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Theoretical Advances and Improvements of "Generalized Approximation Spaces Generation from $\mathbb{I}_j$-Neighborhoods and Ideals with Application to Chikungunya Disease"

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Abstract. In their article, "Generalized approximation spaces generation from $\mathbb{I}_j$-neighborhoods and ideals with application to Chikungunya disease," published in AIMS Mathematics, 9(4) (2024), 10050–10077, T. M. Al-Shami and M. Hosny introduced a novel framework for constructing generalized approximation spaces utilizing $\mathbb{I}_j$-neighborhoods and ideals. This current paper undertakes a critical validation of the published work. Our detailed investigation indicates that several core propositions and methodological instances require careful re-examination and precision, including the formulation of key results, underlying derivations, and illustrative examples. To address these areas, we provide rigorous counterexamples that highlight where further theoretical refinement is necessary. We subsequently correct and formally refine the established findings, offer enhanced mathematical formulations for critical concepts, and present new properties and clarifications. The goal of this paper is to strengthen the theoretical foundation of these generalized approximation spaces, ensuring maximal rigor and practical applicability for future research in the field.

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1. Introduction

- **Literature Review:**

The theory of rough sets is a significant and effective methodology for handling uncertainty and ambiguity in data, as well as for analyzing and extracting knowledge from it. Introduced by Pawlak in 1982 [1, 2], this theory has served as a foundational framework for effective solutions to decision-making problems. Despite its strengths, the constraints of equivalence relation conditions central to this approach have led many researchers to propose generalizations and extensions to broaden its applications.

Several avenues have been explored in this regard. For instance, Yao [3] extended the concept of an equivalence relation to a more general binary relation, introducing the notions of right and left neighborhoods derived from the concepts of after- and before-sets, which were defined in [4]. Similarly, Allam et al. [5, 6] proposed minimal right and minimal left neighborhoods to generate new approximations that generalize Pawlak's rough set theory.

In 2007, Abo Khadra et al. [7] introduced a technique to generalize Pawlak's approximation space using binary relations. This method constructs dual topologies directly from right and left neighborhoods, without modifying the data within the information system. It asserts that the class

$$\mathcal{T} = \{\mathcal{A} \subseteq U : \mho(s) \subseteq \mathcal{A},\ \forall s \in \mathcal{A}\}$$

forms a topology on U, where U represents the universe and $\mho(s)$ is the neighborhood of s. By defining these neighborhoods, various topologies on U can be generated.

This technique was expanded in El-Bably's 2008 Master's thesis [8], which examined additional topological structures and rough set theory generalizations. Later, Abd El-Monsef et al. [9] extended the work of [7, 8] by introducing a new space called j-neighborhood space (j-**NS**), constructed by defining union and intersection neighborhoods from right and minimal right neighborhoods. This development led to eight types of neighborhoods derived from binary relations, which have been widely used in recent studies to define neighborhoods based on these concepts (e.g., [10, 11]), as well as E_j-neighborhoods [12, 13] and initial neighborhoods [14, 15].

Yao [3] presented a special class of neighborhood systems, termed 1-neighborhood systems, which combine right and left neighborhoods without studying topological structures. Kozae et al. [16] further proposed $R < x > R$ as the intersection of minimal right and left neighborhoods to define rough sets, building on the works in [5, 6].

In a subsequent study, Atef et al. [17] introduced the j-adhesion neighborhood space, highlighting the potential of j-**NS** to extend concepts related to adhesion sets. Later, El-Bably et al. [18] addressed inconsistencies in the findings of Atef et al., providing corrected results and additional insights into j-adhesion neighborhoods. Their approach is based on the concept of "adhesion sets" proposed by Ma in 2012 [19], which is relevant to the theory of covering rough sets. Additionally, Nawar et al. [20] introduced j-adhesion neighborhoods in covering-based rough sets, based on the generalized covering approximation space of Abd El-Monsef et al. [21]. Core neighborhood ideas from [22, 23] were also extended in [24] using techniques from [7, 9]. Furthermore, El-Bably and Al-shami [25] introduced core minimal neighborhoods through binary relations, extending Pawlak rough sets to generalized types for applications, such as in lung cancer research.

On the other hand, the application of neighborhood concepts has expanded in new directions, such as advanced neighborhood-based methods used for feature selection—for example, an enhanced evolutionary feature selection method based on extended knowledge from rough approximations [26]. Studies such as An Efficient Approach to Neighborhood Learning in Big Data and Its Application to Medical Diagnosis [27, 28] and Neighborhood Combination Entropy and Multi-Scale Information Fusion-Based Multiple Correlations for Unsupervised Attribute Selection [29, 30] have also employed these methods for feature selection. In addition, NMF-based deep representation algorithms and deep learning algorithms based on NMF for multi-view clustering [31, 32] have further demonstrated their usefulness in this context. Furthermore, these approaches have been applied to decision-making problems [33, 34], $\theta\beta$-ideal approximation spaces [35, 36], primal approximation spaces [37, 38], and medical applications [14, 39].

- **Motivations and Objectives:**

Recently, T. M. Al-Shami and M. Hosny [40] introduced a new type of neighborhood based on the $\mathbb{I}_j$-neighborhood concept [12]. By utilizing the definitions of intersection and union neighborhoods proposed by Abd El-Monsef et al. [9], they constructed eight novel neighborhoods, examined their properties and interrelationships, and presented theoretical results supported by illustrative examples. They further developed various rough set approximations based on $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods, demonstrating their potential through a medical case study. However, the objective of their study was not achieved due to several fundamental errors and a lack of precision.

This paper aims to address the errors, inconsistencies, and inaccuracies identified in their work. The primary objectives of our study are as follows:

(i) Identify and highlight the errors in results such as Theorem 3.4, Proposition 4.3, and Proposition 5.4.

(ii) Provide counterexamples to demonstrate inaccuracies in their findings, such as Examples 1, 2, 3, 4, 5, and 6.

(iii) Correct the erroneous results, definitions, and properties introduced in their research.

(iv) Rectify the incorrect examples and flawed proofs presented in their paper.

(v) Enhance the understanding of the research topic by presenting corrected results and expanding the scope of potential applications.

(vi) Contribute to the advancement of rough set theory and methodologies through refined neighborhood concepts.

- **Organization of the Manuscript:**

The remainder of this paper is organized as follows: Section 2 presents essential concepts and foundational results necessary for understanding the manuscript. This includes corrections to inaccurate definitions, results, and citations from [40] to ensure coherence. Section 3 systematically identifies and analyzes the errors in [40], addressing inaccuracies in results, definitions, and examples, along with the corresponding corrections. Finally, Section 4 provides a summary of our findings and a concluding discussion.

2. Preliminaries

This section establishes fundamental concepts and key results necessary for understanding both the study and the manuscript's content. A primary objective is to rectify inaccuracies in definitions and results, as well as to correct erroneous citations of key concepts from [40], ensuring clarity and coherence throughout the discussion.

A well-established fact is that a binary relation R on a nonempty set (universe) U is a subset of $U \times U$. Throughout this work, we assume U to be a nonempty finite set, with R representing an arbitrary relation unless specified otherwise. For $s, t \in U$, we write sRt if $(s, t) \in R$. Several key concepts in [40] were inaccurately presented. Below, we provide the corrected definitions to ensure precision and correctness.

Definition 1. *[3, 4] A relation R on U is classified as follows:*

(i) Serial: For each $s \in U$, there exists $t \in U$ such that sRt.

(ii) Reflexive: For all $s \in U$, sRs.

(iii) Symmetric: For all $s, t \in U$, sRt if and only if tRs.

(iv) Transitive: For all $s, t, p \in U$, if sRp and pRt, then sRt.

(v) Preorder (or quasi-order): R is reflexive and transitive.

(vi) Equivalence relation: R is reflexive, symmetric, and transitive.

In [40], the authors presented Definition 2.2; however, they omitted the essential concepts of intersection (and minimal intersection) neighborhoods as well as union (and minimal union) neighborhoods, which are fundamental to their definitions and results. To rectify this, we provide the necessary definitions below.

Definition 2. *Let R be a binary relation on a set U, and let $s \in U$. The following types of neighborhoods, based on R, are defined:*

(i) After (right) neighborhood [3, 4]: $\omega_a(s) = \{t \in U : sRt\}$.

(ii) Before (left) neighborhood [3, 4]: $\omega_b(s) = \{t \in U : tRs\}$.

(iii) Minimal-after neighborhood [5, 6]: $\omega_{<a>}(s) = \bigcap\{\omega_a(p) : s \in \omega_a(p)\}$.

(iv) Minimal-before neighborhood [6]: $\omega_{<b>}(s) = \bigcap\{\omega_b(p) : s \in \omega_b(p)\}$.

(v) Intersection neighborhood [3, 9]: $\omega_i(s) = \omega_a(s) \bigcap \omega_b(s)$.

(vi) Union neighborhood [3, 9]: $\omega_u(s) = \omega_a(s) \bigcup \omega_b(s)$.

(vii) Minimal-intersection neighborhood [9, 16]: $\omega_{<i>}(s) = \omega_{<a>}(s) \bigcap \omega_{<b>}(s)$.

(viii) Minimal-union neighborhood [9]: $\omega_{<u>}(s) = \omega_{<a>}(s) \bigcup \omega_{<b>}(s)$.

For simplicity, the set $\{a, b, < a >, < b >, i, u, < i >, < u >\}$ will be collectively denoted by Ω.

Definition 3. *[3, 5, 16] Let R be a binary relation on a finite universe U, and let $F \subseteq U$ be a nonempty subset. For each $j \in \Omega$, the approximation operators based on the ω_j-neighborhoods are defined as:*

- *Lower approximation:*

$$R_\star^{\omega_j}(F) = \{s \in U : \omega_j(s) \subseteq F\}.$$

- *Upper approximation:*

$$R^{\star\omega_j}(F) = \{s \in U : \omega_j(s) \cap F \neq \emptyset\}.$$

- *Boundary region:*

$$BND_R^{\star\omega_j}(F) = R^{\star\omega_j}(F) \setminus R_\star^{\omega_j}(F).$$

- *Accuracy measure:*

$$ACC_R^{\star\omega_j}(F) = \frac{|R_\star^{\omega_j}(F) \cap F|}{|R^{\star\omega_j}(F) \cup F|}, \textit{ provided that } \mid R^{\star\omega_j}(F) \cup F \mid \neq 0.$$

The following definition presents an interesting application of j-**NS**, as defined by [17, 18]. In this definition, the ω_j-neighborhoods are extended to generalized neighborhoods called "adhesion-neighborhoods" (denoted as ρ_j-neighborhoods). These generalized neighborhoods are utilized in identifying generalized rough sets, which serve as generalizations of Pawlak's models and their extensions.

Definition 4. *[17, 18, 20] Let R be a binary relation on a set U, and let $s \in U$. The ρ_j-neighborhood of s collects all elements that share the same ω_j-neighborhood. For each $j \in \Omega$, the neighborhoods are defined as follows:*

(i) *After ρ-neighborhood:* $\rho_a(s) = \{t \in U : \omega_a(t) = \omega_a(s)\}$.

(ii) *Before ρ-neighborhood:* $\rho_b(s) = \{t \in U : \omega_b(t) = \omega_b(s)\}$.

(iii) *Intersection ρ-neighborhood:* $\rho_i(s) = \rho_a(s) \cap \rho_b(s)$.

(iv) *Union ρ-neighborhood:* $\rho_u(s) = \rho_a(s) \cup \rho_b(s)$.

(v) *Minimal-after ρ-neighborhood:* $\rho_{<a>}(s) = \{t \in U : \omega_{<a>}(t) = \omega_{<a>}(s)\}$.

(vi) *Minimal-before ρ-neighborhood:* $\rho_{<b>}(s) = \{t \in U : \omega_{<b>}(t) = \omega_{<b>}(s)\}$.

(vii) *Minimal-intersection ρ-neighborhood:* $\rho_{<i>}(s) = \rho_{<a>}(s) \cap \rho_{<b>}(s)$.

(viii) *Minimal-union ρ-neighborhood:* $\rho_{<u>}(s) = \rho_{<a>}(s) \cup \rho_{<b>}(s)$.

Definition 5. *[17, 18, 20] Let $F \subseteq U$ be a nonempty subset, and let $j \in \Omega$. The approximation operators based on ρ_j-neighborhoods are defined as follows:*

- *Lower approximation:*

$$R_{\star}^{\rho_j}(F) = \{s \in U : \rho_j(s) \subseteq F\}.$$

- *Upper approximation:*

$$R^{\star\rho_j}(F) = \{s \in U : \rho_j(s) \cap F \neq \emptyset\}.$$

- *Boundary region:*

$$BND_R^{\star\rho_j}(F) = R^{\star\rho_j}(F) \setminus R_{\star}^{\rho_j}(F).$$

- *Accuracy measure:*

$$ACC_R^{\star\rho_j}(F) = \frac{|R_{\star}^{\rho_j}(F)|}{|R^{\star\rho_j}(F)|}, \text{ where } \mid R^{\star\rho_j}(F) \mid \neq 0.$$

The following definition introduces a significant extension of the j-**NS** concept, as proposed in [12]. It generalizes the notion of ω_j-neighborhoods into a broader framework known as $\mathbb{I}_j$-neighborhoods, for each $j \in \Omega$. These generalized neighborhoods are employed to define more flexible rough set models that extend beyond Pawlak's classical framework and support diverse applications, such as those discussed in [40].

Definition 6. *[12] The subsequent $\mathbb{I}_j$-neighborhoods of an element $s \in U$, derived from a relation R, are specified as follows:*

(i) *After $\mathbb{I}$-neighborhood:* $\mathbb{I}_a(s) = \{t \in U : \omega_a(t) \cap \omega_a(s) \neq \emptyset\}$.

(ii) *Before $\mathbb{I}$-neighborhood:* $\mathbb{I}_b(s) = \{t \in U : \omega_b(t) \cap \omega_b(s) \neq \emptyset\}$.

(iii) Intersection $\mathbb{I}$*-neighborhood:* $\mathbb{I}_i(s) = \mathbb{I}_a(s) \cap \mathbb{I}_b(s)$.

(iv) Union $\mathbb{I}$*-neighborhood:* $\mathbb{I}_u(s) = \mathbb{I}_a(s) \cup \mathbb{I}_b(s)$.

(v) Minimal-after $\mathbb{I}$*-neighborhood:* $\mathbb{I}_{<a>}(s) = \{t \in U : \omega_{<a>}(t) \cap \omega_{<a>}(s) \neq \emptyset\}$.

(vi) Minimal-before $\mathbb{I}$*-neighborhood:* $\mathbb{I}_{<b>}(s) = \{t \in U : \omega_{<b>}(t) \cap \omega_{<b>}(s) \neq \emptyset\}$.

(vii) Minimal-intersection $\mathbb{I}$*-neighborhood:* $\mathbb{I}_{<i>}(s) = \mathbb{I}_{<a>}(s) \cap \mathbb{I}_{<b>}(s)$.

(viii) Minimal-union $\mathbb{I}$*-neighborhood:* $\mathbb{I}_{<u>}(s) = \mathbb{I}_{<a>}(s) \cup \mathbb{I}_{<b>}(s)$.

In [12], $\mathbb{I}_j$-neighborhoods were examined under the designation "E_j-neighborhoods", $j \in \Omega$.

Definition 7. *[12] For* $\mathbb{I}_j$*-neighborhoods and for each* $j \in \Omega$*, the approximation operators (both lower and upper), boundary region, and accuracy measure for a nonempty subset* F *of* U *are defined as follows:*

- *Lower approximation:*

$$R_{\star}^{\mathbb{I}_j}(F) = \{s \in U : \mathbb{I}_j(s) \subseteq F\}.$$

- *Upper approximation:*

$$R^{\star\mathbb{I}_j}(F) = \{s \in U : \mathbb{I}_j(s) \cap F \neq \emptyset\}.$$

- *Boundary region:*

$$BND_R^{\star\mathbb{I}_j}(F) = R^{\star\mathbb{I}_j}(F) \setminus R_{\star}^{\mathbb{I}_j}(F).$$

- *Accuracy measure:*

$$ACC_R^{\star\mathbb{I}_j}(F) = \frac{|R_{\star}^{\mathbb{I}_j}(F) \cap F|}{|R^{\star\mathbb{I}_j}(F) \cup F|}, \text{ where } | R^{\star\mathbb{I}_j}(F) \cup F | \neq 0.$$

Definition 8. *[41] An ideal* $\mathcal{K}$ *on a nonempty set* U *is a nonempty collection of subsets of* U *that is closed under finite unions and contains all subsets of its elements.*

By using the concept of ideals, R. A. Hosny et al. [13] have succeeded in proposing a notable application of j-**NS**. In this context, the concept of "$\mathbb{I}_j$-neighborhoods" was used alongside the topological structure of "ideals" to play a crucial role in the identification of generalized rough sets, which encompass and extend Pawlak's models and their variations. Furthermore, they find applications in diverse settings.

Definition 9. *[13] Let* R *be a binary relation and* $\mathcal{K}$ *an ideal on a nonempty set* U*. The approximation operators (lower and upper), boundary region, and accuracy measure of a nonempty subset* F *of* U*, derived from* R *and* $\mathcal{K}$ *using* $\mathbb{I}_j$*-neighborhoods, are defined as follows:*

- *Lower approximation:*

$$L_{\star}^{\mathbb{I}_j}(F) = \{s \in U : \mathbb{I}_j(s) \setminus F \in \mathcal{K}\}.$$

- *Upper approximation:*

$$U^{\star \mathbb{I}_j}(F) = \{s \in U : \mathbb{I}_j(s) \cap F \notin \mathcal{K}\}.$$

- *Boundary region:*

$$\triangle_R^{\star \mathbb{I}_j}(F) = U^{\star \mathbb{I}_j}(F) \setminus L_{\star}^{\mathbb{I}_j}(F).$$

- *Accuracy measure:*

$$\mathcal{M}_R^{\star \mathbb{I}_j^{\mathcal{K}}}(F) = \frac{|L_{\star}^{\mathbb{I}_j}(F) \cap F|}{|U^{\star \mathbb{I}_j}(F) \cup F|}, \text{ where } | U^{\star \mathbb{I}_j}(F) \cup F | \neq 0.$$

Definition 10. *[40] Let R be a relation on U, and let $\mathcal{K}$ be an ideal on U. The $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods of an element $s \in U$ are specified as follows:*

(i) $\mathbb{I}_a^{\mathcal{K}}(s) = \{t \in U : \omega_a(t) \cap \omega_a(s) \notin \mathcal{K}\}$.

(ii) $\mathbb{I}_b^{\mathcal{K}}(s) = \{t \in U : \omega_b(t) \cap \omega_b(s) \notin \mathcal{K}\}$.

(iii) $\mathbb{I}_i^{\mathcal{K}}(s) = \mathbb{I}_a^{\mathcal{K}}(s) \cap \mathbb{I}_b^{\mathcal{K}}(s)$.

(iv) $\mathbb{I}_u^{\mathcal{K}}(s) = \mathbb{I}_a^{\mathcal{K}}(s) \cup \mathbb{I}_b^{\mathcal{K}}(s)$.

(v) $\mathbb{I}_{<a>}^{\mathcal{K}}(s) = \{t \in U : \omega_{<a>}(t) \cap \omega_{<a>}(s) \notin \mathcal{K}\}$.

(vi) $\mathbb{I}_{<b>}^{\mathcal{K}}(s) = \{t \in U : \omega_{<b>}(t) \cap \omega_{<b>}(s) \notin \mathcal{K}\}$.

(vii) $\mathbb{I}_{<i>}^{\mathcal{K}}(s) = \mathbb{I}_{<a>}^{\mathcal{K}}(s) \cap \mathbb{I}_{<b>}^{\mathcal{K}}(s)$.

(viii) $\mathbb{I}_{<u>}^{\mathcal{K}}(s) = \mathbb{I}_{<a>}^{\mathcal{K}}(s) \cup \mathbb{I}_{<b>}^{\mathcal{K}}(s)$.

Note that if the ideal is given by $\mathcal{K} = \{\emptyset\}$, then Definition 10 reduces to Definition 6. As a result, the work presented in [40] can be regarded as a genuine extension of the results established in [12].

Furthermore, the notion of the "$\mathbb{I}$-Generalized approximation space" (corrected and abbreviated as $\mathbb{I}_j$-G approximation space) was not explicitly defined in [40], despite being referenced throughout various definitions and results. To address this gap, we formally introduce it as follows:

Definition 11. *Let R be a relation on U, and $\mathcal{K}$ be an ideal on U. The triplet $(U, R, \mathcal{K})$ is named an $\mathbb{I}_j$-G approximation space.*

Theorem 1. *[40] Let $(U, R, \mathcal{K})$ be an $\mathbb{I}_j$-G approximation space, and let $s \in U$. If R is reflexive, then for each $j \in \{a, b, i, u\}$, the following inclusion holds:*

$$\mathbb{I}^{\mathcal{K}}_{<j>}(s) \subseteq \mathbb{I}^{\mathcal{K}}_{j}(s).$$

Definition 12. *[40] Let R be a binary relation and $\mathcal{K}$ an ideal on a nonempty set U. The improved operators (lower and upper), boundary region, and accuracy measure of a nonempty subset F of U, derived from R and $\mathcal{K}$, are defined as follows:*

- *Lower approximation:*

$$R_{\star}^{\mathbb{I}^{\mathcal{K}}_j}(F) = \{s \in U : \mathbb{I}^{\mathcal{K}}_j(s) \setminus F \in \mathcal{K}\}.$$

- *Upper approximation:*

$$R^{\star\mathbb{I}^{\mathcal{K}}_j}(F) = \{s \in U : \mathbb{I}^{\mathcal{K}}_j(s) \cap F \notin \mathcal{K}\}.$$

- *Boundary region:*

$$BND_R^{\star\mathbb{I}^{\mathcal{K}}_j}(F) = R^{\star\mathbb{I}^{\mathcal{K}}_j}(F) \setminus R_{\star}^{\mathbb{I}^{\mathcal{K}}_j}(F).$$

- *Accuracy measure:*

$$ACC_R^{\star\mathbb{I}^{\mathcal{K}}_j}(F) = \frac{|R_{\star}^{\mathbb{I}^{\mathcal{K}}_j}(F) \cap F|}{|R^{\star\mathbb{I}^{\mathcal{K}}_j}(F) \cup F|}, \text{ where } | R^{\star\mathbb{I}^{\mathcal{K}}_j}(F) \cup F | \neq 0.$$

Remark 1. *(i) According to the principles of Pawlak's rough set theory, as established in [1], a set is considered definable (or exact) if its lower and upper approximations coincide. In particular, if the lower approximation of the empty set $\emptyset$ is equal to its upper approximation, which is also $\emptyset$, then $\emptyset$ is a definable set, and its accuracy measure is 1. Conversely, if this condition is not satisfied, $\emptyset$ is classified as an undefinable (or rough) set, meaning its accuracy measure differs from 1. Additionally, the definition of the accuracy measure requires that the upper approximation be nonempty to avoid division by zero. Therefore, if the upper approximation of $\emptyset$ is itself $\emptyset$, the accuracy measure in this case is considered as an indefinite quantity.*

(ii) Accordingly, in Definition 12, the empty set $\emptyset$ is definable if

$$R_{\star}^{\mathbb{I}^{\mathcal{K}}_j}(\emptyset) = R^{\star\mathbb{I}^{\mathcal{K}}_j}(\emptyset) = \emptyset,$$

which implies that

$$ACC_R^{\star\mathbb{I}^{\mathcal{K}}_j}(\emptyset) = 1.$$

Otherwise, $\emptyset$ is an undefinable (rough) set, and the accuracy measure $ACC_R^{\star\mathbb{I}_j^{\mathcal{K}}}(\emptyset)$ is considered an indefinite quantity.

Theorem 2 presents a significant result for deriving a general topology from an arbitrary neighborhood system. This method was initially proposed by Abo Khadra et al. (2007) [7] and later extended by Abd El-Monsef et al. (2014) [9]. Since then, this approach has been widely adopted by researchers to construct various topological frameworks, as demonstrated in Theorem 2.

Theorem 2. *Let U be a universal set and let $j \in \Omega$. Then,*

(i) Fundamental method [7, 9]: The collection

$$\top^{\omega_j} = \{F \subseteq U : \omega_j(s) \subseteq F, \forall s \in F\}$$

forms a topology on U.

(ii) Adhesion topologies [17, 18, 20]: The collection

$$\top^{\rho_j} = \{F \subseteq U : \rho_j(s) \subseteq F, \forall s \in F\}$$

defines a topology on U.

(iii) $\mathbb{I}_j$-topologies [12]: The collection

$$\top^{\mathbb{I}_j} = \{F \subseteq U : \mathbb{I}_j(s) \subseteq F, \forall s \in F\}$$

constitutes a topology on U.

Definition 13. *[12] Let $\top^{\mathbb{I}_j}$ be a topology on U as defined in the theorem above, for all $j \in \Omega$, and let $F \subseteq U$. The interior and closure operators of F in the topological space $(U, \top^{\mathbb{I}_j})$, denoted by $\underline{\top^{\mathbb{I}_j}}(F)$ and $\overline{\top^{\mathbb{I}_j}}(F)$, are referred to as the $\top^{\mathbb{I}_j}$-lower approximation and $\top^{\mathbb{I}_j}$-upper approximation, respectively.*

Definition 14. *[12] The $\top^{\mathbb{I}_j}$-boundary and $\top^{\mathbb{I}_j}$-accuracy of a subset F in a topological space $(U, \top^{\mathbb{I}_j})$ are given as follows:*

- *The $\top^{\mathbb{I}_j}$-boundary of F is given by:*

$$BND^{\top^{\mathbb{I}_j}}(F) = \overline{\top^{\mathbb{I}_j}}(F) \setminus \underline{\top^{\mathbb{I}_j}}(F)$$

- *The $\top^{\mathbb{I}_j}$-accuracy of F is given by:*

$$ACC^{\top^{\mathbb{I}_j}}(F) = \frac{|\underline{\top^{\mathbb{I}_j}}(F)|}{|\overline{\top^{\mathbb{I}_j}}(F)|}, \text{ where } |\overline{\top^{\mathbb{I}_j}}(F)| \neq 0.$$

Theorem 3. *[40] Let $(U, R, \mathcal{K})$ be an $\mathbb{I}_j$-G approximation space, and let $s \in U$. For all $j \in \Omega$, the collection*

$$\tau^{\mathbb{I}_j^{\mathcal{K}}} = \{F \subseteq U : \mathbb{I}_j^{\mathcal{K}}(s) \setminus F \in \mathcal{K}, \forall s \in F\}$$

forms a topology on U.

If $\mathcal{K} = \{\emptyset\}$ in Theorem 5.2 [40], then the resulting topologies coincide with those established in Theorem 2.11 of [12]. Consequently, the findings in [40] serves as a meaningful extension of the work in [12].

Theorem 4. *[42] Let $(U, R, \mathcal{K})$ be an ρ_j-G approximation space and $s \in U$. Then, for all $j \in \Omega$, the collection*

$$\tau^{\rho_j^{\mathcal{K}}} = \{F \subseteq U : \rho_j(s) \setminus F \in \mathcal{K}, \forall s \in F\}$$

constitutes a topology on U.

3. Main results

This section constitutes the core of our research, where we critically examine the errors identified in the study by T. Al-Shami and M. Hosny [40]. We provide multiple counterexamples to substantiate these inaccuracies and propose necessary corrections. Additionally, we refine and amend errors in the proofs of certain results while addressing inconsistencies in the examples and comparison tables presented in their study.

The following theorem, originally stated as Theorem 3.4 in [40], contains errors requiring correction.

Theorem 5. *[40] Let $(U, R, \mathcal{K})$ be an $\mathbb{I}_j$-G approximation space. If $s \in U$, then the following properties hold:*

(i) $\mathbb{I}_i^{\mathcal{K}}(s) \subseteq \mathbb{I}_a^{\mathcal{K}}(s) \cap \mathbb{I}_b^{\mathcal{K}}(s) \subseteq \mathbb{I}_a^{\mathcal{K}}(s) \cup \mathbb{I}_b^{\mathcal{K}}(s) \subseteq \mathbb{I}_u^{\mathcal{K}}(s)$.

(ii) $\mathbb{I}_{<i>}^{\mathcal{K}}(s) \subseteq \mathbb{I}_{<a>}^{\mathcal{K}}(s) \cap \mathbb{I}_{<b>}^{\mathcal{K}}(s) \subseteq \mathbb{I}_{<a>}^{\mathcal{K}}(s) \cup \mathbb{I}_{<b>}^{\mathcal{K}}(s) \subseteq \mathbb{I}_{<u>}^{\mathcal{K}}(s)$.

(iii) If R is reflexive, then $\rho_j(s) \cup \omega_j(s) \subseteq \mathbb{I}_j^{\mathcal{K}}(s)$ for all $j \in \Omega$.

(iv) If R is serial, then $\rho_j(s) \subseteq \mathbb{I}_j^{\mathcal{K}}(s)$ for all $j \in \Omega$.

(v) If R is transitive, then $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq \mathbb{I}_{<j>}^{\mathcal{K}}(s)$ for each $j \in \{a, b, i, u\}$.

Remark 2. *Firstly, statements (1) and (2) are incorrect, as the equalities cannot be replaced by subset symbols according to Definition 2. The following examples demonstrate errors in the remaining statements of the preceding theorem.*

Example 1. *Consider $U = \{p, q, s, t\}$ and the reflexive relation*

$$R = \{(p,s),(p,t),(q,t),(t,q),(p,p),(q,q),(s,s),(t,t)\}$$

on U. The associated neighborhoods for the element t are given as follows:

- $\omega_a(t) = \{q,t\}$, $\omega_b(t) = \{p,q,t\}$, $\omega_i(t) = \{q,t\}$, $\omega_u(t) = \{p,q,t\}$,
- $\omega_{<a>}(t) = \{t\}$, $\omega_{<b>}(t) = \{q,t\}$, $\omega_{<i>}(t) = \{t\}$, $\omega_{<u>}(t) = \{q,t\}$,
- $\rho_a(t) = \{q,t\}$, $\rho_b(t) = \{t\}$, $\rho_i(t) = \{t\}$, $\rho_u(t) = \{q,t\}$,
- $\rho_{<a>}(t) = \{t\}$, $\rho_{<b>}(t) = \{q,t\}$, $\rho_{<i>}(t) = \{t\}$, $\rho_{<u>}(t) = \{q,t\}$.

Now, let $\mathcal{K} = \{\emptyset,\{t\}\}$. The corresponding $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods are:

- $\mathbb{I}_a^{\mathcal{K}}(t) = \{q,t\}$, $\mathbb{I}_b^{\mathcal{K}}(t) = U$, $\mathbb{I}_i^{\mathcal{K}}(t) = \{q,t\}$, $\mathbb{I}_u^{\mathcal{K}}(t) = U$.
- $\mathbb{I}_{<a>}^{\mathcal{K}}(t) = \emptyset$, $\mathbb{I}_{<b>}^{\mathcal{K}}(t) = \{q,t\}$. $\mathbb{I}_{<i>}^{\mathcal{K}}(t) = \emptyset$, $\mathbb{I}_{<u>}^{\mathcal{K}}(t) = \{q,t\}$.

It is evident that for $j \in \{<a>,<i>\}$, neither $\rho_j(t) \subseteq \mathbb{I}_j^{\mathcal{K}}(t)$ nor $\omega_j(t) \subseteq \mathbb{I}_j^{\mathcal{K}}(t)$, thereby demonstrating a counterexample to the corresponding claims.

Example 2. *Consider the set $U = \{p,q,s,t\}$ with the serial relation*

$$R = \{(p,p),(s,p),(t,p),(q,t),(t,q),(t,t)\}.$$

The corresponding neighborhoods under $\omega_a, \rho_a, \mathbb{I}_a$ are given as follows:

- $\omega_a(p) = \{p\}$, $\omega_a(q) = \{t\}$, $\omega_a(s) = \{p\}$, $\omega_a(t) = \{p,q,t\}$.
- $\rho_a(p) = \{p,s\}$, $\rho_a(q) = \{q\}$, $\rho_a(s) = \{p,s\}$, $\rho_a(t) = \{t\}$.
- $\mathbb{I}_a(p) = \{p,s,t\}$, $\mathbb{I}_a(q) = \{q,t\}$, $\mathbb{I}_a(s) = \{p,s,t\}$, $\mathbb{I}_a(t) = U$.

Now, let $\mathcal{K} = \{\emptyset,\{t\}\}$, then the $\mathbb{I}_a^{\mathcal{K}}$-neighborhoods are:

- $\mathbb{I}_a^{\mathcal{K}}(p) = \{p,s,t\}$, $\mathbb{I}_a^{\mathcal{K}}(q) = \emptyset$, $\mathbb{I}_a^{\mathcal{K}}(s) = \{p,s,t\}$, $\mathbb{I}_a^{\mathcal{K}}(t) = \{p,s,t\}$.

It is evident that $\rho_a(q) \nsubseteq \mathbb{I}_a^{\mathcal{K}}(q)$, thereby demonstrating a counterexample to the claim.

Example 3. *Consider $U = \{p,q,s,t\}$ with the transitive relation*

$$R = \{(p,s),(p,t),(p,q),(t,q),(t,t)\}.$$

The corresponding neighborhoods under $\omega_b, \omega_{<b>}, \rho_b, \rho_{<b>}$ are as follows:

- $\omega_b(p) = \emptyset$, $\omega_b(q) = \{p,t\}$, $\omega_b(s) = \{p\}$, $\omega_b(t) = \{p,t\}$.
- $\omega_{<b>}(p) = \{p\}$, $\omega_{<b>}(q) = \emptyset$, $\omega_{<b>}(s) = \emptyset$, $\omega_{<b>}(t) = \{p,t\}$.
- $\rho_b(p) = \{p\}$, $\rho_b(q) = \{q,t\}$, $\rho_b(s) = \{s\}$, $\rho_b(t) = \{q,t\}$.
- $\rho_{<b>}(p) = \{p\}$, $\rho_{<b>}(q) = \{q,s\}$, $\rho_{<b>}(s) = \{q,s\}$, $\rho_{<b>}(t) = \{t\}$.

Now, let $\mathcal{K} = \{\emptyset, \{t\}\}$. The corresponding $\mathbb{I}_b^{\mathcal{K}}$- and $\mathbb{I}_{<b>}^{\mathcal{K}}$-neighborhoods are:

- $\mathbb{I}_b^{\mathcal{K}}(p) = \emptyset$, $\mathbb{I}_b^{\mathcal{K}}(q) = \{q, s, t\}$, $\mathbb{I}_b^{\mathcal{K}}(s) = \{q, s, t\}$, $\mathbb{I}_b^{\mathcal{K}}(t) = \{q, s, t\}$.
- $\mathbb{I}_{<b>}^{\mathcal{K}}(p) = \{p, t\}$, $\mathbb{I}_{<b>}^{\mathcal{K}}(q) = \emptyset$, $\mathbb{I}_{<b>}^{\mathcal{K}}(s) = \emptyset$, $\mathbb{I}_{<b>}^{\mathcal{K}}(t) = \{p, t\}$.

Consequently,

$$\mathbb{I}_j^{\mathcal{K}}(x) \neq \mathbb{I}_{<j>}^{\mathcal{K}}(x), \ \forall x \in U.$$

This demonstrates a discrepancy between $\mathbb{I}_j^{\mathcal{K}}$ and $\mathbb{I}_{<j>}^{\mathcal{K}}$ contradicting any assumption of their equivalence.

Example 4. *Consider $U = \{p, q, s\}$ equipped with the preorder relation*

$$R = \{(p, p), (q, q), (s, s), (p, q), (p, s), (q, s)\}.$$

The corresponding neighborhoods under ω_a, ρ_a are given by:

- $\omega_a(p) = U$, $\omega_a(q) = \{q, s\}$, $\omega_a(s) = \{s\}$.
- $\rho_a(p) = \{p\}$, $\rho_a(q) = \{q\}$, *and* $\rho_a(s) = \{s\}$.

Now, let $\mathcal{K} = \{\emptyset, \{s\}\}$. The corresponding neighborhoods under $\mathbb{I}_a^{\mathcal{K}}$ are given by:

- $\mathbb{I}_a^{\mathcal{K}}(p) = \{p, q\}$, $\mathbb{I}_a^{\mathcal{K}}(q) = \{p, q\}$, $\mathbb{I}_a^{\mathcal{K}}(s) = \emptyset$.

It follows that $\omega_a(x) \nsubseteq \mathbb{I}_a^{\mathcal{K}}(x)$, for all $x \in U$. Additionally, $\rho_a(s) \nsubseteq \mathbb{I}_a^{\mathcal{K}}(s)$.

Remark 3. *(i) For all $j \in \Omega$, and $x \in U$, the inclusion $\rho_j(x) \subseteq \mathbb{I}_j^{\mathcal{K}}(x)$ does not necessarily hold when R is serial, reflexive, symmetric, transitive, a preorder, or a similarity relation, as demonstrated in Examples 1, 2, 3, and 4.*

(ii) For all $j \in \Omega$ and $x, y \in U$, the condition $\omega_j(x) \cap \omega_j(y) \neq \emptyset$ does not imply that $\omega_j(x) \cap \omega_j(y) \notin \mathcal{K}$, as shown in Examples 1, 2, 3, and 4.

(iii) For all $j \in \Omega$ and $x, y \in U$, the intersection $\mathbb{I}_j^{\mathcal{K}}(x) \cap \mathbb{I}^{\mathcal{K}} j(y) \neq \emptyset$ does not necessarily imply that $\mathbb{I}_j^{\mathcal{K}}(x) \cap \mathbb{I}_j^{\mathcal{K}}(y) \notin \mathcal{K}$, as evidenced in Examples 1, 2, 3, and 4.

Accordingly, the revised formulation of Theorem 3.4 from [40] is stated as follows:

Theorem 6. *Let $(U, R, \mathcal{K})$ be an $\mathbb{I}_j$-G approximation space, and let $s, t \in U$. The following statements are valid:*

(i) For all $j \in \Omega$, $t \in \mathbb{I}_j^{\mathcal{K}}(s)$ if and only if $s \in \mathbb{I}_j^{\mathcal{K}}(t)$.

(ii) If R is reflexive, then $\mathbb{I}_{<j>}^{\mathcal{K}}(s) \subseteq \mathbb{I}_j^{\mathcal{K}}(s)$, for each $j \in \{a, b, i, u\}$.

(iii) If R is preorder, then $\mathbb{I}_{<j>}^{\mathcal{K}}(s) = \mathbb{I}_j^{\mathcal{K}}(s)$, for each $j \in \{a, b, i, u\}$.

(iv) If R is reflexive, then for all $j \in \Omega$, the following equalities hold:

$$\rho_j(s) \cap \omega_j(s) = \rho_j(s) \text{ and } \rho_j(s) \cup \omega_j(s) = \omega_j(s).$$

(v) If R is symmetric, then

$$\mathbb{I}_i^{\mathcal{K}}(s) = \mathbb{I}_a^{\mathcal{K}}(s) = \mathbb{I}_b^{\mathcal{K}}(s) = \mathbb{I}_u^{\mathcal{K}}(s)$$

and

$$\mathbb{I}_{<i>}^{\mathcal{K}}(s) = \mathbb{I}_{<a>}^{\mathcal{K}}(s) = \mathbb{I}_{<b>}^{\mathcal{K}}(s) = \mathbb{I}_{<u>}^{\mathcal{K}}(s).$$

Proof. The proofs of statements 1, 2, and 5 can be found in [40]. We now proceed to establish the remaining assertions that are not explicitly stated in [40], as follows:
3. According to [43], since R is a preorder, it follows that $\omega_{<j>}(s) = \omega_j(s)$ for all $j \in \Omega$. Therefore, by applying statements 1 and 2, we conclude that $\mathbb{I}_{<j>}^{\mathcal{K}}(s) = \mathbb{I}_j^{\mathcal{K}}(s)$.
4. As noted in [18], the reflexivity of R ensures that $\rho_j(s) \subseteq \omega_j(s)$ for all $j \in \Omega$. Thus, it follows that

$$\rho_j(s) \cap \omega_j(s) = \rho_j(s) \text{ and } \rho_j(s) \cup \omega_j(s) = \omega_j(s), \text{ for all } j \in \Omega.$$

As noted in Remark 3, the proof of item (8) in Theorem 3.4 from [40] contains inaccuracies. To address this, we present the following lemma, accompanied by a detailed proof—absent in [40]—which serves both as a correction to item (8) of Theorem 3.4 in [40] and as a supporting result for the proof of Theorem 7:

Lemma 1. *Let R be a symmetric and transitive relation on U. Then, for all $s, t \in U$ and each $j \in \Omega$,*

$$\omega_j(s) \cap \omega_j(t) \neq \emptyset \quad \text{if and only if} \quad \omega_j(s) = \omega_j(t).$$

Proof. We prove the case $j = a$; the argument for other values of j is identical.
Assume $\omega_a(s) \cap \omega_a(t) \neq \emptyset$, and let

$$q \in \omega_a(s) \cap \omega_a(t).$$

It suffices to show that

$$\omega_a(s) = \omega_a(q) \quad \text{and} \quad \omega_a(t) = \omega_a(q).$$

(i) $\omega_a(s) = \omega_a(q)$**.** Since $q \in \omega_a(s)$, we have

$$sRq, \tag{1}$$
$$qRs \quad \text{(by the symmetry of } R\text{)}. \tag{2}$$

If $x \in \omega_a(q)$, then qRx, and by (1) and the transitivity of R, it follows that sRx, so $x \in \omega_a(s)$. Hence,

$$\omega_a(q) \subseteq \omega_a(s).$$

Conversely, if $y \in \omega_a(s)$, then sRy, and by (2) and transitivity, qRy, so $y \in \omega_a(q)$. Thus,

$$\omega_a(s) \subseteq \omega_a(q),$$

and therefore $\omega_a(s) = \omega_a(q)$.

(ii) $\omega_a(t) = \omega_a(q)$**.** The same reasoning applies, since $q \in \omega_a(t)$ implies tRq and qRt, and the transitivity argument shows that $\omega_a(t) = \omega_a(q)$.

(iii) Converse Direction. On the other hand, it is easy to see, using the same reasoning, that if $\omega_j(s) = \omega_j(t)$, then $\omega_j(s) \cap \omega_j(t) \neq \emptyset$.

Now, the proof of item (8) in Theorem 3.4 from [40] contains errors. Therefore, we provide a revised and corrected version as follows:

Theorem 7. *[40] Let R be a symmetric and transitive relation on U, and let $\mathcal{K}$ be an ideal on U. For any $s \in U$ and $j \in \Omega$, the following hold:*

(i) $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq \omega_j(s)$.

(ii) If $s \in \mathbb{I}_j^{\mathcal{K}}(t)$ *then* $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq \mathbb{I}_j^{\mathcal{K}}(t)$.

Proof. We again treat the case $j = a$.

1. Let $p \in \mathbb{I}_a^{\mathcal{K}}(s)$. By definition,

$$\omega_a(p) \cap \omega_a(s) \notin \mathcal{K},$$

so in particular

$$\omega_a(p) \cap \omega_a(s) \neq \emptyset.$$

Hence there exists $q \in \omega_a(p) \cap \omega_a(s)$ with pRq and sRq. Symmetry and transitivity of R imply sRp, i.e. $p \in \omega_a(s)$. Thus

$$\mathbb{I}_a^{\mathcal{K}}(s) \subseteq \omega_a(s).$$

2. Suppose $s \in \mathbb{I}_a^{\mathcal{K}}(t)$, so

$$\omega_a(s) \cap \omega_a(t) \notin \mathcal{K} \quad \Longrightarrow \quad \omega_a(s) \cap \omega_a(t) \neq \emptyset.$$

By Lemma 1, $\omega_a(s) = \omega_a(t)$ and hence

$$\omega_a(s) \notin \mathcal{K}, \quad \omega_a(t) \notin \mathcal{K}.$$

Now let $p \in \mathbb{I}_a^{\mathcal{K}}(s)$, so

$$\omega_a(p) \cap \omega_a(s) \notin \mathcal{K} \quad \Longrightarrow \quad \omega_a(p) \cap \omega_a(s) \neq \emptyset.$$

Again by Lemma 1, $\omega_a(p) = \omega_a(s)$. Therefore

$$\omega_a(p) \cap \omega_a(t) = \omega_a(s) \notin \mathcal{K},$$

and hence $p \in \mathbb{I}_a^{\mathcal{K}}(t)$, completing the proof.

The converse of Theorem 7 (1) does not hold in general, as demonstrated by the following example.

Example 5. *Consider the set $U = \{p, q, s, t\}$ with the relation $R = \{(t, t)\}$, which is both symmetric and transitive. Let $\mathcal{K} = \{\emptyset, \{t\}\}$ be an ideal on U. Then, we obtain $\omega_a(t) = \{t\}$ and $\mathbb{I}_a^{\mathcal{K}}(t) = \emptyset$. Consequently, it follows that $\omega_a(t) \nsubseteq \mathbb{I}_a^{\mathcal{K}}(t)$.*

Table 1 in [40] introduces various kinds of neighborhoods, including ω_j-neighborhoods, ρ_j-neighborhoods, $\mathbb{I}_j$-neighborhoods, and $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods, for each $j \in \Omega$. In this work, we address and correct several errors in Table 1, that create ambiguity in the intended meaning. Specifically, while a neighborhood of a point in the universe can be equal to the empty set, it should not be represented as $\{\emptyset\}$.

Furthermore, Table 2 in [40] provides a comparative analysis of the lower and upper operators, as well as accuracy measure results for any subset of U, with respect to $\{a, b, i, u\}$. According to Remark 1, we identify and clarify several observations and necessary corrections related to this table.

(i) Since $R_\star^{\mathbb{I}_a^{\mathcal{K}}}(\emptyset) = \{q, s, t\}$, it follows that $R^{\star\mathbb{I}_a^{\mathcal{K}}}(\emptyset) = \emptyset$. Consequently, $ACC_R^{\star\mathbb{I}_a^{\mathcal{K}}}(\emptyset)$ is not zero but rather an indefinite quantity. By the same manner, $ACC_R^{\star\mathbb{I}_b^{\mathcal{K}}}(\emptyset)$, $ACC_R^{\star\mathbb{I}_i^{\mathcal{K}}}(\emptyset)$, and $ACC_R^{\star\mathbb{I}_u^{\mathcal{K}}}(\{p\})$ should also be considered indefinite quantities.

(ii) Since $R_\star^{\mathbb{I}_u^{\mathcal{K}}}(\emptyset) = R^{\star\mathbb{I}_u^{\mathcal{K}}}(\emptyset) = \emptyset$, it follows that $ACC_R^{\star\mathbb{I}_u^{\mathcal{K}}}(\emptyset)$ should be 1, not zero.

(iii) Given that $R_\star^{\mathbb{I}_b^{\mathcal{K}}}(\emptyset) = \{q\}$, which is not equal to $\emptyset$, it follows that $ACC_R^{\star\mathbb{I}_b^{\mathcal{K}}}(\emptyset)$ is also an indefinite quantity.

(iv) $ACC_R^{\star\mathbb{I}_i^{\mathcal{K}}}(\emptyset) = 1$.

(v) Since $R_\star^{\mathbb{I}_u^{\mathcal{K}}}(\{p\}) = \{q\}$, this set is nonempty.

Item (3) of Proposition 4.3 in [40], which asserts that $R^{\star\mathbb{I}_a^{\mathcal{K}}}(U) \supseteq U$, is incorrect. As illustrated in Table 2 of [40], it is observed that $R^{\star\mathbb{I}_a^{\mathcal{K}}}(U) = \{p\}$, which does not satisfy $R^{\star\mathbb{I}_a^{\mathcal{K}}}(U) \supseteq U$. Consequently, several observations and necessary corrections have been recorded based on this table.

Remark 4.4 in [40] further asserts that the inclusion relations in Proposition 4.3 (3) are generally proper. However, this claim is also inaccurate, as the inclusion $R^{\star\mathbb{I}_a^{\mathcal{K}}}(U) \subseteq U$ does not hold in this context.

According to Remark 1, Table 3 in [40] provides a comparative analysis between the results derived from the proposed Definition 4.1 in [40] and those obtained from Definition 2.7 in [12] for the case $j = a$. This comparison reveals several observations and necessary corrections, as outlined below:

(i) Given that $R_\star^{\mathbb{I}_a^{\mathcal{K}}}(\emptyset) = \{q, s, t\}$, $R^{\star\mathbb{I}_a^{\mathcal{K}}}(\emptyset) = \emptyset$, it follows that $ACC_R^{\star\mathbb{I}_a^{\mathcal{K}}}(\emptyset)$ is not zero but an indefinite quantity.

(ii) $ACC_R^{\star\mathbb{I}_a}(\emptyset)$ should be 1, not zero.

According to Remark 1, Table 4 in [40] presents a comparative analysis of results derived from the proposed Definition 4.1 from [40], alongside those of Definition 2 from [3], Definition 3 from [5] and Definitions 4.3, 4.5 in [17] for the case $j = b$. Several omissions and errors were identified in this comparison, as outlined below.

(i) Since $R_\star^{\mathbb{I}_b^{\mathcal{K}}}(\emptyset) = \{q\}$, which is not equal to $\emptyset$, it follows that $\emptyset$ is rough, implying that $ACC_R^{\star\mathbb{I}_b^{\mathcal{K}}}(\emptyset)$ is undefined.

(ii) $R_\star^{w_b}(\emptyset) = \emptyset$, rather than $\{q\}$. Thus, $R_\star^{w_b}(\emptyset) = R^{\star\omega_b}(\emptyset) = \emptyset$, indicating that $\emptyset$ is exact and hence $ACC_R^{\star w_b}(\emptyset) = 1$.

(iii) $R_\star^{w_b}(\{p\}) = \{p, s\}$, which differs from $\{p, q, s\}$.

(iv) $R_\star^{w_b}(\{q\}) = \emptyset$, rather than $\{q\}$, and consequently, $ACC_R^{\star w_b}(\{q\}) = 0$.

(v) $R_\star^{w_b}(\{s\}) = \emptyset$, differing from $\{q\}$.

(vi) $R_\star^{w_b}(\{p, q\}) = \{p, s\}$, which is not equal to U. Also, $ACC_R^{\star\omega_b}(\{p, q\}) = \frac{1}{4}$.

(vii) $R_\star^{w_b}(\{p, s\}) = \{p, s\}$, differing from $\{p, q, s\}$.

(viii) $R_\star^{w_b}(\{q, s\}) = \emptyset$, which does not equal to $\{q\}$. Furthermore, $ACC_R^{\star w_b}(\{q, s\}) = 0$.

(ix) $R_\star^{w_b}(\{p, q, s\}) = \{p, s\}$, differing from U. Moreover, $ACC_R^{\star w_b}(\{p, q, s\}) = \frac{1}{2}$.

(x) $R_\star^{\rho_b}(\emptyset) = \emptyset$, differing from $\{t\}$. Therefore, $R_\star^{\rho_b}(\emptyset) = R^{\star\rho_b}(\emptyset) = \emptyset$, implying that $\emptyset$ is an exact set, and hence $ACC_R^{\star\rho_b}(\emptyset) = 1$.

(xi) $R^{\star\rho_b}(U) = U$, rather than $\{p, q, s\}$.

(xii) $R_\star^{\rho_b}(\{p\}) = \emptyset$, instead of $\{t\}$, while $R^{\star\rho_b}(\{p\}) = \{p, s\}$, differing from $\{p, q\}$.

(xiii) $R_\star^{\rho_b}(\{q\}) = \{q\}$, not $\{q, t\}$.

(xiv) $R_\star^{\rho_b}(\{s\}) = \emptyset$, rather than $\{t\}$.

(xv) $R^{\star\rho_b}(\{t\}) = \{t\}$, not $\emptyset$.

(xvi) $R_\star^{\rho_b}(\{p, q\}) = \{q\}$, rather than $\{q, t\}$.

(xvii) $R_\star^{\rho_b}(\{p, s\}) = \{p, s\}$, differing from $\{p, s, t\}$.

(xviii) $R_\star^{\rho_b}(\{p, t\}) = \{t\}$, rather than $\{p, s, t\}$, while $R^{\star\rho_b}(\{p, t\}) = \{p, s, t\}$, differing from $\{p, s\}$. In addition, $ACC_R^{\star\rho_b}(\{p, t\}) = \frac{1}{3}$.

(xix) $R_{\star}^{\rho_b}(\{q,s\}) = \{q\}$, which does not equal to $\{q,t\}$.

(xx) $R^{\star\rho_b}(\{q,t\}) = \{q,t\}$, rather than $\{q\}$.

(xxi) $R^{\star\rho_b}(\{s,t\}) = \{p,s,t\}$, not $\{p,s\}$.

(xxii) $R_{\star}^{\rho_b}(\{p,q,s\}) = \{p,q,s\}$, differing from U.

(xxiii) $R^{\star\rho_b}(\{p,q,t\}) = U$, instead of $\{p,q\}$. Additionally, $ACC_R^{\star\rho_b}(\{p,q,t\}) = \frac{1}{2}$.

(xxiv) $R_{\star}^{\rho_b}(\{p,s,t\}) = \{p,s,t\}$, rather than $\{p,s\}$, and $R^{\star\rho_b}(\{p,s,t\}) = \{p,s,t\}$, differing from $\{p,s\}$. Moreover, $ACC_R^{\star\rho_b}(\{p,s,t\}) = 1$.

(xxv) $R^{\star\rho_b}(\{q,s,t\}) = U$, differing from $\{p,q\}$.

Remark 4. *In the proof of Theorem 5.1 in [40], the symbol $\mathcal{I}$ should be modified into the symbol $\mathcal{K}$.*

Items (5) and (8) of Proposition 5.4 in [40] are not generally valid, as evidenced by the following example.

Example 6. *Consider the reflexive relation $R = \{(p,p),(p,s),(p,t),(q,t),(q,q),(s,s),(t,q),(t,t)\}$ on $U = \{p,q,s,t\}$. If the ideal is given by $\mathcal{K} = \{\emptyset,\{s\},\{t\},\{s,t\}\}$, then the associated topologies are as follows:*

$$\tau^{\mathbb{I}^{\mathcal{K}}_{<a>}} = 2^U \text{ (the power set of } U\text{)},$$

$$\tau^{\mathbb{I}^{\mathcal{K}}_{a}} = \{\{p\},\{q\},\{s\},\{p,q\},\{p,s\},\{q,s\},\{q,t\},\{p,q,s\},\{p,q,t\},\{q,s,t\},\emptyset,U\}.$$

Thus, we observe that $\tau^{\mathbb{I}^{\mathcal{K}}_{<a>}} \nsubseteq \tau^{\mathbb{I}^{\mathcal{K}}_{a}}$.

The next proposition presents a revised formulation to rectify item (5) of Proposition 5.4 in [40]:

Proposition 1. *Let $(U,R,\mathcal{K})$ be an $\mathbb{I}_j$-G approximation space, and let $s \in U$. If R is reflexive, then for each $j \in \{a,b,i,u\}$, the inclusion $\tau^{\mathbb{I}^{\mathcal{K}}_{j}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{<j>}}$ holds.*

Proof. Let $F \in \tau^{\mathbb{I}^{\mathcal{K}}_{j}}$ and suppose $s \in F$. By definition, this implies that $\mathbb{I}^{\mathcal{K}}_{j}(s) \setminus F \in \mathcal{K}$. Since R is reflexive, it follows from Theorem 1 that $\mathbb{I}^{\mathcal{K}}_{<j>}(s) \subseteq \mathbb{I}^{\mathcal{K}}_{j}(s)$. Therefore, we obtain $\mathbb{I}^{\mathcal{K}}_{<j>}(s) \setminus F \in \mathcal{K}$, which implies that $F \in \tau^{\mathbb{I}^{\mathcal{K}}_{<j>}}$. Thus, the desired inclusion holds.

Remark 5. *If R is reflexive, then for every $x \in U$ and for all $j \in \Omega$, the inclusion $\rho_j(x) \subseteq \mathbb{I}^{\mathcal{K}}_{j}(x)$ does not hold in general. As a result, it follows that $\tau^{\rho^{\mathcal{K}}_{j}} \nsubseteq \tau^{\mathbb{I}^{\mathcal{K}}_{j}}$ $\forall j \in \Omega$.*

According to Remark 1, Table 5 in [40] presents a comparative analysis of the boundary region and accuracy measure results for a set F based on Definition 5.7 in [40], where $j \in \{a, b, i, u\}$. The values of $ACC_R^{\tau^{\mathbb{I}_a^{\mathcal{K}}}}(\emptyset)$ (and similarly $ACC_R^{\tau^{\mathbb{I}_b^{\mathcal{K}}}}(\emptyset), ACC_R^{\tau^{\mathbb{I}_i^{\mathcal{K}}}}(\emptyset), and ACC_R^{\tau^{\mathbb{I}_u^{\mathcal{K}}}}(\emptyset)$) should be 1, rather than 0. Besides, the indices $j = r, j = l$ have been revised to $j = a, j = b$ respectively.

Table 6 in [40] presents a comparison of the boundary region and accuracy measure results for a set F based on the proposed Definition 5.7 and Definition 2.13 from [12] for $j = a$. The notation $\underline{\tau^{\mathbb{I}_b}}(L)$ (and similarly $\overline{\tau^{\mathbb{I}_b}}(L), BND^{\tau^{\mathbb{I}_b}}(L), ACC^{\tau^{\mathbb{I}_b}}(L)$) should be corrected to $\underline{\tau^{\mathbb{I}_a}}(L)$ (and similarly $\overline{\tau^{\mathbb{I}_a}}(L), BND^{\tau^{\mathbb{I}_a}}(L), ACC^{\tau^{\mathbb{I}_a}}(L)$). Additionally, the values of $ACC_R^{\tau^{\mathbb{I}_a^{\mathcal{K}}}}(\emptyset)$, and $ACC_R^{\tau^{\mathbb{I}_a}}(\emptyset)$ should be considered indefinite.

4. Conclusion and Discussion

This paper presents a systematic critique of the work by T. M. Al-Shami and M. Hosny [40], identifying and resolving fundamental flaws in their formalization of $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods within rough set theory. Through rigorous analysis, counterexamples, and reconstructed proofs, we have thoroughly examined and rectified critical inaccuracies in their definitions, theoretical results, and illustrative examples. The proposed corrections provide dual benefits: they strengthen the theoretical foundations of neighborhood systems while enhancing their practical applicability within rough set–based frameworks. Our primary contribution lies in dismantling the problematic claims of the original work through comprehensive error identification and replacement with validated and consistent alternatives. The revised framework substantially improves the internal coherence and reliability of neighborhood-based approximation methods, particularly through corrected examples that now accurately support the theoretical propositions. Moreover, the refined methodology paves the way for new research directions in uncertainty quantification, offering researchers more robust analytical tools for complex data interpretation, as demonstrated through improved medical application scenarios.

In this study, several major errors have been identified in Theorem 3.4, Proposition 4.3, and Proposition 5.4 of [40]. These inaccuracies were demonstrated through various examples, including Examples 1, 2, 3, 4, 5, and 6. Corrections and clarifications were subsequently provided, such as the revised proof of Theorem 3.4, which is now accurately reformulated and presented in Theorems 6 and 7 of this paper. Additional corrected versions of erroneous results, including Theorem 6 and Proposition 1, have also been proposed. Errors within several concepts, definitions, examples, and tables from [40] have been thoroughly identified and corrected. These include revisions to Definitions 1, 2, 11, and 12, as well as major modifications to five tables and the resolution of twenty-five distinct inconsistencies across multiple examples. Furthermore, new theoretical results and properties, including Remark 3 and Lemma 1, have been introduced and justified to reinforce the consistency of the proposed framework.

Overall, the findings of this study highlight the necessity of critically examining recently proposed methodologies and results in evolving fields such as rough set theory. The corrections and refinements provided herein ensure that the theoretical basis of neighborhood systems remains sound, rigorous, and dependable for subsequent research and applications. Moreover, the enhanced understanding of neighborhood-based rough approximations broadens their applicability in decision-making, data analysis, and knowledge discovery across diverse scientific and engineering domains.

In our future work, we will apply the techniques introduced in the present paper to various fields, such as applications of fuzzy topological spaces [44], rough lattices [45], the study of SC-compactness of rough sets using ideal neighborhoods [46], and the improvement of rough approximation and ideal methods introduced in [47].

Finally, to facilitate easy reference, we provide a nomenclature section listing the symbols along with their abbreviations.

List of symbols and abbreviations

U	A nonempty universe set.
$\emptyset$	The empty set.
$\mathcal{K}$	The ideal.
j-**NS**	j-Neighborhood Space, for each $j \in \Omega$, where $\Omega = \{a, b, < a >, < b >, i, u, < i >, < u >\}$.
$\omega_j(s)$	The j-neighborhood of a point s, for each $j \in \Omega$.
$\rho_j(s)$	The ρ_j-neighborhood of a point s, for each $j \in \Omega$.
$\mathbb{I}_j(s)$	The $\mathbb{I}_j$-neighborhood of a point s, for each $j \in \Omega$.
$\mathbb{I}_j^{\mathcal{K}}(s)$	The $\mathbb{I}_j^{\mathcal{K}}$-neighborhood of a point s, for each $j \in \Omega$.
$\mathbb{I}_j$-G approximation space	$\mathbb{I}_j$-Generalized approximation space.
$\top^{h_j}$	h_j-topology generated by h_j-neighborhood, where $h \in \{\omega, \rho, \mathbb{I}, \mathbb{I}^{\mathcal{K}}\}$.
$R_\star^{h_j}(F)$	The lower approximation of a subset $F \subseteq U$, where $h \in \{\omega, \rho, \mathbb{I}, \mathbb{I}^{\mathcal{K}}\}$.
$R^{\star h_j}(F)$	The upper approximation of a subset $F \subseteq U$, where $h \in \{\omega, \rho, \mathbb{I}, \mathbb{I}^{\mathcal{K}}\}$.
$BND_R^{\star h_j}(F)$	The boundary region of a subset $F \subseteq U$, where $h \in \{\omega, \rho, \mathbb{I}, \mathbb{I}^{\mathcal{K}}\}$.
$ACC_R^{\star h_j}(F)$	The accuracy measure of a subset $F \subseteq U$, where $h \in \{\omega, \rho, \mathbb{I}, \mathbb{I}^{\mathcal{K}}\}$.

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The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of the content of this article. However, AI tools have been utilized solely for grammar and spelling correction support. All the information and data in this paper have been ethically reviewed, edited, and studied. No AI-generated data or results are included.

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