a KS-semigroup homomorphism and μ a non-zero fuzzy KS-ideal of Y satisfying condition (C). Then μ^f is a fuzzy KS-ideal of X. Now, let $x, y \in X$. Then,

$$\mu^f(xy) = \mu(f(xy)) = \mu(f(x)f(y)) \geq \mu(f(x)) = \mu^f(x)$$

and

$$\mu^f(xy) = \mu(f(xy)) = \mu(f(x)f(y)) \geq \mu(f(y)) = \mu^f(y). \quad \square$$

Theorem 20. *Let $f : X \to Y$ be an epimorphism and μ be a non-zero fuzzy KS-ideal of Y satisfying condition* (C). *Then $X/\mu^f \cong Y/\mu$.*

Proof. Let $f : X \to Y$ be an epimorphism and μ be a non-zero fuzzy KS-ideal of Y satisfying condition (C). Then by Lemma 3, μ^f is a non-zero fuzzy KS-ideal of X satisfying condition (C). By Theorem 11, X/μ^f and Y/μ are KS-semigroups. Consider a mapping $\varphi : X/\mu^f \to Y/\mu$ defined by $\varphi(\mu_x^f) = \mu_{f(x)}$ for each $x \in X$. Let $\mu_x^f, \mu_y^f \in X/\mu^f$ such that $\mu_x^f = \mu_y^f$. Then $x \sim_{\mu^f} y$, that is, $\mu^f(x * y) = \mu^f(0)$ and $\mu^f(y * x) = \mu^f(0)$. Since f is a homomorphism, $\mu(f(x) * f(y)) = \mu(f(x * y)) = \mu^f(x * y) = \mu^f(0)$ and $\mu(f(y)*f(x)) = \mu(f(y*x)) = \mu^f(y*x) = \mu^f(0)$. Thus, $f(x) \sim_\mu f(y)$, that is, $\mu_{f(x)} = \mu_{f(y)}$. Hence, φ is well-defined. Now, let $\mu_x^f, \mu_y^f \in X/\mu^f$. Then

$$\varphi(\mu_x^f * \mu_y^f) = \varphi(\mu_{x*y}^f) = \mu_{f(x*y)} = \mu_{f(x)*f(y)} = \mu_{f(x)} * \mu_{f(y)} = \varphi(\mu_x^f) * \varphi(\mu_y^f)$$

and

$$\varphi(\mu_x^f \mu_y^f) = \varphi(\mu_{xy}^f) = \mu_{f(xy)} = \mu_{f(x)f(y)} = \mu_{f(x)}\mu_{f(y)} = \varphi(\mu_x^f)\varphi(\mu_y^f).$$

Thus, by Definition 14(i), φ is a homomorphism.

Now, since f is an epimorphism, for each $y \in Y$, there exists $x \in X$ such that $y = f(x)$. Thus, $\varphi(\mu_x^f) = \mu_{f(x)} = \mu_y$. Hence φ is onto. Suppose $\varphi(\mu_x^f) = \varphi(\mu_y^f)$ for $\mu_x^f, \mu_y^f \in X/\mu^f$. Then

$$\begin{aligned}
\mu_{f(x)} = \mu_{f(y)} &\Rightarrow f(x) \sim_\mu f(y) \\
&\Rightarrow \mu(f(x) * f(y)) > 0 \text{ and } \mu(f(y) * f(x)) > 0 \\
&\Rightarrow \mu(f(x * y)) > 0 \text{ and } \mu(f(y * x)) > 0 \\
&\Rightarrow \mu^f(x * y) > 0 \text{ and } \mu^f(y * x) > 0 \\
&\Rightarrow x \sim_{\mu^f} y \\
&\Rightarrow \mu_x^f = \mu_y^f.
\end{aligned}$$

Thus, φ is one-to-one. Hence, φ is an isomorphism. Therefore, $X/\mu^f \cong Y/\mu$. $\square$

6. Quotient Structure of Product KS-semigroup via fuzzy KS-ideals

In this section, we investigate the quotient structure of product KS-semigroup induced by fuzzy KS-ideals.

Definition 16. *Let $(X, *_X, \cdot_X, 0_X)$ and $(Y, *_Y, \cdot_Y, 0_Y)$ be KS-semigroups. Define the operations $*$ and $\cdot$ in $X \times Y$ by*

$$(x_1, y_1) * (x_2, y_2) = (x_1 *_X x_2, y_1 *_Y y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot_X x_2, y_1 \cdot_Y y_2).$$

Then $(X \times Y, *, \cdot, 0)$ is also KS-semigroup, where $0 = (0_X, 0_Y)$ and it is called product KS-semigroup.

Definition 17. [4] *Let μ and ν be fuzzy sets on the sets X and Y, respectively. The Cartesian product $\mu \times \nu : X \times Y \to [0, 1]$ is defined by $\mu \times \nu(x, y) = \min\{\mu(x), \nu(y)\}$ for all $(x, y) \in X \times Y$.*

In [7], if μ and ν are fuzzy KS-ideals of a KS-semigroup X, then $\mu \times \nu$ is also a fuzzy KS-ideal in $X \times X$. The following result is a general case.

Proposition 6. *Let μ and ν be fuzzy KS-ideals of KS-semigroups X and Y, respectively. Then $\mu \times \nu$ is also a fuzzy KS-ideal of $X \times Y$.*

Proof. Let μ and ν be fuzzy KS-ideals of KS-semigroups X and Y, respectively.

(i) For any $(x, y) \in X \times Y$,

$$\mu \times \nu(0_X, 0_Y) = \min\{\mu(0_X), \nu(0_Y)\} \geq \min\{\mu(x), \mu(y)\} = \mu \times \nu(x, y).$$

(ii) Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then

$$\begin{aligned}
\mu \times \nu(x_1, x_2) &= \min\{\mu(x_1), \nu(x_2)\} \\
&\geq \min\{\min\{\mu(x_1 *_X y_1), \mu(y_1)\}, \min\{\nu(x_2 *_Y y_2), \nu(y_2)\}\} \\
&= \min\{\min\{\mu(x_1 *_X y_1), \nu(x_2 *_Y y_2)\}, \min\{\mu(y_1), \nu(y_2)\}\} \\
&= \min\{\mu \times \nu(x_1 *_X y_1, x_2 *_Y y_2), \mu \times \nu(y_1, y_2)\} \\
&= \min\{\mu \times \nu((x_1, x_2) * (y_1, y_2)), \mu \times \nu(y_1, y_2)\}.
\end{aligned}$$

(iii) Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then

$$\begin{aligned}
\mu \times \nu((x_1, x_2)(y_1, y_2)) &= \mu \times \nu(x_1 \cdot_X y_1, x_2 \cdot_Y y_2) \\
&= \min\{\mu(x_1 \cdot_X y_1), \nu(x_2 \cdot_Y y_2)\} \\
&\geq \min\{\min\{\mu(x_1), \mu(y_1)\}, \min\{\nu(x_2), \nu(y_2)\}\} \\
&= \min\{\min\{\mu(x_1), \nu(x_2)\}, \min\{\mu(y_1), \nu(y_2)\}\} \\
&= \min\{\mu \times \nu(x_1, x_2), \mu \times \nu(y_1, y_2)\}.
\end{aligned}$$

Therefore, $\mu \times \nu$ is a fuzzy KS-ideal of $X \times Y$. □

Corollary 2. *Let μ and ν be non-zero fuzzy KS-ideals of KS-semigroups X and Y, respectively, both satisfying condition* (C). *Then $\mu \times \nu$ is a non-zero fuzzy KS-ideal of $X \times Y$ satisfying condition* (C).

Proof. Let μ and ν be non-zero fuzzy KS-ideals of KS-semigroups X and Y, respectively, both satisfying condition (C). Then there exist $x \in X$ and $y \in Y$ such that $\mu(x) \neq 0$ and $\nu(y) \neq 0$. Thus, $\mu \times \nu(x, y) = \min\{\mu(x), \nu(y)\} \neq 0$. Thus, $\mu \times \nu$ is non-zero. By Proposition 6, $\mu \times \nu$ is a fuzzy KS-ideal of $X \times Y$. Moreover, since μ and ν satisfy condition (C), for all $(x_1, x_2), (y_1, y_2) \in X \times Y$,

$$\begin{aligned} \mu \times \nu((x_1, x_2)(y_1, y_2)) &= \mu \times \nu(x_1 \cdot_X y_1, x_2 \cdot_Y y_2) \\ &= \min\{\mu(x_1 \cdot_X y_1), \nu(x_2 \cdot_Y y_2)\} \\ &\geq \min\{\mu(x_1), \nu(x_2)\} \\ &= \mu \times \nu(x_1, x_2) \end{aligned}$$

and

$$\begin{aligned} \mu \times \nu((x_1, x_2)(y_1, y_2)) &= \mu \times \nu(x_1 \cdot_X y_1, x_2 \cdot_Y y_2) \\ &= \min\{\mu(x_1 \cdot_X y_1), \nu(x_2 \cdot_Y y_2)\} \\ &\geq \min\{\mu(y_1), \nu(y_2)\} \\ &= \mu \times \nu(y_1, y_2). \end{aligned}$$

Therefore, $\mu \times \nu$ satisfies condition (C). □

The following corollary follows from Theorems 16, 17 and 18.

Corollary 3. *Let μ and ν be fuzzy KS-ideals of KS-semigroups X and Y, respectively.*

(i) *If $\mu \times \nu$ is a non-zero fuzzy commutative KS-ideal of $X \times Y$ satisfying condition* (C), *then $X \times Y/\mu \times \nu$ is a commutative KS-semigroup.*

(ii) *If $\mu \times \nu$ is a non-zero fuzzy KS-p-ideal of $X \times Y$ satisfying condition* (C), *then $X \times Y/\mu \times \nu$ is a positive implicative KS-semigroup.*

(iii) *If $\mu \times \nu$ is a non-zero fuzzy implicative KS-ideal of $X \times Y$ satisfying condition* (C), *then $X \times Y/\mu \times \nu$ is an implicative KS-semigroup.*

Recall that in [5], if $f : X \to Y$ is an epimorphism of KS-semigroup, then $X/\ker f \cong Y$.

Theorem 21. *Let μ and ν be non-zero fuzzy KS-ideals of KS-semigroups X and Y, respectively, both satisfying condition* (C). *Then*

$$\frac{X \times Y}{\mu \times \nu} \cong X/\mu \times Y/\nu.$$

Proof. Let μ and ν be non-zero fuzzy KS-ideals of KS-semigroups X and Y, respectively, both satisfying condition (C). Define a map $\psi : X \times Y \to X/\mu \times Y/\nu$ by

$\psi(x,y) = (\mu_x, \nu_y)$. Let $(x_1,y_1),(x_2,y_2) \in X \times Y$ such that $(x_1,y_1) = (x_2,y_2)$. Then $x_1 = x_2$ and $y_1 = y_2$. Thus,

$$\psi(x_1,y_1) = (\mu_{x_1}, \nu_{y_1}) = (\mu_{x_2}, \nu_{y_2}) = \psi(x_2,y_2).$$

Thus, ψ is well-defined.

Now, let $(x_1,y_1),(x_2,y_2) \in X \times Y$. Then

$$\begin{aligned}\psi((x_1,y_1) * (x_2,y_2)) &= \psi(x_1 *_X x_2, y_1 *_Y y_2)\\ &= (\mu_{x_1 *_X x_2}, \nu_{y_1 *_Y y_2})\\ &= (\mu_{x_1} *_X \mu_{x_2}, \nu_{y_1} *_Y \nu_{y_2})\\ &= (\mu_{x_1}, \nu_{y_1}) * (\mu_{x_2}, \nu_{y_2})\\ &= \psi(x_1,y_1) * \psi(x_2,y_2)\end{aligned}$$

and

$$\begin{aligned}\psi((x_1,y_1)(x_2,y_2)) &= \psi(x_1 \cdot_X x_2, y_1 \cdot_Y y_2)\\ &= (\mu_{x_1 \cdot_X x_2}, \nu_{y_1 \cdot_Y y_2})\\ &= (\mu_{x_1} \cdot_X \mu_{x_2}, \nu_{y_1} \cdot_Y \nu_{y_2})\\ &= (\mu_{x_1}, \nu_{y_1})(\mu_{x_2}, \nu_{y_2})\\ &= \psi(x_1,y_1)\psi(x_2,y_2).\end{aligned}$$

Thus, ψ is a homomorphism. Moreover, for each $(\mu_x, \nu_y) \in X/\mu \times Y/\nu$, there exists $(x,y) \in X \times Y$ such that $\psi(x,y) = (\mu_x, \nu_y)$. Thus, ψ is an epimorphism. By First Isomorphism Theorem, we have

$$\frac{X \times Y}{\ker \psi} \cong X/\mu \times Y/\nu.$$

Let $K = \ker \psi$. Then $\dfrac{X \times Y}{\ker \psi} = \dfrac{X \times Y}{K} = \{K_{(x,y)} : (x,y) \in X \times Y\}$. Similarly, $\dfrac{X \times Y}{\mu \times \nu} = \{\mu \times \nu_{(x,y)} : (x,y) \in X \times Y\}$.

Claim: $K_{(x,y)} \cong \mu \times \nu_{(x,y)}$.

For $(\alpha, \beta) \in K_{(x,y)}$,

$$\begin{aligned}(\alpha,\beta) \in K_{(x,y)} &\Leftrightarrow (\alpha,\beta) \sim_K (x,y)\\ &\Leftrightarrow (\alpha,\beta) * (x,y) \in K,\ (x,y) * (\alpha,\beta) \in K\\ &\Leftrightarrow \psi((\alpha,\beta) * (x,y)) = (\mu_0, \nu_0) = \psi((x,y) * (\alpha,\beta)),\\ &\qquad \text{where } \mu_0 \text{ and } \nu_0 \text{ are fixed elements in } X/\mu \text{ and } Y/\nu\text{, respectively}\\ &\Leftrightarrow \psi(\alpha * x, \beta * y) = (\mu_0, \nu_0) = \psi(x * \alpha, y * \beta)\\ &\Leftrightarrow (\mu_{\alpha * x}, \nu_{\beta * y}) = (\mu_0, \nu_0) = (\mu_{x * \alpha}, \nu_{y * \beta})\end{aligned}$$

$$\begin{aligned} &\Leftrightarrow \quad \mu_{\alpha * x} = \mu_{x*\alpha} = \mu_0,\ \nu_{\beta * y} = \nu_{y*\beta} = \nu_0 \\ &\Leftrightarrow \quad \mu_\alpha = \mu_x,\ \nu_\beta = \nu_y. \end{aligned}$$

Also, for $(\alpha, \beta) \in \mu \times \nu_{(x,y)}$,

$$\begin{aligned} (\alpha,\beta) \in \mu \times \nu_{(x,y)} \quad &\Leftrightarrow \quad (\alpha,\beta) \sim_{\mu\times\nu} (x,y) \\ &\Leftrightarrow \quad \mu \times \nu((\alpha,\beta) * (x,y)) > 0 \text{ and } \mu \times \nu((x,y) * (\alpha,\beta)) > 0 \\ &\Leftrightarrow \quad \mu \times \nu(\alpha * x, \beta * y) > 0 \text{ and } \mu \times \nu(x * \alpha, y * \beta) > 0 \\ &\Leftrightarrow \quad \min\{\mu(\alpha * x), \nu(\beta * y)\} > 0 \text{ and } \min\{\mu(x * \alpha), \nu(y * \beta)\} > 0 \\ &\Leftrightarrow \quad \mu(\alpha * x) > 0, \nu(\beta * y) > 0 \text{ and } \mu(x * \alpha) > 0, \nu(y * \beta) > 0 \\ &\Leftrightarrow \quad \mu(\alpha * x) > 0 \text{ and } \mu(x * \alpha) > 0, \quad \nu(\beta * y) > 0 \text{ and } \nu(y * \beta) > 0 \\ &\Leftrightarrow \quad \alpha \sim_\mu x, \quad \beta \sim_\nu y \\ &\Leftrightarrow \quad \mu_\alpha = \mu_\beta, \quad \nu_\beta = \nu_y. \end{aligned}$$

Thus, $(\alpha, \beta) \in K_{(x,y)} \Leftrightarrow (\alpha, \beta) \in \mu \times \nu_{(x,y)}$. This proves the claim. Hence,

$$\frac{X \times Y}{\mu \times \nu} \cong \frac{X \times Y}{\ker \psi} \cong X/\mu \times Y/\nu.$$

□

Lemma 4. *Let J and J be ideals of KS-semigroups X and Y, respectively. Then $J \times I$ is an ideal of $X \times Y$. Moreover, $X \times Y/J \times I \cong X/J \times Y/I$.*

Proof. Let J and J be ideals of KS-semigroups X and Y, respectively. Let $(x, y) \in X \times Y$ and $(a, b) \in J \times I$. Since J and I are ideals, $(x, y) \cdot (a, b) = (x \cdot_X a, y \cdot_Y b) \in J \times I$. Similarly, $(a, b) \cdot (x, y) \in J \times I$. Now, let $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $(x_1 *_X x_2, y_1 *_Y y_2) = (x_1, y_1) * (x_2, y_2) \in J \times I$ and $(x_2, y_2) \in J \times I$. Since J and I are ideals, $(x_1, y_1) \in J \times I$. Thus, $J \times I$ is an ideal of $X \times Y$.

Define $\varphi : X \times Y \to X/J \times Y/I$ by $\varphi(x, y) = (J_x, I_y)$. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $(x_1, y_1) = (x_2, y_2)$. Then $x_1 = x_2$ and $y_1 = y_2$ so that $x_1 \sim_J x_2$ and $y_1 \sim_I y_2$. Thus, $J_{x_1} = J_{x_2}$ and $I_{y_1} = I_{y_2}$, that is, $(J_{x_1}, I_{y_1}) = (J_{x_2}, I_{y_2})$. This shows that φ is well-defined. Now, let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned} \varphi((x_1, y_1) * (x_2, y_2)) \quad &= \quad \varphi(x_1 *_X x_2, y_1 *_Y y_2) \\ &= \quad (J_{x_1 *_X x_2}, I_{y_1 *_Y y_2}) \\ &= \quad (J_{x_1} *_{X/J} J_{x_2}, I_{y_1} *_{Y/I} I_{y_2}) \\ &= \quad (J_{x_1}, I_{y_1}) * (J_{x_2}, I_{y_2}) \\ &= \quad \varphi(x_1, y_1) * \varphi(x_2, y_2) \end{aligned}$$

and

$$\varphi((x_1, y_1) \cdot (x_2, y_2)) \quad = \quad \varphi(x_1 \cdot_X x_2, y_1 \cdot_Y y_2)$$

$$
\begin{aligned}
&= (J_{x_1 \cdot_X x_2}, I_{y_1 \cdot_Y y_2}) \\
&= (J_{x_1} \cdot_{X/J} J_{x_2}, I_{y_1} \cdot_{Y/I} I_{y_2}) \\
&= (J_{x_1}, I_{y_1}) \cdot (J_{x_2}, I_{y_2}) \\
&= \varphi(x_1, y_1) \cdot \varphi(x_2, y_2).
\end{aligned}
$$

Thus, φ is a homomorphism. For any $(J_x, I_y) \in X/J \times Y/I$, there exists $(x, y) \in X \times Y$ such that $\varphi(x, y) = (J_x, I_y)$. This means that φ is an epimorphism. Moreover,

$$
\begin{aligned}
\ker \varphi &= \{(x, y) \in X \times Y : \varphi(x, y) = (J_0, I_0)\} \\
&= \{(x, y) \in X \times Y : (J_x, I_y) = (J_0, I_0)\} \\
&= \{(x, y) \in X \times Y : J_x = J_0 \text{ and } I_y = I_0\} \\
&= \{(x, y) \in X \times Y : x \sim_J 0 \text{ and } \ y \sim_I 0\} \\
&= \{(x, y) \in X \times Y : x = x * 0 \in J \text{ and } y = y * 0 \in I\} \\
&= J \times I.
\end{aligned}
$$

By First Isomorphism Theorem,

$$
X \times Y/J \times I = X \times Y/\ker \varphi \cong X/J \times Y/I. \quad \square
$$

Theorem 22. *Let μ and ν be non-zero fuzzy KS-ideals of KS-semigroups X and Y, respectively, both satisfying condition* (C). *If J and I are ideals of X and Y, respectively, then*

$$
\frac{X \times Y/\mu \times \nu}{J \times I/\mu \times \nu} \cong X/J \times Y/I.
$$

Proof. Let μ and ν be non-zero fuzzy KS-ideals of KS-semigroups X and Y, respectively, both satisfying condition (C). Let J, I be ideals of X and Y, respectively. Then by Lemma 4, $J \times I$ is an ideal of $X \times Y$. By Proposition 6, $\mu \times \nu$ is a fuzzy KS-ideal in $X \times Y$, so is in $J \times I$. Thus, by Theorem 21,

$$
\frac{X \times Y}{\mu \times \nu} \cong X/\mu \times Y/\nu \ \text{ and } \ \frac{J \times I}{\mu \times \nu} \cong J/\mu \times I/\nu.
$$

Hence, by Theorem 15 and Lemma 4,

$$
\frac{X \times Y/\mu \times \nu}{J \times I/\mu \times \nu} \cong \frac{X/\mu \times Y/\nu}{J/\mu \times I/\nu} \cong \frac{X/\mu}{J/\mu} \times \frac{Y/\nu}{I/\nu} \cong X/J \times Y/I.
$$

$\square$

7. Conclusion

In this paper, we studied new types of fuzzy ideals, namely, fuzzy commutative KS-ideals and fuzzy implicative KS-ideals. We compared them with existing fuzzy KS-ideals and investigated some of their properties. We defined binary relation on a KS-semigroup using fuzzy KS-ideals with certain condition and showed that it is a congruence relation on a KS-semigroup. Using this congruence relation, we constructed a quotient KS-semigroup induced by fuzzy KS-ideals and showed that it is a generalization of the quotient KS-semigroup via ideals. We investigated the homomorphic properties of this generalized quotient and its properties with respect to fuzzy commutative KS-ideals, fuzzy implicative KS-ideals and fuzzy KS-p-ideals. Moreover, the properties of this generalized structure were examined using fuzzy KS-ideals and product KS-semigroups.

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