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Fekete-Szegö Inequality Estimate for Analytic Functions Using Sălăgean-Difference Operator and Leaf-Like Domain

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Abstract. This paper investigates the Fekete-Szegö inequality for subclasses of analytic functions in the unit disk, including starlike, convex, bounded turning, and close-to-convex functions of complex order. Employing the Sălăgean-difference operator, we derive sharp bounds for the functional $|b_3 - \gamma b_2^2|$ and extend these results to leaf-like domains. Our findings generalize classical inequalities, offering new insights into coefficient constraints and geometric properties in complex analysis.

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Key Words and Phrases: Analytic function, Univalent functions,Fekete-Szegö inequality,Leaf like domain

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1. Introduction

The theory of univalent functions, a cornerstone of geometric function theory, explores the properties of analytic functions that are injective in the open unit disk. A fundamental problem in this field is to estimate the coefficients of such functions, as these coefficients encode critical geometric and analytic information about their mappings. Among the classical results, the Fekete-Szegö inequality stands out as a powerful tool for a normalized analytic function. The Fekete-Szegö inequality was first proposed by Hungarian Mathematicians Michael Fekete and Gaber Szegö in 1933 (see[1]). Since then, the various authors were investigated and obtained the Fekete-Szegö inequalities for different subclasses (see[2][3][4],[5],[6],[7],[8],[9],[10])This inequality has been extensively studied for various subclasses of univalent functions, such as starlike, convex, and close-to-convex functions, due to its applications in understanding extremal problems and conformal mappings.

In recent decades, differential and integral operators have enriched the study of univalent functions by generalizing classical subclasses and introducing new geometric constraints. One such operator is the Sălăgeandifferential operator, introduced by Sălăgean

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in 1983 (see[11]), which iteratively applies weighted combinations of the function and its derivatives to generate new classes of analytic functions. Building on this foundation, Al-Oboudi (see [12]) proposed a generalized difference operator, later adapted to form the Salagean-difference operator, denoted D_λ^β.

Parallel to operator-based studies, the exploration of non-standard domains has gained traction in complex analysis.The leaf-like domains, defined by mappings exhibit unique boundary properties that deviate from the circular or convex shapes typically associated with starlike or convex functions. These domains, named for their resemblance to a leaf?s contour, challenge traditional coefficient bounds and inspire new inequalities tailored to their geometry. The interplay between differential operators and such domains is particularly intriguing, as it allows researchers to investigate how operator-induced transformations affect mappings onto complex regions.

The research conducted by Srivastava et al.(see [13]),Murugusundaramoorthy (see [14]), Orhan and Cotîrlă ([15]), Al-Sadi (see[16]), Panigrahi et al.([17]) advances the theoretical foundations of these subclasses. Several scholars, including Al-Sadi (see[16]) and Srivastava et al.(see [13]), focused on deriving constraints for the initial coefficients, which are crucial for understanding the evolution and structure of these functions. The Fekete-Szegö functional was the focus of several investigations, including those by Srivastava et al. (see [13]) and Panigrahi et al. (see[17]), which provided upper estimates and inequalities for these specialized subclasses. And the Geometric structures like leaf-shaped domains Panigrahi et al.(see[17]) and crescent-shaped areas Murugusundaramoorthy,(see[14]) demonstrate how these functions can be connected to specific geometric curves and figures, impacting their analytical behavior.

Recently Kavita P et al.(see[18]) obtained significant inequality for starlike, bounded turning, and close-to-convex functions by using Hohlov operator and taking leaf like domain in account.

Motivated by these developments, this paper derives new Fekete-Szegöinequality estimates for subclasses of analytic functions defined using the Sǎlǎgean-difference operator D_λ^β and associated with leaf-like domains. Specifically, we consider functions in classes such as starlike, convex, bounded turning, and close-to-convex functions of complex order, extending classical results to these generalized settings. Our main objective is to obtain sharp bounds for the functional $|b_3 - \gamma b_2^2|$ for functions $h(\zeta)$ satisfying $D_\lambda^\beta h(\zeta) \in \mathcal{S}_\wp^*$ where $\mathcal{S}_\wp^*$, denotes a starlike class relative to a specific subordination. Additionally, we explore how these bounds adapt to mappings onto leaf-like domains, offering insights into their geometric implications.

The novelty of this work lies in its integration of the Sǎlǎgean-difference operator with leaf-like domains, a combination that has not been extensively studied in the context of the Fekete-Szegö problem. By leveraging known results on coefficient bounds (see [19][20]) and introducing new techniques for handling the operators symmetry, we establish inequalities that generalize and sharpen existing estimates. These findings not only enhance our understanding of univalent function behavior but also pave the way for applications in conformal mapping and operator theory.

Let $\mathcal{A}$ be the family of function h which are analytic in the open unit dick $\Delta = \{\zeta :$

$|\zeta| < 1\}$ normalized by the conditions $h(0) = 0$ and $h'(0) = 1$ with the series expansion of the form

$$h(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n \tag{1}$$

Let S be a subclass of $\mathcal{A}$ consists of "schilit functions". The sufficient conditions for a function $h \in \mathcal{A}$ to belong to the classes S^* (starlike functions) and S^c (convex functions) are well established results in geometric function theory. These foundational conditions can be trace back to the pioneering contribution of W.odzimierżerański, Robertson, and Hummel in the mid-20th century. Alexander (see[21]) introduced the necessary and sufficient condition for a function $h \in \mathcal{A}$ to be in S^* is that

$$Re\left\{\frac{\zeta h'(\zeta)}{h(\zeta)}\right\} > 0, \quad (\zeta \in \Delta)$$

Similarly, the necessary and sufficient condition for a function $h \in \mathcal{A}$ to be in S^c is that

$$Re\left\{1 + \frac{\zeta h''(\zeta)}{h'(\zeta)}\right\} > 0, \quad (\zeta \in \Delta)$$

In 1964, Robertson [22] introduced a generalized class of starlike and convex functions of complex order. Subsequently, Miller and Mocanu [23] incorporated similar conditions in their work to extend classical results related to starlike and convex functions. These conditions were formulated using differential subordinations, a powerful technique for analyzing functional inequalities within geometric function theory.

These classes have been thoroughly examined, and various characteristics have been identified, such as coefficient bounds, growth and distortion theorems, radii of starlikeness and convexity, as well as properties related to convolution, Hadamard products, and subordination. These functions facilitate the development of new function classes, aid in modeling intricate geometric forms, and contribute to a deeper understanding of geometric properties. Let $\alpha \in \mathbb{C}$, a function $h \in \mathcal{A}$ is in the class of starlike functions of complex order α and denoted by $\mathrm{S}^*(\alpha)$, [24] if and only if

$$\frac{h(\zeta)}{\zeta} \neq 0 \quad and \quad Re\left(1 + \frac{1}{\alpha}\left\{\frac{\zeta h'(\zeta)}{h(\zeta)} - 1\right\}\right) > 0, \quad (\zeta \in \Delta) \tag{2}$$

A function $h \in \mathcal{A}$ is in the class of convex functions of complex order α and denoted by $\mathcal{C}(\alpha)$,[25] if and only if

$$h'(\zeta) \neq 0 \quad and \quad Re\left(1 + \frac{1}{\alpha}\left\{\frac{\zeta h''(\zeta)}{h'(\zeta)}\right\}\right) > 0, \quad (\zeta \in \Delta) \tag{3}$$

A function $h \in \mathcal{A}$ is in the class of convex functions of complex order α and denoted by $\mathcal{K}(\alpha)$,[25] if and only if

$$Re\left(1 + \frac{1}{\alpha}(h'(\zeta) - 1)\right) > 0 \quad (\zeta \in \Delta) \tag{4}$$

For a function $h \in \mathcal{A}$ the class of analytic functions with $h(0) = 0$ and $h'(0) = 1$, the operator is defined as $\mathcal{D}_\lambda^\beta : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$\mathcal{D}_\lambda^0 h(\zeta) = h(\zeta)$$

$$\begin{aligned}\mathcal{D}_\lambda^1 h(\zeta) &= \zeta h'(\zeta) + \frac{\lambda}{2}[h(\zeta) - h(-\zeta) - 2\zeta] \quad (\lambda \in \mathbb{R}) \\ &= \zeta + \sum_{n=2}^{\infty} \left[n + \frac{\lambda}{2}(1 + (-1)^{n+1}) \right] b_n \zeta^n\end{aligned}$$

$$\mathcal{D}_\lambda^2 h(\zeta) = \mathcal{D}^1(\mathcal{D}_\lambda^1 h(\zeta))$$

In general, for $\beta \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$,

$$\mathcal{D}_\lambda^\beta h(\zeta) = \mathcal{D}_\lambda^1(\mathcal{D}_\lambda^{\beta-1} h(\zeta)) = \zeta + \sum_{n=2}^{\infty} \left[n + \frac{\lambda}{2}(1 + (-1)^{n+1}) \right]^\beta b_n \zeta^n \quad (\zeta \in \Delta) \tag{5}$$

This operator, characterized by its parameter λ and β order offers a versatile framework for studying function behavior under symmetric transformations, bridging classical differential operators and modern geometric constraints. The Operator D_λ^β is known as the Sălăgean-difference operator in literature (see [26],[27]). This operator is a modified Dunkel operator of complex variables(see [28],[29]). When $\lambda = 0, \mathcal{D}_\lambda = \mathcal{D}_0^\beta = \mathcal{D}^\beta$ is known as the Sălăgean-differential operator (see [11]).

Example 1.

$$h(\zeta) = \zeta + \frac{\zeta^2}{2} + \frac{\zeta^3}{8} + \frac{\zeta^4}{48} + \frac{\zeta^5}{384} + \cdots$$

Then

$$\mathcal{D}_1^1 h(\zeta) = \zeta + \zeta^2 + \frac{\zeta^3}{2} + \frac{\zeta^4}{12} + \frac{\zeta^5}{64} + \cdots$$

Example 2.

$$h(\zeta) = \zeta + \frac{2\zeta^2}{5} + \frac{3\zeta^3}{25} + \frac{4\zeta^4}{125} + \cdots$$

Then

$$h(\zeta) = \zeta + \frac{4\zeta^2}{5} + \frac{12\zeta^3}{25} + \frac{16\zeta^4}{125} + \cdots$$

2. Preliminaries

Let us define the bounded turning function with Sălăgean- difference operator as $\mathrm{R}_\wp$ which contains all the function $h \in \mathcal{A}$ and satisfying

$$Re\big((D_\lambda^\beta h(\zeta))'\big), \quad (\zeta \in \Delta) \tag{6}$$

Similarly starlike function with Săl̆agean- difference operator $S^*_{\wp}$ which maps $|\Delta| < 1$ conformally on to starlike domain and statisfying

$$Re\left(\frac{\zeta(D^{\beta}_{\lambda}h(\zeta))'}{D^{\beta}_{\lambda}h(\zeta)}\right) \quad (\zeta \in \Delta) \tag{7}$$

Let us define starlike function of complex order with Săl̆agean- difference operator $S^*_{\wp}(\alpha)$ which maps $|\Delta| < 1$ conformally onto starlike domain of complex order and satisfying

$$Re\left(1 + \frac{1}{\alpha}\left[\frac{\zeta(D^{\beta}_{\lambda}h(\zeta))'}{D^{\beta}_{\lambda}h(\zeta)} - 1\right]\right) > 0 \quad (\zeta \in \Delta) \tag{8}$$

Let us define convex function of complex order with Săl̆agean- difference operator $S^c_{\wp}(\alpha)$, which maps $|\zeta| < 1$ conformally onto convex domain of complex order and satisfying

$$Re\left(1 + \frac{1}{\alpha}\left[\frac{\zeta(D^{\beta}_{\lambda}h(\zeta))''}{(D^{\beta}_{\lambda}h(\zeta))'} - 1\right]\right) > 0 \quad (\zeta \in \Delta) \tag{9}$$

Let us define close to convex function with Săl̆agean- difference operator $K_{\wp}(\alpha)$, which maps $|\zeta| < 1$ conformally onto closed convex domain of complex order and satisfying

$$Re\left\{1 + \frac{1}{\alpha}[(D^{\beta}_{\lambda}h(\zeta))' - 1]\right\}, \quad (\zeta \in \Delta) \tag{10}$$

Raina and Sokol [30] and Haripriya [31] explored the function $h(\zeta) = \zeta + (1 + \zeta^3)^{\frac{1}{3}}$, which has symmetry with respect to the real axis. Real part of this function is positive with conditions $h(0) = h'(0) = 1$, and it maps the unit disc onto analytic and univalent region which has the shape of leaf-like domain.This leaf-like domain can model complex shapes with smooth boundaries. In general, a "leaf" is a smooth submanifold or region of a manifold that resembles "sheets" within a system with layers. Particular subsets, or "leaves," inside a manifold are referred to as leaf-like domains when discussing foliations or decompositions of the manifold into simpler structures. In certain geometric contexts, such as the study of dynamical systems or the theory of foliations, examining the behavior of the manifold within these leaves can provide crucial insights into the general topology and geometry of the space.

The result of following Lemmas are applied in our main theorems.

Lemma 1. *Let P denote the class of function denoted by p such that $p(\zeta) = d_1\zeta + d_2\zeta^2 + d_3\zeta^3 + \cdots$ be an analytic function in the region R with the property that $p(0) = 1$ then $|d_n| \leq 2$ for all $n \geq 1$ and $|d_2 - \frac{d_1^2}{2}|$. P is the class of all such function which has the property of positive real part.*

Lemma 2. *Let the analytic function $p(\zeta) = d_1\zeta + d_2\zeta^2 + d_3\zeta^3 + \cdots$ which have the positive real part, then $|d_2 - \alpha d_1^2| \leq 2max1, |2\alpha - 1|$ here α is the complex number.*

Functions $p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2}$ and $p(\zeta) = \frac{1+\zeta}{1-\zeta}$ provides the sharp results.

3. Main Results

Theorem 1. *If $h \in \mathcal{A}$ is the form given by (1) belongs $S^*_{\wp}$ and γ is a real number then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{1}{2(3+\lambda)^\beta} & \text{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{1}{2(3+\lambda)^\beta}\left|\frac{2\gamma(3+\lambda)^\beta}{2^{2\beta}} - 1\right| & \text{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases} \tag{11}$$

Proof. If $h \in S^*M_\alpha$, then for the schwarz function w with $w(0) = 0$ and $|w(\zeta)| \leq 1$ and concept of sub-ordination property of equation (7), we have

$$\frac{\zeta(D^\beta_\lambda h(\zeta))'}{D^\beta_\lambda h(\zeta)} = w(\zeta) + \sqrt[3]{1 + (w(\zeta))^3} \tag{12}$$

we have

$$p(\zeta) = \frac{1 + w(\zeta)}{1 - w(\zeta)} = 1 + d_1\zeta + d_2\zeta^2 + d_3\zeta^3 + \cdots$$

$$w(\zeta) = \frac{1 + p(\zeta)}{1 - p(\zeta)}$$

on simplifying right hand side of equation (25) we get

$$w(\zeta) + \sqrt[3]{1 + (w(\zeta))^3} = 1 + \frac{d_1}{2}\zeta + \left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)\zeta^2 + \left(\frac{d_3}{2} - \frac{d_1 d_2}{2} + \frac{d_1^3}{6}\right)\zeta^3 + \left(\frac{d_4}{2} - \frac{d_2^2}{4} - \frac{d_1 d_3}{2} + \frac{d_1^2 d_2}{2} - \frac{d_1^4}{8}\right)\zeta^4 + \cdots \tag{13}$$

From left hand side of (25) we get

$$\frac{\zeta(D^\beta_\lambda h'(\zeta))}{D^\beta_\lambda h(\zeta)} = 1 + 2^\beta b_2\zeta + \left(2(3+\lambda)^\beta b_3 - 2^{2\beta}b_2^2\right)\zeta^2 + \left(3(4^\beta)b_4 - 3(2^\beta)(3+\lambda)^\beta b_2 b_3 + 2^{3\beta}b_2^3\right)\zeta^3 + \left(4(5+\lambda)^\beta b_5 - 4(2^\beta)4^\beta b_2 b_4 - 2(3+\lambda)^{2\beta}b_3^2 - 2^{4\beta}b_2^4 + 4(2^{2\beta})(3+\lambda)^\beta b_2^2 b_3\right)\zeta^4 + \cdots \tag{14}$$

Now from (12),(13) and (14)

$$1 + 2^\beta b_2\zeta + \left(2(3+\lambda)^\beta b_3 - 2^{2\beta}b_2^2\right)\zeta^2 + \left(3(4^\beta)b_4 - 3(2^\beta)(3+\lambda)^\beta b_2 b_3 + 2^{3\beta}b_2^3\right)\zeta^3$$
$$+\left(4(5+\lambda)^\beta b_5 - 4(2^\beta)4^\beta b_2 b_4 - 2(3+\lambda)^{2\beta}b_3^2 - 2^{4\beta}b_2^4 + 4(2^{2\beta})(3+\lambda)^\beta b_2^2 b_3\right)\zeta^4 + \cdots$$
$$= 1 + \frac{d_1}{2}\zeta + \left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)\zeta^2 + \left(\frac{d_3}{2} - \frac{d_1 d_2}{2} + \frac{d_1^3}{6}\right)\zeta^3 +$$

$$\left(\frac{d_4}{2} - \frac{d_2^2}{4} - \frac{d_1 d_3}{2} + \frac{d_1^2 d_2}{2} - \frac{d_1^4}{8}\right)\zeta^4 + \cdots$$

On equating the coefficients, we get

$$b_2 = \frac{d_1}{2(2^\beta)}$$

$$b_3 = \frac{d_2}{4(3+\lambda)^\beta}$$

$$b_3 - \gamma b_2^2 = \frac{1}{4(3+\lambda)^\beta}\left(d_2 - \frac{\gamma(3+\lambda)^\beta d_1^2}{2^{2\beta}}\right)$$

Applying the lemma (2) we get

$$|b_3 - \gamma b_2^2| \leq \frac{1}{2(3+\lambda)^\beta} max\left\{1, \left|\frac{2\gamma(3+\lambda)^\beta}{2^{2\beta}}\right|\right\} \tag{15}$$

This Completes the proof

Theorem 2. *If $h \in \mathcal{A}$ is the form given by (1) belongs $RM_\wp$ and γ is a real number then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{1}{3(3+\lambda)^\beta} & if\ p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{1}{3(3+\lambda)^\beta}\left|\frac{3\gamma(3+\lambda)^\beta}{4(2)^{2\beta}}\right| & if\ p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases} \tag{16}$$

Proof. If $RM_\wp$, then for the schwarz function w with $w(0) = 0$ and $|w(\zeta)| \leq 1$

$$(D_\lambda^\beta h(\zeta))' = w(\zeta) + \sqrt[3]{1 + (w(\zeta))^3} \tag{17}$$

$$(D_\lambda^\beta h(\zeta))' = 1 + 2(2^\beta)b_2\zeta + 3(3+\lambda)^\beta b_3\zeta^2 + 4(4^\beta)b_4\zeta^3 + 5(5+\lambda)^\beta b_5\zeta^5 + \cdots \tag{18}$$

From equation (13) and (18) we have

$$1 + 2(2^\beta)b_2\zeta + 3(3+\lambda)^\beta b_3\zeta^2 + 4(4^\beta)b_4\zeta^3 + 5(5+\lambda)^\beta b_5\zeta^5 + \cdots$$
$$= 1 + \frac{d_1}{2}\zeta + \left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)\zeta^2 + \left(\frac{d_3}{2} - \frac{d_1 d_2}{2} + \frac{d_1^3}{6}\right)\zeta^3 +$$
$$\left(\frac{d_4}{2} - \frac{d_2^2}{4} - \frac{d_1 d_3}{2} + \frac{d_1^2 d_2}{2} - \frac{d_1^4}{8}\right)\zeta^4 + \cdots$$

On contrasting similar terms

$$b_2 = \frac{d_1}{4(2^\beta)}$$

$$b_3 = \frac{1}{3(3+\lambda)^\beta}\left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)$$

on streamlining and using lemma (2) we get

$$|b_3-\gamma b_2^2|\leq\frac{1}{3(3+\lambda)^\beta}max\left\{1,\left|\frac{3\gamma(3+\lambda)^\beta}{4(2^{2\beta})}\right|\right\} \tag{19}$$

Theorem 3. *If $h\in\mathcal{A}$ is the form given by (1) belongs $S^*M_\wp(\alpha)$ and γ is a real number then*

$$|b_3-\gamma b_2^2|\leq\begin{cases}\frac{\alpha}{2(3+\lambda)^\beta} & if\ p(\zeta)=\frac{1+\zeta^2}{1-\zeta^2}\\ \frac{\alpha}{2(3+\lambda)^\beta}\left|\left(\frac{2\gamma(3+\lambda)^\beta}{2^{2\beta}}-1\right)\alpha\right| & if\ p(\zeta)=\frac{1+\zeta^2}{1-\zeta^2}\end{cases} \tag{20}$$

Proof.

$$1+\frac{1}{\alpha}\left(\frac{\zeta(D_\lambda^\beta h(\zeta))'}{D_\lambda^\beta h(\zeta)}-1\right)=w(\zeta)+\sqrt[3]{1+(w(\zeta))^3} \tag{21}$$

and

$$1+\frac{1}{\alpha}\left(\frac{\zeta(D_\lambda^\beta h(\zeta))'}{D_\lambda^\beta h(\zeta)}-1\right) \tag{22}$$

$$=1+\frac{1}{\alpha}\Big[2^\beta b_2\zeta+\big(2(3+\lambda)^\beta b_3-2^{2\beta}b_2^2\big)\zeta^2+\Big(3(4^\beta)b_4-3(2^\beta)(3+\lambda)^\beta b_2b_3+2^{3\beta}b_2^3\Big)\zeta^3$$
$$+\big(4(5+\lambda)^\beta b_5-4(2^\beta)4^\beta b_2b_4-2(3+\lambda)^{2\beta}b_3^2-2^{4\beta}b_2^4+4(2^{2\beta})(3+\lambda)^\beta b_2^2b_3\big)\zeta^4+\cdots\Big]$$

from equation (13),(21) and (22)

$$1+\frac{1}{\alpha}\Big[2^\beta b_2\zeta+\big(2(3+\lambda)^\beta b_3-2^{2\beta}b_2^2\big)\zeta^2+\Big(3(4^\beta)b_4-3(2^\beta)(3+\lambda)^\beta b_2b_3+2^{3\beta}b_2^3\Big)\zeta^3$$
$$+\big(4(5+\lambda)^\beta b_5-4(2^\beta)4^\beta b_2b_4-2(3+\lambda)^{2\beta}b_3^2-2^{4\beta}b_2^4+4(2^{2\beta})(3+\lambda)^\beta b_2^2b_3\big)\zeta^4+\cdots\Big]$$
$$=1+\frac{d_1}{2}\zeta+\left(\frac{d_2}{2}-\frac{d_1^2}{4}\right)\zeta^2+\left(\frac{d_3}{2}-\frac{d_1d_2}{2}+\frac{d_1^3}{6}\right)\zeta^3+$$
$$\left(\frac{d_4}{2}-\frac{d_2^2}{4}-\frac{d_1d_3}{2}+\frac{d_1^2d_2}{2}-\frac{d_1^4}{8}\right)\zeta^4+\cdots$$

By equating the like term we have

$$b_2=\frac{d_1\alpha}{2(2^\beta)}$$

$$b_3=\frac{\alpha}{4(3+\lambda)^\beta}\left(d_2-\frac{d_1^2}{2}(1-\alpha)\right)$$

On simplifying by lemma (2) we get

$$|b_3 - \gamma b_2^2| \leq \frac{\alpha}{2(3+\lambda)^\beta} max\left\{1, \left|\left(\frac{2\gamma(3+\lambda)^\beta}{2^{2\beta}} - 1\right)\alpha\right|\right\}$$

As a result, we get the desired outcomes.

Theorem 4. *If $h \in \mathcal{A}$ is the form given by (1) belongs $KM_\wp$ and γ is a real number then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{\alpha}{3(3+\lambda)^\beta} & if\ p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{\alpha}{3(3+\lambda)^\beta}\left|\frac{3\gamma\alpha(3+\lambda)^\beta}{4(2)^{2\beta}} - 1\right| & if\ p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Proof. If $KM_\wp$, then for the schwarz function w with $w(0) = 0$ and $|w(\zeta)| \leq 1$

$$1 + \frac{1}{\alpha}\left((D_\lambda^\beta h(\zeta))' - 1\right) = w(\zeta) + \sqrt[3]{1 + (w(\zeta))^3} \tag{23}$$

$$1+\frac{1}{\alpha}\left((D_\lambda^\beta h(\zeta))'-1\right) = 1+\frac{1}{\alpha}\left(2(2^\beta)b_2\zeta+3(3+\lambda)^\beta b_3\zeta^2+4(4^\beta)b_4\zeta^3+5(5+\lambda)^\beta b_5\zeta^5+\cdots\right) \tag{24}$$

From equation (13),(23) and (24)

$$\begin{aligned} 1 + \frac{1}{\alpha}&\left(2(2^\beta)b_2\zeta + 3(3+\lambda)^\beta b_3\zeta^2 + 4(4^\beta)b_4\zeta^3 + 5(5+\lambda)^\beta b_5\zeta^5 + \cdots\right) \\ &= 1 + \frac{d_1}{2}\zeta + \left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)\zeta^2 + \left(\frac{d_3}{2} - \frac{d_1 d_2}{2} + \frac{d_1^3}{6}\right)\zeta^3 + \\ &\left(\frac{d_4}{2} - \frac{d_2^2}{4} - \frac{d_1 d_3}{2} + \frac{d_1^2 d_2}{2} - \frac{d_1^4}{8}\right)\zeta^4 + \cdots \end{aligned}$$

On contrasting similar terms

$$b_2 = \frac{d_1\alpha}{4(2^\beta)}$$

$$b_3 = \frac{\alpha}{6(3+\lambda)^\beta}\left(d_2 - \frac{d_1^2}{2}\right)$$

on streamlining and using lemma (2) we get

$$|b_3 - \gamma b_2^2| \leq \frac{\alpha}{3(3+\lambda)^\beta} max\left\{1, \left|\frac{3\gamma\alpha(3+\lambda)^\beta}{4(2^{2\beta})}\right|\right\}$$

As a result, we get the desired outcomes.

Theorem 5. *If $h \in \mathcal{A}$ is the form given by (1) belongs $S^c M_{\wp}(\alpha)$ and γ is a real number then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{\alpha}{6(3+\lambda)^\beta} & \text{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{\alpha}{6(3+\lambda)^\beta}\left|\frac{3\gamma\alpha(3+\lambda)^\beta}{2(2^{2\beta})} - \alpha\right| & \text{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases} \tag{25}$$

Proof.

$$1 + \frac{1}{\alpha}\left(\frac{\zeta(D_\lambda^\beta h(\zeta))''}{(D_\lambda^\beta h(\zeta))'}\right) = w(\zeta) + \sqrt[3]{1 + (w(\zeta))^3} \tag{26}$$

and

$$1 + \frac{1}{\alpha}\left(\frac{\zeta(D_\lambda^\beta h(\zeta))''}{(D_\lambda^\beta h(\zeta))'}\right) = \tag{27}$$

$$1 + \frac{1}{\alpha}\Big[2(2^\beta)b_2\zeta + \Big(6(3+\lambda)^\beta b_3 - 4(2^{2\beta})b_2^2\Big)\zeta^2 +$$

$$\Big(12(4^\beta)a_4 - 18(2^\beta)(3+\lambda)^\beta b_2 b_3 + 8(2^{3\beta})b_2^3\Big)\zeta^3 + \cdots\Big]$$

From equation (13),(26) and (27), we have

$$b_2 = \frac{d_1\alpha}{4(2^\beta)}$$

$$b_3 = \frac{\alpha}{6(3+\lambda)^\beta}\left(\frac{d_2}{2} - \frac{d_1^2}{4}(1-\alpha)\right)$$

On simplifying by lemma (2) we get

$$|b_3 - \gamma b_2^2| \leq \frac{\alpha}{6(3+\lambda)^\beta} max\left\{1, \left|\left(\frac{3\gamma\alpha(3+\lambda)^\beta}{2(2^{2\beta})} - \alpha\right)\right|\right\}$$

As a result, we get the desired outcomes.

Remark 1. *case-I: If $p(\zeta) = \frac{1+\zeta}{1-\zeta}$ then in this case $d_1 = d_2 = d_3 = \cdots = 2$.*
Case-II: If $p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2}$ then in this case $d_1 = d_3 = d_5 = \cdots = 0$ and $d_2 = d_4 = d_6 = \cdots = 2$.

On taking consideration of these above instance we get the results of above theorems.

4. special cases

Remark 2. *If we take $\lambda = 0$ in $D_\lambda^\beta h(\zeta)$, it will reduce to Sălăgean-deferential Operator and ,$\beta = 1$ then $2^\beta = 2$, $\frac{1}{(3+\lambda)^\beta} = \frac{1}{3}$ so that in theorem (1),(2),(3),(4),(5) we find the coresponding results of Sălăgean deference Operator.*

Corollary 1. *Let $\lambda = 0, \beta = 1$, If $h \in S^*M_\wp$, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{1}{6} & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{1}{6}\left|\frac{3\gamma}{2} - 1\right| & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Corollary 2. *Let $\lambda = 0, \beta = 1$, If $h \in RM_\wp$, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{1}{9} & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{1}{9}\left|\frac{9\gamma}{16}\right| & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Corollary 3. *Let $\lambda = 0, \beta = 1$, If $h \in S^*M_\wp(\alpha)$, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{\alpha}{6} & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{\alpha}{6}\left|\left(\frac{3\gamma}{2} - 1\right)\alpha\right| & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Corollary 4. *Let $\lambda = 0, \beta = 1$, If $h \in KM_\wp(\alpha)$, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{\alpha}{9} & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{\alpha}{9}\left|\frac{9\gamma\alpha}{16} - 1\right| & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Corollary 5. *Let $\lambda = 0, \beta = 1$. If $h \in S^cM_\wp(\alpha)$, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{\alpha}{18} & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{\alpha}{18}\left|\frac{9\gamma\alpha}{8} - \alpha\right| & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Remark 3. *If we take $\lambda = 1$, in $D_\lambda^\beta h(\zeta)$ then it will reduce to Al-Oboudi differential operator, $\beta = 1$ then $2^\beta = 2$ and $\frac{1}{(3+\lambda)^\beta} = \frac{1}{4}$ so that in theorem (1), (2), (3), (4), (5) we find the coresponding results of Al-Oboudi differential operator.*

Corollary 6. *Let $\lambda = 1, \beta = 1$, If $h \in S^*M_\wp$, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{1}{8} & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{1}{8}\left|2\gamma - 1\right| & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Corollary 7. *Let $\lambda = 1, \beta = 1$, If $h \in RM_\wp$, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{1}{12} & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{1}{12}\left|\frac{3\gamma}{4}\right| & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Corollary 8. *Let $\lambda = 1, \beta = 1$, If $h \in S^*M_\wp(\alpha)$, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{\alpha}{8} & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{\alpha}{8}\left|(2\gamma - 1)\alpha\right| & \textit{if } p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Corollary 9. *Let* $\lambda = 1, \beta = 1$ *, If* $h \in KM_{\wp}(\alpha)$*, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{\alpha}{12} & if\, p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{\alpha}{12}\left|\frac{3\gamma\alpha}{4} - 1\right| & if\, p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

Corollary 10. *Let* $\lambda = 1, \beta = 1$ *. If* $h \in S^c M_{\wp}(\alpha)$*, then*

$$|b_3 - \gamma b_2^2| \leq \begin{cases} \frac{\alpha}{24} & if\, p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \\ \frac{\alpha}{24}\left|\frac{3\gamma\alpha}{2} - \alpha\right| & if\, p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \end{cases}$$

5. Conclusion

In conclusion, this work expands our understanding of the Fekete-Szegö inequality by applying it to a larger class of holomorphic functions, specifically starlike, bounded turning, and close-to-convex functions of complex order. By considering the Sălăgean-difference operator and leaf-like domains, we have created new inequalities that expand on conventional findings. These findings improve our understanding of how these functions behave in geometric function theory and complex analysis.Additionally, we examine specific instances of the deferential operator and provide strict limitations on the coefficients, providing useful information for further study.

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Conflict of Interest

The authors declare that there are no conflicts of interest.

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