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Complex Intuitionistic Fuzzy Quasi-Associative Ideals of BCI-algebras

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Abstract. This paper explores the application of complex intuitionistic fuzzy sets to the study of quasi-associative ideals in BCI-algebras. We introduce a new concept—complex intuitionistic fuzzy quasi-associative ideals in BCI-algebras—and analyze their fundamental properties. The relationships between complex intuitionistic fuzzy ideals and complex intuitionistic fuzzy quasi-associative ideals are thoroughly investigated. Furthermore, we present key characterizations of these quasi-associative ideals, providing deeper insights into their structure and role within BCI-algebraic systems. Finally, we prove that every complex intuitionistic fuzzy b-ideal is a complex intuitionistic fuzzy quasi-associative ideal in BCI-algebras.

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Key Words and Phrases: BCI-algebra, complex intuitionistic fuzzy quasi-associative ideal, complex intuitionistic fuzzy b-ideal

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1. Introduction

The concept of fuzzy sets (FSs), introduced by Zadeh in 1965, revolutionized the modeling of uncertainty and vagueness in mathematical structures [1]. By extending classical set theory, FSs enabled the representation of partial membership, leading to widespread applications in control systems, decision-making, and beyond. Rosenfeld (1971) advanced the theory further by introducing fuzzy groups, which integrated group theory with fuzzy logic to study algebraic structures under uncertainty [2].

The foundational work of Imai and Iséki (1966) established the axiom systems for BCK-algebras, bridging logical frameworks with algebraic formalism [3, 4]. Iséki and

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Tanaka (1978) later provided a comprehensive introduction to BCK-algebra theory, laying the groundwork for exploring their structural properties and mathematical applications [5]. Zhang [6] introduced quasi-associative ideals in BCI-algebras.

The integration of fuzzy logic with BCK-algebras was pioneered by Xi (1991), who introduced fuzzy BCK-algebras to study algebraic structures under uncertainty [7]. Ahmad (1993) extended this framework to fuzzy BCI-algebras, deepening the understanding of fuzzy algebraic systems [8]. Jun and Song (2012) contributed to the field by examining falling fuzzy quasi-associative ideals in BCI-algebras, thereby enriching the study of non-associative fuzzy structures [9]. In contrast, Lele and Moutari (2007) explored n-fold quasi-associative ideals in the same context [10].

Atanassov (1986) generalized FSs by introducing intuitionistic fuzzy sets (IFSs), which incorporate degrees of membership, non-membership, and hesitation to model uncertainty more robustly [11]. Jun and Kim (2000) applied this framework to BCK-algebras by investigating intuitionistic fuzzy ideals (IFIds), adding a new dimension to their algebraic structure [12]. Ramot et al. (2002, 2003) further expanded the scope of fuzzy logic by proposing complex fuzzy sets (CFSs) and complex fuzzy logic, where phase components enable richer representations of uncertainty [13, 14].

Alkouri and Salleh (2012) introduced complex intuitionistic fuzzy sets ($\mathcal{CIF}$-sets), combining the advantages of IFSs and CFSs to model uncertainty with greater precision [15]. Deepika et al. [16] introduced hybrid quasi-ideals in ternary semigroups. Balamurugan et al. [15, 17] investigated complex Linear Diaphantine fuzzy ideals in BCK-algebras.

The structure of the paper is organized as follows:

- Section 2 reviews fundamental definitions of BCI-algebras and $\mathcal{CIF}$-sets.
- Section 3 introduces and analyzes complex intuitionistic fuzzy quasi-associative ideals in BCI-algebras.
- Section 4 investigates complex intuitionistic fuzzy b-ideals and their related properties.
- Section 5 concludes with findings and future perspectives.

2. Preliminaries

To facilitate the understanding of the main results, this section briefly recalls fundamental concepts related to BCI-algebras and $\mathcal{CIF}$-sets. We begin by reviewing the axioms and structural properties of BCI-algebras, followed by essential definitions concerning $\mathcal{CIF}$-sets, including their algebraic behavior and membership representations. These preliminaries lay the groundwork for the subsequent development of complex intuitionistic fuzzy quasi-associative ideals.

Definition 1. *[5] A BCI-algebra is an algebra* $(\mathcal{X};*,0)$ *of type* $(2,0)$ *that obeys the axioms for all* $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$,
(BCI-1) $((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)) * (\ulcorner\tilde{\varkappa}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) = 0$,

($\mathcal{BCI}$-2) $(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)) * \ulcorner\tilde{\vartheta}\urcorner = 0$,
($\mathcal{BCI}$-3) $\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varrho}\urcorner = 0$,
($\mathcal{BCI}$-4) $\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner = 0,\ \ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varrho}\urcorner = 0 \ \Rightarrow\ \ulcorner\tilde{\varrho}\urcorner = \ulcorner\tilde{\vartheta}\urcorner$.

Definition 2. *[3, 4] A void subset $\mathfrak{C}$ is an ideal if the following hold for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner \in \mathcal{X}$,*
($\mathcal{I}$-1) $0 \in \mathfrak{C}$,
($\mathcal{I}$-2) $\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner \in \mathfrak{C} \ \Rightarrow\ \ulcorner\tilde{\varrho}\urcorner \in \mathfrak{C}$.

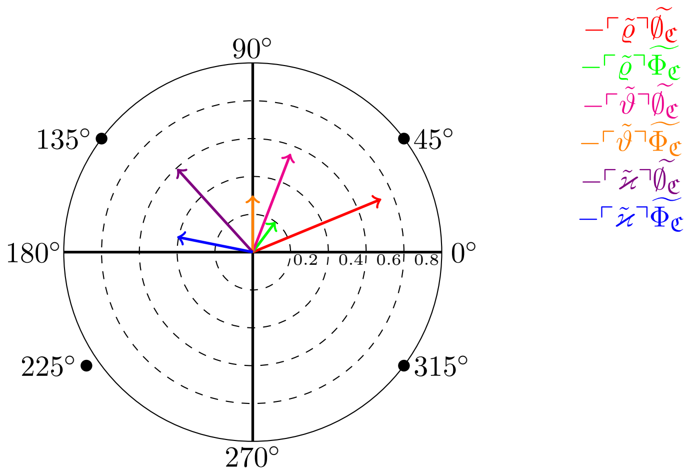
Definition 3. *[6] A void subset $\mathfrak{C}$ is quasi-associative ideal if the following hold for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner \in \mathcal{X}$,*
($\mathcal{QAI}$-1) $0 \in \mathfrak{C}$,
($\mathcal{QAI}$-2) $\ulcorner\tilde{\vartheta}\urcorner \in \mathfrak{C},\ \ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) \in \mathfrak{C} \ \Rightarrow\ \ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner \in \mathfrak{C}$.

Definition 4. *[18] A $\mathcal{CIF}$-set $\mathfrak{C}$ defined on universal set $\mathcal{X}$ is characterized as follows:*

$$A = \{(\ulcorner\tilde{\varrho}\urcorner, \widetilde{\emptyset}_{\mathfrak{C}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}),\ \forall \ulcorner\tilde{\varrho}\urcorner \in \mathcal{X}\},$$

where $\widetilde{\emptyset}_{\mathfrak{C}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}$ is truth function, $\widetilde{\emptyset}_{\mathfrak{C}}(\ulcorner\tilde{\varrho}\urcorner) : \mathcal{X} \to [0,1]$ and $\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) \in [0, 2\pi]$ is a periodic function, $\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}$ is falsity function, $\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) : \mathcal{X} \to [0,1]$ and $\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) \in [0, 2\pi]$ is a periodic function. Finally $0 \leq \widetilde{\emptyset}_{\mathfrak{C}}(\ulcorner\tilde{\varrho}\urcorner) + \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) \leq 1$ for all $\ulcorner\tilde{\varrho}\urcorner \in \mathcal{X}$.

Example 1. *Let $\mathcal{X} = \{\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner\}$. Then, $\mathcal{CIF}$-set $\mathfrak{C} = \{(\ulcorner\tilde{\varrho}\urcorner, 0.7e^{\frac{i\pi}{6}}, 0.2e^{\frac{i\pi}{4}})$, $(\ulcorner\tilde{\vartheta}\urcorner, 0.6e^{\frac{i\pi}{3}}, 0.3e^{\frac{i\pi}{2}}), (\ulcorner\tilde{\varkappa}\urcorner, 0.5e^{\frac{i2\pi}{3}}, 0.4e^{\frac{i4\pi}{5}})\}$.*



3. Complex Intuitionistic Fuzzy Quasi-Associative Ideals

In this section, we introduce the concept of complex intuitionistic fuzzy quasi-associative ideals ($\mathcal{CIFQA}$-ideals) within the framework of BCI-algebras. By extending the classical notion of quasi-associative ideals through the lens of complex intuitionistic fuzzy logic, we establish new algebraic structures equipped to handle both magnitude and phase uncertainty. We define these ideals formally and explore their basic properties and examples to illustrate their significance and distinct behavior.

Definition 5. *A $\mathcal{CIF}$-set $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}} e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}} e^{i\widetilde{\theta_{\mathfrak{C}}}})$ forms a complex intuitionistic fuzzy subalgebra ($\mathcal{CIF}$-subalgebra) if it satisfies the following:*
($\mathcal{CIFS}$-1) $\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$,
($\mathcal{CIFS}$-2) $\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$,
for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner \in \mathcal{X}$.

Definition 6. *A $\mathcal{CIF}$-set $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}} e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}} e^{i\widetilde{\theta_{\mathfrak{C}}}})$ forms a complex intuitionistic fuzzy ideal ($\mathcal{CIF}$-ideal) if it satisfies the following:*
($\mathcal{CIFI}$-1) $\widetilde{\emptyset_{\mathfrak{C}}}(0) e^{i\widetilde{\omega_{\mathfrak{C}}}(0)} \geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)},\ \widetilde{\Phi_{\mathfrak{C}}}(0) e^{i\widetilde{\theta_{\mathfrak{C}}}(0)} \leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}$,
($\mathcal{CIFI}$-2) $\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$,
($\mathcal{CIFI}$-3) $\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$,
for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner \in \mathcal{X}$.

Lemma 1. *Let $\mathfrak{C}$ be a $\mathcal{CIF}$-ideal of $\mathcal{X}$. For $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner \in \mathcal{X}$ with $\ulcorner\tilde{\varrho}\urcorner \leq \ulcorner\tilde{\vartheta}\urcorner$ holds in $\mathcal{X}$. The following are equivalent:*

(1) $\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)},\ \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}$,

(2) $\widetilde{\emptyset_{\mathfrak{C}}}(0) e^{i\widetilde{\omega_{\mathfrak{C}}}(0)} = \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)},\ \widetilde{\Phi_{\mathfrak{C}}}(0) e^{i\widetilde{\theta_{\mathfrak{C}}}(0)} = \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}$.

Proof. Straightforward.

Definition 7. *A $\mathcal{CIF}$-set $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}} e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}} e^{i\widetilde{\theta_{\mathfrak{C}}}})$ forms a complex intuitionistic fuzzy quasi-associative ideal ($\mathcal{CIFQA}$-ideal) of $\mathcal{X}$ if it satisfies the following:*
($\mathcal{CIFQAI}$-1) $\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))}$,
$\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$,
($\mathcal{CIFQAI}$-2) $\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)} \leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))}$,
$\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$,
for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$.

Example 2. *Let $\mathcal{X} = \{0, \tilde{\tau}, \tilde{\upsilon}, \tilde{\zeta}\}$ be a BCI-algebra with cayley Table 1. Let $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}} e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}} e^{i\widetilde{\theta_{\mathfrak{C}}}})$*

Table 1: Cayley Table for $(\mathcal{X}, *)$

$*$	0	$\tilde{\tau}$	$\tilde{\upsilon}$	$\tilde{\zeta}$
0	0	$\tilde{\tau}$	$\tilde{\upsilon}$	$\tilde{\zeta}$
$\tilde{\tau}$	$\tilde{\tau}$	0	$\tilde{\zeta}$	$\tilde{\upsilon}$
$\tilde{\upsilon}$	$\tilde{\upsilon}$	$\tilde{\zeta}$	0	$\tilde{\tau}$
$\tilde{\zeta}$	$\tilde{\zeta}$	$\tilde{\upsilon}$	$\tilde{\tau}$	0

be a $\mathcal{CIF}$-set is given by Table 2.

Table 2: The values of $(\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$

$\mathcal{X}$	$\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}$	$\widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}}$
0	$0.7e^{\frac{i\pi}{3}}$	$0.2e^{\frac{i\pi}{2}}$
$\tilde{\tau}$	$0.5e^{\frac{i\pi}{4}}$	$0.3e^{\frac{i\pi}{3}}$
$\tilde{\upsilon}$	$0.3e^{\frac{i\pi}{3}}$	$0.5e^{\frac{i\pi}{4}}$
$\tilde{\zeta}$	$0.3e^{\frac{i\pi}{5}}$	$0.3e^{\frac{i\pi}{3}}$

Our calculations demonstrate that $\mathcal{C}$ is a $\mathcal{CIFQA}$-ideal in $\mathcal{X}$.

Theorem 1. *In a BCI-algebra, every $\mathcal{CIFQA}$-ideal is necessarily a $\mathcal{CIF}$-ideal.*

Proof. Let $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$ be a $\mathcal{CIFQA}$-ideal of $\mathcal{X}$. Put $\ulcorner\tilde{\varkappa}\urcorner = 0$ in Definition 7, $\mathcal{CIFQAI}$-2 and $\mathcal{CIFQAI}$-3, we have

$$\begin{aligned}
\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} &= \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * 0)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * 0)} \\
&\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * 0))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * 0))}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\
\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} &\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\},
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} &= \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * 0)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * 0)} \\
&\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * 0))e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * 0))}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\
\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} &\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\},
\end{aligned}$$

for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner \in \mathcal{X}$. Hence, $A = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$ is a $\mathcal{CIF}$-ideal of $\mathcal{X}$.

Remark 1. *The converse of Theorem 1 fails to hold in general.*

Example 3. *Let $\mathcal{X} = \{0, \tilde{\tau}, \tilde{\upsilon}, \tilde{\zeta}, \tilde{\sigma}\}$ be a BCI-algebra in Table 3. Let $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$*

Table 3: $(\mathcal{X}, *)$

$*$	0	$\tilde{\tau}$	$\tilde{\upsilon}$	$\tilde{\zeta}$	$\tilde{\sigma}$
0	0	0	$\tilde{\sigma}$	$\tilde{\zeta}$	$\tilde{\upsilon}$
$\tilde{\tau}$	$\tilde{\tau}$	0	$\tilde{\sigma}$	$\tilde{\zeta}$	$\tilde{\upsilon}$
$\tilde{\upsilon}$	$\tilde{\upsilon}$	$\tilde{\upsilon}$	0	$\tilde{\sigma}$	$\tilde{\zeta}$
$\tilde{\zeta}$	$\tilde{\zeta}$	$\tilde{\zeta}$	$\tilde{\upsilon}$	0	$\tilde{\sigma}$
$\tilde{\sigma}$	$\tilde{\sigma}$	$\tilde{\sigma}$	$\tilde{\zeta}$	$\tilde{\upsilon}$	0

be a $\mathcal{CIF}$-set in $\mathcal{X}$ which is given by Table 4. Using the routine calculation, we obtained that $\mathfrak{C}$ is a $\mathcal{CIF}$-ideal of $\mathcal{X}$, but not a $\mathcal{CIFQA}$-ideal as

$$\widetilde{\emptyset_{\mathfrak{C}}}(\tilde{\sigma} * \tilde{\upsilon})e^{i\widetilde{\omega_{\mathfrak{C}}}(\tilde{\sigma} * \tilde{\upsilon})} = \widetilde{\emptyset_{\mathfrak{C}}}(\tilde{\zeta})e^{i\widetilde{\omega_{\mathfrak{C}}}(\tilde{\zeta})}$$

Table 4: The values of $(\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$

$\mathcal{X}$	$\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}$	$\widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}}$
0	$0.7e^{\frac{i\pi}{2}}$	$0.2e^{\frac{i\pi}{7}}$
$\tilde{\tau}$	$0.5e^{\frac{i\pi}{3}}$	$0.3e^{\frac{i\pi}{5}}$
$\tilde{\upsilon}$	$0.3e^{\frac{i\pi}{5}}$	$0.4e^{\frac{i\pi}{4}}$
$\tilde{\zeta}$	$0.3e^{\frac{i\pi}{4}}$	$0.4e^{\frac{i\pi}{3}}$
$\tilde{\sigma}$	$0.3e^{\frac{i\pi}{3}}$	$0.4e^{\frac{i\pi}{2}}$

$$\begin{aligned} &= 0.3e^{\frac{i\pi}{4}} \ngeq 0.7e^{\frac{i\pi}{2}} \\ &= \widetilde{\emptyset_{\mathfrak{C}}}(0)e^{i\widetilde{\omega_{\mathfrak{C}}}(0)} \\ &= \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\tilde{\sigma} * (0 * \tilde{\upsilon}))e^{i\widetilde{\omega_{\mathfrak{C}}}(\tilde{\sigma}*(0*\tilde{\upsilon}))}, \widetilde{\emptyset_{\mathfrak{C}}}(0)e^{i\widetilde{\omega_{\mathfrak{C}}}(0)}\} \end{aligned}$$

and

$$\begin{aligned} \widetilde{\Phi_{\mathfrak{C}}}(\tilde{\sigma} * \tilde{\upsilon})e^{i\widetilde{\theta_{\mathfrak{C}}}(d*\tilde{\upsilon})} &= \widetilde{\Phi_{\mathfrak{C}}}(\tilde{\zeta})e^{i\widetilde{\theta_{\mathfrak{C}}}(\tilde{\zeta})} \\ &= 0.4e^{\frac{i\pi}{3}} \nleq 0.2e^{\frac{i\pi}{7}} \\ &= \widetilde{\Phi_{\mathfrak{C}}}(0)e^{i\widetilde{\theta_{\mathfrak{C}}}(0)} \\ &= \max\{\widetilde{\Phi_{\mathfrak{C}}}(\tilde{\sigma} * (0 * \tilde{\upsilon}))e^{i\widetilde{\theta_{\mathfrak{C}}}(\tilde{\sigma}*(0*\tilde{\upsilon}))}, \widetilde{\Phi_{\mathfrak{C}}}(0)e^{i\widetilde{\theta_{\mathfrak{C}}}(0)}\}. \end{aligned}$$

Remark 2. *Since Example 3 shows that $\mathcal{CIF}$-ideals need not be $\mathcal{CIFQA}$-ideals, we determine the exact conditions under which this inclusion holds in Theorem 2.*

Theorem 2. *If the relations $(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner \leq \ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)$ is holds for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$ and $\mathcal{X}$ is quasi-associative in $\mathcal{BCI}$-algebra, then every $\mathcal{CIF}$-ideal is a $\mathcal{CIFQA}$-ideal.*

Proof. Let $A = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$ be $\mathcal{CIF}$-ideal of $\mathcal{X}$, and $\mathcal{X}$ be the quasi-associative $(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner \leq \ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)$ is valid for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$. By Lemma 1, for every $\mathcal{CIF}$-ideal $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$. Then $\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}$ is order-reversing and $\widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}}$ is order-preserving, we have

$$\begin{aligned} \widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner)} &\geq \emptyset_A(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner)} \\ \widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner)} &\leq \widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner)}. \end{aligned}$$

By $\mathcal{CIFI}$-2 and $\mathcal{CIFI}$-3 in Definition 6, we have

$$\begin{aligned} &\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)} \\ &\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)*\ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\ &\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\ &\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(\ulcorner\tilde{\vartheta}\urcorner*\ulcorner\tilde{\varkappa}\urcorner))}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \end{aligned}$$

and

$$
\begin{aligned}
&\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)} \\
&\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\
&\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\
&\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}.
\end{aligned}
$$

Hence, $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$ is a $\mathcal{CIFQA}$-ideal of $\mathcal{X}$.

Corollary 1. *A $\mathcal{BCI}$-algebra $\mathcal{X}$ has the property that all $\mathcal{CIF}$-ideals are $\mathcal{CIFQA}$-ideals if and only if it satisfies $0 * (0 * \ulcorner\tilde{\varrho}\urcorner) = 0 * \ulcorner\tilde{\varrho}\urcorner$.*

Proof. Straightforward.

Proposition 1. *If $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$ is a $\mathcal{CIFQA}$-ideal of $\mathcal{X}$, then the following hold:*
(1) $\ulcorner\tilde{\varrho}\urcorner \leq \ulcorner\tilde{\vartheta}\urcorner \Rightarrow \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}$, $\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}$,
(2) $\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} = \widetilde{\emptyset_{\mathfrak{C}}}(0)e^{i\widetilde{\omega_{\mathfrak{C}}}(0)} \Rightarrow \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}$,
$\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} = \widetilde{\Phi_{\mathfrak{C}}}(0)e^{i\widetilde{\theta_{\mathfrak{C}}}(0)} \Rightarrow \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}$,
(3) $\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$,
$\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$
(4) $\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varkappa}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varkappa}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}\}$,
$\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varkappa}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varkappa}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}\}$
(5) $\widetilde{\emptyset_{\mathfrak{C}}}((0 * \ulcorner\tilde{\varrho}\urcorner) * \ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((0 * \ulcorner\tilde{\varrho}\urcorner) * \mathfrak{l})} = \widetilde{\emptyset_{\mathfrak{C}}}(0)e^{i\widetilde{\omega_{\mathfrak{C}}}(0)}$,
$\widetilde{\Phi_{\mathfrak{C}}}((0 * \ulcorner\tilde{\varrho}\urcorner) * \ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((0 * \ulcorner\tilde{\varrho}\urcorner) * \ulcorner\tilde{\varrho}\urcorner)} = \widetilde{\Phi_{\mathfrak{C}}}(0)e^{i\widetilde{\theta_{\mathfrak{C}}}(0)}$,
for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$.

Proof. (1) If $\ulcorner\tilde{\varrho}\urcorner \leq \ulcorner\tilde{\vartheta}\urcorner$, then $\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner = 0$. Then

$$
\begin{aligned}
\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} &= \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * 0)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * 0)} \\
&\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * 0))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * 0))}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\
&\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\
&\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(0)e^{i\widetilde{\omega_{\mathfrak{C}}}(0)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\
\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} &\geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} &= \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * 0)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * 0)} \\
&\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * 0))e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * 0))}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\
&\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}
\end{aligned}
$$

$$\begin{aligned} &\leq \max\{\widetilde{\Phi_{\mathfrak{c}}}(0)e^{i\widetilde{\theta_{\mathfrak{c}}}(0)}, \widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\ \widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner)} &\leq \widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)}. \end{aligned}$$

(2) This is analogous to (1).

(3) Let $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner \in \mathcal{X}$. Then

$$\begin{aligned} &\widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \\ &\geq \min\{\widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\vartheta}\urcorner))e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\vartheta}\urcorner))}, \widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\ &\geq \min\{\widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * 0)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * 0)}, \widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\ &\geq \min\{\widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner)}, \widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \end{aligned}$$

and

$$\begin{aligned} &\widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \\ &\leq \max\{\widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\vartheta}\urcorner))e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\vartheta}\urcorner))}, \widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\ &\leq \max\{\widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * 0)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * 0)}, \widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)}\} \\ &\leq \max\{\widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner)}, \widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}. \end{aligned}$$

(4) Using $\mathcal{BCI}$-1 and (1) above, we have

$$\widetilde{\emptyset_{\mathfrak{c}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\omega_{\mathfrak{c}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)} \geq \widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}$$

and

$$\widetilde{\Phi_{\mathfrak{c}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\omega_{\mathfrak{c}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)} \leq \widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}.$$

By using Theorem 1, we have

$$\begin{aligned} &\widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \\ &\geq \min\{\widetilde{\emptyset_{\mathfrak{c}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\omega_{\mathfrak{c}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}, \widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}\} \\ &\geq \min\{\widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}\} \end{aligned}$$

and

$$\begin{aligned} &\widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)} \\ &\leq \max\{\widetilde{\Phi_{\mathfrak{c}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}, \widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}\} \\ &\leq \max\{\widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\Phi_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{c}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}\}. \end{aligned}$$

(5) Put $\ulcorner\tilde{\varrho}\urcorner = 0 * \ulcorner\tilde{\varrho}\urcorner$, $\ulcorner\tilde{\vartheta}\urcorner = 0$, $\ulcorner\tilde{\varkappa}\urcorner = \ulcorner\tilde{\varrho}\urcorner$ in $\mathcal{CIFQAI}$-2 and $\mathcal{CIFQAI}$-3, we have

$$\widetilde{\emptyset_{\mathfrak{c}}}((0 * \ulcorner\tilde{\varrho}\urcorner) * \ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{c}}}((0 * \ulcorner\tilde{\varrho}\urcorner) * \ulcorner\tilde{\varrho}\urcorner)}$$

$$\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((0*\ulcorner\tilde{\varrho}\urcorner)*(0*\ulcorner\tilde{\varrho}\urcorner))e^{i\widetilde{\omega_{\mathfrak{C}}}(0*\ulcorner\tilde{\varrho}\urcorner)*(0*\ulcorner\tilde{\varrho}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(0)e^{i\widetilde{\omega_{\mathfrak{C}}}(0)}\}$$
$$= \widetilde{\emptyset_{\mathfrak{C}}}(0)e^{i\widetilde{\omega_{\mathfrak{C}}}(0)}$$

and

$$\widetilde{\Phi_{\mathfrak{C}}}((0*\ulcorner\tilde{\varrho}\urcorner)*\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((0*\ulcorner\tilde{\varrho}\urcorner)*\ulcorner\tilde{\varrho}\urcorner)}$$
$$\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}((0*\ulcorner\tilde{\varrho}\urcorner)*(0*\ulcorner\tilde{\varrho}\urcorner))e^{i\widetilde{\theta_{\mathfrak{C}}}(0*\ulcorner\tilde{\varrho}\urcorner)*(0*\ulcorner\tilde{\varrho}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(0)e^{i\widetilde{\theta_{\mathfrak{C}}}(0)}\}$$
$$= \widetilde{\Phi_{\mathfrak{C}}}(0)e^{i\widetilde{\theta_{\mathfrak{C}}}(0)}.$$

It follows from $\mathcal{CIFI}$-1 that

$$\widetilde{\emptyset_{\mathfrak{C}}}((0*\ulcorner\tilde{\varrho}\urcorner)*\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((0*\ulcorner\tilde{\varrho}\urcorner)*\ulcorner\tilde{\varrho}\urcorner)} = \widetilde{\emptyset_{\mathfrak{C}}}(0)e^{i\widetilde{\omega_{\mathfrak{C}}}(0)}$$

and

$$\widetilde{\Phi_{\mathfrak{C}}}((0*\ulcorner\tilde{\varrho}\urcorner)*\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((0*\ulcorner\tilde{\varrho}\urcorner)*\ulcorner\tilde{\varrho}\urcorner)} = \widetilde{\Phi_{\mathfrak{C}}}(0)e^{i\widetilde{\theta_{\mathfrak{C}}}(0)}.$$

Remark 3. *In Proposition 1, every $\mathcal{CIFQA}$-ideal is a $\mathcal{CIF}$-subalgebra.*

Proposition 2. *Let $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$ be a $\mathcal{CIF}$-ideal. Then the following statements are equivalent:*

(1) $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$ *is a $\mathcal{CIFQA}$-ideal of $\mathcal{X}$.*

(2) $\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)} \geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))}$, $\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)} \leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))}$.

(3) $\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner)} \geq \widetilde{\emptyset_{\mathfrak{C}}}(l*(m*n))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner))}$, $\widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner)} \leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)*\ulcorner\tilde{\varkappa}\urcorner))}$,

for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$.

Proof. (1) $\Rightarrow$ (2) Assume that $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$ is a $\mathcal{CIFQA}$-ideal. Then by using $\mathcal{CIFI}$-1, $\mathcal{CIFQAI}$-1 and $\mathcal{CIFQAI}$-2, we have

$$\begin{aligned}\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)} &\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))}, \widetilde{\emptyset_{\mathfrak{C}}}(0)e^{i\widetilde{\omega_{\mathfrak{C}}}(0)}\}\\ &\geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)} &\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))}, \widetilde{\Phi_{\mathfrak{C}}}(0)e^{i\widetilde{\theta_{\mathfrak{C}}}(0)}\}\\ &\leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(0*\ulcorner\tilde{\vartheta}\urcorner))}.\end{aligned}$$

Hence, (2) is proved.

(2) $\Rightarrow$ (3) Assume that (2) is satisfied. By using $(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner = (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner$ and $\mathcal{BCI}$-1, $\mathcal{BCI}$-3, we have

$$
\begin{aligned}
&((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (0 * \ulcorner\tilde{\varkappa}\urcorner)) * (\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))\\
&= ((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))) * (0 * \ulcorner\tilde{\varkappa}\urcorner)\\
&\leq ((\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner) * (0 * \ulcorner\tilde{\varkappa}\urcorner)\\
&\leq ((\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner) * (0 * \ulcorner\tilde{\varkappa}\urcorner)\\
&\leq (0 * \ulcorner\tilde{\varkappa}\urcorner) * (0 * \ulcorner\tilde{\varkappa}\urcorner)\\
&\leq 0.
\end{aligned}
$$

Hence, $(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * (0 * \ulcorner\tilde{\varkappa}\urcorner) \leq \ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)$. By Lemma 1, we get

$$
\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)} \geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner))}
$$

and

$$
\widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)} \leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner))}.
$$

Hence, (3) is proved.

(3) $\Rightarrow$ (1) Assume that (3) is satisfied. By using $\mathcal{CIFI}$-2 and $\mathcal{CIFI}$-3 in Definition 6, $(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner = (\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner$ and (3), we have

$$
\begin{aligned}
&\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}\\
&\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}\\
&\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}\\
&\geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}\\
&\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(m)e^{i\widetilde{\theta_{\mathfrak{C}}}(m)}\}\\
&\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\vartheta}\urcorner) * \ulcorner\tilde{\varkappa}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}\\
&\leq \max\{\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * (\ulcorner\tilde{\vartheta}\urcorner * \ulcorner\tilde{\varkappa}\urcorner))}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}.
\end{aligned}
$$

Thus, $\mathcal{CIFQAI}$-1 and $\mathcal{CIFQAI}$-2 are satisfied. Hence, $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}}e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}}e^{i\widetilde{\theta_{\mathfrak{C}}}})$ is a $\mathcal{CIFQA}$-ideal.

4. Complex Intuitionistic Fuzzy b-Ideals

This section focuses on the formulation and analysis of complex intuitionistic fuzzy b-ideals ($\mathcal{CIFB}$-ideals) in BCI-algebras. These ideals represent a refined class of $\mathcal{CIF}$-ideals, characterized by stronger absorption properties under fuzzy composition. We define $\mathcal{CIFB}$-ideals rigorously and examine their interrelationships with previously established ideal types, highlighting their role in the broader structure of fuzzy algebraic systems.

Definition 8. *A $\mathcal{CIF}$-set $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}} e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}} e^{i\widetilde{\theta_{\mathfrak{C}}}})$ forms a complex intuitionistic fuzzy b-ideal ($\mathcal{CIFB}$-ideal) of $\mathcal{X}$ if it satisfies the following:*
($\mathcal{CIFBI}$-1) $\widetilde{\emptyset_{\mathfrak{C}}}(0) e^{i\widetilde{\omega_{\mathfrak{C}}}(0)} \geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}$, $\widetilde{\Phi_{\mathfrak{C}}}(0) e^{i\widetilde{\theta_{\mathfrak{C}}}(0)} \leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)}$,
($\mathcal{CIFBI}$-2) $\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$,
($\mathcal{CIFBI}$-3) $\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \leq \max\{\widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}$,
for all $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$.

Theorem 3. *Let $\mathfrak{C} = (\widetilde{\emptyset_{\mathfrak{C}}} e^{i\widetilde{\omega_{\mathfrak{C}}}}, \widetilde{\Phi_{\mathfrak{C}}} e^{i\widetilde{\theta_{\mathfrak{C}}}})$ be a $\mathcal{CIF}$-set. Then the following conditions are equivalent:*

(1) $\mathfrak{C}$ *is a $\mathcal{CIF}$-ideal of $\mathcal{X}$.*

(2) $\mathfrak{C}$ *is a $\mathcal{CIFQA}$-ideal of $\mathcal{X}$.*

(3) $\mathfrak{C}$ *is a $\mathcal{CIFB}$-ideal of $\mathcal{X}$.*

Proof. The diagram below illustrates the equivalence between the three types of ideals.

$$\begin{tikzcd}
\mathcal{CIF}\text{-ideal} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathcal{CIFQA}\text{-ideal} \arrow[dl, leftrightarrow] \\
\mathcal{CIFB}\text{-ideal} &
\end{tikzcd}$$

We prove the equivalences by showing the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Assume that $\mathfrak{C}$ is a $\mathcal{CIF}$-ideal. We show it is a $\mathcal{CIFQA}$-ideal. For any $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$, by Definition 6, we have

$$\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * \ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}.$$

Setting $\ulcorner\tilde{\vartheta}\urcorner = 0$, we get

$$\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * 0) e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * 0}, \widetilde{\emptyset_{\mathfrak{C}}}(0) e^{i\widetilde{\omega_{\mathfrak{C}}}(0)}\}.$$

Since $(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) * 0 = \ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner$ and $\widetilde{\emptyset_{\mathfrak{C}}}(0) e^{i\widetilde{\omega_{\mathfrak{C}}}(0)} \geq \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}$ for any $\ulcorner\tilde{\vartheta}\urcorner \in \mathcal{X}$, this simplifies to

$$\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner) e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(0) e^{i\widetilde{\omega_{\mathfrak{C}}}(0)}\}$$

$$= \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner * \ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)}.$$

Thus, $\mathcal{CIFQAI}$-1 is satisfied. Similarly, for the non-membership function,

$$\widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)} \leq \max\{\widetilde{\Phi_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)*\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)*\ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}.$$

Setting $\ulcorner\tilde{\vartheta}\urcorner = 0$ and using $\widetilde{\Phi_{\mathfrak{C}}}(0)e^{i\widetilde{\theta_{\mathfrak{C}}}(0)} \leq \widetilde{\Phi_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\theta_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}$, we obtain the required condition.

(2) $\Rightarrow$ (3) Assume that $\mathfrak{C}$ is a $\mathcal{CIFQA}$-ideal. We show it is a $\mathcal{CIFB}$-ideal. For any $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$, by using Definition 7, we have

$$\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(\ulcorner\tilde{\vartheta}\urcorner*\ulcorner\tilde{\varkappa}\urcorner))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(\ulcorner\tilde{\vartheta}\urcorner*\ulcorner\tilde{\varkappa}\urcorner))}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}.$$

Setting $\ulcorner\tilde{\varkappa}\urcorner = 0$, we get

$$\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*0)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*0)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(\ulcorner\tilde{\vartheta}\urcorner*0))e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*(\ulcorner\tilde{\vartheta}\urcorner*0))}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}.$$

Since $\ulcorner\tilde{\varrho}\urcorner * 0 = \ulcorner\tilde{\varrho}\urcorner$ and $\ulcorner\tilde{\vartheta}\urcorner * 0 = \ulcorner\tilde{\vartheta}\urcorner$, this simplifies to

$$\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}.$$

This is the condition for $\mathcal{CIFB}$-ideal when $\ulcorner\tilde{\varkappa}\urcorner = 0$. For general $\ulcorner\tilde{\varkappa}\urcorner$, the proof follows similarly by expanding the terms. The non-membership condition is analogous.

(3) $\Rightarrow$ (1) Assume that $\mathfrak{C}$ is a $\mathcal{CIFB}$-ideal. We show it is a $\mathcal{CIF}$-ideal. For any $\ulcorner\tilde{\varrho}\urcorner, \ulcorner\tilde{\vartheta}\urcorner, \ulcorner\tilde{\varkappa}\urcorner \in \mathcal{X}$, by Definition 8, we have

$$\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)*\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*\ulcorner\tilde{\varkappa}\urcorner)*\ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}.$$

Setting $\ulcorner\tilde{\varkappa}\urcorner = 0$, we get

$$\widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\varrho}\urcorner)} \geq \min\{\widetilde{\emptyset_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*0)*\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}((\ulcorner\tilde{\varrho}\urcorner*0)*\ulcorner\tilde{\vartheta}\urcorner)}, \widetilde{\emptyset_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)e^{i\widetilde{\omega_{\mathfrak{C}}}(\ulcorner\tilde{\vartheta}\urcorner)}\}.$$

This is the condition for $\mathcal{CIF}$-ideal. The non-membership condition is similarly verified. Thus, the three conditions are equivalent.

5. Conclusion

In this study, we established a comprehensive framework connecting complex intuitionistic fuzzy ideals ($\mathcal{CIF}$-ideals), quasi-associative ideals ($\mathcal{CIFQA}$-ideals), and b-ideals ($\mathcal{CIFB}$-ideals) within BCI-algebras. We formally introduced the notion of $\mathcal{CIFQA}$-ideals and demonstrated that every $\mathcal{CIFQA}$-ideal is inherently a $\mathcal{CIF}$-ideal. However, through counterexamples, we showed that the converse does not generally hold. We further derived necessary and sufficient conditions under which all three classes of ideals coincide, and established that these conditions depend on the quasi-associativity of the underlying algebraic structure. These findings reveal a hierarchical lattice among the ideals and illustrate the bridging role of $\mathcal{CIFQA}$-ideals between general fuzzy ideals and the more restrictive b-ideals.

Future research could explore extensions of these ideal structures to more generalized algebraic systems such as BCI-semigroups, n-ary BCI-algebras, or their neutrosophic and soft set analogues. Additionally, applications in decision support systems, logic programming, and computational models under hybrid uncertainty could benefit from the theoretical foundations laid in this work.

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