EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6340
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

The Conformable Double Laplace-Shehu Transform

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Abstract. We present a new transform called the conformable double Laplace-Shehu transform. This tool helps in solving fractional partial differential equations. These equations come up often in science and engineering. The transform is built using the idea of the conformable derivative. We explain the basic rules of the transform and show how it can be used. To show its use, we solve two equations that are well known. These are the wave and heat equations.
2020 Mathematics Subject Classifications: 44A05
Key Words and Phrases: Laplace transform, Shehu transform, double transform, conformable double Laplace-Shehu transform

1. Introduction

Fractional partial differential equations are used to model many problems in physics, electric circuits, fluid flow, optics, and biology. The conformable derivative, as introduced in [1], keeps most of the main ideas of classical derivatives while working for fractional cases.

Several methods have been developed for solving conformable fractional equations. The conformable double Laplace transform was discussed in [2, 3], while the conformable double Sumudu transform appeared in [4]. More work on these types of transforms can be found in [5–7].

Later, a new method called the double Laplace-Shehu transform was proposed in [8]. It was applied successfully to different types of partial differential equations. Further studies related to integral transforms are available in [9–16].

In this paper, we presented a new method called the conformable double Laplace-Shehu transform (CL-SH) for solving a specific type of partial differential equations. We first outline the basic properties of this transform such as the conditions that make it valid for use and how it interacts with mathematical derivatives then we demonstrate through practical examples how it can be applied to solve these equations. This method offers a different perspective for studying mathematical problems and may pave the way for developing new ideas in applied mathematics.

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6340

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2. Conformable Fractional Derivative

We give the main definitions and results for conformable fractional derivatives in this section.

Definition 1. *[1] Let $0 < \gamma \leq 1$ and $r : [0, \infty) \rightarrow \mathbb{R}$. The conformable fractional derivative of order γ is defined as:*

$$\frac{d^{\gamma}}{d\lambda^{\gamma}} r(\lambda) = \lim_{\upsilon \to 0} \frac{r(\lambda + \upsilon\lambda^{1-\gamma}) - r(\lambda)}{\upsilon}$$

where $\lambda \geq 0$, and $\frac{\partial^{\gamma}}{\partial\lambda^{\gamma}}$ is referred to as the fractional derivative of order γ.

Definition 2. *[17] Let $0 < \gamma_1, \gamma_2 \leq 1$ and $r(\lambda, \mu) : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$. The conformable partial derivatives of orders γ_1 and γ_2 of the function $r(\lambda, \mu)$ are defined as:*

$$\frac{\partial^{\gamma_1}}{\partial\lambda^{\gamma_1}} r(\lambda, \mu) = \lim_{\upsilon \to 0} \frac{r(\lambda + \upsilon\lambda^{1-\gamma_1}, \mu) - r(\lambda, \mu)}{\upsilon}$$

$$\frac{\partial^{\gamma_2}}{\partial\mu^{\gamma_2}} r(\lambda, \mu) = \lim_{\upsilon \to 0} \frac{r(\lambda, \mu + \upsilon\mu^{1-\gamma_2}) - r(\lambda, \mu)}{\upsilon}$$

where $\lambda, \mu \geq 0$, $\frac{\partial^{\gamma_1}}{\partial\lambda^{\gamma_1}}$ and $\frac{\partial^{\gamma_2}}{\partial\mu^{\gamma_2}}$ are referred to as fractional derivatives of orders γ_1 and γ_2, respectively.

Theorem 1. *[18]Suppose that $r(\lambda, \mu)$ is differentiable at a point $\lambda, \mu \geq 0$, $0 < \gamma_1, \gamma_2 \leq 1$, then:*

$$\frac{\partial^{\gamma_1} r}{\partial\lambda^{\gamma_1}} = \lambda^{1-\gamma_1} \frac{\partial r}{\partial\lambda},$$

$$\frac{\partial^{\gamma_2} r}{\partial\mu^{\gamma_2}} = \mu^{1-\gamma_2} \frac{\partial r}{\partial\mu}.$$

3. The Conformable Double Laplace-Shehu transform

This section defines the CL-SH and explains its basic properties. We start by introducing the individual conformable Laplace and Shehu transforms, then define the combined form. We prove that the CL-SH is a linear operator. We also give the conditions under which the transform exists. Several examples for basic functions are included. Finally, we show how the CL-SH interacts with conformable partial derivatives through a set of key identities.

3.1. Definition and basic properties of the conformable double Laplace-Shehu transform

Definition 3. *Let $r(\lambda,\mu)$ be a continuous function on $[0,\infty)\times[0,\infty)$. Then*
1- The conformable Laplace transformation (CL) of $r(\lambda,\mu)$, denoted by $L_\lambda^\gamma[r(\lambda,\mu)]$, is defined as:

$$A(\varsigma) = L_\lambda^\gamma(r(\lambda,\mu)) = \int_0^\infty e^{-\varsigma\frac{\lambda^\gamma}{\gamma}} r(\lambda,\mu)\lambda^{\gamma-1}d\lambda,\ \varsigma \in \mathbb{C}$$

2- The conformable Shehu transformation (CSH) of $r(\lambda,\mu)$, denoted by $L_\mu^\gamma[r(\lambda,\mu)]$, is defined as:

$$B(\upsilon) = H_\mu^\gamma(r(\lambda,\mu)) = \int_0^\infty e^{-\tau\frac{\mu^\gamma}{\upsilon\gamma}} r(\lambda,\mu)\mu^{\gamma-1}d\mu,\ \upsilon \in \mathbb{C}$$

3- The CL-SH of $r(\lambda,\mu)$, denoted by $L_\lambda^{\gamma_1}H_\mu^{\gamma_2}[r(\lambda,\mu)]$, is defined as:

$$R(\varsigma,\upsilon) = L_\lambda^{\gamma_1}H_\mu^{\gamma_2}[r(\lambda,\mu)] = \int_0^\infty\int_0^\infty e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)} r(\lambda,\mu)\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu.$$

Theorem 2. *Assume that $r:[0,\infty)\times[0,\infty)\to\mathbb{R}$ such that $R(\varsigma,\upsilon) = L_\lambda^{\gamma_1}H_\mu^{\gamma_2}[r(\frac{\lambda^{\gamma_1}}{\gamma_1},\frac{\mu^{\gamma_2}}{\gamma_2})]$ exist, then*

$$L_\lambda^{\gamma_1}H_\mu^{\gamma_2}[r(\frac{\lambda^{\gamma_1}}{\gamma_1},\frac{\mu^{\gamma_2}}{\gamma_2})] = L_\lambda H_\mu[r(\lambda,\mu)],$$

where

$$L_\lambda H_\mu[r(\lambda,\mu)] = \int_0^\infty\int_0^\infty e^{-\left(\varsigma\lambda+\frac{\tau\mu}{\upsilon}\right)} r(\lambda,\mu)\,d\lambda\,d\mu.$$

Proof.

$$L_\lambda^{\gamma_1}H_\mu^{\gamma_2}[r(\frac{\lambda^{\gamma_1}}{\gamma_1},\frac{\mu^{\gamma_2}}{\gamma_2})] = \int_0^\infty\int_0^\infty e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)} r(\frac{\lambda^{\gamma_1}}{\gamma_1},\frac{\mu^{\gamma_2}}{\gamma_2})\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu \tag{1}$$

Substitute $z=\frac{\lambda^{\gamma_1}}{\gamma_1}$ and $w=\frac{\mu^{\gamma_2}}{\gamma_2}$ in Equation 1, we have

$$\begin{aligned} L_\lambda^{\gamma_1}H_\mu^{\gamma_2}[r(\frac{\lambda^{\gamma_1}}{\gamma_1},\frac{\mu^{\gamma_2}}{\gamma_2})] &= \int_0^\infty\int_0^\infty e^{-\left(\varsigma z+\frac{\tau w}{\upsilon}\right)} r(z,w)dzdw \\ &= \int_0^\infty\int_0^\infty e^{-\left(\varsigma\lambda+\frac{\tau\mu}{\upsilon}\right)} r(\lambda,\mu)\,d\lambda\,d\mu \\ &= L_\lambda H_\mu[r(\lambda,\mu)] \end{aligned}$$

Lemma 1. *$L_\lambda^{\gamma_1}H_\mu^{\gamma_2}(r(\lambda,\mu))$ is a linear transformation.*

Proof. for nonzero constants α and β, we have

$$\begin{aligned}
&L_{\lambda}^{\gamma_1}W_{\mu}^{\gamma_2}(\alpha r_1(\lambda,\mu)+\beta r_2(\lambda,\mu)) \\
&= \int_0^\infty\int_0^\infty e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)}(\alpha r_1(\lambda,\mu)+\beta r_2(\lambda,\mu))\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu, \\
&= \alpha\int_0^\infty\int_0^\infty e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)}r_1(\lambda,\mu)\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu+\beta\int_0^\infty\int_0^\infty e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)}r_2(\lambda,\mu)\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu \\
&= \alpha L_{\lambda}^{\gamma_1}W_{\mu}^{\gamma_2}(r_1(\lambda,\mu))+\beta L_{\lambda}^{\gamma_1}W_{\mu}^{\gamma_2}(r_2(\lambda,\mu)).
\end{aligned}$$

If $r(\lambda,\mu)$ can be written as $r(\lambda,\mu)=p(\lambda)q(\mu)$ for some continuous functions p and q, then $L_{\lambda}^{\gamma_1}H_{\mu}^{\gamma_2}(r(\lambda,\mu))=L_{\lambda}^{\gamma_1}(p(\lambda))H_{\mu}^{\gamma_2}(q(\mu))$. In fact

$$\begin{aligned}
L_{\lambda}^{\gamma_1}H_{\mu}^{\gamma_2}(r(\lambda,\mu)) &= L_{\lambda}^{\gamma_1}H_{\mu}^{\gamma_2}(p(\lambda)q(\mu)) \\
&= \int_0^\infty\int_0^\infty e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)}p(\lambda)q(\mu)\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu \\
&= \left(\int_0^\infty e^{-\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}}p(\lambda)\lambda^{\gamma_1-1}d\lambda\right)\left(\int_0^\infty e^{-\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}}q(\mu)\mu^{\gamma_2-1}d\mu\right) \\
&= L_{\lambda}^{\gamma_1}(p(\lambda))H_{\mu}^{\gamma_2}(q(\mu)).
\end{aligned}$$

Definition 4. *Let $0<\gamma_1,\gamma_2\le 1$. Then a function $r(\lambda,\mu)$ is said to be of conformable exponential orders α and β on $0\le\lambda<\infty$ and $0\le\mu<\infty$. If there exist $K,N,M>0$ such that $|r(\lambda,\mu)|\le Ke^{\alpha\frac{\lambda^{\gamma_1}}{\gamma_1}+\beta\frac{\mu^{\gamma_2}}{\gamma_2}}$, for all $\frac{\lambda^{\gamma_1}}{\gamma_1}>N$, $\frac{\mu^{\gamma_2}}{\gamma_2}>M$.*

Theorem 3. *Let $0<\gamma_1,\gamma_2\le 1$ and $r(\lambda,\mu)$ be a continuous function on the region $[0,\infty)\times[0,\infty)$ of conformable exponential orders α and β. Then $R(\varsigma,\upsilon)=L_{\lambda}^{\gamma_1}H_{\mu}^{\gamma_2}[r(\lambda,\mu)]$ exists for ς,υ whenever $\mathrm{Re}(\varsigma)>\alpha$ and $\mathrm{Re}\left(\frac{\tau}{\upsilon}\right)>\beta$.*

We have

$$\begin{aligned}
|R(\varsigma,\upsilon)| &= \left|\int_0^\infty\int_0^\infty e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)}r(\lambda,\mu)\lambda^{\gamma_1-1}\mu^{\gamma_2-1}\,d\lambda d\mu\right| \\
&\le \int_0^\infty\int_0^\infty e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)}|r(\lambda,\mu)|\,\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu \\
&\le K\int_0^\infty\int_0^\infty e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)}e^{\alpha\frac{\lambda^{\gamma_1}}{\gamma_1}+\beta\frac{\mu^{\gamma_2}}{\gamma_2}}\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu
\end{aligned}$$

$$\begin{aligned} &= K\left(\int_0^\infty e^{-(\varsigma-\alpha)\frac{\lambda^{\gamma_1}}{\gamma_1}}\lambda^{\gamma_1-1}d\lambda\right)\left(\int_0^\infty e^{-(\frac{1}{\upsilon}-\beta)\frac{\mu^{\gamma_2}}{\gamma_2}}\mu^{\gamma_2-1}d\mu\right) \\ &= \frac{K\upsilon}{(\varsigma-\alpha)(\tau-\beta\upsilon)}, \end{aligned}$$

where $\mathrm{Re}(\varsigma) > \alpha$ and $\mathrm{Re}\left(\frac{\tau}{\upsilon}\right) > \beta$.

3.2. The conformable double Laplace-Shehu transform for some basic functions

(i)

$$L_\lambda^{\gamma_1}H_\mu^{\gamma_2}[c] = L_\lambda H_\mu[c] = \frac{c\upsilon}{\varsigma\tau}, \quad c \in \mathbb{R},$$

(ii)

$$L_\lambda^{\gamma_1}H_\mu^{\gamma_2}\left[\left(\frac{\lambda^{\gamma_1}}{\gamma_1}\right)^\alpha\left(\frac{\mu^{\gamma_2}}{\gamma_2}\right)^\beta\right]$$

$$\begin{aligned} &= L_\lambda H_\mu[\lambda^\alpha\mu^\beta] \\ &= \frac{\upsilon^{\beta+1}}{\varsigma^{\alpha+1}\tau^{\beta+1}}\Gamma(\alpha+1)\Gamma(\beta+1),\ \mathrm{Re}(\varsigma) > 0 \text{ and } \mathrm{Re}(\alpha) > -1, \end{aligned}$$

(iii)

$$L_\lambda^{\gamma_1}H_\mu^{\gamma_2}\left[e^{\alpha\frac{\lambda^{\gamma_1}}{\gamma_1}+\beta\frac{\mu^{\gamma_2}}{\gamma_2}}\right]$$

$$= L_\lambda H_\mu[e^{\alpha\lambda+\beta\mu}] = \frac{\upsilon}{(\varsigma-\alpha)(\tau-\beta\upsilon)}, \mathrm{Re}(\varsigma) > \mathrm{Re}(\alpha).$$

3.3. Derivatives properties

Now, we present some basic properties of the CL-SH

Let $R(\varsigma,\upsilon) = L_\lambda^{\gamma_1}H_\mu^{\gamma_2}(r(\lambda,\mu))$ where $r(\lambda,\mu)$ is a continuous function on $[0,\infty)\times[0,\infty)$. Then

(i)

$$L_\lambda^{\gamma_1}H_\mu^{\gamma_2}\left(\frac{\partial^{\gamma_1}r(\lambda,\mu)}{\partial\lambda^{\gamma_1}}\right) = \varsigma R(\varsigma,\upsilon) - H_\mu^{\gamma_2}(r(0,\mu)), \tag{2}$$

(ii)

$$L_\lambda^{\gamma_1}H_\mu^{\gamma_2}\left(\frac{\partial^{2\gamma_1}r(\lambda,\mu)}{\partial\lambda^{2\gamma_1}}\right) = \varsigma^2R(\varsigma,\upsilon) - \varsigma H_\mu^{\gamma_2}(r(0,\mu)) - H_\mu^{\gamma_2}(\frac{\partial^{\gamma_1}r(0,\mu)}{\partial\lambda^{\gamma_1}}), \tag{3}$$

(iii)

$$L_{\lambda}^{\gamma_1} H_{\mu}^{\gamma_2}\left(\frac{\partial^{\gamma_2} r(\lambda,\mu)}{\partial\mu^{\gamma_2}}\right)=\frac{\tau}{\upsilon}R(\varsigma,\upsilon)-L_{\lambda}^{\gamma_1}(r(\lambda,0)), \tag{4}$$

(iv)

$$L_{\lambda}^{\gamma_1} H_{\mu}^{\gamma_2}\left(\frac{\partial^{2\gamma_2} r(\lambda,\mu)}{\partial\mu^{2\gamma_2}}\right)=\frac{\tau^2}{\upsilon^2}R(\varsigma,\upsilon)-\frac{\tau}{\upsilon}L_{\lambda}^{\gamma_1}(r(\lambda,0))-L_{\lambda}^{\gamma_1}\left(\frac{\partial^{\gamma_2} r(\lambda,0)}{\partial\mu^{\gamma_2}}\right). \tag{5}$$

Proof. Proof of Equation 2 $L_{\lambda}^{\gamma_1} W_{\mu}^{\gamma_2}\left(\frac{\partial^{\gamma_1} r(\lambda,\mu)}{\partial\lambda^{\gamma_1}}\right)=\int_0^{\infty}\int_0^{\infty} e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)}\frac{\partial^{\gamma_1} r(\lambda,\mu)}{\partial\lambda^{\gamma_1}}\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu.$
By Theorem 1, we have $\frac{\partial^{\gamma_1} r(\lambda,\mu)}{\partial\lambda^{\gamma_1}}=\lambda^{1-\gamma_1}\frac{\partial r(\lambda,\mu)}{\partial\lambda}$. So,
$L_{\lambda}^{\gamma_1} W_{\mu}^{\gamma_2}\left(\frac{\partial^{\gamma_1} r(\lambda,\mu)}{\partial\lambda^{\gamma_1}}\right)=\int_0^{\infty} e^{-\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}}\mu^{\gamma_2-1}\int_0^{\infty} e^{-\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}}\frac{\partial r(\lambda,\mu)}{\partial\lambda}d\lambda d\mu.$
By integrating by parts, we get

$$L_{\lambda}^{\gamma_1} W_{\mu}^{\gamma_2}\left(\frac{\partial^{\gamma_1} r(\lambda,\mu)}{\partial\lambda^{\gamma_1}}\right)=\int_0^{\infty} e^{-\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}}\mu^{\gamma_2-1}\left(-r(0,\mu)+\varsigma\int_0^{\infty} e^{-\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}}r(\lambda,\mu)\lambda^{\gamma_1-1}\,d\lambda\right)d\mu$$

$$=-\int_0^{\infty} e^{-\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}}r(0,\mu)\mu^{\gamma_2-1}d\mu+\varsigma\int_0^{\infty}\int_0^{\infty} e^{-\left(\varsigma\frac{\lambda^{\gamma_1}}{\gamma_1}+\tau\frac{\mu^{\gamma_2}}{\upsilon\gamma_2}\right)}r(\lambda,\mu)\lambda^{\gamma_1-1}\mu^{\gamma_2-1}d\lambda d\mu$$

$$=\varsigma R(\varsigma,\upsilon)-W_{\mu}^{\gamma_2}(r(0,\mu)).$$

The proof of Equations 3, 4 and 5 can be obtained in the same manner.

In Table 1, we have the CL-SH of some basic functions.

Table 1: Table of conformable double Laplace-Shehu transform

$r(\lambda,\mu)$	$L_{\lambda}^{\gamma_1} H_{\mu}^{\gamma_2}(r(\lambda,\mu))$
c	$\frac{c\upsilon}{\varsigma\tau}$, $\mathrm{Re}(\varsigma)>0$
$\left(\frac{\lambda^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{\mu^{\gamma_2}}{\gamma_2}\right)^{\beta}$	$\frac{\upsilon^{\beta+1}}{\varsigma^{\alpha+1}\tau^{\beta+1}}\Gamma(\alpha+1)\Gamma(\beta+1)$, $\mathrm{Re}(\varsigma)>0$ and $\mathrm{Re}(\alpha)>-1$
$e^{\alpha\frac{\lambda^{\gamma_1}}{\gamma_1}+\beta\frac{\mu^{\gamma_2}}{\gamma_2}}$	$\frac{\upsilon}{(\varsigma-\alpha)(\tau-\beta\upsilon)}$, $\mathrm{Re}(\varsigma)>\mathrm{Re}(\alpha)$
$e^{i\left(\alpha\frac{\lambda^{\gamma_1}}{\gamma_1}+\beta\frac{\mu^{\gamma_2}}{\gamma_2}\right)}$	$\frac{i\upsilon}{(\varsigma-i\alpha)(\tau-i\beta\upsilon)}$, $\mathrm{Im}(\alpha)+\mathrm{Re}(\varsigma)>0$
$\sin\left(\alpha\frac{\lambda^{\gamma_1}}{\gamma_1}+\beta\frac{\mu^{\gamma_2}}{\gamma_2}\right)$	$\frac{\upsilon(\tau\alpha+\varsigma\upsilon\beta)}{(\varsigma^2+\alpha^2)(\tau^2+\beta^2\upsilon^2)}$, $\lvert\mathrm{Im}(\alpha)\rvert<\mathrm{Re}(\varsigma)$
$\cos\left(\alpha\frac{\lambda^{\gamma_1}}{\gamma_1}+\beta\frac{\mu^{\gamma_2}}{\gamma_2}\right)$	$\frac{\upsilon(\varsigma\tau-\upsilon\alpha\beta)}{(\varsigma^2+\alpha^2)(\tau^2+\beta^2\upsilon^2)}$, $\lvert\mathrm{Im}(\alpha)\rvert<\mathrm{Re}(\varsigma)$
$\sinh\left(\alpha\frac{\lambda^{\gamma_1}}{\gamma_1}+\beta\frac{\mu^{\gamma_2}}{\gamma_2}\right)$	$\frac{\upsilon(\tau\alpha+\varsigma\upsilon\beta)}{(\varsigma^2-\alpha^2)(\tau^2-\beta^2\upsilon^2)}$, $\mathrm{Re}(\varsigma)>\mathrm{Re}(\alpha)$ and $\mathrm{Re}(\varsigma)+\mathrm{Re}(\alpha)>0$
$\cosh\left(\alpha\frac{\lambda^{\gamma_1}}{\gamma_1}+\beta\frac{\mu^{\gamma_2}}{\gamma_2}\right)$	$\frac{\upsilon(\varsigma\tau-\upsilon\alpha\beta)}{(\varsigma^2-\alpha^2)(\tau^2-\beta^2\upsilon^2)}$, $\mathrm{Re}(\varsigma)>\mathrm{Re}(\alpha)$ and $\mathrm{Re}(\varsigma)+\mathrm{Re}(\alpha)>0$
$p(\lambda)q(\mu)$	$L_{\lambda}^{\gamma_1}(p(\lambda))H_{\mu}^{\gamma_2}(q(\mu))$

4. Applications

In this section, we apply the CL-SH transform to solve some conformable partial differential equations.

Example 1. *Consider the conformable wave equation*

$$\frac{\partial^{2\gamma_1} r(\lambda,\mu)}{\partial\lambda^{2\gamma_1}} + 4\frac{\partial^{2\gamma_2} r(\lambda,\mu)}{\partial\mu^{2\gamma_2}} = 6\frac{\lambda^{\gamma_1}}{\gamma_1},\ \textit{where}\ \lambda,\mu \geq 0 \tag{6}$$

With initial conditions (ICs)
$r(\lambda,0) = \left(\frac{\lambda^{\gamma_1}}{\gamma_1}\right)^3$, $\frac{\partial^{\gamma_2} r(\lambda,0)}{\partial\mu^{\gamma_2}} = \cos\left(2\frac{\lambda^{\gamma_1}}{\gamma_1}\right)$,
and boundary conditions (BCs)
$r(0,\mu) = \sinh\left(\frac{\mu^{\gamma_2}}{\gamma_2}\right)$, $\frac{\partial^{\gamma_1} r(0,\mu)}{\partial\lambda^{\gamma_1}} = 0$.

Solution 1. *By applying the CL to the ICs and the CSH to the BCs, we get*
$L_\lambda^{\gamma_1}\left(\left(\frac{\lambda^{\gamma_1}}{\gamma_1}\right)^3\right) = \frac{6}{\varsigma^4}$, $L_\lambda^{\gamma_1}\left(\cos\left(2\frac{\lambda^{\gamma_1}}{\gamma_1}\right)\right) = \frac{\varsigma}{\varsigma^2+4}$, $H_\mu^{\gamma_2}\left(\sinh\left(\frac{\mu^{\gamma_2}}{\gamma_2}\right)\right) = \frac{\upsilon^2}{\tau^2-\upsilon^2}$, $H_\mu^{\gamma_2}(0) = 0$.
Apply the CL-SH to Equation 6, we get

$$\varsigma^2 R - \frac{\varsigma\upsilon^2}{\tau^2-\upsilon^2} + \frac{4\tau^2}{\upsilon^2}R - \frac{24\tau}{\varsigma^4\upsilon} - \frac{4\varsigma}{\varsigma^2+4} = \frac{6\upsilon}{\varsigma^2\tau}$$

So,

$$\begin{aligned} R(\varsigma,\upsilon) &= \frac{\frac{\varsigma\upsilon^2}{\tau^2-\upsilon^2} + \frac{24\tau}{\varsigma^4\upsilon} + \frac{4\varsigma}{\varsigma^2+4} + \frac{6\upsilon}{\varsigma^2\tau}}{\varsigma^2 + \frac{4\tau^2}{\upsilon^2}} \\ &= \frac{\frac{\varsigma^3\upsilon^2+4\varsigma\tau^2}{(\varsigma^2+4)(\tau^2-\upsilon^2)} + \frac{24\tau^2+6\varsigma^2\upsilon^2}{\varsigma^4\tau\upsilon}}{\frac{\varsigma^2\upsilon^2+4\tau^2}{\upsilon^2}} \end{aligned}$$

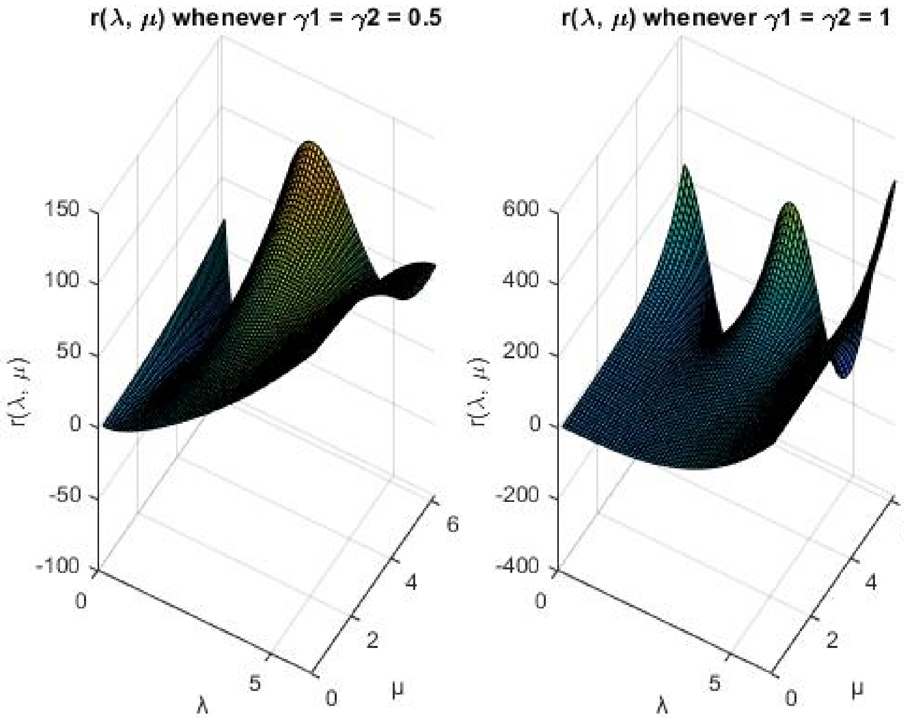
By simplify,

$$R(\varsigma,\upsilon) = \frac{\varsigma\upsilon^2}{(\varsigma^2+4)(\tau^2-\upsilon^2)} + \frac{6\upsilon}{\varsigma^4\tau}.$$

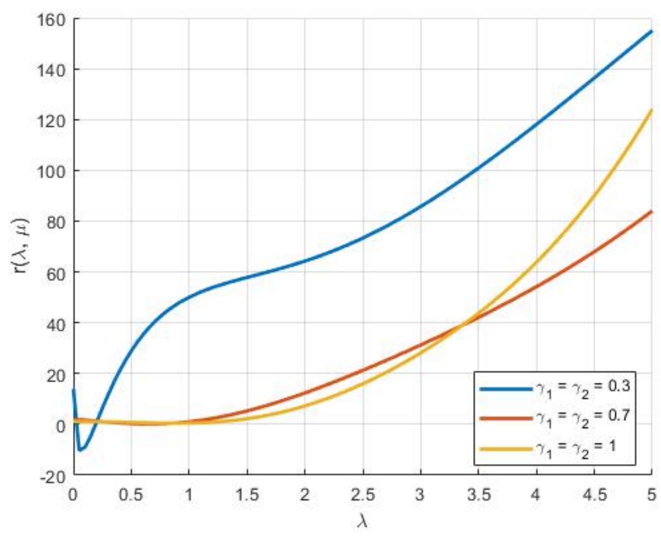
So,

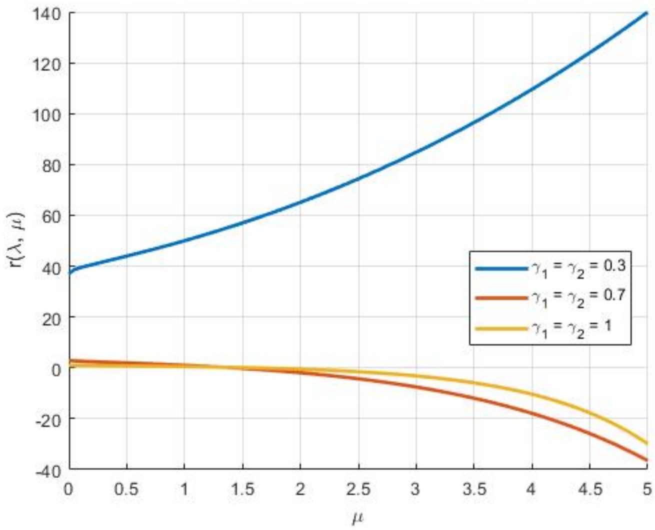
$$r(\lambda,\mu) = \left(L_\lambda^{\gamma_1}\right)^{-1}\left(H_\mu^{\gamma_2}\right)^{-1}\left(\frac{\varsigma\upsilon^2}{(\varsigma^2+4)(\tau^2-\upsilon^2)} + \frac{6\upsilon}{\varsigma^4\tau}\right) = \cos\left(2\frac{\lambda^{\gamma_1}}{\gamma_1}\right)\sinh\left(\frac{\mu^{\gamma_2}}{\gamma_2}\right) + \left(\frac{\lambda^{\gamma_1}}{\gamma_1}\right)^3.$$

The following figures show the 3D representation of the solution at $\gamma_1 = \gamma_2 = 0.5, 1$.



The following two figures illustrates the 2D graph of the solution with respect to λ and μ at $\gamma_1 = \gamma_2 = 0.3, 0.7, 1$.





Example 2. *Consider the conformable heat equation*

$$\frac{\partial^{\gamma_1} r(\lambda,\mu)}{\partial \lambda^{\gamma_1}} = \frac{\partial^{2\gamma_2} r(\lambda,\mu)}{\partial \mu^{2\gamma_2}} - 3, \textit{ where } \lambda,\mu \geq 0 \tag{7}$$

With IC
$r(\lambda,0) = -3\frac{\lambda^{\gamma_1}}{\gamma_1}$, $\frac{\partial^{\gamma_2} r(\lambda,0)}{\partial \mu^{\gamma_2}} = e^{-\frac{\lambda^{\gamma_1}}{\gamma_1}}$,
and BCs
$r(0,\mu) = \sin\left(\frac{\mu^{\gamma_2}}{\gamma_2}\right)$.

Solution 2. *By applying the CL to the IC and the CSH to the BCs, we get*
$L_\lambda^{\gamma_1}\left(-3\frac{\lambda^{\gamma_1}}{\gamma_1}\right) = \frac{-3}{\varsigma^2}$, $L_\lambda^{\gamma_1}\left(e^{-\frac{\lambda^{\gamma_1}}{\gamma_1}}\right) = \frac{1}{\varsigma+1}$, $H_\mu^{\gamma_2}\left(\sin\left(\frac{\mu^{\gamma_2}}{\gamma_2}\right)\right) = \frac{\upsilon^2}{\tau^2+\upsilon^2}$.
Apply the CL-SH to Equation 7, we get

$$\varsigma R - \frac{\upsilon^2}{\tau^2+\upsilon^2} = \frac{\tau^2}{\upsilon^2} R + \frac{3\tau}{\varsigma^2\upsilon} - \frac{1}{\varsigma+1} - \frac{3\upsilon}{\varsigma\tau}.$$

So,

$$R(\varsigma,\upsilon) = \frac{\frac{\upsilon^2}{\tau^2+\upsilon^2} + \frac{3\tau}{\varsigma^2\upsilon} - \frac{1}{\varsigma+1} - \frac{3\upsilon}{\varsigma\tau}}{\varsigma - \frac{\tau^2}{\upsilon^2}}.$$

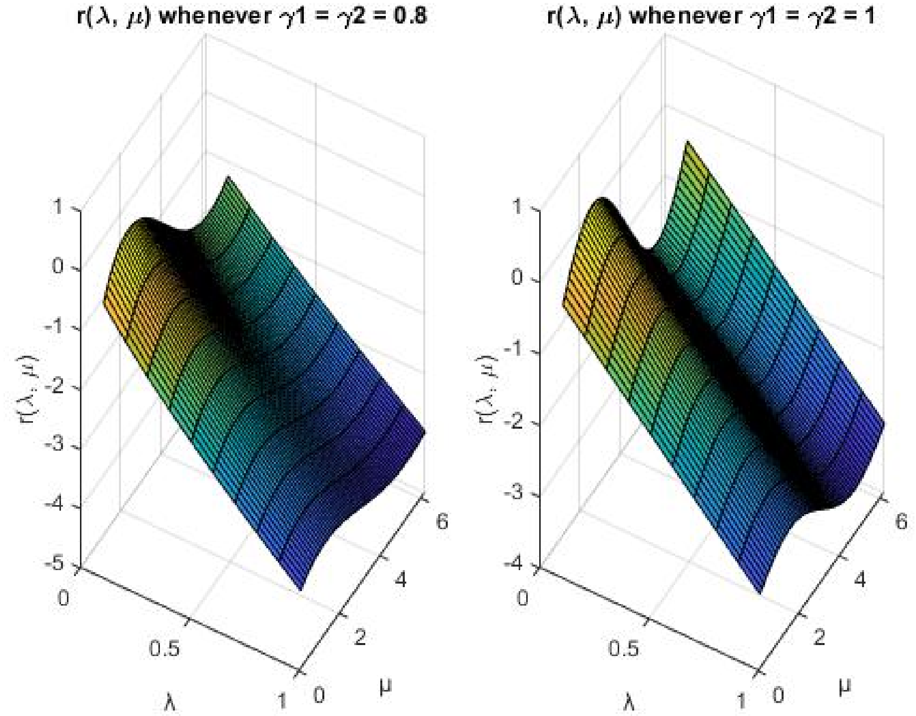
By simplify,

$$R(\varsigma,\upsilon) = \frac{\upsilon^2}{(\varsigma+1)(\tau^2+\upsilon^2)} - \frac{3\upsilon}{\varsigma^2\tau}.$$

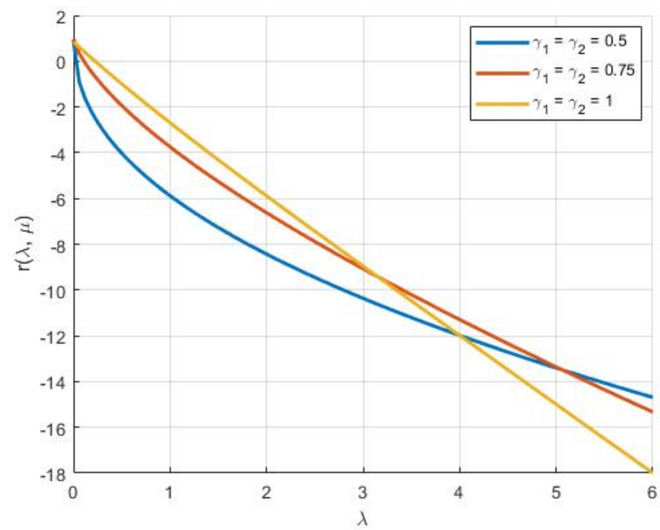
So,

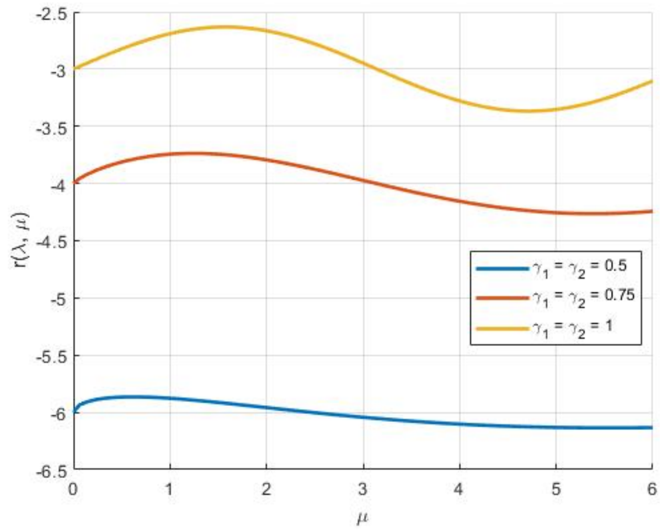
$$r(\lambda,\mu) = \left(L_\lambda^{\gamma_1}\right)^{-1}\left(H_\mu^{\gamma_2}\right)^{-1}\left(\frac{\upsilon^2}{(\varsigma+1)(\tau^2+\upsilon^2)} - \frac{3\upsilon}{\varsigma^2\tau}\right) = e^{-\frac{\lambda^{\gamma_1}}{\gamma_1}}\sin\left(\frac{\mu^{\gamma_2}}{\gamma_2}\right) - 3\frac{\lambda^{\gamma_1}}{\gamma_1}.$$

The following figures show the 3D representation of the solution at $\gamma_1 = \gamma_2 = 0.8, 1$.



The following two figures illustrates the 2D graph of the solution with respect to λ *and* μ *at* $\gamma_1 = \gamma_2 = 0.5, 0.75, 1$.





5. Conclusion

We introduced a new double transform based on the conformable approach. We showed that it works well for solving fractional differential equations. The examples we gave proved that the method is simple and gives correct results. This shows that the CL-SH is a helpful tool in this area. Future work may include other equations and systems.

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