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Some Geometric Results for a New Subfamily of Regular Functions in the Generalized Janowski Domain

Tamer M. Seoudy[1,*], Amnah E. Shammaky[2]

[1] *Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt*
[2] *Department of Mathematics, Faculty of Science, Jazan University, Jazan 45142, Saudi Arabia*

Abstract. Let $\mathcal{SK}[P,Q;\alpha,\beta]$ denote the subfamily of normalized regular function $g(\xi)=\xi+d_2\xi^2+d_3\xi^3+d_4\xi^4+...$ in the open unit disk Δ holding the next subordination condition:

$$\frac{\alpha g(\xi)+\beta\xi g'(\xi)}{\alpha\xi+\beta g(\xi)} \prec \frac{1+[Q+(1-\gamma)(P-Q)]\xi}{1+M\xi},$$

where $0\leq\alpha,\beta\leq 1$, $-1\leq Q<P\leq 1$, $0\leq\gamma<1$ and $\xi\in\Delta$. In this article, we study some geometric properties such as convolution results, coefficient estimates, upper bounds for the initial coefficients of the first four coefficients, and the Fekete-Szegö inequalities for this subfamily.

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1. Introduction

Let us represent the family of regular (analytic) functions inside the symmetrical unit disc $\Delta=\{\xi\in\mathbb{C}:|\xi|<1\}$ by $\mathcal{A}(\Delta)$ and let $\mathcal{H}$ be the subfamily of $\mathcal{A}(\Delta)$ consisting of all functions g in Δ that has the following form:

$$g(\xi)=\xi+\sum_{j=2}^{\infty}d_j\xi^j \quad (\xi\in\Delta). \tag{1}$$

Also, let Ω be the family of functions $w\in\mathcal{A}(\Delta)$ satisfying $w(0)=0$ and $|w(\xi)|<1$ for all $\xi\in\Delta$. For $h_1(\xi),h_2(\xi)\in\mathcal{A}(\Delta)$, we say that $h_1(\xi)$ is subordinate to $h_2(\xi)$, written as $h_1(\xi)\prec h_2(\xi)$ if there exists a function $w(\xi)\in\Omega$ such that $h_1(\xi)=(h_2\circ w)(\xi)$ for all $\xi\in\Delta$. Moreover, if $h_1(\xi)$ is univalent function in Δ, then $h_1(\xi)\prec h_2(\xi)$ if and only if $h_1(0)=h_2(0)$ and $h_1(\Delta)\subset h_2(\Delta)$.

*Corresponding author.
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Email addresses: tms00@fayoum.edu.eg (T. M. Seoudy), aeshamakhi@jazan.edu.sa (A. E. Shammaky)

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For functions $g, h \in \mathcal{H}$, where g is given by (1) and h has the following form:

$$h(\xi) = \xi + \sum_{j=2}^{\infty} e_j \xi^j, \tag{2}$$

then, the convolution of g and h is defined by

$$(g * h)(\xi) = \xi + \sum_{j=2}^{\infty} d_j e_j \xi^j = (h * g)(\xi). \tag{3}$$

Definition 1. *Let us introduce the new subfamily $\mathcal{SK}[P, Q; \alpha, \beta]$ of $\mathcal{H}$ by using the subordination principle between two analytic functions as follows:*

$$\mathcal{SK}[P, Q, \gamma; \alpha, \beta] = \left\{ g \in \mathcal{H} : \frac{\alpha g(\xi) + \beta \xi g'(\xi)}{\alpha \xi + \beta g(\xi)} \prec \frac{1 + [Q + (1-\gamma)(P-Q)]\xi}{1 + M\xi} \right\}, \tag{4}$$

where $0 \le \alpha, \beta \le 1$, $-1 \le Q < P \le 1$, $0 \le \gamma < 1$ and $\xi \in \Delta$.

Remark 1. *The function*

$$\psi(\xi) = \frac{1 + [Q + (1-\gamma)(P-Q)]\xi}{1 + Q\xi}$$

is a bilinear transformation that maps the symmetrical unit disk Δ onto the symmetrical disk with respect to positive real axis, which is centered at $\frac{1-[Q+(1-\gamma)(P-Q)]Q}{1-Q^2}$ $(Q \ne \pm 1)$ and, with a radius $\frac{(1-\gamma)(P-Q)}{1-Q^2}$ $(Q \ne \pm 1)$.

(i) Substituting $\alpha = \gamma = 0$ in (4), we get the subfamily

$$\mathcal{S}[P, Q] = \left\{ g \in \mathcal{H} : \frac{\xi g'(\xi)}{g(\xi)} \prec \frac{1 + P\xi}{1 + Q\xi} \right\},$$

which was introduced by Janowski [1, 2]. Furthermore, this subfamily has been studied by the many authors (see [3–13]). By taking $\beta = \gamma = 0$ in (4), we obtain the next subfamily

$$\mathcal{K}[P, Q] = \left\{ g \in \mathcal{H} : \frac{g(\xi)}{\xi} \prec \frac{1 + P\xi}{1 + Q\xi} \right\};$$

(ii) Putting $P = 1$ and $Q = -1$ in (4), we obtain the next subfamily

$$\mathcal{SK}[\gamma; \alpha, \beta] = \left\{ g \in \mathcal{H} : \Re\left\{ \frac{\alpha g(\xi) + \beta \xi g'(\xi)}{\alpha \xi + \beta g(\xi)} \right\} > \gamma \right\}, \tag{5}$$

by taking $\alpha = 0$ in (5), we obtain the next subfamily

$$\mathcal{S}(\gamma) = \left\{ g \in \mathcal{H} : \Re\left\{ \frac{\xi g'(\xi)}{g(\xi)} \right\} > \gamma \right\},$$

where $\mathcal{S}(\gamma)$ is the family of starlike of order γ in Δ (see [14–19]) and by putting $\beta = 0$ in (5), we obtain the next subfamily

$$\mathcal{K}(\gamma) = \left\{ g \in \mathcal{H} : \Re\left\{ \frac{g(\xi)}{\xi} \right\} > \gamma \right\};$$

(iii) Substituting the value of $P = \rho$ and $Q = -\rho$ with $0 < \rho \leq 1$ in (4), we get the following subfamily

$$\mathcal{SK}(\gamma, \rho; \alpha, \beta) = \left\{ g \in \mathcal{H} : \left| \frac{\frac{\alpha g(\xi)+\beta \xi g'(\xi)}{\alpha \xi + \beta g(\xi)} - 1}{\frac{\alpha g(\xi)+\beta \xi g'(\xi)}{\alpha \xi + \beta g(\xi)} + 1 - 2\gamma} \right| < \rho \right\} \tag{6}$$

by substituting $\alpha = 0$ in (6), we have

$$\mathcal{S}(\gamma, \rho) = \left\{ g \in \mathcal{H} : \left| \frac{\frac{\xi g'(\xi)}{g(\xi)} - 1}{\frac{\xi g'(\xi)}{g(\xi)} + 1 - 2\gamma} \right| < \rho \right\},$$

where the subfamily $\mathcal{S}(\gamma, \rho)$ was introduced in [20] and also, by putting $\alpha = 0$ in (6), we get the following subfamily

$$\mathcal{K}(\gamma, \rho) = \left\{ g \in \mathcal{H} : \left| \frac{\frac{g(\xi)}{\xi} - 1}{\frac{g(\xi)}{\xi} + 1 - 2\gamma} \right| < \rho \right\}.$$

In order to obtain the results of this study, we will mention the next definition and the next lemmas.

Definition 2. *The family of regular functions $\phi(\xi)$ in Δ with the positive real part in Δ, and of the form*

$$\phi(\xi) = 1 + \sum_{j=1}^{\infty} \delta_j \xi^j \quad (\xi \in \Delta) \tag{7}$$

is denoted by Φ, that calls the well-known Carathéodory family.

Lemma 1. *If ϕ given by (7) belongs to Φ, then*

$$|\delta_j| \leq 2, \quad for \quad j \in \mathbb{N} = \{1, 2, 3, ...\}, \tag{8}$$

for u, v are complex numbers we have

$$\left|\delta_2 - u\delta_1^2\right| \leq 2\max\{1; |2u - 1|\} \tag{9}$$

and

$$|\delta_{i+j} - v\delta_i\delta_j| \leq 2\max\{1; |1 - 2v|\}. \tag{10}$$

The inequality (8) is Carathéodory's result (see [21, 22]), while the inequality (9) may be detected in [23], and the inequality (10) is from [24].

Lemma 2. *[25] If ϕ given by (7) belongs to Φ, then*

$$\left|\delta_2 - u\delta_1^2\right| \leq \begin{cases} -4u+2 & if \quad u \leq 0, \\ 2 & if \quad 0 \leq u \leq 1, \\ 4u-2 & if \quad u \geq 1. \end{cases} \tag{11}$$

Also the upper bound (11) can be improved as follows for $0 \leq u \leq \frac{1}{2}$, we have

$$\left|\delta_2 - u\delta_1^2\right| + u\left|\delta_1\right|^2 \leq 2,$$

and for $\frac{1}{2} \leq u \leq 1$, we have

$$\left|\delta_2 - u\delta_1^2\right| + (1-u)\left|\delta_1\right|^2 \leq 2.$$

In the following sections, for the function subfamily $\mathcal{SK}[P,Q;\alpha,\beta]$ and its special subfamilies $\mathcal{S}[P,Q]$ and $\mathcal{K}[P,Q]$, we study the well-known results, like convolution properties, necessary and sufficient conditions, coefficient estimates, and Fekete-Szegö inequalities.

2. Convolution Properties

Unless otherwise stated, we suppose throughout this article that $0 \leq \theta < 2\pi$; $0 \leq \alpha, \beta \leq 1$; $\alpha + \beta \neq 0$; $-1 \leq Q < P \leq 1$; $0 \leq \gamma < 1$ and $g \in \mathcal{H}$ has the series form (1).

Theorem 1. *The function $g \in \mathcal{SK}[P,Q;\alpha,\beta]$ if and only if*

$$\frac{1}{\xi}\left\{g(\xi) * \frac{\xi - \left(\frac{\alpha+(\alpha+\beta)R}{\alpha+\beta}\right)\xi^2 + \left(\frac{\alpha R}{\alpha+\beta}\right)\xi^3}{(1-\xi)^2}\right\} \neq 0, \tag{12}$$

where R is defined by

$$R = R(\theta, P, Q, \gamma) = \frac{\mathrm{e}^{-i\theta} + Q}{(1-\gamma)(P-Q)} + 1. \tag{13}$$

Proof. It is clear to check the next:

$$\left.\begin{aligned} g(\xi) &= g(\xi) * \frac{\xi}{1-\xi}, \\ \xi g'(\xi) &= g(\xi) * \frac{\xi}{(1-\xi)^2}. \end{aligned}\right\} \tag{14}$$

To show that (12) is true, we will write (4) as follows

$$\frac{\alpha g(\xi) + \beta\xi g'(\xi)}{\alpha\xi + \beta g(\xi)} = \frac{1 + [Q + (1-\gamma)(P-Q)]\,w(\xi)}{1 + Qw(\xi)}, \tag{15}$$

where $w(\xi) \in \Omega$, hence we have

$$\frac{\alpha g(\xi) + \beta \xi g'(\xi)}{\alpha \xi + \beta g(\xi)} \neq \frac{1 + [Q + (1-\gamma)(P-Q)] \mathrm{e}^{i\theta}}{1 + Q\mathrm{e}^{i\theta}},$$

which is equivalent to

$$\frac{1}{\xi} \left\{ \left(1 + Q\mathrm{e}^{i\theta}\right) \left[\alpha g(\xi) + \beta \xi g'(\xi)\right] - \left(1 + [Q + (1-\gamma)(P-Q)] \mathrm{e}^{i\theta}\right) \left[\alpha \xi + \beta g(\xi)\right] \right\} \neq 0. \tag{16}$$

By using (14) in (16), we get

$$\begin{aligned}
&\frac{1}{\xi} \left\{ \begin{array}{c} g(\xi) * \left(\frac{\alpha\xi}{1-\xi} + \frac{\beta\xi}{(1-\xi)^2}\right)\left(1 + Q\mathrm{e}^{i\theta}\right) \\ -g(\xi) * \left(\alpha\xi + \frac{\beta\xi}{1-\xi}\right)\left(1 + [Q + (1-\gamma)(P-Q)] \mathrm{e}^{i\theta}\right) \end{array} \right\} \\
= &\frac{1}{\xi} \left\{ \begin{array}{c} g(\xi) * \left(\frac{(\alpha+\beta)\xi - \alpha\xi^2}{(1-\xi)^2}\right)\left(1 + Q\mathrm{e}^{i\theta}\right) \\ -g(\xi) * \left(\frac{(\alpha+\beta)\xi - (2\alpha+\beta)\xi^2 + \alpha\xi^3}{(1-\xi)^2}\right)\left(1 + [Q + (1-\gamma)(P-Q)] \mathrm{e}^{i\theta}\right) \end{array} \right\} \\
= &-\frac{(\alpha+\beta)(1-\gamma)(P-Q)\mathrm{e}^{i\theta}}{\xi} \left\{ g(\xi) * \frac{\xi - \frac{\alpha + (\alpha+\beta)\left(\frac{\mathrm{e}^{-i\theta}+Q}{(1-\gamma)(P-Q)}+1\right)}{\alpha+\beta}\xi^2 + \frac{\alpha\left(\frac{\mathrm{e}^{-i\theta}+Q}{(1-\gamma)(P-Q)}+1\right)}{\alpha+\beta}\xi^3}{(1-\xi)^2} \right\} \\
\neq &\ 0
\end{aligned}$$

which explains the necessary condition (12) for this theorem.

Reversely, assume that $g \in \mathcal{H}$ satisfies (12). Since the first part has been proven, assumption (12) is equivalent to (16), so we get that

$$\frac{\alpha g(\xi) + \beta \xi g'(\xi)}{\alpha \xi + \beta g(\xi)} \neq \frac{1 + [Q + (1-\gamma)(P-Q)] \mathrm{e}^{i\theta}}{1 + Q\mathrm{e}^{i\theta}}, \tag{17}$$

if we assume to $\varphi(\xi) = \frac{\alpha g(\xi)+\beta\xi g'(\xi)}{\alpha\xi+\beta g(\xi)}$ and $\psi(\xi) = \frac{1+[Q+(1-\gamma)(P-Q)]\xi}{1+Q\xi}$, the relationship (17) indicates that $\varphi(\Delta) \cap \psi(\partial\Delta) = \emptyset$. Therefore, the simply connected domain $\varphi(\Delta)$ lies inside a connected component of $\mathbb{C} \backslash \psi(\partial\Delta)$. Thus, using the fact that the function $\psi(\xi)$ is univalent and $\varphi(0) = \psi(0)$, it follows that $\varphi(\xi) \prec \psi(\xi)$, which means that $g \in \mathcal{SK}[P, Q, \gamma; \alpha, \beta]$. This finishes Theorem 1.

Letting $P = 1$ and $Q = -1$ in Theorem 1, we get the following result.

Corollary 1. *The function $g \in \mathcal{SK}[\gamma; \alpha, \beta]$ if and only if*

$$\frac{1}{\xi} \left\{ g(\xi) * \frac{\xi - \left(\frac{\alpha+(\alpha+\beta)T}{\alpha+\beta}\right)\xi^2 + \left(\frac{\alpha T}{\alpha+\beta}\right)\xi^3}{(1-\xi)^2} \right\} \neq 0,$$

where T is given by

$$T = T(\theta, \gamma) = \frac{\mathrm{e}^{-i\theta} - 1}{2(1-\gamma)} + 1. \tag{18}$$

Putting $\alpha = \gamma = 0$ in Theorem 1, we have the next corollary.

Corollary 2. *The function $g \in \mathcal{S}[P,Q]$ if and only if*

$$\frac{1}{\xi}\left\{g(\xi) * \frac{\xi - R\xi^2}{(1-\xi)^2}\right\} \neq 0,$$

where R is given by (13).

Putting $\beta = \gamma = 0$ in Theorem 1, we get the next.

Corollary 3. *The function $g \in \mathcal{K}[P,Q]$ if and only if*

$$\frac{1}{\xi}\left\{g(\xi) * \frac{\xi - R\xi^2}{1-\xi}\right\} \neq 0,$$

where R is given by (13).

Theorem 2. *The function $g \in \mathcal{SK}[P,Q,\gamma;\alpha,\beta]$ if and only if*

$$1 - \sum_{j=2}^{\infty} \frac{\alpha\left(\mathrm{e}^{-i\theta}+Q\right) + \beta\left[(j-1)\left(\mathrm{e}^{-i\theta}+Q\right) - (1-\gamma)(P-Q)\right]}{(\alpha+\beta)(1-\gamma)(P-Q)} d_j \xi^{j-1} \neq 0. \tag{19}$$

Proof. From Theorem 1, we find that $g(\xi) \in \mathcal{SK}[P,Q,\gamma;\alpha,\beta]$ if and only if

$$\frac{1}{\xi}\left[g(\xi) * \frac{\xi - \frac{\alpha+(\alpha+\beta)R}{\alpha+\beta}\xi^2 + \frac{\alpha R}{\alpha+\beta}\xi^3}{(1-\xi)^2}\right] \neq 0 \tag{20}$$

for all R given by (13). The left hand side of (20) can be written as

$$\frac{1}{\xi}\left\{g(\xi) * \left[\frac{\alpha R}{\alpha+\beta}\xi + \frac{\left(\frac{\alpha+(\beta-\alpha)R}{\alpha+\beta}\right)\xi}{1-\xi} + \frac{\left(\frac{\beta(1-R)}{\alpha+\beta}\right)\xi}{(1-\xi)^2}\right]\right\}$$

$$\begin{aligned}
&= \frac{1}{\xi}\left\{\left(\frac{\alpha R}{\alpha+\beta}\right)\xi + \left(\frac{\alpha+(\beta-\alpha)R}{\alpha+\beta}\right)g(\xi) + \left(\frac{\beta(1-R)}{\alpha+\beta}\right)\xi g'(\xi)\right\} \\
&= 1 - \sum_{j=2}^{\infty} \frac{\alpha(R-1) + \beta(j(R-1)-R)}{\alpha+\beta} d_j \xi^{j-1} \\
&= 1 - \sum_{j=2}^{\infty} \frac{\alpha\left(\mathrm{e}^{-i\theta}+Q\right) + \beta\left[(j-1)\left(\mathrm{e}^{-i\theta}+Q\right) - (1-\gamma)(P-Q)\right]}{(\alpha+\beta)(1-\gamma)(P-Q)} d_j \xi^{j-1}.
\end{aligned}$$

Thus, the proof of Theorem 2 is completed.

Putting $P = 1$ and $Q = -1$ in Theorem 2, we obtain the following result for $\mathcal{SK}[\gamma;\alpha,\beta]$.

Corollary 4. *The function $g \in \mathcal{SK}[\gamma;\alpha,\beta]$ if and only if*

$$1-\sum_{j=2}^{\infty}\frac{\alpha\left(\mathrm{e}^{-i\theta}-1\right)+\beta\left[(j-1)\left(\mathrm{e}^{-i\theta}-1\right)-2(1-\gamma)\right]}{2(\alpha+\beta)(1-\gamma)}d_j\xi^{j-1}\neq 0.$$

Putting $\alpha=\gamma=0$ in Theorem 2, we obtain the following result for $\mathcal{S}[P,Q]$.

Corollary 5. *The function $g \in \mathcal{S}[P,Q]$ if and only if*

$$1-\sum_{j=2}^{\infty}\frac{(j-1)\left(\mathrm{e}^{-i\theta}+Q\right)-P+Q}{P-Q}d_j\xi^{j-1}\neq 0.$$

Putting $\beta=\gamma=0$ in Theorem 2, we obtain the following result for $\mathcal{K}[P,Q]$.

Corollary 6. *The function $g \in \mathcal{K}[P,Q]$ if and only if*

$$1-\sum_{j=2}^{\infty}\frac{\mathrm{e}^{-i\theta}+Q}{P-Q}d_j\xi^{j-1}\neq 0.$$

Theorem 3. *If the function $g(\xi)$ satisfy the inequality*

$$\sum_{j=2}^{\infty}\{\alpha(1-Q)+\beta[(j-1)(1-Q)+(1-\gamma)(P-Q)]\}|d_j|\leq(\alpha+\beta)(P-Q), \tag{21}$$

then $g \in \mathcal{SK}[P,Q,\gamma;\alpha,\beta]$.

Proof. Note that

$$\left|1-\sum_{j=2}^{\infty}\frac{\alpha\left(\mathrm{e}^{-i\theta}+Q\right)+\beta\left[(j-1)\left(\mathrm{e}^{-i\theta}+Q\right)-(1-\gamma)(P-Q)\right]}{(\alpha+\beta)(1-\gamma)(P-Q)}d_j\xi^{j-1}\right|$$

$$\begin{aligned}&\geq\ 1-\left|\sum_{j=2}^{\infty}\frac{\alpha\left(\mathrm{e}^{-i\theta}+Q\right)+(\beta)\left[(j-1)\left(\mathrm{e}^{-i\theta}+Q\right)-(1-\gamma)(P-Q)\right]}{(\alpha+\beta)(1-\gamma)(P-Q)}d_j\xi^{j-1}\right|\\&\geq\ 1-\sum_{j=2}^{\infty}\frac{\alpha(1-Q)+\beta[(j-1)(1-Q)+(1-\gamma)(P-Q)]}{(\alpha+\beta)(1-\gamma)(P-Q)}|d_j|>0.\end{aligned}$$

Thus, the inequality (21) holds and our result follows from Theorem 3.

Putting $P=1$ and $Q=-1$ in Theorem 3, we obtain the following result for $\mathcal{SK}[\gamma;\alpha,\beta]$.

Corollary 7. *If the function $g(\xi)$ satisfy the inequality*

$$\sum_{j=2}^{\infty} [\alpha + \beta (j - \gamma)] |d_j| \leq (\alpha + \beta) (1 - \gamma),$$

then $g \in \mathcal{SK}[\gamma; \alpha, \beta]$.

Putting $\alpha = \gamma = 0$ in Theorem 3, we obtain the following result for $\mathcal{S}[P, Q]$.

Corollary 8. *If $g(\xi)$ satisfy the inequality*

$$\sum_{j=2}^{\infty} [(j - 1)(1 - Q) + P - Q] |d_j| \leq P - Q,$$

then $g \in \mathcal{S}[P, Q]$.

Putting $\beta = \gamma = 0$ in Theorem 3, we obtain the following result for $\mathcal{K}[P, Q]$.

Corollary 9. *If $g(\xi)$ satisfy the inequality*

$$\sum_{j=2}^{\infty} (1 - Q) |d_j| \leq P - Q,$$

then $g \in \mathcal{K}[P, Q]$.

3. Initial Coefficient Estimates and Fekete–Szegő Problems

The Fekete-Szegő problems are a major area of research in complex analysis concerning the maximum value of functionals of the form $\left|d_3 - \mu d_2^2\right|$ over families of univalent functions found by Fekete and Szegő [26] (see also, [27–32]). In the following theorems, we determine the upper bounds for the coefficients $|d_2|$, $|d_3|$, $|d_4|$ and the Fekete–Szegő functional for the subfamily $\mathcal{SK}[P, Q, \gamma; \alpha, \beta]$.

Theorem 4. *If $g \in \mathcal{SK}[P, Q, \gamma; \alpha, \beta]$, then*

$$\begin{aligned}
|d_2| &\leq (1 - \gamma)(P - Q), \\
|d_3| &\leq \frac{(\alpha + \beta)(1 - \gamma)(P - Q)}{\alpha + 2\beta} \max\left\{1; \left|Q - \frac{\beta (1 - \gamma)(P - Q)}{\alpha + \beta}\right|\right\}, \\
|d_4| &\leq \tfrac{(\alpha+\beta)(1-\gamma)(P-Q)}{(\alpha+3\beta)} \left[\begin{array}{c} \max\left\{1; \left|\frac{(2\alpha+3\beta)\beta(1-\gamma)(P-Q)}{(\alpha+\beta)(\alpha+2\beta)} - 1 - 2Q\right|\right\} \\ + \left|1 + Q - \frac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\right| \left|1 + Q - \frac{\beta(1-\gamma)(P-Q)}{(\alpha+2\beta)}\right| \end{array}\right].
\end{aligned}$$

Proof. If $g \in \mathcal{SK}[P,Q,\gamma;\alpha,\beta]$, then there exists a function $w(\xi) \in \Omega$ in Δ such that

$$\frac{\alpha g(\xi) + \beta \xi g'(\xi)}{\alpha \xi + \beta g(\xi)} = \frac{1 + [Q + (1-\gamma)(P-Q)]\, w(\xi)}{1 + Qw(\xi)} = J(w(\xi)). \tag{22}$$

Since $g(\xi)$ is given by (1), it follows that

$$\begin{aligned}\frac{\alpha g(\xi) + \beta \xi g'(\xi)}{\alpha \xi + \beta g(\xi)} &= 1 + d_2\xi + \left[\tfrac{\alpha+2\beta}{\alpha+\beta} d_3 - \tfrac{\beta}{\alpha+\beta} d_2^2\right]\xi^2 \\ &+ \left[\tfrac{\alpha+3\beta}{\alpha+\beta} d_4 - \tfrac{(2\alpha+3\beta)\beta}{(\alpha+\beta)^2} d_2 d_3 + \tfrac{\beta^2}{(\alpha+\beta)^2} d_2^3\right]\xi^3 + \ldots\, .\end{aligned} \tag{23}$$

Define the function $\phi(\xi)$ by

$$\phi(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + \delta_1\xi + \delta_2\xi^2 + \delta_3\xi^3 + \ldots\, ,$$

since $w(\xi) \in \Omega$, we see that $\phi \in \Phi$ and

$$w(\xi) = \frac{\phi(\xi) - 1}{\phi(\xi) + 1} = \frac{\delta_1\xi + \delta_2\xi^2 + \delta_3\xi^3 + \ldots}{2 + \delta_1\xi + \delta_2\xi^2 + \delta_3\xi^3 + \ldots}. \tag{24}$$

According to (24), we have

$$\begin{aligned}J(w(\xi)) = \frac{1 + [Q + (1-\gamma)(P-Q)]\, w(\xi)}{1 + Qw(\xi)} &= 1 + \tfrac{(1-\gamma)(P-Q)}{2}\delta_1\xi + \tfrac{(1-\gamma)(P-Q)}{2}\left[\delta_2 - \tfrac{(1+Q)}{2}\delta_1^2\right]\xi^2 \\ &+ \frac{(1-\gamma)(P-Q)}{2}\left[\delta_3 - (1+Q)\,\delta_1\delta_2 + \frac{(1+Q)^2}{4}\delta_1^3\right]\xi^3 + \ldots\, .\end{aligned} \tag{25}$$

Equating the corresponding coefficients of (23) and (25), we obtain

$$d_2 = \frac{(1-\gamma)(P-Q)\,\delta_1}{2}, \tag{26}$$

$$d_3 = \frac{(\alpha+\beta)(1-\gamma)(P-Q)}{2(\alpha+2\beta)}\left\{\delta_2 - \frac{1}{2}\left[1 + Q - \frac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\right]\delta_1^2\right\} \tag{27}$$

$$d_4 = \tfrac{(\alpha+\beta)(1-\gamma)(P-Q)}{2(\alpha+3\beta)}\left\{\begin{array}{c}\delta_3 - \left[1 + Q - \frac{(2\alpha+3\beta)\beta(1-\gamma)(P-Q)}{2(\alpha+\beta)(\alpha+2\beta)}\right]\delta_1\delta_2 \\ + \left\{\left(\frac{1+Q}{2}\right)^2 - \frac{(2\alpha+3\beta)\beta(1+Q)(1-\gamma)(P-Q)}{4(\alpha+\beta)(\alpha+2\beta)} + \frac{\beta^2(P-Q)^2(1-\gamma)^2}{4(\alpha+\beta)(\alpha+2\beta)}\right\}\delta_1^3\end{array}\right\} \tag{28}$$

Using (26), we have

$$|d_2| = \frac{(1-\gamma)(P-Q)}{2}|\delta_1|,$$

and from (8), we have $|\delta_1| \le 2$, therefore

$$|d_2| \le (1-\gamma)(P-Q).$$

The relation (27) yields to

$$|d_3| = \frac{(\alpha+\beta)(1-\gamma)(P-Q)}{2(\alpha+2\beta)} \left| \delta_2 - \frac{1}{2}\left[1+Q-\frac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\right]\delta_1^2 \right|$$

and according to (9), we get

$$\begin{aligned} |d_3| &\leq \frac{(\alpha+\beta)(1-\gamma)(P-Q)}{2(\alpha+2\beta)} .2. \max\left\{1; \left|2.\frac{1}{2}\left[1+Q-\frac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\right]-1\right|\right\} \\ &= \frac{(\alpha+\beta)(1-\gamma)(P-Q)}{\alpha+2\beta} \max\left\{1; \left|Q-\frac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\right|\right\}. \end{aligned}$$

The equality (28) leads to

$$|d_4| = \tfrac{(\alpha+\beta)(1-\gamma)(P-Q)}{2(\alpha+3\beta)} \left| \begin{array}{c} \delta_3 - \left[1+Q-\frac{(2\alpha+3\beta)\beta(1-\gamma)(P-Q)}{2(\alpha+\beta)(\alpha+2\beta)}\right]\delta_1\delta_2 \\ +\left\{\left(\frac{1+Q}{2}\right)^2 - \frac{(2\alpha+3\beta)\beta(1+Q)(1-\gamma)(P-Q)}{4(\alpha+\beta)(\alpha+2\beta)} + \frac{\beta^2(P-Q)^2(1-\gamma)^2}{4(\alpha+\beta)(\alpha+2\beta)}\right\}\delta_1^3 \end{array} \right|$$

and using the triangle inequality we have

$$|d_4| \leq \tfrac{(\alpha+\beta)(1-\gamma)(P-Q)}{2(\alpha+3\beta)} \left\{ \begin{array}{c} \left|\delta_3 - \left[1+Q-\frac{(2\alpha+3\beta)\beta(1-\gamma)(P-Q)}{2(\alpha+\beta)(\alpha+2\beta)}\right]\delta_1\delta_2\right| \\ +\left|\left(\frac{1+Q}{2}\right)^2 - \frac{(2\alpha+3\beta)\beta(1+Q)(1-\gamma)(P-Q)}{4(\alpha+\beta)(\alpha+2\beta)} + \frac{\beta^2(P-Q)^2(1-\gamma)^2}{4(\alpha+\beta)(\alpha+2\beta)}\right| |\delta_1|^3 \end{array} \right\},$$

from (8) and (10), the above inequality implies that

$$|d_4| \leq \tfrac{(\alpha+\beta)(1-\gamma)(P-Q)}{2(\alpha+3\beta)} \left[\begin{array}{c} \max\left\{1; \left|\frac{(2\alpha+3\beta)\beta(1-\gamma)(P-Q)}{(\alpha+\beta)(\alpha+2\beta)} - 1 - 2Q\right|\right\} \\ +\left|1+Q-\frac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\right|\left|1+Q-\frac{\beta(1-\gamma)(P-Q)}{\alpha+2\beta}\right| \end{array} \right].$$

This completes the proof of Theorem 4.

Putting $P=1$ and $Q=-1$ in Theorem 4, we get the following.

Corollary 10. *If $g \in \mathcal{SK}[\gamma;\alpha,\beta]$, then*

$$\begin{aligned} |d_2| &\leq 2(1-\gamma), \\ |d_3| &\leq \frac{2[2\beta(1-\gamma)+\alpha+\beta](1-\gamma)}{\alpha+2\beta}, \\ |d_4| &\leq \frac{2[2\beta(1-\gamma)+\alpha+\beta][2\beta(1-\gamma)+\alpha+2\beta](1-\gamma)}{(\alpha+2\beta)(\alpha+3\beta)}. \end{aligned}$$

Putting $\alpha=\gamma=0$ in Theorem 4, we get the following.

Corollary 11. *If $g \in \mathcal{S}[P,Q]$, then*

$$\begin{aligned} |d_2| &\leq P-Q, \\ |d_3| &\leq \frac{P-Q}{2}\max\{1;|2Q-P|\}, \\ |d_4| &\leq \tfrac{P-Q}{3}\left[\max\left\{1;\left|1+\tfrac{7}{2}Q-\tfrac{3}{2}P\right|\right\}+|1+2Q-P|\left|1+\frac{3}{2}Q-\frac{1}{2}P\right|\right]. \end{aligned}$$

Putting $\beta=\gamma=0$ in Theorem 4, we get the next.

Corollary 12. *If $g\in\mathcal{K}[P,Q]$, then*

$$\begin{aligned}|d_2| &\leq P-Q,\\ |d_3| &\leq P-Q,\\ |d_4| &\leq (P-Q)\left[\max\{1;|1+2Q|\}+(1+Q)^2\right].\end{aligned}$$

Theorem 5. *If $g\in\mathcal{SK}[P,Q,\gamma;\alpha,\beta]$, then*

$$\left|d_3-\mu d_2^2\right|\leq\frac{(\alpha+\beta)(1-\gamma)(P-Q)}{\alpha+2\beta}\max\left\{1;\left|Q-\tfrac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right|\right\}. \tag{29}$$

Proof. If $g\in\mathcal{SK}[P,Q,\gamma;\alpha,\beta]$, then from (26) and (27) we get

$$d_3-\mu d_2^2=\frac{(\alpha+\beta)(1-\gamma)(P-Q)}{2(\alpha+2\beta)}\left\{\delta_2-u\delta_1^2\right\}, \tag{30}$$

where

$$u=\frac{1}{2}\left[1+Q-\frac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right]. \tag{31}$$

Applying (9) to (30), it follows that

$$\begin{aligned}\left|d_3-\mu d_2^2\right| &\leq \tfrac{(\alpha+\beta)(1-\gamma)(P-Q)}{2(\alpha+2\beta)}.2\max\left\{1;\left|2.\frac{1}{2}\left[1+Q-\tfrac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\left(1-\tfrac{\alpha+2\beta}{\beta}\mu\right)\right]-1\right|\right\}\\ &= \tfrac{(\alpha+\beta)(1-\gamma)(P-Q)}{\alpha+2\beta}\max\left\{1;\left|Q-\tfrac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\left(1-\tfrac{\alpha+2\beta}{\beta}\mu\right)\right|\right\}.\end{aligned}$$

This completes the proof of Theorem 5.

Putting $P=1$ and $Q=-1$ in Theorem 5, we get the following.

Corollary 13. *If $g\in\mathcal{SK}[\gamma;\alpha,\beta]$, then*

$$\left|d_3-\mu d_2^2\right|\leq\frac{2(\alpha+\beta)(1-\gamma)}{\alpha+2\beta}\max\left\{1;\left|Q-\frac{2\beta(1-\gamma)}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right|\right\}.$$

Putting $\alpha=\gamma=0$ in Theorem 5, we get the following.

Corollary 14. *If $g\in\mathcal{S}[P,Q]$, then*

$$\left|d_3-\mu d_2^2\right|\leq\tfrac{P-Q}{2}\max\{1;|Q-(P-Q)(1-2\mu)|\}.$$

Putting $\beta=\gamma=0$ in Theorem 5, we get the following.

Corollary 15. *If $g\in\mathcal{K}[P,Q]$, then*

$$\left|d_3-\mu d_2^2\right|\leq P-Q.$$

Theorem 6. *Let*

$$\begin{aligned}\sigma_1 &= \frac{\beta(1-\gamma)(P-Q)-(\alpha+\beta)(1+Q)}{(\alpha+2\beta)(1-\gamma)(P-Q)}, \sigma_2 = \frac{\beta(1-\gamma)(P-Q)+(\alpha+\beta)(1-Q)}{(\alpha+2\beta)(1-\gamma)(P-Q)},\\ \sigma_3 &= \frac{\beta(1-\gamma)(P-Q)-(\alpha+\beta)Q}{(\alpha+2\beta)(1-\gamma)(P-Q)}.\end{aligned}$$

If $g \in \mathcal{SK}[P,Q,\gamma;\alpha,\beta]$, *then*

$$\left|d_3-\mu d_2^2\right| \le \begin{cases} -\frac{(\alpha+\beta)(1-\gamma)(P-Q)}{(\alpha+2\beta)}\left[Q-\frac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right] & (\mu \le \sigma_1),\\ \frac{(\alpha+\beta)(1-\gamma)(P-Q)}{(\alpha+2\beta)} & (\sigma_1 \le \mu \le \sigma_2),\\ \frac{(\alpha+\beta)(1-\gamma)(P-Q)}{(\alpha+2\beta)}\left[Q-\frac{\beta(1-\gamma)(P-Q)}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right] & (\mu \ge \sigma_2).\end{cases}$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, *then*

$$\left|d_3-\mu d_2^2\right|+\frac{\alpha+\beta}{\alpha+2\beta}\left[\frac{1+Q}{(1-\gamma)(P-Q)}-\frac{\beta}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right]|d_2|^2 \le \frac{(\alpha+\beta)(1-\gamma)(P-Q)}{\alpha+2\beta}.$$

If $\sigma_3 \le \mu \le \sigma_2$, *then*

$$\left|d_3-\mu d_2^2\right|+\frac{\alpha+\beta}{\alpha+2\beta}\left[\frac{1-Q}{(1-\gamma)(P-Q)}+\frac{\beta}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right]|d_2|^2 \le \frac{(\alpha+\beta)(1-\gamma)(P-Q)}{\alpha+2\beta}.$$

Proof. Using Lemma 2 to (30) and (31), we can derive our results, which are confirmed by Theorem 6.

Taking $P=1$ and $Q=-1$ in Theorem 6, we derive the following.

Corollary 16. *Let*

$$\sigma_4 = \frac{\beta}{\alpha+2\beta}, \sigma_5 = \frac{\beta(1-\gamma)+\alpha+\beta}{(\alpha+2\beta)(1-\gamma)}, \sigma_6 = \frac{2\beta(1-\gamma)+\alpha+\beta}{2(\alpha+2\beta)(1-\gamma)}.$$

If $g \in \mathcal{SK}[\gamma;\alpha,\beta]$, *then*

$$\left|d_3-\mu d_2^2\right| \le \begin{cases} \frac{2(\alpha+\beta)(1-\gamma)}{\alpha+2\beta}\left[1+\frac{2\beta(1-\gamma)}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right] & (\mu \le \sigma_4),\\ \frac{2(\alpha+\beta)(1-\gamma)}{\alpha+2\beta} & (\sigma_4 \le \mu \le \sigma_5),\\ -\frac{2(\alpha+\beta)(1-\gamma)}{\alpha+2\beta}\left[1+\frac{2\beta(1-\gamma)}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right] & (\mu \ge \sigma_5).\end{cases}$$

Further, if $\sigma_4 \le \mu \le \sigma_6$, *then*

$$\left|d_3-\mu d_2^2\right|-\frac{\beta}{\alpha+2\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)|d_2|^2 \le \frac{2(\alpha+\beta)(1-\gamma)}{(\alpha+2\beta)}.$$

If $\sigma_3 \le \mu \le \sigma_2$, *then*

$$\left|d_3-\mu d_2^2\right|+\frac{\alpha+\beta}{\alpha+2\beta}\left[\frac{1}{1-\gamma}+\frac{\beta}{\alpha+\beta}\left(1-\frac{\alpha+2\beta}{\beta}\mu\right)\right]|d_2|^2 \le \frac{2(\alpha+\beta)(1-\gamma)}{(\alpha+2\beta)}.$$

Putting $\alpha = \gamma = 0$ in Theorem 6, we get the following.

Corollary 17. *Let*

$$\sigma_{10} = \frac{P-2Q-1}{2\left(P-Q\right)}, \sigma_{11} = \frac{P-2Q+1}{2\left(P-Q\right)}, \sigma_{12} = \frac{P-2Q}{2\left(P-Q\right)}.$$

If $g \in \mathcal{S}\left[P,Q\right]$, *then*

$$\left|d_3 - \mu d_2^2\right| \leq \begin{cases} -\frac{(P-Q)[Q-(P-Q)(1-2\mu)]}{2} & \left(\mu \leq \sigma_{10}\right), \\ \frac{P-Q}{2} & \left(\sigma_{10} \leq \mu \leq \sigma_{11}\right), \\ \frac{(P-Q)[Q-(P-Q)(1-2\mu)]}{2} & \left(\mu \geq \sigma_{11}\right). \end{cases}$$

Further, if $\sigma_{10} \leq \mu \leq \sigma_{12}$, *then*

$$\left|d_3 - \mu d_2^2\right| + \frac{1}{2}\left(\frac{1+Q}{P-Q} - 1 + 2\mu\right)\left|d_2\right|^2 \leq \frac{P-Q}{2}.$$

If $\sigma_{12} \leq \mu \leq \sigma_{11}$, *then*

$$\left|d_3 - \mu d_2^2\right| + \frac{1}{2}\left(\frac{1-Q}{P-Q} + 1 - 2\mu\right)\left|d_2\right|^2 \leq \frac{P-Q}{2}.$$

Putting $\beta = \gamma = 0$ in Theorem 5, we obtain the following.

Corollary 18. *Let*

$$\sigma_{13} = -\frac{1+Q}{P-Q}, \sigma_{14} = \frac{1-Q}{P-Q}, \sigma_{15} = -\frac{Q}{P-Q}.$$

If $g \in \mathcal{K}\left[P,Q\right]$, *then*

$$\left|d_3 - \mu d_2^2\right| \leq \begin{cases} -\left(P-Q\right)Q & \left(\mu \leq \sigma_{13}\right), \\ P-Q & \left(\sigma_{13} \leq \mu \leq \sigma_{14}\right), \\ \left(P-Q\right)Q & \left(\mu \geq \sigma_{14}\right). \end{cases}$$

Further, if $\sigma_{13} \leq \mu \leq \sigma_{15}$, *then*

$$\left|d_3 - \mu d_2^2\right| + \frac{1+Q}{P-Q}\left|d_2\right|^2 \leq P-Q.$$

If $\sigma_{15} \leq \mu \leq \sigma_{14}$, *then*

$$\left|d_3 - \mu d_2^2\right| + \frac{1-Q}{P-Q}\left|d_2\right|^2 \leq P-Q.$$

4. Conclusion

This paper mainly focuses on finding the convolution results and coefficient estimates for a new subfamily $\mathcal{SK}\left[P,Q,\gamma;\alpha,\beta\right]$ of analytic functions linked to the generalized Janowski domain. We also studied the upper bounds of the first four coefficients and the Fekete-Szegö inequalities for the functions in this subfamily. We note that the results of this study naturally include many of the known results for these subfamilies, which are listed in the Introduction section. For future studies, we can derive important geometric properties by introducing the same subfamily in the case of multivalent (or meromorphic) regular functions and applying the same techniques used in this paper.

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