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Generalized SWAP and iSWAP and Solutions to the Yang–Baxter Equation in All Dimensions

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Abstract. We introduce an infinite family of universal quantum logic gates that includes not only higher-dimensional versions of the usual SWAP and **i**SWAP gates, but also their previously known extensions. This family consists of permutation-like matrices with nonzero entries of the form $e^{\mathbf{i}\alpha_i}$, where the α_i are arbitrary real numbers. Moreover, we show that these gates, which we refer to as αSWAP, provide unitary solutions to the constant Yang–Baxter equation in all dimensions.

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1. Introduction

The d-level qudit-based computing, where $d \geq 3$, has been at the center of attention in recent research. This is partly due to the advantages of d-level qudits ($d > 2$) over conventional qubits ($d = 2$) in quantum computing and quantum information. In parallel, there have been developments in higher-dimensional quantum logic gates, both theoretically and in terms of implementation [1–12].

In dimension two, the familiar swap gate (SWAP) and iswap gate (**i**SWAP) are among the most important quantum gates in quantum computing [9–11, 13, 14]. The use of SWAP and **i**SWAP gates is crucial in implementing quantum algorithms, particularly for moving information to resolve the neighborhood constraints of qubit topology. The **i**SWAP gate is also entangling and hence a universal gate. The study of these gates and their generalizations, from various perspectives and their physical implementations, remains an active area of research [6, 8–11, 15–17].

On a related topic, the Yang–Baxter equation and its solutions have played a fundamental role in several areas of physics and mathematics [18–22]. Unitary solutions to the constant Yang–Baxter equation for $d = 2$ are well-known sources of quantum logic gates [23–25], and are therefore important tools in quantum information theory. They also play a crucial role in exploring the relationship between quantum entanglement and topological

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entanglement [26–30]. The search for new solutions to the Yang–Baxter equation and their applications is a common endeavor in both mathematics and physics [6, 21, 22, 26, 31–35].

The goal of this paper is to introduce an infinite family of universal quantum logic gates that not only extends the usual SWAP and **i**SWAP gates to all dimensions $d \geq 2$, beyond their previously known extensions [15], but also provides an infinite family of unitary solutions to the constant Yang–Baxter equation. The organization of the paper is as follows. In Section 2, we recall some preliminary notions and notations. In Section 3, specifically in Definition 1, we give an explicit description of the quantum gate named αSWAP and present concrete examples. In Subsection 3.1, we prove that αSWAP is a universal gate for quantum computing. In Subsection 3.2, we prove that αSWAP is a unitary solution to the Yang–Baxter equation. We conclude with final remarks in Section 4.

2. Preliminaries and Notations

In what follows, to avoid confusion, we use i to denote subscripts, as in α_i, and we use $\mathbf{i}$ to denote the imaginary unit $\sqrt{-1}$.

Recall that in quantum computing, quantum logic gates are represented by unitary matrices acting on a Hilbert space $\mathscr{H}$ of dimension d, i.e., $\mathscr{H} = \mathbb{C}^d$. A qubit is the fundamental unit of quantum information in dimension $d = 2$, while a qudit generalizes this concept to a higher-dimensional Hilbert space with $d \geq 2$. Throughout this paper we use the standard basis

$$\Big\{|0\rangle,\, |1\rangle,\, |2\rangle,\, \cdots,\, |d-1\rangle\Big\}$$

for $\mathbb{C}^d$. We also use the computational basis for 2-qudit states. For instance, in dimension $d = 2$, the computational basis for 2-qubit states is

$$\Big\{|00\rangle,\, |01\rangle,\, |10\rangle,\, |11\rangle\Big\}.$$

In dimension $d = 3$, the computational basis for 2-qutrit states is

$$\Big\{|00\rangle,\, |01\rangle,\, |02\rangle,\, |10\rangle,\, |11\rangle,\, |12\rangle,\, |20\rangle,\, |21\rangle,\, |22\rangle\Big\}.$$

In dimension $d = 2$, i.e., for qubit systems, two very important examples of quantum logic gates are the swap gate (SWAP), denoted here by S, and the iswap gate (**i**SWAP), denoted here by $\mathbf{i}S$. They are defined as follows, for all $u, v \in \mathscr{H}$:

$$\begin{gathered} SWAP : \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{H} \\ S(u \otimes v) = v \otimes u \end{gathered} \tag{1}$$

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

$$\begin{aligned} \mathbf{i}SWAP &: \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{H} \\ \mathbf{i}S(u \otimes v) &= \mathbf{i}(v \otimes u), \quad \text{when} \quad v \neq u, \\ \mathbf{i}S(u \otimes u) &= u \otimes u \end{aligned} \tag{3}$$

$$\mathbf{i}SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{i} & 0 \\ 0 & \mathbf{i} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

An n-qudit quantum logic gate is a unitary operator on $(\mathbb{C}^d)^{\otimes n}$. A quantum logic gate is said to be entangling if it can produce entangled states from unentangled ones. A 2-qudit gate G (i.e., a gate acting on two d-level qudits) is universal if the set consisting of G together with all 1-qudit gates is *sufficient* to generate all qudit gates. It is well known that a quantum logic gate acting on two d-level qudits is universal for quantum computing [13] if and only if it is entangling [30].

On a related note, entangling (unitary) solutions of the Yang-Baxter equation [18–20] have proven useful in quantum information theory, primarily as effective sources of universal quantum logic gates [23, 25, 30]. They have also been shown to produce non-constant knot invariants [26], via the Turaev invariant [28]. Consequently, they play an important role in linking quantum and topological entanglement [23, 26].

Let V be any complex vector space, and let I denote the identity map on V. A linear operator $R : V \otimes V \to V \otimes V$ is said to be a solution to the constant braided Yang–Baxter equation (BYBE) if [18–20]

$$R_{23}R_{12}R_{23} = R_{12}R_{23}R_{12}. \tag{5}$$

Similarly, a linear map $\hat{R} : V \otimes V \to V \otimes V$ is a solution to the constant quantum Yang–Baxter equation (QYBE) if

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23} = \hat{R}_{23}\hat{R}_{13}\hat{R}_{12}. \tag{6}$$

In the above relations, the operators R_{12}, R_{23}, and R_{13} all belong to $End(V \otimes V \otimes V)$ and are defined as follows:

$$R_{12} := R \otimes I,$$

$$R_{23} := I \otimes R,$$
$$R_{13}(u, v) := (I \otimes S)R_{12}(I \otimes S),$$

where the map $S : V \otimes V \to V \otimes V$ is the usual swap gate (SWAP), sometimes called the flip map.

Remark 1. Equations (5) and (6) represent two sides of the same coin in the following sense. For any solution R to equation (5), the transformation $\hat{R} = RS$ (or SR) yields a solution to equation (6), and vice versa. Finding all solutions to these equations for dimensions $d > 2$ is extremely challenging and remains an open problem [18, 19, 25, 29].

3. αSWAP, a universal quantum logic gate and a solution to the Yang–Baxter equation

In this section we introduce an infinite family of universal quantum logic gates that extends the usual SWAP and **i**SWAP gates to all dimensions $d \geq 2$, beyond their previously found extensions in [15]. We name these gates αSWAP. Here, α represents a d-tuple, $(\alpha_1, \alpha_2, \cdots, \alpha_d)$, in which the α_i's are arbitrary real numbers. Moreover, we show that αSWAP provides solutions to the constant Yang–Baxter equations (5) and (6) in all dimensions.

Definition 1. *For any $d \geq 2$ and for any set of real numbers $\alpha_1, \alpha_2, \cdots, \alpha_{d^2}$, we define the quantum gate αSWAP (denoted by αS for short) by the following relations:*

$$\alpha S_{i,j} = e^{\mathbf{i}\alpha_i}, \qquad \text{for} \quad i = (t-1)d + s \quad \text{and} \quad j = (s-1)d + t, \tag{7}$$

where $1 \leq t \leq d$ and $1 \leq s \leq d$. We let $\alpha S_{i,j} = 0$ for all other values of i, j. Here, $\alpha S_{i,j}$ denotes the entry of the αSWAP located at row i and column j.

For both $d = 2$ and $d = 3$, the gate αSWAP is illustrated below.

$$\alpha S = \begin{pmatrix} e^{\mathbf{i}\alpha_1} & 0 & 0 & 0 \\ 0 & 0 & e^{\mathbf{i}\alpha_2} & 0 \\ 0 & e^{\mathbf{i}\alpha_3} & 0 & 0 \\ 0 & 0 & 0 & e^{\mathbf{i}\alpha_4} \end{pmatrix}$$

$$\alpha S = \begin{pmatrix} e^{\mathbf{i}\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\mathbf{i}\alpha_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\mathbf{i}\alpha_3} & 0 & 0 \\ 0 & e^{\mathbf{i}\alpha_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\mathbf{i}\alpha_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\mathbf{i}\alpha_6} & 0 \\ 0 & 0 & e^{\mathbf{i}\alpha_7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\mathbf{i}\alpha_8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\mathbf{i}\alpha_9} \end{pmatrix}$$

Remark 2. To give the reader some intuition behind the relations in (7), let us compute the indices for $d = 3$. Consider $1 \leq t \leq 3$ and $1 \leq s \leq 3$.

If we fix $t = 1$ and compute the indices for $s = 1, 2, 3$, we obtain:

$$\alpha S_{11} = e^{\mathbf{i}\alpha_1}, \quad \alpha S_{24} = e^{\mathbf{i}\alpha_2}, \quad \alpha S_{37} = e^{\mathbf{i}\alpha_3}$$

corresponding to $s = 1$, $s = 2$, and $s = 3$, respectively.

Similarly, fixing $t = 2$ and computing the indices for $s = 1, 2, 3$, we get:

$$\alpha S_{42} = e^{\mathbf{i}\alpha_4}, \quad \alpha S_{55} = e^{\mathbf{i}\alpha_5}, \quad \alpha S_{68} = e^{\mathbf{i}\alpha_6}.$$

Finally, if we fix $t = 3$ and compute the indices for $s = 1, 2, 3$, we have:

$$\alpha S_{73} = e^{\mathbf{i}\alpha_7}, \quad \alpha S_{86} = e^{\mathbf{i}\alpha_8}, \quad \alpha S_{99} = e^{\mathbf{i}\alpha_9}.$$

The calculations for $d = 2$ are similar and even simpler.

Note that the condition $t = s$ corresponds to the diagonal (nonzero) entries αS_{ii}, where $i = j = (t-1)d + t$, $1 \leq t \leq d$. In the case of $d = 3$, these entries are αS_{11}, αS_{55}, and αS_{99}, whereas for $d = 2$, they are only αS_{11} and αS_{44}.

Remark 3. For clearer illustration, let us write the action of αSWAP (denoted by αS for short) on the computational basis for 2-qubit and 2-qutrit gates.

For $d = 2$, the action of αS on the computational basis states is given by:

$$\begin{aligned}
\alpha S(|00\rangle) &= e^{\mathbf{i}\alpha_1}|00\rangle, \\
\alpha S(|01\rangle) &= e^{\mathbf{i}\alpha_2}|10\rangle, \\
\alpha S(|10\rangle) &= e^{\mathbf{i}\alpha_3}|01\rangle, \\
\alpha S(|11\rangle) &= e^{\mathbf{i}\alpha_4}|11\rangle.
\end{aligned}$$

Similarly, for $d = 3$, the action of αS on the computational basis states is:

$$\begin{aligned}
\alpha S(|00\rangle) &= e^{\mathbf{i}\alpha_1}|00\rangle, \\
\alpha S(|01\rangle) &= e^{\mathbf{i}\alpha_2}|10\rangle, \\
\alpha S(|02\rangle) &= e^{\mathbf{i}\alpha_3}|20\rangle, \\
\alpha S(|10\rangle) &= e^{\mathbf{i}\alpha_4}|01\rangle, \\
\alpha S(|11\rangle) &= e^{\mathbf{i}\alpha_5}|11\rangle, \\
\alpha S(|12\rangle) &= e^{\mathbf{i}\alpha_6}|21\rangle, \\
\alpha S(|20\rangle) &= e^{\mathbf{i}\alpha_7}|02\rangle, \\
\alpha S(|21\rangle) &= e^{\mathbf{i}\alpha_8}|12\rangle, \\
\alpha S(|22\rangle) &= e^{\mathbf{i}\alpha_9}|22\rangle.
\end{aligned}$$

Now let us proceed to some examples. The xSWAP gate introduced in [15] is, in fact, a special case of the above definition, as illustrated in the following example.

Example 1. *Let $\alpha_i = 0$ for the diagonal (nonzero) entries, i.e., for $i = j = (t-1)d + t$ with $1 \leq t \leq d$, and let $\alpha_i = \alpha$ otherwise. If we set $x = e^{\mathbf{i}\alpha}$, then this gate coincides with the xSWAP from [15]. The xSWAP, in turn, includes the usual SWAP for $x = 1$ (i.e., $\alpha = 0$) and the $\mathbf{i}$SWAP for $x = \mathbf{i}$ (i.e., $\alpha = \frac{\pi}{2}$) in all dimensions. Below, we illustrate the xSWAP matrices for $d = 2$ and $d = 3$.*

$$xSWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad xSWAP = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By choosing different values for the α_i's, the αSWAP generalizes both the SWAP and $\mathbf{i}$SWAP gates beyond the xSWAP example given above. A few such examples follow.

Example 2. *Let $\alpha_i = (-1)^{\frac{i}{2}}(\frac{\pi}{2})$ for even i, and let $\alpha_i = \pi$ for odd i. Below, we illustrate this particular αSWAP for $d = 2$ and $d = 3$.*

$$\alpha SWAP = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{i} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{i} \end{pmatrix}, \quad \alpha SWAP = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \mathbf{i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{i} & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Example 3. *Let $\alpha_i = \dfrac{\pi}{4}$ for diagonal (nonzero) entries, i.e., for $i = j = (t-1)d + t$, and let $\alpha_i = \dfrac{\pi}{3}$ otherwise. We illustrate this αSWAP for $d = 2$ and $d = 3$.*

$$\alpha SWAP = \begin{pmatrix} \frac{\sqrt{2}}{2}(1+\mathbf{i}) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1+\sqrt{3}\,\mathbf{i}) & 0 \\ 0 & \frac{1}{2}(1+\sqrt{3}\,\mathbf{i}) & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2}(1+\mathbf{i}) \end{pmatrix}$$

$$\alpha SWAP = \begin{pmatrix} \frac{\sqrt{2}}{2}(1+\mathbf{i}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1+\sqrt{3}\,\mathbf{i}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1+\sqrt{3}\,\mathbf{i}) & 0 & 0 \\ 0 & \frac{1}{2}(1+\sqrt{3}\,\mathbf{i}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2}(1+\mathbf{i}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1+\sqrt{3}\,\mathbf{i}) & 0 \\ 0 & 0 & \frac{1}{2}(1+\sqrt{3}\,\mathbf{i}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1+\sqrt{3}\,\mathbf{i}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2}(1+\mathbf{i}) \end{pmatrix}$$

3.1. αSWAP is a universal quantum logic gate

From Definition 1, it is clear that αSWAP is unitary. From [30] it is well known that a 2-qudit gate (i.e., a gate operating on two d-level qudits) is universal (i.e., forms a universal set together with all 1-qudit gates) if and only if it is entangling (i.e., it can create entangled states from some non-entangled ones). However, the entangling property of αSWAP is not obvious. This fact is well known and easy to prove only for **i**SWAP in dimension two.

In this section we will prove that αSWAP is almost always a universal quantum gate. We use the non-entangling criterion from [26]. To show that any operator $R : \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is entangling, it suffices to show that neither R nor RS can be factored as $X \otimes Y$, where X and Y are arbitrary operators on $\mathscr{H}$, i.e., $X, Y : \mathscr{H} \to \mathscr{H}$. Here S is the usual SWAP gate with the same dimension as R.

Theorem 1. *For any $d \geq 2$ the αSWAP from Definition 1 is entangling, hence universal [30], if the following condition is satisfied:*

$$e^{\mathbf{i}\alpha_1} e^{\mathbf{i}\alpha_{d^2}} \neq e^{\mathbf{i}\alpha_d} e^{\mathbf{i}\alpha_{d^2-d+1}} \tag{8}$$

Proof. We prove this theorem by showing that the condition (8) is sufficient to ensure that neither αSWAP nor (αSWAP)S can be factored as $X \otimes Y$ [26]. In fact, as we will see, αSWAP can never be factored.

To see this, suppose the contrary: assume that αSWAP can be written as $X \otimes Y$ for two $d \times d$ matrices X and Y with entries denoted by x_{ij} and y_{ij}, respectively. By focusing solely on the diagonal entries at positions $(1, 1)$, (d, d), $(d^2 - d + 1, d^2 - d + 1)$, and (d^2, d^2) in both αSWAP and $X \otimes Y$, we obtain the following equalities:

On one hand,

$$(x_{11}y_{11})(x_{dd}y_{dd}) = e^{\mathbf{i}\alpha_1} e^{\mathbf{i}\alpha_{d^2}},$$

and on the other hand,

$$(x_{11}y_{dd})(x_{dd}y_{11}) = (0)(0) = 0.$$

However, the left sides of these expressions must be equal. Thus, assuming that αSWAP can be factored as $X \otimes Y$ leads to the contradiction

$$e^{\mathbf{i}\alpha_1} e^{\mathbf{i}\alpha_{d^2}} = 0.$$

Therefore, αSWAP cannot be factored. Below, we illustrate this argument by highlighting the relevant entries with boxes:

$$\alpha\text{SWAP} = \left(\begin{array}{ccc|ccc|ccc} \boxed{e^{\mathbf{i}\alpha_1}} & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & \boxed{0} & * & * & * & * & * & * \\ \hline * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ \hline * & * & * & * & * & * & \boxed{0} & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & \boxed{e^{\mathbf{i}\alpha_{d^2}}} \end{array}\right)$$

$$= \begin{pmatrix} x_{11} & * & * \\ * & * & * \\ * & * & x_{dd} \end{pmatrix} \otimes \begin{pmatrix} y_{11} & * & * \\ * & * & * \\ * & * & y_{dd} \end{pmatrix}$$

$$= \left(\begin{array}{ccc|ccc|ccc} \boxed{x_{11}y_{11}} & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & \boxed{x_{11}y_{dd}} & * & * & * & * & * & * \\ \hline * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ \hline * & * & * & * & * & * & \boxed{x_{dd}y_{11}} & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & \boxed{x_{dd}y_{dd}} \end{array}\right)$$

For $(\alpha\text{SWAP})S$, a similar argument (illustrated below with matrices and boxed entries) shows that assuming $(\alpha\text{SWAP})S = X \otimes Y$ leads to the equality

$$e^{\mathbf{i}\alpha_1} e^{\mathbf{i}\alpha_{d^2}} = e^{\mathbf{i}\alpha_d} e^{\mathbf{i}\alpha_{d^2-d+1}}.$$

Therefore, if

$$e^{\mathbf{i}\alpha_1} e^{\mathbf{i}\alpha_{d^2}} \neq e^{\mathbf{i}\alpha_d} e^{\mathbf{i}\alpha_{d^2-d+1}},$$

then $(\alpha\text{SWAP})S$ cannot be factored.

$$
(\alpha\text{SWAP})S = \left(\begin{array}{cccc|ccc|cccc}
\boxed{e^{\mathbf{i}\alpha_1}} & & * & * & * & * & * & & * & * & * \\
 & * & * & * & * & * & * & & * & * & * \\
 & * & * & \boxed{e^{\mathbf{i}\alpha_d}} & * & * & * & & * & * & * \\
\hline
 & * & * & * & * & * & * & & * & * & * \\
 & * & * & * & * & * & * & & * & * & * \\
 & * & * & * & * & * & * & & * & * & * \\
\hline
 & * & * & * & * & * & * & \boxed{e^{\mathbf{i}\alpha_{d^2-d+1}}} & & * & * \\
 & * & * & * & * & * & * & & * & * & * \\
 & * & * & * & * & * & * & & * & * & \boxed{e^{\mathbf{i}\alpha_{d^2}}}
\end{array}\right)
$$

$$
= \begin{pmatrix} x_{11} & * & * \\ * & * & * \\ * & * & x_{dd} \end{pmatrix} \otimes \begin{pmatrix} y_{11} & * & * \\ * & * & * \\ * & * & y_{dd} \end{pmatrix}
$$

$$
= \left(\begin{array}{cccc|ccc|cccc}
\boxed{x_{11}y_{11}} & & * & * & * & * & * & & * & * & * \\
 & * & * & * & * & * & * & & * & * & * \\
 & * & * & \boxed{x_{11}y_{dd}} & * & * & * & & * & * & * \\
\hline
 & * & * & * & * & * & * & & * & * & * \\
 & * & * & * & * & * & * & & * & * & * \\
 & * & * & * & * & * & * & & * & * & * \\
\hline
 & * & * & * & * & * & * & \boxed{x_{dd}y_{11}} & & * & * \\
 & * & * & * & * & * & * & & * & * & * \\
 & * & * & * & * & * & * & & * & * & \boxed{x_{dd}y_{dd}}
\end{array}\right)
$$

This completes the proof.

Remark 4. The criterion in Theorem 1 shows that αSWAP is almost always entangling, hence universal. This includes the gates from Example 1 (except for $x = 1$) and Example 2.

Remark 5. To directly prove a 2-qudit gate G is universal, one must show that the set containing G and all 1-qudit gates can generate all other qudit gates. Proving this directly is not always easy, even in dimension $d = 2$. This is why, in the proof of Theorem 1, we instead used the powerful Brylinskis' criterion [30] and the non-entangling criterion from [26].

However, below we illustrate a worked example showing that αSWAP explicitly generates entanglement, i.e., turns an unentangled state representation into an entangled one.

Example 4. *Consider the following gate for* $d = 2$ *from Example 2:*

$$
\alpha\text{SWAP} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{i} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{i} \end{pmatrix}
$$

If we apply this gate to the unentangled state

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle),$$

the outcome is the entangled state

$$\alpha\text{SWAP}|\psi\rangle = \frac{1}{2}(-|00\rangle - |01\rangle - \mathbf{i}|10\rangle + \mathbf{i}|11\rangle).$$

3.2. αSWAP is a solution to the constant Yang–Baxter equation

Theorem 2. *For any $d \geq 2$, the gate $R = \alpha\text{SWAP}$ from Definition 1 is a solution to the Yang–Baxter equation* (5)*. In other words, the gate $\hat{R} = (\alpha\text{SWAP})S$, where S denotes the usual SWAP gate, is a solution to the Yang–Baxter equation* (6) *(refer to Remark 1).*

Proof. For notational simplicity, we define $a_i = e^{\mathbf{i}\alpha_i}$. Given the straightforward structure of αSWAP from Definition 1, specifically that the only nonzero entries of αSWAP are a_i located at row $i = (t-1)d + s$ and column $j = (s-1)d + t$ with $1 \leq t \leq d$ and $1 \leq s \leq d$, we observe that the only nonzero blocks in $R_{12} = \alpha\text{SWAP} \otimes I$ are of the form $\text{diag}(a_i, a_i, \ldots, a_i)$, located at the block positions indexed by $i = (t-1)d + s$ and $j = (s-1)d + t$, where $1 \leq t \leq d$ and $1 \leq s \leq d$.

For instance, when $d = 2$, R_{12} takes the form:

$$R_{12} = \left(\begin{array}{cc|cc|cc|cc} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 \\ \hline 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 \end{array}\right)$$

Similarly, the only nonzero blocks in $R_{23} = I \otimes \alpha\text{SWAP}$ are its diagonal blocks, each being a copy of αSWAP. For example, when $d = 2$, R_{23} is given by:

$$R_{23} = \left(\begin{array}{cccc|cccc} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 \end{array}\right)$$

Recall the general identity for any three matrices X, Y, and Z, where the matrix product is defined: the (i, j)-entry of their product is given by;

$$(XYZ)_{i,j} = \sum_{k,l} (X)_{i,k}(Y)_{k,l}(Z)_{l,j} \tag{9}$$

Using equation (9), it is not very difficult to verify that for $R = \alpha\text{SWAP}$, both sides of equation (5) are equal, i.e., $R_{23}R_{12}R_{23} = R_{12}R_{23}R_{12}$. For example, when $d = 2$, both sides of equation (5) evaluate to:

$$\begin{pmatrix} a_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1a_2^2 & 0 & 0 & 0 \\ 0 & 0 & a_1a_2a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2^2a_4 & 0 \\ 0 & a_1a_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2a_3a_4 & 0 & 0 \\ 0 & 0 & 0 & a_3^2a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4^3 \end{pmatrix}$$

This completes the proof.

Remark 6. It is clear from the above proof that the result of Theorem 2 holds for any arbitrary set of complex numbers $a_1, a_2, \cdots, a_d$.

Examples of such solutions include (but are not limited to) the gates from Examples 1, 2, and 3. In what follows, we present another simple but interesting example.

Example 5. *For this example, we let $\alpha_i = 0$ when $i = (t-1)d + t$ for $1 \le t \le d$, and otherwise, if i is even then $\alpha_i = 0$, and if i is odd then $\alpha_i = \pi$. Below we illustrate this example for dimensions two and three.*

$$\alpha\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\alpha\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Concluding remarks

Due to the advantages of qudit-based computing ($d > 2$) compared to qubit-based computing ($d = 2$), there has been an ever-increasing interest in the applications of d-level qudits in quantum computing and quantum information in recent decades. Unitary solutions to the Yang–Baxter equation (5) are well-known candidates for (universal) quantum logic gates. On the other hand, the search for new solutions to the Yang–Baxter equation or its applications is a common endeavor in both mathematics and physics.

In the present paper, we introduce an infinite family of universal quantum logic gates, αSWAP which, on one hand, includes the usual SWAP and **i**SWAP gates extended to all dimensions $d \geq 2$, along with previously found generalizations. On the other hand, it provides unitary solutions to the constant Yang–Baxter equation. The α in αSWAP represents a d-tuple $(\alpha_1, \alpha_2, \cdots, \alpha_d)$ for arbitrary real numbers α_i.

Let us conclude by mentioning some future research directions related to the present work. One possible direction is to generalize αSWAP to more general (and not necessarily unitary) solutions of the Yang–Baxter equation. It would also be interesting to investigate how αSWAP can be expressed as a linear combination of tensor products of generalized (higher-dimensional) Pauli matrices. These research directions will be pursued in a sequel to this paper.

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