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An Exploration of Compactness and Separation Axioms in Generalized Primal Topological Spaces

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Abstract. The research explores S_g^*-compactness together with S_g^*-connectedness in generalized primal topological spaces to enhance theoretical knowledge of non-classical topological systems. This paper provides an extensive analysis of these two concepts to show their characteristics and potential applications. The research examines the relationship dynamics between T_0, T_1, and T_2 separation axioms and these concepts throughout their expanded theoretical framework. Examining S_g^*-compactness and S_g^*-connectedness independently provides an advanced understanding of generalized primal spaces, although they diverge from standard separation properties. This study simultaneously supports theoretical research of these domains while building essential foundations for math investigations in this field.

2020 Mathematics Subject Classifications: 54A05, 54A10

Key Words and Phrases: Generalized topological space, Hausdorff space, compactness, primal topological space

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1. Introduction

Topology, as a core branch of modern mathematics, offers powerful tools for understanding ideas such as convergence, continuity, and separation axioms in different types of

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spaces. The theoretical and practical world bases its work on compactness and connectedness—influential concepts which lead topology to its most important advancements. The recent study conducted by Shahbaz et al. [1, 2] investigates various continuity, compactness, and connectedness frameworks that generate novel insights which define modern topological research. Their research on generalized topological spaces leads the way toward investigating particular cases of spaces especially within generalized primal topological spaces.

Topological ideas face increasing interest from scholars to rethink traditional interpretations through abstract flexible frameworks within the last few years. Generalized primal topological spaces emerged because of heightened interest and created new ways to understand familiar concepts.

Historical fundamental principles about open and closed sets form the basis of developing these spaces. The generalized closed sets theory of Levine from 1970 provided innovative methods for examining covering properties and separation even though his original research investigated generalized topology but these principles align with contemporary primal environment research. Császár developed a set of theoretical generalized open sets [3] that built the essential framework for open set analysis for non-traditional mathematical spaces.

Maki, Balachandran and Devi [4], Maki, Rao, and Gani [5], and Navalagi along with Page [6], have made essential scholarly contributions to the field through their semi-generalized closed sets and their connected structures of semi-open and pre-open sets. These fundamental theoretical foundations have set the base for analyzing analogous features within primal spaces with generalization.

The field of generalized primal topology continues to evolve through Choquet's initial research on grills and filters from 1961 which remains influential in current investigations. Structural analysis of primal spaces received clearer explanations through research conducted by Acharjee, Özkoç, and Issaka [7] and Özkoç and Köstel [8]. Saadi and Malki [9, 10] developed our knowledge of these spaces through their examinations of different open set categories. A combination of above-mentioned aspects results in the adaptability of primal spaces being demonstrated also through extension to the soft structures [11]. Besides these, new operator-based solutions have been suggested to reinforce the primal topology framework [12].

The most significant advancement emerged through Missier and Jesti's research which proposed S_g^*-open sets [13]. The research initiated by Missier and Jesti [13] created a whole family of connected functions for improved set operation analysis in generalized primal spaces. The initial concept of μ-compactness created by Thomas and John [14] started in general settings before it helped researchers understand compactness definitions in specific spaces.

The investigation examines basic space differentiation criteria known as T_0, T_1 and T_2 separation axioms because they serve as fundamental tools to distinguish space types.

These axioms introduced by Urysohn [15] then improved by Freudenthal and Est [16] continue to play an important role in contemporary topological classifying systems. Stone [17] proved that any topological space can become a T_0 space through the process of point merging to merge indistinguishable points. Meanwhile Youngs [18] examined separation conditions between T_0 and T_1.

The current research established S_g^*-compact and S_g^*-connected spaces as its main focus inside generalized primal topological worlds. The paper studies how spaces change under the influence of T_0, T_1 and T_2 separation axioms in their internal structure. Thus this piece develops the concepts of S_g^*-T_0, S_g^*-T_1 and S_g^*-T_2 spaces to demonstrate their alignment with S_g^*-set characteristics. The study explores theoretical growth through its results which establish foundations for future research in generalized primal topology.

2. Preliminaries

This section provides some findings and definitions from the literature in order to clarify the main part.

Definition 2.1. *[19] An empty set does not exist yet the set $\mathcal{V} \neq \emptyset$. A collection $\mathcal{G}_\tau \subseteq 2^{\mathcal{V}}$ satisfies the criteria to be considered a generalized topology ($\mathcal{GT}$) on $\mathcal{V}$ if it contains the empty set and all possible unions of non-empty subclasses within $\mathcal{G}_\tau$ are also contained in $\mathcal{G}_\tau$.*

The pair made up of ($\mathcal{V}$, $\mathcal{G}_\tau$) represents a $\mathcal{GTS}$ (Generalized Topological Space).

Remark 2.1. *[3] Each member of the set $\mathcal{G}_\tau$ is identified as open in the space. Any set $\mathcal{E}$ present in the context of ($\mathcal{V}$, $\mathcal{G}_\tau$) forms an essential part of our consideration. It becomes a closed set whenever the complement of $\mathcal{E}$ relative to the set $\mathcal{V}$ is open.*

The closure of a set $\mathcal{E}$ receives the notation $Cl_g(\mathcal{E})$ through intersection of every closed set which contains $\mathcal{E}$. Interior of a set known as $Int_g(\mathcal{E})$ consist of every open subset found within $\mathcal{E}$.

Definition 2.2. *[3] According to $\mathcal{GTS}$ a ψ operator maps elements x from $\mathcal{V}$ to sets in $2^{2^{\mathcal{V}}}$ and it fulfills the condition where $x \in \mathcal{F}$ whenever $\mathcal{F}$ belongs to the image of x. According to Definition a generalized neighbourhood of point x in set $\mathcal{V}$ refers to the element $\mathcal{F} \in \psi(x)$.*

The collection of every generalized neighbourhood that exists within $\mathcal{V}$ gets symbolized by $\Psi(\mathcal{V})$.

Definition 2.3. *[4–6, 20]*

(i) Consider a $\mathcal{GTS}$ ($\mathcal{V}$, $\mathcal{G}_\tau$). If a set $\mathcal{E}$ in a $\mathcal{GTS}$ has an open container set $\mathcal{F}$ that satisfies the condition $\mathcal{F} \subseteq \mathcal{E} \subseteq Cl(\mathcal{F})$ or if $\mathcal{E} \subseteq Cl(Int(\mathcal{E}))$ then it is known as generalized $\mathcal{G}_\tau$-semi-open.

(ii) *The complement of a generalized $\mathcal{G}_\tau$-semi-open set results in a generalized $\mathcal{G}_\tau$-semi-closed set. All generalized $\mathcal{G}_\tau$-semi-open sets within ($\mathcal{V}$, $\mathcal{G}_\tau$) form the set known as $SO(\mathcal{V}, \mathcal{G}_\tau)$.*

(iii) *In a generalized $\mathcal{G}_\tau$-semi-open setting the overall collection of such sets contained within $\mathcal{E}$ forms the generalized semi-interior which is denoted as $s_gInt(\mathcal{E})$.*

(iv) *A generalized semi-closure consists of the intersection between all generalized semi-closed sets of $\mathcal{V}$ that contain $\mathcal{E}$. This set bears the notation $s_gCl(\mathcal{E})$.*

Definition 2.4. *[14] If ($\mathcal{V}$, $\mathcal{G}_\tau$) has a finite subcover for every open cover, then a generalized topological space ($\mathcal{V}$, $\mathcal{G}_\tau$) is called $\mathcal{G}_\tau$-compact.*

Definition 2.5. *[1] Assume ($\mathcal{V}$ $\mathcal{G}_{\tau_1}$) and ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$) as $\mathcal{GTS}$. A mapping $\mathfrak{j} : (\mathcal{V}, \mathcal{G}_{\tau_1}) \to (\mathcal{Z}, \mathcal{G}_{\tau_2})$ is classified as $\mathcal{G}_\tau$-S_g^*-irresolute when the preimage of every $\mathcal{G}_\tau$-S_g^*-open set in $(\mathcal{Z}, \mathcal{G}_{\tau_2})$ is a $\mathcal{G}_\tau$-S_g^*-open set in ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$).*

Main Results

3. $\mathcal{G}_{pt}$-S_g^*-Compact Space in $\mathcal{GPTS}$

This particular section evolves to generalized primal topological spaces. The extension of the previous ideas is emphasized in this part, which also uses the Kuratowski closure operator to examine closure features in primal topological spaces.

Definition 3.1. *[21] Assume that $\mathcal{V} \neq \emptyset$. A grill on $\mathcal{V}$ is a family $\mathcal{G} \subseteq 2^{\mathcal{V}}$ if the following criteria are met:*

(i) *$\emptyset$ is not a member of $\mathcal{G}$.*

(ii) *For $\mathcal{D}$, $\mathcal{E} \subseteq \mathcal{V}$ having $\mathcal{D} \subseteq \mathcal{E}$ implies $\mathcal{E} \in \mathcal{G}$ if $\mathcal{D} \in \mathcal{G}$.*

(iii) *For $\mathcal{D}$, $\mathcal{E} \subseteq \mathcal{V}$, then $\mathcal{D} \cup \mathcal{E} \in \mathcal{G}$, whenever $\mathcal{D} \in \mathcal{G}$ or $\mathcal{E} \in \mathcal{G}$.*

Definition 3.2. *[8] Assume that $\mathcal{V} \neq \emptyset$. A primal on $\mathcal{V}$ is a collection $\mathcal{P}$ of $2^{\mathcal{V}}$ if the following criteria are met:*

(i) *$\mathcal{V} \notin \mathcal{P}$.*

(ii) *If $\mathcal{D} \in \mathcal{P}$ and $\mathcal{E} \subseteq \mathcal{D}$ then $\mathcal{E} \in \mathcal{P}$.*

(iii) *If $\mathcal{D} \cap \mathcal{E} \in \mathcal{P}$, then $\mathcal{D} \in \mathcal{P}$ or $\mathcal{E} \in \mathcal{P}$.*

A pair ($\mathcal{V}$, $\mathcal{G}_\tau$) with a primal $\mathcal{P}$ on $\mathcal{V}$ is termed generalized primal topological space ($\mathcal{GPTS}$) symbolized as ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$). The members of ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) are known as $\mathcal{G}_{pt}$-open sets, and their complements are considered $\mathcal{G}_{pt}$-closed sets.

Definition 3.3. *[9] Assume $\mathcal{A} \subseteq \mathcal{V}$. Let an operator $(.)^\circ : 2^{\mathcal{V}} \to 2^{\mathcal{V}}$ in $\mathcal{GPTS}$ is defined as $\mathcal{A}^\circ$ ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) $= \{x \in \mathcal{V} : \mathcal{A}^c \cup O^c \in \mathcal{P}, \forall\ O \in \psi(x)\}$ where O is generalized primal neighbourhood of x in $\mathcal{V}$ and the collection of all generalized neighbourhood of $\mathcal{V}$ is termed as $\Psi(\mathcal{V})$.*

Definition 3.4. *Let $\mathcal{A} \subseteq \mathcal{V}$ in $\mathcal{GPTS}$. The generalized Kuratowski closure operator Cl° is defined as $Cl^\circ(\mathcal{A}) = \mathcal{A} \cup \mathcal{A}^\circ$, with the condition: $Cl^\circ(\mathcal{A} \cup \mathcal{B}) \supseteq Cl^\circ(\mathcal{A}) \cup Cl^\circ(\mathcal{B})$.*

Remark 3.1. *The operator Cl° satisfies the following properties:*

(i) ***Extensivity:*** *$\mathcal{A} \subseteq Cl^\circ(\mathcal{A})$.*

(ii) ***Monotonicity:*** *If $\mathcal{A} \subseteq \mathcal{B}$, then $Cl^\circ(\mathcal{A}) \subseteq Cl^\circ(\mathcal{B})$.*

(iii) ***Idempotency:*** *$Cl^\circ(Cl^\circ(\mathcal{A})) = Cl^\circ(\mathcal{A})$.*

(iv) ***Generalized Subset Union Property:*** *$Cl^\circ(\mathcal{A} \cup \mathcal{B}) \supseteq Cl^\circ(\mathcal{A}) \cup Cl^\circ(\mathcal{B})$.*

In standard topological spaces, Cl° reduces to the classical Kuratowski closure operator.

Proof. 1. Extensivity

By definition, the generalized Kuratowski closure operator $Cl^\circ(\mathcal{A})$ consists of all points in $\mathcal{A}$. All generalized limit points of $\mathcal{A}$, i.e., points where every generalized primal neighbourhood intersects $\mathcal{A}$. Since every point in $\mathcal{A}$ is trivially in its closure, thus $\mathcal{A} \subseteq Cl^\circ(\mathcal{A})$.

2. Monotonicity

Suppose $\mathcal{A} \subseteq \mathcal{B}$. Any generalized primal neighbourhood of a point x that intersects $\mathcal{A}$ also intersects $\mathcal{B}$. Hence, any generalized limit point of $\mathcal{A}$ must also be a generalized limit point of $\mathcal{B}$, implying $Cl^\circ(\mathcal{A}) \subseteq Cl^\circ(\mathcal{B})$.

3. Idempotency

Expanding $Cl^\circ(Cl^\circ(\mathcal{A}))$.
Applying the closure operator twice, $Cl^\circ(Cl^\circ(\mathcal{A})) = Cl^\circ(\mathcal{A} \cup \mathcal{A}^\circ)$. Using the definition of closure again. By the definition of the closure operator $Cl^\circ(\mathcal{A} \cup \mathcal{A}^\circ) = (\mathcal{A} \cup \mathcal{A}^\circ) \cup (\mathcal{A} \cup \mathcal{A}^\circ)^\circ$. Since closure always includes the interior, $(\mathcal{A} \cup \mathcal{A}^\circ)^\circ = \mathcal{A}^\circ$. By substituting this, we get $Cl^\circ(Cl^\circ(\mathcal{A})) = (\mathcal{A} \cup \mathcal{A}^\circ) \cup \mathcal{A}^\circ$. Since $\mathcal{A}^\circ$ is already included in $Cl^\circ(\mathcal{A})$, implies $Cl^\circ(Cl^\circ(\mathcal{A})) = Cl^\circ(\mathcal{A})$.

4. Generalized Finite Union Property

Consider $x \in Cl^\circ(\mathcal{A}) \cup Cl^\circ(\mathcal{B})$. This means x is either in $Cl^\circ(\mathcal{A})$ or $Cl^\circ(\mathcal{B})$, so every generalized primal neighbourhood of x intersects either $\mathcal{A}$ or $\mathcal{B}$. Therefore, every generalized primal neighbourhood of x intersects $\mathcal{A} \cup \mathcal{B}$, implying $Cl^\circ(\mathcal{A} \cup \mathcal{B}) \supseteq Cl^\circ(\mathcal{A}) \cup Cl^\circ(\mathcal{B})$. □

Definition 3.5. *(i) Assume $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{P})$ as a $\mathcal{GPTS}$. If there exists an open set $\mathcal{F}$ in $\mathcal{V}$ such that $\mathcal{F} \subseteq \mathcal{E} \subseteq Cl^\circ(\mathcal{F})$ or equivalently if $\mathcal{E} \subseteq Cl^\circ(Int(\mathcal{E}))$, then the subset $\mathcal{E}$ of $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{P})$ is known as generalized primal semi-open [10].*

(ii) The generalized primal semi-closed set exists as the complement of generalized primal semi-open sets. All generalized primal semi-open sets make up the collection known as $\mathcal{G}_{pt}$-SO in $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{P})$ [10].

(iii) *The union of all $\mathcal{G}_{pt}$-SO sets of $\mathcal{V}$ contained in $\mathcal{E}$ is called $\mathcal{G}_{pt}$-semi−interior of $\mathcal{E}$ (briefly $\mathcal{G}_{pt}$-sInt($\mathcal{E}$).*

(iv) *The intersection of all $\mathcal{G}_{pt}$-SC sets of $\mathcal{V}$ containing $\mathcal{E}$ is $\mathcal{G}_{pt}$-semi-closure of $\mathcal{E}$ (briefly $\mathcal{G}_{pt}$-sCl($\mathcal{E}$)).*

Definition 3.6. *[1]*

(i) *Consider ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) be a $\mathcal{GPTS}$. If $\mathcal{G}_{pt}$-sCl($\mathcal{E}$) $\subseteq \mathcal{F}$ whenever $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{F}$ is $\mathcal{G}_{pt}$- semi−open in $\mathcal{V}$, then the subset $\mathcal{E}$ of $\mathcal{V}$ is known as generalized primal semi-generalized closed (briefly $\mathcal{G}_{pt}$- Sg-closed).*

(ii) *If Cl($\mathcal{E}$) $\subseteq \mathcal{F}$ whenever $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{F}$ is $\mathcal{G}_{pt}$- semi−open in $\mathcal{V}$, then $\mathcal{E}$ as a subset of a space $\mathcal{V}$ is generalized primal semi−star generalized closed (briefly $\mathcal{G}_{pt}$-S^*-closed).*

(iii) *The complement of $\mathcal{G}_{pt}$- Sg-closed set ($\mathcal{G}_{pt}$-S^*g-closed set) is generalized primal semi-generalized open (generalized primal semi-star generalized open). It is represented by $\mathcal{G}_{pt}$-Sg-open ($\mathcal{G}_{pt}$-S^*g-open) appropriately.*

(iv) *The generalized primal semi-generalized interior (briefly $\mathcal{G}_{pt}$-$sInt^*$($\mathcal{E}$)) of $\mathcal{E}$ is indicated as the union of all $\mathcal{G}_{pt}$-Sg-open sets of $\mathcal{V}$ contained in $\mathcal{E}$.*

(v) *The intersection of all $\mathcal{G}_{pt}$- Sg−closed sets in $\mathcal{V}$ containing $\mathcal{E}$ when $\mathcal{E}$ is a subset of $\mathcal{V}$ is called generalized primal semi−generalized closure (briefly $\mathcal{G}_{pt}$-sCl^*($\mathcal{E}$)) of $\mathcal{E}$.*

Definition 3.7. *Assume a $\mathcal{GPTS}$ ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) and a subset $\mathcal{E}$ of ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$), a collection $\{ \mathcal{E}_{\alpha_i} : i \in \curlywedge \}$ of $\mathcal{G}_{pt}$-S_g^*-open set in $\mathcal{G}_{pt}$ is referred to as $\mathcal{G}_{pt}$-S_g^*-open cover of $\mathcal{E}$ if $\mathcal{E} \subset \cup_{i \in \curlywedge} \mathcal{E}_{\alpha_i}$.*

Definition 3.8. *If every $\mathcal{G}_{pt}$-S_g^*-open cover of ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) has a finite subcover, then the $\mathcal{GPTS}$ is called $\mathcal{G}_{pt}$-S_g^*-compact.*

Definition 3.9. *The subset $\mathcal{E}$ of $\mathcal{GPTS}$ is named as $\mathcal{G}_{pt}$-S_g^*-compact relative of $\mathcal{V}$ if there exists $\curlywedge_o$ of $\curlywedge$ as a finite subset which satisfies $\mathcal{E} \subset \cup \{ Z_i : i \in \curlywedge_o \}$ for each $\{ Z_i : i \in \curlywedge \}$ consisting of $\mathcal{G}_{pt}$-S_g^*-open subset of $\mathcal{V}$ such that $\mathcal{E} \subset \cup \{ Z_i : i \in \curlywedge \}$.*

Definition 3.10. *Assume $\mathcal{E}$ as a subset of $\mathcal{GPTS}$. A subset $\mathcal{E}$ of $\mathcal{V}$ is named $\mathcal{G}_{pt}$-S_g^*-compact when it maintains this property as a subspace of $\mathcal{V}$.*

Theorem 3.1. *(i) Every $\mathcal{G}_{pt}$-S_g^*-compact space is $\mathcal{G}_{pt}$-compact.*

(ii) *The property of being $\mathcal{G}_{pt}$-semi-compact implies $\mathcal{G}_{pt}$-S_g^*-compactness.*

Proof. (i) Let $\mathcal{U} = \{u_i : i \in \Lambda,\ u_i \in \mathcal{G}_{pt}\}$ be an $\mathcal{G}_{pt}$-open cover of $\mathcal{X}$, so $\mathcal{X} = \bigcup_{i\in\Lambda} u_i$. Because every $\mathcal{G}_{pt}$-open set is $\mathcal{G}_{pt}$-S_g^*, $\mathcal{U}$ is also a $\mathcal{G}_{pt}$-S_g^*-open cover of $\mathcal{X}$. If $\mathcal{X}$ is $\mathcal{G}_{pt}$-S_g^*-compact, then by definition every $\mathcal{G}_{pt}$-S_g^*-open cover of $\mathcal{X}$ has a finite subcover, hence there exist a finite subset $\Lambda_0 = \{i_1, \ldots, i_n\} \subseteq \Lambda$ such that $\mathcal{X} = \bigcup_{k=1}^{n} u_{i_k}$. Therefore $\mathcal{X}$ admits a finite subcover of the $\mathcal{G}_{pt}$-open cover.

(ii) Since every $\mathcal{G}_{pt}$-S_g^*-open set is a $\mathcal{G}_{pt}$-semi-open set, the result follows similarly to part (i).

□

Example 3.1. *Let $\mathcal{V}$ be the set of all bounded spherical regions in 3-dimensional Euclidean space:*

$$\mathcal{V} = \{S \subseteq \mathbb{R}^3 \mid S \text{ is a bounded spherical region}\}.$$

where S is defined as:

$$S = \{(l, m, n) \in \mathbb{R}^3 \mid \sqrt{(l-a)^2 + (m-b)^2 + (n-c)^2} \leq r\},$$

with $r > 0$ as the radius of the spherical region and a, b, $c \in \mathbb{R}$. A bounded spherical region is a subset of $\mathbb{R}^3$ consisting of points within a sphere of finite radius.

$$\mathcal{G}_\tau = \{\mathcal{E} \subseteq \mathcal{V} \setminus S_{r_1} \mid S_{r_1} = \{S \in \mathcal{V} \mid \text{radius}(S) = r_1\}\},$$

where $r_1 > 0$ is a fixed radius, and S_{r_1} is the set of all spherical regions in $\mathcal{V}$ with radius exactly r_1, forms a generalized topology on $\mathcal{V}$. We now prove that $\mathcal{G}_\tau$ satisfies the axioms of a generalized topology.

A collection $\mathcal{G}_\tau$ forms a generalized topology if it satisfies the following conditions:

Condition 1: *$\emptyset \in \mathcal{G}_\tau$ The empty set trivially belongs to $\mathcal{G}_\tau$ since there is no restriction preventing $\emptyset$ from being included.*
Condition 2: Arbitrary Unions of Elements of $\mathcal{G}_\tau$ Remain in $\mathcal{G}_\tau$ *Let $\{\mathcal{E}_{\alpha_i}\}_{i \in I}$ be a family of sets in $\mathcal{G}_\tau$, meaning each $\mathcal{E}_{\alpha_i} \subseteq \mathcal{V} \setminus S_r$. Consider their union:*

$$\mathcal{E} = \bigcup_{i \in I} \mathcal{V}_i.$$

Since each $\mathcal{E}_{\alpha_i}$ excludes all spherical regions of radius exactly r, their union $\mathcal{E}$ must also exclude all such regions. That is, $\mathcal{E} \subseteq \mathcal{V} \setminus S_r$. Thus, $\mathcal{S}_\alpha$ satisfies the definition of $\mathcal{G}_\tau$, ensuring that arbitrary unions remain in $\mathcal{G}_\tau$. Since $\mathcal{G}_\tau$ satisfies both required conditions, so it forms a generalized topology on $\mathcal{V}$ and define a collection $\mathcal{G}_{pt} \subseteq 2^{\mathcal{V}}$ as:

$$\mathcal{G}_{pt} = \{\mathcal{E} \subseteq \mathcal{V} \mid \text{all spherical region in } \mathcal{E} \text{ have radius strictly less than } r, \text{ where } r > 0\}.$$

To determine whether $\mathcal{E}$ is $\mathcal{G}_{pt}$-semi-compact, consider a family $\{Z_i : i \in \lambda\}$ of $\mathcal{G}_{pt}$-S_g^*-open sets satisfying $\mathcal{E} \subseteq \bigcup\{Z_i : i \in \lambda\}$. Within this covering, a finite subcover can always be extracted. Hence, $\mathcal{E}$ is $\mathcal{G}_{pt}$-S_g^*-semi-compact.

Next, let $\{Z_i : i \in \lambda\}$ be a collection of $\mathcal{G}_{pt}$-S_g^*-open sets such that $\mathcal{E} \subseteq \bigcup\{Z_i : i \in \lambda\}$. Each point in $\mathcal{E}$ must be contained within at least one Z_i, and thus a finite subset $\lambda_o \subseteq \lambda$ is sufficient to cover $\mathcal{E}$. Consequently, $\mathcal{E}$ is $\mathcal{G}_{pt}$-S_g^*-compact.

All open covers from $\mathcal{G}_{pt}$ naturally have finite subcovers in the context of $\mathcal{E}$. Thus, $\mathcal{E}$ is $\mathcal{G}_{pt}$-compact.

Theorem 3.2. *A $\mathcal{G}_{pt}$-S_g^*-compact space makes every of its $\mathcal{G}_{pt}$-S_g^*-closed subsets compact relative to ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$).*

Proof. Consider $\mathcal{E}$ as a $\mathcal{G}_{pt}$-S_g^*-closed subset of $\mathcal{G}_{pt}$-S_g^*-compact space ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) implies $\mathcal{E}^c$ is $\mathcal{G}_{pt}$-S_g^*-open in ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$). The collection $\{ Z_i : \mathrm{i} \in \curlywedge \}$ serves as a cover of the set $\mathcal{E}$ since each member of the collection is an element from the family of $\mathcal{G}_{pt}$-S_g^*-open subsets of $\mathcal{V}$ so that $\mathcal{E} \subset \cup \{ Z_i : \mathrm{i} \in \curlywedge \}$ implies $\mathcal{E}^c \subset \cup \{ Z_i : \mathrm{i} \in \curlywedge \} = \mathcal{V}$. Therefore, ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is $\mathcal{G}_{pt}$-S_g^*-compact then there exists a finite subset $\mathcal{E}_o$ of $\mathcal{E}$ so that $\mathcal{E} \subset \mathcal{E}^c \cup \{ Z_i : \mathrm{i} \in \curlywedge \} = \mathcal{V}$. Then, $\mathcal{E} \subset \cup \{ Z_i : \mathrm{i} \in \curlywedge \}$ and therefore $\mathcal{E}$ is $\mathcal{G}_{pt}$-S_g^*-compact relative to $\mathcal{V}$. □

Example 3.2. *Assume $\mathcal{V} = \mathbb{R}$ and ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) be defined as follows: $U \in \mathcal{G}_\tau$ if and only if either $U = \emptyset$ or $1 \in U$, see Example 10 in [22]. Let $\mathcal{G}_{pt}$ be defined on $\mathbb{R}$ as follows: $U \in \mathcal{G}_{pt}$ if and only if $1 \notin U$. Then, ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is a generalized primal topology. Now, consider the subset $\mathbb{N} \subset \mathbb{R}$. Let $\mathscr{S}$ is index set and $\{V_\eta\}_{\eta\in\mathscr{S}}$ be a $\mathcal{G}_{pt}$-S_g^*-open cover of $\mathbb{N}$ such that $V_\eta \neq \emptyset$ for every $\eta \in \mathscr{S}$. This implies that:*

$$\mathbb{N} \subseteq \bigcup_{\eta\in\mathscr{S}} V_\eta.$$

Let $\mathscr{S}_0 = \{V_i\}_{i=1}^n \subseteq \{V_\eta\}_{\eta\in\mathscr{S}}$. Then, for any $x \in \mathbb{N} \setminus \bigcup_{i=1}^n V_i$, it must follow that $\mathbb{R} \setminus \left[\mathbb{N} \setminus \bigcup_{i=1}^n V_i\right] \notin \mathcal{G}_{pt}$. Thus, there exists a finite subcover $\mathscr{S}_0$ that covers $\mathbb{N}$, proving that $\mathbb{N}$ is $\mathcal{G}_{pt}$-S_g^*-compact relative to ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$).

Theorem 3.3. *Consider a $\mathcal{G}_{pt}$-S_g^*-continuous surjective map $\mathfrak{f}$: ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) $\to$ ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) from $\mathcal{V}$ to $\mathcal{Z}$. If ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) is $\mathcal{G}_{pt}$-S_g^*-compact, then ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) is $\mathcal{G}_{pt}$-compact.*

Proof. Consider $\{ \mathcal{E}_{\alpha_i} : \mathrm{i} \in \curlywedge \}$, an open cover of $\mathcal{Z}$. As $\mathfrak{f}$ is $\mathcal{G}_{pt}$-S_g^*-continuous implies $\{ \mathfrak{f}^{-1} (\mathcal{E}_{\alpha_i}) : \mathrm{i} \in \curlywedge \}$ is a $\mathcal{G}_{pt}$-S_g^*-open cover of $\mathcal{V}$. Furthermore, there exists a finite $\mathcal{G}_{pt}$-subcover $\{ \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_1}), \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_2}), \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_3}),..., \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_n})\}$ as $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-compact. The surjectiveness of $\mathcal{V}$ implies $\{ \mathcal{E}_{\alpha_1}, \mathcal{E}_{\alpha_2}, \mathcal{E}_{\alpha_3},..., \mathcal{E}_{\alpha_n}\}$ is a finite $\mathcal{G}_{pt}$-subcover of $\mathcal{Z}$ implies $\mathcal{Z}$ is $\mathcal{G}_{pt}$-compact. □

Theorem 3.4. *Consider a $\mathcal{G}_{pt}$-S_g^*-irresolute surjective map $\mathfrak{f}$: ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) $\to$ ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) from $\mathcal{GPTS}$ $\mathcal{V}$ into a $\mathcal{GPTS}$ $\mathcal{Z}$. If ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) is $\mathcal{G}_{pt}$-S_g^*-compact, then ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) is $\mathcal{G}_{pt}$-S_g^*-compact.*

Proof. Let $\{ \mathcal{E}_{\alpha_i} : \mathrm{i} \in \curlywedge \}$, a $\mathcal{G}_{pt}$-S_g^*-open cover of $\mathcal{Z}$. As $\mathfrak{f}$ is $\mathcal{G}_{pt}$-S_g^*-irresolute implies $\{ \mathfrak{f}^{-1} (\mathcal{E}_{\alpha_i}) : \mathrm{i} \in \curlywedge \}$ is a $\mathcal{G}_{pt}$-S_g^*-open cover of $\mathcal{V}$. Furthermore, there exists a finite $\mathcal{G}_{pt}$-subcover $\{ \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_1}), \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_2}), \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_3}),..., \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_n})\}$ as $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-compact. Now, $\mathfrak{f}$ is onto implies $\{ \mathcal{E}_{\alpha_1}, \mathcal{E}_{\alpha_2}, \mathcal{E}_{\alpha_3},..., \mathcal{E}_{\alpha_n}\}$ is a finite $\mathcal{G}_{pt}$-subcover of $\mathcal{Z}$ implies $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-compact. □

Theorem 3.5. *Let* $\mathfrak{f}$*:* $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$ *a* $\mathcal{G}_{pt}$*-*S_g^**-irresolute map and a subset* $\mathcal{D}$ *of* $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha)$ *is* $\mathcal{G}_{pt}$*-*S_g^**-compact relative to* $\mathcal{V}$*, then the image* $\mathfrak{f}(\mathcal{D})$ *is* $\mathcal{G}_{pt}$*-*S_g^**-compact relative to* $\mathcal{Z}$*.*

Proof. Any collection of $\mathcal{G}_{pt}$-S_g^*-open subsets $\{\mathcal{E}_{\alpha_i} : i \in \curlywedge\}$ of $\mathcal{Z}$ containing the image of $\mathcal{D}$ under the mapping $\mathfrak{f}$ becomes part of the union function. Then, $\mathcal{D} \subset \cup \{ \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_i}) : \mathrm{i} \in \curlywedge \}$ holds. $\mathcal{D}$ is $\mathcal{G}_{pt}$-S_g^*-compact relative to $\mathcal{V}$ by hypothesis, then there exists $\curlywedge_o$ a finite subset of $\curlywedge$ such that $\mathcal{D} \subset \cup \{ \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_i}) : \mathrm{i} \in \curlywedge_o \}$ implies $\mathfrak{f}(\mathcal{D}) \subset \cup \{ \mathcal{E}_{\alpha_i} : \mathrm{i} \in \curlywedge_o \}$. Thus, $\mathfrak{f}(\mathcal{D})$ is $\mathcal{G}_{pt}$-S_g^*-compact relative to $\mathcal{Z}$. □

Theorem 3.6. *If a surjective map* $\mathfrak{f}$*:* $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$ *is strongly* $\mathcal{G}_{pt}$*-*S_g^**-continuous and* $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha)$ *is* $\mathcal{G}_{pt}$*-compact, then* $(\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$ *is* $\mathcal{G}_{pt}$*-*S_g^**-compact.*

Proof. Let a $\mathcal{G}_{pt}$-S_g^*-open cover $\{ \mathcal{E}_{\alpha_i} : \mathrm{i} \in \curlywedge \}$ of $\mathcal{Z}$. As $\mathfrak{f}$ exhibits strong $\mathcal{G}_{pt}$-S_g^*-continuity it implies that $\{ \mathfrak{f}^{-1} (\mathcal{E}_{\alpha_i}) : \mathrm{i} \in \curlywedge \}$ creates an open cover of $\mathcal{V}$. Since $\mathcal{V}$ has the $\mathcal{G}_{pt}$-compactness property it contains the finite subcover $\{ \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_1}), \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_2}), \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_3}),..., \mathfrak{f}^{-1}(\mathcal{E}_{\alpha_n})\}$. The surjectiveness of map $\mathfrak{f}$ leads to the finite subcover $\{ \mathcal{E}_{\alpha_1}, \mathcal{E}_{\alpha_2}, \mathcal{E}_{\alpha_3},..., \mathcal{E}_{\alpha_n}\}$ of collection $\mathcal{Z}$ which makes $\mathcal{Z}$ $\mathcal{G}_{pt}$-S_g^*-compact. □

Example 3.3. *Let* $\mathcal{V}$ *be the set of all bounded spherical regions in* $\mathbb{R}^3$*, where a spherical region* S *is defined as:*

$$S = \{(l, m, n) \in \mathbb{R}^3 \mid \sqrt{(l-a)^2 + (m-b)^2 + (n-c)^2} \leq r\},$$

with $r > 0$ *as the radius of the spherical region and* $a, b, c \in \mathbb{R}$, $\tau_{g_1} = \{\mathcal{E} \subseteq \mathcal{V} \mid \mathcal{E} \subseteq \mathcal{V} \setminus \{S \mid \textit{radius}(S) = r\}$ *for some fixed* $r > 0\}$ *(see example 3.1) and define a collection* $\mathcal{P}_\alpha \subseteq 2^{\mathcal{V}}$ *as:*

$$\mathcal{P}_\alpha = \{\mathcal{E} \subseteq \mathcal{V} \mid \textit{all spherical region in } \mathcal{E} \textit{ have radius strictly less than } r, \textit{ where } r > 0\}.$$

Let $\mathcal{Z}$ *be the set of all bounded circular regions in* $\mathbb{R}^2$*, where a circular region* C *is defined as:*

$$C = \{(l, m) \in \mathbb{R}^2 \mid \sqrt{(l-a)^2 + (m-b)^2} \leq r_1\},$$

with (a, b) is the center of the circular region $r_1 > 0$ *as the radius of the circular region and a, b* $\in \mathbb{R}$, $\tau_{g_2} = \{\mathcal{E} \subseteq \mathcal{V} \mid \mathcal{E} \subseteq \mathcal{V} \setminus \{C \mid \textit{radius}(C) = r_1\}$ *for some fixed* $r_1 > 0\}$ *is generalized topology, similarly step (see example 3.1) and* $\mathcal{P}_\beta$ *be a primal on* $\mathcal{Z}$ *such that*

$$\mathcal{P}_\beta = \{\mathcal{E} \subseteq \mathcal{V} \mid \mathcal{E} \textit{ does not contain any point on the boundary } \}.$$

A surjective map $\mathfrak{f} : \mathcal{V} \to \mathcal{Z}$ *from a spherical region to a circular region is defined as:*

$$h(l, m, n) = \left(\frac{r}{r_1} l, \frac{r}{r_1} m \right).$$

This map is surjective, as every element of $\mathcal{Z}$ has a pre-image in $\mathcal{V}$. Assume that $\mathfrak{f}$ is strongly $\mathcal{G}_{pt}$-S_g^*-continuous, meaning it preserves the primal structure and satisfies the required continuity. Since $\mathcal{V}$ is $\mathcal{G}_{pt}$-compact, and $\mathfrak{f}$ is a surjective map that is strongly $\mathcal{G}_{pt}$-S_g^*-continuous. Thus, $\mathcal{Z}$ inherits the $\mathcal{G}_{pt}$-S_g^*-compactness.

Theorem 3.7. *If a surjective map $\mathfrak{f}$: ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) $\rightarrow$ ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) is perfectly $\mathcal{G}_{pt}$-S_g^*-continuous and ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) is $\mathcal{G}_{pt}$-compact, then ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) is $\mathcal{G}_{pt}$-S_g^*-compact.*

Proof. As every perfectly $\mathcal{G}_{pt}$-S_g^*-continuous is strongly $\mathcal{G}_{pt}$-S_g^*-continuous. Theorem 3.6 leads to the obtained result. □

3.1. $\mathcal{G}_{pt}$-S_g^*-Connected Space in $\mathcal{GPTS}$

Definition 3.11. *Assume a $\mathcal{GPTS}$ and two disjoint non-empty open sets $\mathcal{D}$ and $\mathcal{E}$ in $\mathcal{V}$, then ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is known as $\mathcal{G}_{pt}$-disconnection if $\mathcal{D} \cup \mathcal{E} = \mathcal{V}$.*

Definition 3.12. *A $\mathcal{GPTS}$, ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is known as $\mathcal{G}_{pt}$-connected space if ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) has no $\mathcal{G}_{pt}$-disconnection.*

Definition 3.13. *The condition for a space to become $\mathcal{G}_{pt}$-S_g^*-connected exists when two non-empty $\mathcal{G}_{pt}$-S_g^*-open sets $\mathcal{D}$ and $\mathcal{E}$ in ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) cannot be disjoint as $\mathcal{D} \cup \mathcal{E} = \mathcal{V}$. Otherwise called $\mathcal{G}_{pt}$-S_g^*-disconnected space if $\mathcal{D} \cup \mathcal{E} = \mathcal{V}$.*

Example 3.4. *Assuming $\mathcal{V} = \{ j_1, k_1, l_1 \}$, $\mathcal{G}_\tau = \{ \emptyset, \{j_1, k_1\} \}$ and $\mathcal{P} = \{ \emptyset, \{j_1\}, \{l_1\}, \{j_1, l_1\} \}$. In this $\mathcal{GPTS}$ ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$)$O = \{ \emptyset, \{j_1, k_1\} \}$ and ($\mathcal{G}_\tau$, $\mathcal{P}$)$S_g^*O = \{ \emptyset, \{j_1, k_1\}, \mathcal{V} \}$. There does not exists two disjoint non-empty $\mathcal{G}_{pt}$-S_g^*-open sets $\mathcal{D}$ and $\mathcal{E}$ in ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) such that $\mathcal{D} \cup \mathcal{E} = \mathcal{V}$ implies ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is $\mathcal{G}_{pt}$-S_g^*-connected space.*

Theorem 3.8. *A space which satisfies $\mathcal{G}_{pt}$-S_g^*-connectedness contains the condition of $\mathcal{G}_{pt}$-connectedness.*

Proof. Assume a $\mathcal{G}_{pt}$-S_g^*-connected space ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$). Consider ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is not a $\mathcal{G}_{pt}$-connected space, then there exist nonempty open subsets $\mathcal{D}$ and $\mathcal{E}$ in $\mathcal{V}$ such that their union equals $\mathcal{V}$ itself. Every open set holds the status of being both $\mathcal{G}_{pt}$-S_g^*-open and $\mathcal{G}_{pt}$-open in the defined topology. The pair of open sets $\mathcal{D}$ and $\mathcal{E}$ belongs to the class of $\mathcal{G}_{pt}$-S_g^*-open sets since $\mathcal{V} = \mathcal{D} \cup \mathcal{E}$. The finding of two nonempty open sets in $\mathcal{V}$ leads to a contradiction when applying the definition of $\mathcal{G}_{pt}$-S_g^*-connected space. A space represented by ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) serves as a connected space under the $\mathcal{G}_{pt}$ structure. □

Remark 3.2. *An illustration disproves the invalidity of the converse statement derived from above.*

Example 3.5. *Assuming $\mathcal{V} = \{j_1, k_1, l_1\}$, $\mathcal{G}_\tau = \{ \emptyset, \{j_1\}, \{k_1\}, \{j_1, k_1\} \}$ and $\mathcal{P} = \{ \emptyset, \{j_1\}, \{l_1\}, \{j_1, l_1\} \}$. In this $\mathcal{GPTS}$, ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$)$O = \{ \emptyset, \{j_1\}, \{k_1\}, \{j_1, k_1\}\}$ and ($\mathcal{G}_\tau$, $\mathcal{P}$)$S_g^*O = \{ \emptyset, \{j_1\}, \{k_1\}, \{j_1, k_1\}, \{j_1, l_1\}, \{k_1, l_1\}\}$. There does not exists two disjoint non-empty open sets $\mathcal{D}$ and $\mathcal{E}$ in ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) such that $\mathcal{D} \cup \mathcal{E} = \mathcal{V}$ implies ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$)*

is $\mathcal{G}_{pt}$-S_g^*-connected space. But two disjoint non-empty $\mathcal{G}_\tau$-S_g^*-open sets $\{j_1\}$ and $\{k_1, l_1\}$ exists such that $\{j_1\} \cup \{k_1, l_1\} = \mathcal{V}$ implies ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is not $\mathcal{G}_{pt}$-S_g^*-connected space.

Theorem 3.9. *Assume a $\mathcal{GPTS}$, then the subsequent are equivalent.*

i. $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-connected.

ii. $\emptyset$ and $\mathcal{V}$ are only $\mathcal{G}_{pt}$-S_g^*-open set and $\mathcal{G}_{pt}$-S_g^*-closed set in $\mathcal{V}$.

iii. Any $\mathcal{G}_{pt}$-S_g^*-continuous mapping $\mathfrak{f}$: ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) $\to$ ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) is a constant map where ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) at least two-point discrete space.

Proof. (i) then (ii): Let a $\mathcal{G}_{pt}$-S_g^*-open set and $\mathcal{G}_{pt}$-S_g^*-closed set $\mathcal{E}$ in ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) implies $\mathcal{E}^c$ is also $\mathcal{G}_{pt}$-S_g^*-open set and $\mathcal{G}_{pt}$-S_g^*-closed set in ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$). Then, $\mathcal{E} \cup \mathcal{E}^c = \mathcal{V}$ as $\mathcal{E}$ and $\mathcal{E}^c$ are both disjoint $\mathcal{G}_{pt}$-S_g^*-open sets which implies contradiction. As ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is $\mathcal{G}_{pt}$-S_g^*-connected space. Hence, $\mathcal{V}$ is $\emptyset$ or $\mathcal{V}$.

(ii) then (i): Assume $\mathcal{E}$ and $\mathcal{D}$ as disjoint $\mathcal{G}_{pt}$-S_g^*-open sets in ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) and $\mathcal{E} \cup \mathcal{D} = \mathcal{V}$. Since $\mathcal{E}^c = \mathcal{D}$, then $\mathcal{E}$ is $\mathcal{G}_{pt}$-S_g^*-open set ($\mathcal{G}_{pt}$-S_g^*-closed set). Then, by hypothesis, $\mathcal{E}$ is $\emptyset$ or $\mathcal{V}$, this contradicts. Hence, ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is $\mathcal{G}_{pt}$-S_g^*-connected space.

(ii) then (iii): A mapping $\mathfrak{f}$: ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) $\to$ ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) that is both $\mathcal{G}_{pt}$-S_g^*-continuous and constant functions to at least two-point discrete space works as an assumption. The pre-image of any set point under $\mathfrak{f}$ satisfies both $\mathcal{G}_{pt}$-S_g^*-open and $\mathcal{G}_{pt}$-S_g^*-closed properties throughout the elements of $\mathcal{Z}$. The domain set $\mathcal{V}$ equals the union of these pre-image sets $\mathfrak{f}^{-1}$ ($\{x\}$), while each set point x belongs to $\mathcal{Z}$ making the pre-images both $\mathcal{G}_{pt}$-S_g^*-open and $\mathcal{G}_{pt}$-S_g^*-closed in $\mathcal{V}$. The failure for $\mathfrak{f}$ to be a proper mapping occurs when each inverse image set $\mathfrak{f}^{-1}$ ($\{x\}$) equals either empty set or the entire space $\mathcal{V}$ for all elements x in $\mathcal{Z}$. Among all points x in $\mathcal{Z}$ there exists at least one where the pre-image $\mathfrak{f}^{-1}$ ($\{x\}$) consists of $\mathcal{V}$ so $\mathfrak{f}$ functions as a constant.

(iii) then (ii): Asume $\mathcal{E}$ is $\mathcal{G}_{pt}$-S_g^*-open and $\mathcal{G}_{pt}$-S_g^*-closed in $\mathcal{GPTS}$ where $\mathcal{E}$ is non-empty set. Consider $\mathcal{G}_{pt}$-S_g^*-continuous mapping $\mathfrak{f}$: ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) $\to$ ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) defined as $\mathfrak{f}$ ($\mathcal{E}$) $= \{x\}$ and $\mathfrak{f}$ ($\mathcal{E}^c$) $= \{y\}$, where $x, y \in \mathcal{Z}$ and $x \neq y$. By hypothesis, $\mathfrak{f}$ is constant map and $\mathcal{E} = \mathcal{V}$. □

Theorem 3.10. *Let $\mathfrak{f}$: ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) $\to$ ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) is a surjective, $\mathcal{G}_{pt}$-S_g^*-continuous mapping and ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) is $\mathcal{G}_{pt}$-S_g^*-connected space then ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) is $\mathcal{G}_{pt}$-connected space.*

Proof. The space ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) fails to be a $\mathcal{G}_{pt}$-connected space. The space contains two non-empty disjoint open sets $\mathcal{E}$ and $\mathcal{D}$ which exist within $\mathcal{Z}$. Such as $\mathcal{E} \cup \mathcal{D} = \mathcal{Z}$. The fact that $\mathfrak{f}$ demonstrates $\mathcal{G}_{pt}$-S_g^*-continuity creates a relationship between $\mathfrak{f}^{-1}(\mathcal{E}) \cup \mathfrak{f}^{-1}(\mathcal{D}) = \mathcal{V}$ and $\mathfrak{f}^{-1}(\mathcal{E})$, $\mathfrak{f}^{-1}(\mathcal{D})$ which implies a contradiction. Thus, $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-connected. Both conditions demonstrate that the space $\mathcal{Z}$ is $\mathcal{G}_{pt}$-connected. □

Theorem 3.11. *Let $\mathfrak{f}$: $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$ is a surjective, $\mathcal{G}_{pt}$-S_g^*-irresolute function and $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha)$ is $\mathcal{G}_{pt}$-S_g^*-connected space, then $(\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$ is $\mathcal{G}_{pt}$-connected space.*

Proof. Assume $(\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$ is $\mathcal{G}_{pt}$-S_g^*-connected space. Let $\mathcal{E}$ and $\mathcal{D}$ be two non-empty $\mathcal{G}_{pt}$-S_g^*-open sets in $\mathcal{Z}$. Such as $\mathcal{E} \cup \mathcal{D} = \mathcal{Z}$. As $\mathfrak{f}$ is surjective, $\mathcal{G}_{pt}$-S_g^*-continuous so $\mathfrak{f}^{-1}(\mathcal{E}) \cup \mathfrak{f}^{-1}(\mathcal{D}) = \mathcal{V}$ and $\mathfrak{f}^{-1}(\mathcal{E})$, $\mathfrak{f}^{-1}(\mathcal{D})$ is $\mathcal{G}_{pt}$-S_g^*-open disjoint subsets in $\mathcal{V}$. this implies a contradiction. So, $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-connected. Therefore, $\mathcal{Z}$ is also $\mathcal{G}_{pt}$-S_g^*-connected space. □

Theorem 3.12. *Suppose $\mathfrak{f}$: $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$ is a strongly $\mathcal{G}_{pt}$-S_g^*-continuous function and $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha)$ is $\mathcal{G}_{pt}$-connected space, then its image is $\mathcal{G}_{pt}$-S_g^*-connected space.*

Proof. Consider $\mathfrak{f}$: $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$ is a strongly $\mathcal{G}_{pt}$-S_g^*-continuous map and $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha)$ is $\mathcal{G}_{pt}$-connected space. Let $(\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$ is not $\mathcal{G}_{pt}$-S_g^*-connected space for $\mathcal{G}_{pt}$-S_g^*-open sets $\mathcal{E}$ and $\mathcal{D}$ in $\mathcal{Z}$. Such as $\mathcal{E} \cup \mathcal{D} = \mathcal{Z}$. As $\mathfrak{f}$ is strongly $\mathcal{G}_{pt}$-S_g^*-continuous so $\mathfrak{f}^{-1}(\mathcal{E}) \cup \mathfrak{f}^{-1}(\mathcal{D}) = \mathcal{V}$ and $\mathfrak{f}^{-1}(\mathcal{E})$, $\mathfrak{f}^{-1}(\mathcal{D})$ is open disjoint sets in $\mathcal{V}$. This implies a contradiction. Thus, $\mathcal{V}$ is $\mathcal{G}_{pt}$-connected. Hence, $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-connected space. □

3.2. $\mathcal{G}_{pt}$-S_g^*-Separation Axioms

Definition 3.14. *A $\mathcal{GPTS}$ $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{P})$ is called $\mathcal{G}_{pt}$-S_g^* Tc space if every $\mathcal{G}_{pt}$-S_g^*-closed is closed.*

Definition 3.15. *Let $\mathcal{E}$ be a subset of $\mathcal{V}$ which is $\mathcal{G}_{pt}$-T_0 space when every explicit point $\mathfrak{r}, \mathfrak{s}$ of $\mathcal{V}$ satisfies conditions: either $\mathfrak{s} \notin \mathcal{M}_\alpha$ and $\mathfrak{r} \in \mathcal{M}_\alpha$ or $\mathfrak{r} \notin \mathcal{M}_\alpha$, $\mathfrak{s} \in \mathcal{M}_\alpha$, where $\mathcal{M}_\alpha$ is an $\mathcal{G}_{pt}$-open set of $\mathcal{V}$.*

Definition 3.16. *Let $\mathcal{M}_\alpha \subseteq \mathcal{V}$. $\mathcal{E}$ is $\mathcal{G}_{pt}$-T_1 space if for every explicit point $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{V}$, $\mathfrak{s} \notin \mathcal{M}_\alpha$, $\mathfrak{r} \in \mathcal{M}_\alpha$ and $\mathfrak{r} \notin \mathcal{N}_\alpha$, $\mathfrak{s} \in \mathcal{N}_\alpha$, where $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ are $\mathcal{G}_{pt}$-open sets of $\mathcal{V}$.*

Definition 3.17. *Let $\mathcal{E} \subseteq \mathcal{V}$. $\mathcal{E}$ is known as $\mathcal{G}_{pt}$-T_2 space if for every explicit point $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{V}$, $\mathfrak{s} \notin \mathcal{M}_\alpha$, $\mathfrak{r} \in \mathcal{M}_\alpha$ and $\mathfrak{r} \notin \mathcal{N}_\alpha$, $\mathfrak{s} \in \mathcal{N}_\alpha$, where $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ are disjoint $\mathcal{G}_{pt}$-open sets of $\mathcal{V}$.*

Definition 3.18. *The function $\mathfrak{j} : \mathcal{V} \to \mathcal{Z}$ performs as an $\mathcal{G}_{pt}$-S_g^*-continuous operator. A function is $\mathcal{G}_{pt}$-S_g^*-continuous when its inverse images is $\mathcal{G}_{pt}$-S_g^*-open in $(\mathcal{V}, \mathcal{G}_{\tau_1}, \mathcal{P}_\alpha)$ for every open set in $(\mathcal{Z}, \mathcal{G}_{\tau_2}, \mathcal{P}_\beta)$.*

3.2.1. $\mathcal{G}_{pt}$-S_g^*-T_0, $\mathcal{G}_{pt}$-S_g^*-T_1, $\mathcal{G}_{pt}$-S_g^*-T_2 Spaces

Definition 3.19. *A subset $\mathcal{E}$ of $\mathcal{V}$ is called $\mathcal{G}_{pt}$-S_g^*-T_0 space if for any two different points $\mathfrak{r}$ and $\mathfrak{s}$ satisfies either $\mathfrak{s} \notin \mathcal{M}_\alpha$ and $\mathfrak{r} \in \mathcal{M}_\alpha$ or $\mathfrak{r} \notin \mathcal{M}_\alpha$ and $\mathfrak{s} \in \mathcal{M}_\alpha$, where $\mathcal{M}_\alpha$ is $\mathcal{G}_{pt}$-S_g^*-open set.*

Definition 3.20. *A subset $\mathcal{E}$ of $\mathcal{V}$ is called $\mathcal{G}_{pt}$-S_g^*-T_1 space if for any two distinct point $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{V}$, $\mathfrak{s} \notin \mathcal{M}_\alpha$, $\mathfrak{r} \in \mathcal{M}_\alpha$ and $\mathfrak{r} \notin \mathcal{N}_\alpha$, $\mathfrak{s} \in \mathcal{N}_\alpha$, where $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ are $\mathcal{G}_{pt}$-S_g^*-open sets of $\mathcal{V}$.*

Definition 3.21. *Let $\mathcal{E} \subseteq \mathcal{V}$. $\mathcal{E}$ is called $\mathcal{G}_{pt}$-S_g^*-T_2 ($\mathcal{G}_{pt}$-S_g^*-Housdorff) space if there exist two disjoint $\mathcal{G}_{pt}$-S_g^*-open sets $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ for any two different points $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{V}$, $\mathfrak{s} \notin \mathcal{M}_\alpha$, $\mathfrak{r} \in \mathcal{M}_\alpha$ and $\mathfrak{r} \notin \mathcal{N}_\alpha$, $\mathfrak{s} \in \mathcal{N}_\alpha$.*

Theorem 3.13. *i. If $\mathcal{V}$ is $\mathcal{G}_{pt}$-T_0, then $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_0.*

ii. If $\mathcal{V}$ is $\mathcal{G}_{pt}$-T_1, then $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_0 and $\mathcal{G}_{pt}$-S_g^*-T_1.

iii. If $\mathcal{V}$ is $\mathcal{G}_{pt}$-T_2, then $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_2.

iv. If $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_2, then $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_0.

v. If $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_2, then $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_1.

Proof. i) Given, $\mathcal{V}$ is $\mathcal{G}_{pt}$-T_0 space. An open set $\mathcal{F}$ exists which covers every pair of points $\mathfrak{r}$, $\mathfrak{s}$ belonging to $\mathcal{V}$ such that $\mathfrak{s} \notin \mathcal{F}$ while $\mathfrak{r} \in \mathcal{F}$ or $\mathfrak{r} \notin \mathcal{F}$ but $\mathfrak{s} \in \mathcal{F}$. The family set $\mathcal{F}$ belongs to the collection $\mathcal{G}_{pt}$-S_g^*O($\mathcal{V}$) while satisfying two conditions: $\mathfrak{s} \notin \mathcal{F}$ and $\mathfrak{r} \in \mathcal{F}$ or $\mathfrak{r} \notin \mathcal{F}$ and $\mathfrak{s} \in \mathcal{F}$. Thus, $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_0 space.

The demonstration of all proposed points ii), iii), iv), and v) is also possible in the same way. □

Remark 3.3. *An illustration disproves the invalidity of the converse statement derived from above.*

Example 3.6. *Let $\mathcal{V} = \{j_1, k_1, l_1\}$ and $\mathcal{G}_\tau = \{ \emptyset, \mathcal{V}, \{l_1\}, \{j_1, k_1\} \}$, $\mathcal{P} = \{ \emptyset, \{j_1\}, \{l_1\}, \{j_1, l_1\} \}$. In ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$)$O = \{ \emptyset, \mathcal{V}, \{l_1\}, \{j_1, k_1\} \}$ and $\mathcal{G}_{pt}$-$S_g^* O = P(\mathcal{V})$. Hence, ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is*

- *$\mathcal{G}_{pt}$-S_g^*-T_0 but not $\mathcal{G}_{pt}$-T_0. No open set exists with $\mathfrak{s} \notin \mathcal{M}_\alpha$, $\mathfrak{r} \in \mathcal{M}_\alpha$ or $\mathfrak{r} \notin \mathcal{M}_\alpha$, $\mathfrak{s} \in \mathcal{M}_\alpha$, where $\mathcal{M}_\alpha$ is an open set of $\mathcal{V}$ for explicit points $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{V}$.*
- *$\mathcal{G}_{pt}$-S_g^*-T_1 space but not $\mathcal{G}_{pt}$-T_1 space. No two open sets $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ exist with $\mathfrak{s} \notin \mathcal{M}_\alpha$, $\mathfrak{r} \in \mathcal{M}_\alpha$ and $\mathfrak{r} \notin \mathcal{N}_\alpha$, $\mathfrak{s} \in \mathcal{N}_\alpha$ for any explicit points $\mathfrak{s}$ and $\mathfrak{r}$ of $\mathcal{V}$.*
- *$\mathcal{G}_{pt}$-S_g^*-T_2 space but not $\mathcal{G}_{pt}$-T_2 space. No two distinct open sets $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ exist with $\mathfrak{s} \notin \mathcal{M}_\alpha$, $\mathfrak{r} \in \mathcal{M}_\alpha$ and $\mathfrak{r} \notin \mathcal{N}_\alpha$, $\mathfrak{s} \in \mathcal{N}_\alpha$ for any explicit points $\mathfrak{s}$ and $\mathfrak{r}$ of $\mathcal{V}$.*

Theorem 3.14. *If $\mathfrak{f}$ is bijective, strongly $\mathcal{G}_{pt}$-S_g^*-open and $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_0, then $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-T_0 space.*

Proof. Take $\mathfrak{r}_2$ and $\mathfrak{s}_2$ of $\mathcal{Z}$ with $\mathfrak{r}_2 \neq \mathfrak{s}_2$. By hypothesis, $\mathfrak{r}_2 = \mathfrak{f}(\mathfrak{r}_\alpha)$ and $\mathfrak{s}_2 = \mathfrak{f}(\mathfrak{s}_1)$ where $\mathfrak{r}_\alpha$ and $\mathfrak{s}_1$ are the explicit points of $\mathcal{V}$. By hypothesis, $\mathcal{M}_\alpha \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{V}$) with $\mathfrak{r}_\alpha \in \mathcal{M}_\alpha$ and $\mathfrak{s}_1 \notin \mathcal{M}_\alpha$. Therefore, $\mathfrak{f}(\mathfrak{r}_\alpha) \in \mathfrak{f}(\mathcal{M}_\alpha)$ and $\mathfrak{f}(\mathfrak{s}_1) \notin \mathfrak{f}(\mathcal{M}_\alpha)$. $\mathfrak{f}(\mathcal{M}_\alpha) \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{Z}$) as $\mathcal{V}$ is strongly $\mathcal{G}_{pt}$-S_g^*-open. Thus, $\mathfrak{f}(\mathcal{M}_\alpha)$ is $\mathcal{G}_{pt}$-S_g^*-open set in $\mathcal{Z}$ with $\mathfrak{r}_2 \in \mathfrak{f}(\mathcal{M}_\alpha)$ and $\mathfrak{s}_2 \notin \mathfrak{f}(\mathcal{M}_\alpha)$. So, $\mathcal{Z}$ is a $\mathcal{G}_{pt}$-S_g^*-T_0 space. □

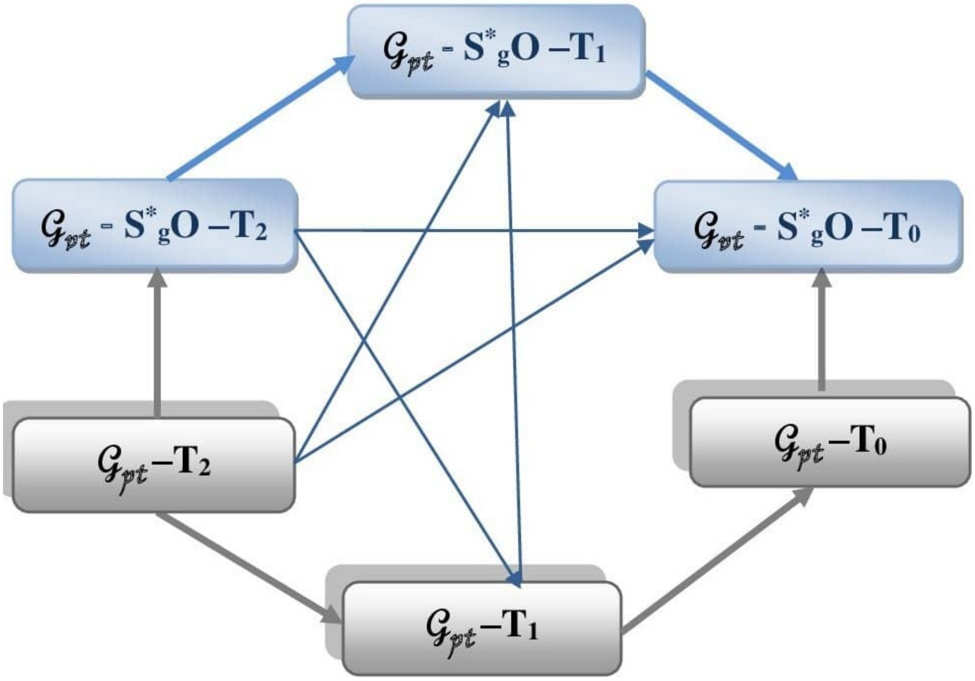

Figure 1: Illustrates the above theorems and considerations.

Theorem 3.15. *If $\mathfrak{f}$ is $\mathcal{G}_{pt}$-S_g^*-irresolute, injective, and $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-T_0, then $\mathcal{X}$ is $\mathcal{G}_{pt}$-S_g^*-T_0 space.*

Proof. Let $\mathfrak{r}_\alpha$ and $\mathfrak{s}_1$ distinct points of $\mathcal{V}$. Since $\mathfrak{f}$ is injective implies $\mathfrak{f}(\mathfrak{s}_1) \neq \mathfrak{f}(\mathfrak{r}_\alpha)$. As $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-T_0 there exists $\mathcal{F} \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{Z}$) such that $\mathfrak{f}(\mathfrak{r}_\alpha) \in \mathcal{F}$, $\mathfrak{f}(\mathfrak{s}_1) \notin \mathcal{F}$ or exists $\mathcal{F}_\beta \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{Z}$) such that $\mathfrak{f}(\mathfrak{s}_1) \in \mathcal{F}_\beta$, $\mathfrak{f}(\mathfrak{r}_\alpha) \notin \mathcal{F}_\beta$ with $\mathfrak{f}(\mathfrak{s}_1) \neq \mathfrak{f}(\mathfrak{r}_\alpha)$. As $\mathfrak{f}$ is $\mathcal{G}_{pt}$-S_g^*-irresolute then $\mathfrak{f}^{-1}(\mathcal{F}) \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{V}$), there exists $\mathfrak{f}^{-1}(\mathfrak{r}_\alpha) \in \mathcal{F}$, $\mathfrak{f}^{-1}(\mathfrak{s}_1) \notin \mathcal{F}$ or $\mathfrak{f}^{-1}(\mathcal{F}_\beta) \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{V}$) implies $\mathfrak{f}^{-1}(\mathfrak{s}_1) \in \mathcal{F}_\beta$, $\mathfrak{f}^{-1}(\mathfrak{r}_\alpha) \notin \mathcal{F}_\beta$. Hence, $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_0 space. □

Theorem 3.16. *Assume $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_1 iff $\mathfrak{s}_1 \in \mathcal{V}$ singleton $\{\mathfrak{s}_1\} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$.*

Proof. Assume $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_1, $\mathfrak{r}_\alpha \in \mathcal{V}$. Then, $\mathfrak{s}_1 \in \mathcal{V}$ - $\{\mathfrak{r}_\alpha\}$ implies $\mathfrak{r}_\alpha \neq \mathfrak{s}_1 \in \mathcal{V}$. But $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_1 space implies there exists $\mathcal{F}$, $\mathcal{F}_\beta \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{V}$) implies $\mathfrak{r}_\alpha \notin \mathcal{F}$, $\mathfrak{s}_1 \in \mathcal{F}_\beta \subseteq$ ($\mathcal{V}$ - $\{\mathfrak{r}_\alpha\}$). Also $\mathfrak{s}_1 \in \mathcal{F}_\beta \subseteq$ ($\mathcal{V}$ - $\{\mathfrak{r}_\alpha\}$) implies ($\mathcal{V}$ - $\{\mathfrak{r}_\alpha\}$) $\in \mathcal{G}_{pt}$-S_g^*O($\mathcal{V}$). Thus, $\{\mathfrak{r}_\alpha\}$ is $\mathcal{G}_{pt}$-S_g^*-closed. Conversely, Consider the distinct elements $\mathfrak{r}_\alpha \neq \mathfrak{s}_1 \in \mathcal{V}$ where sets $\{\mathfrak{r}_\alpha\}$ and $\{\mathfrak{s}_1\}$ form $\mathcal{G}_{pt}$-S_g^*-closed sets and their complement $\{\mathfrak{r}_\alpha\}^c$ represents an $\mathcal{G}_{pt}$-S_g^*-open subset. Certainly, $\{\mathfrak{r}_\alpha\} \notin \{\mathfrak{r}_\alpha\}^c$ and $\{\mathfrak{s}_1\} \in \{\mathfrak{r}_\alpha\}^c$. Similarly $\{\mathfrak{s}_1\}^c$ is $\mathcal{G}_{pt}$-S_g^*-open, $\{\mathfrak{s}_1\} \notin \{\mathfrak{s}_1\}^c$ and $\{\mathfrak{r}_\alpha\} \in \{\mathfrak{s}_1\}^c$. Thus, $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_1 space. □

Theorem 3.17. *Assume $\mathfrak{f} : \mathcal{V} \to \mathcal{Z}$. Then the subsequent results hold:*

i) If $\mathfrak{f}$ is injective, $\mathcal{G}_{pt}$-S_g^*-continuous, $\mathcal{Z}$ is $\mathcal{G}_{pt}$-T_1, then $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_1.

ii) If $\mathfrak{f}$ is injective, $\mathcal{G}_{pt}$-S_g^*-continuous, $\mathcal{Z}$ is $\mathcal{G}_{pt}$-T_2, then $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_2.

iii) If $\mathfrak{f}$ is injective, $\mathcal{G}_{pt}$-S_g^*-irresolute, $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-T_2, then $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_2.

Proof.

i) Assume $\mathfrak{r}_\alpha \neq \mathfrak{s}_1$, where $\mathfrak{r}_\alpha$, $\mathfrak{s}_1 \in \mathcal{V}$, then $\mathfrak{f}(\mathfrak{r}_\alpha) = \mathfrak{r}_2$ and $\mathfrak{f}(\mathfrak{s}_1) = \mathfrak{s}_2$. Additionally, $\mathfrak{f}(\mathfrak{r}_\alpha) \neq \mathfrak{f}(\mathfrak{s}_1)$. As ($\mathcal{Z}$, $\mathcal{G}_{\tau_2}$, $\mathcal{P}_\beta$) $\mathcal{G}_{pt}$-T_1 implies $\mathfrak{r}_2 \in \mathcal{M}_\alpha$, $\mathfrak{s}_2 \notin \mathcal{M}_\alpha$ and $\mathfrak{s}_2 \in \mathcal{N}_\alpha$, $\mathfrak{r}_2 \notin \mathcal{N}_\alpha$. Then $\mathfrak{r}_\alpha \in \mathfrak{f}^{-1}(\mathcal{M}_\alpha)$, $\mathfrak{r}_\alpha \notin \mathfrak{f}^{-1}(\mathcal{N}_\alpha)$ and $\mathfrak{s}_1 \in \mathfrak{f}^{-1}(\mathcal{N}_\alpha)$, $\mathfrak{s}_1 \notin \mathfrak{f}^{-1}(\mathcal{M}_\alpha)$. According to the definition of $\mathcal{G}_{pt}$-S_g^*-continuity, $\mathfrak{f}^{-1}(\mathcal{M}_\alpha)$ and $\mathfrak{f}^{-1}(\mathcal{N}_\alpha) \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{V}$). For $\mathfrak{r}_\alpha \neq \mathfrak{s}_1$, $\mathfrak{r}_\alpha$, $\mathfrak{s}_1 \in \mathcal{V}$ implies $\mathfrak{r}_\alpha \in \mathfrak{f}^{-1}(\mathcal{M}_\alpha)$, $\mathfrak{r}_\alpha \notin \mathfrak{f}^{-1}(\mathcal{N}_\alpha)$ and $\mathfrak{s}_1 \in \mathfrak{f}^{-1}(\mathcal{N}_\alpha)$, $\mathfrak{s}_1 \notin \mathfrak{f}^{-1}(\mathcal{M}_\alpha)$. Thus, ($\mathcal{V}$, $\mathcal{G}_{\tau_1}$, $\mathcal{P}_\alpha$) is $\mathcal{G}_{pt}$-S_g^*-T_1 space. In the same way, ii) and iii) can be proven. □

Theorem 3.18. *The subsequent statements are equivalent.*

(i) $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_2.

(ii) If $\mathfrak{r}_\alpha \in \mathcal{V}$, then $\mathfrak{r}_\alpha \neq \mathfrak{s}_1$, there exists $\mathcal{U}_\alpha$ containing $\mathfrak{r}_\alpha$ and $\mathfrak{s}_1 \notin \mathcal{G}_{pt}$-$S_g^*$Cl($\mathcal{U}_\alpha$).

Proof. (1) implies (2) Take $\mathfrak{r}_\alpha \in \mathcal{V}$ and $\mathfrak{s}_1 \in \mathcal{V}$ with $\mathfrak{r}_\alpha \neq \mathfrak{s}_1$ there exists disjoint set $\mathcal{U}_\alpha$ and $\mathcal{V} \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{V}$) such that $\mathfrak{r}_\alpha \in \mathcal{U}_\alpha$ and $\mathfrak{s}_1 \in \mathcal{V}$. Then, $\mathfrak{r}_\alpha \in \mathcal{U}_\alpha \subseteq \mathcal{V}^c$ and $\mathcal{V}^c \in \mathcal{G}_{pt}$-$S_g^*$C($\mathcal{V}$) and $\mathfrak{s}_1 \notin \mathcal{V}^c$ implies $\mathfrak{s}_1 \notin \mathcal{G}_{pt}$-$S_g^*$Cl($\mathcal{U}_\alpha$). (2) implies (1) Consider $\mathfrak{r}_\alpha \in \mathcal{V}$ and $\mathfrak{s}_1 \in \mathcal{V}$ with $\mathfrak{r}_\alpha \neq \mathfrak{s}_1$ implies there exists $\mathcal{G}_{pt}$-S_g^*-open $\mathcal{U}_\alpha$ containing $\mathfrak{r}_\alpha$ such that $\mathfrak{s}_1 \notin \mathcal{G}_{pt}$-$S_g^*$Cl($\mathcal{U}_\alpha$) implies $\mathfrak{s}_1 \in$ ($\mathcal{V}$ - ($\mathcal{G}_{pt}$-S_g^*Cl($\mathcal{U}_\alpha$))). ($\mathcal{V}$ - ($\mathcal{G}_{pt}$-S_g^*Cl($\mathcal{U}_\alpha$))) $\in \mathcal{G}_{pt}$-S_g^*O($\mathcal{V}$) and $\mathfrak{r}_\alpha \notin$ ($\mathcal{V}$ - ($\mathcal{G}_{pt}$-S_g^*Cl($\mathcal{U}_\alpha$))). Furthermore, $\mathcal{U}_\alpha \cap$ ($\mathcal{V}$ - ($\mathcal{G}_{pt}$-S_g^*Cl($\mathcal{U}_\alpha$))) $= \emptyset$. So $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-T_2. □

3.2.2. $\mathcal{G}_{pt}$-S_g^*-Regular Space

Definition 3.22. *If for all $\mathcal{F} \in \mathcal{G}_{pt}$-$S_g^*$C($\mathcal{V}$) and $\mathfrak{r} \notin \mathcal{F}$, there exists disjoint open sets $\mathcal{E}$ and $\mathcal{D}$ such that $\mathcal{F} \subseteq \mathcal{E}$, $\mathfrak{r} \in \mathcal{D}$.*

Theorem 3.19. *The condition of being $\mathcal{G}_{pt}$-S_g^*-regular implies $\mathcal{G}_{pt}$-regularity.*

Proof. Let $\mathcal{V}$ be a $\mathcal{G}_{pt}$-S_g^*-regular space. Take $\mathcal{F} \in \mathcal{G}_{pt}$-$S_g^*$C($\mathcal{V}$) and $\mathfrak{r} \notin \mathcal{F}$. As $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-regular, there exists a pair of disjoint open sets $\mathcal{E}$ and $\mathcal{D}$ such that $\mathcal{F} \subseteq \mathcal{E}$, $\mathfrak{r} \in \mathcal{D}$. Hence, $\mathcal{V}$ is a $\mathcal{G}_{pt}$-regular. □

Example 3.7. *Let $\mathcal{V}$ be the set of all bounded spherical regions in $\mathbb{R}^3$, where a spherical region S is defined as:*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \leq r\},$$

with $r > 0$ as the radius of the spherical region and a, b, $c \in \mathbb{R}$, $\tau_g = \{\mathcal{E} \subseteq \mathcal{V} \mid \mathcal{E} \subseteq \mathcal{V} \setminus \{S \mid radius(S) = r\}$ for some fixed $r > 0\}$ (see example 3.1) and define a collection $\mathcal{G}_{pt} \subseteq 2^{\mathcal{V}}$ as:

$$\mathcal{G}_{pt} = \{\mathcal{E} \subseteq \mathcal{V} \mid \textit{all spherical region in } \mathcal{E} \textit{ have radius strictly less than } r, \textit{ where } r > 0\}.$$

Let $x \in \mathcal{V}$ and $C \subseteq \mathcal{V}$ be a $\mathcal{G}_{pt}$-S_g^*-closed set such that $x \notin C$. Since C is $\mathcal{G}_{pt}$-S_g^*-closed, its complement $\mathcal{V} \setminus C$ is $\mathcal{G}_{pt}$-S_g^*-open in $\mathcal{V}$. Let an $\mathcal{G}_{pt}$-S_g^*-open set U such that $x \in U$ and $U \cap C = \emptyset$. Therefore, $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{G}_{pt})$, satisfies the condition for $\mathcal{G}_{pt}$-S_g^*-regularity. Now, we check if $\mathcal{V}$ is $\mathcal{G}_{pt}$-regular: Let $x \in \mathcal{V}$ and $C \subset \mathcal{V}$ be a closed set such that $x \notin C$. Since every closed set is $\mathcal{G}_{pt}$-S_g^*- closed set. By construction, such a set U exists and satisfies the $\mathcal{G}_{pt}$-regularity condition which implies $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{G}_{pt})$ is $\mathcal{G}_{pt}$-regular. Thus, $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{G}_{pt})$ is both $\mathcal{G}_{pt}$-S_g^*-regular and $\mathcal{G}_{pt}$-regular.

Remark 3.4. *Every $\mathcal{G}_{pt}$-regular is a not $\mathcal{G}_{pt}$-S_g^*-regular space.*

Example 3.8. *Let $\mathcal{V} = \{j_1, k_1, l_1\}$ and $\mathcal{G}_\tau = \{\ \emptyset,\ \mathcal{V},\ \{l_1\},\ \{j_1, k_1\}\ \}$, $\mathcal{P} = \{\ \emptyset,\ \{j_1\}, \{l_1\}, \{j_1, l_1\}\ \}$. Hence, ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is $\mathcal{G}_{pt}$-regular but not $\mathcal{G}_{pt}$-S_g^*-regular space. For $\{k_1\} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$ and $\mathfrak{r} \notin \{k_1\}$, there does not exists disjoint open sets $\mathcal{E}$ and $\mathcal{D}$ such that $\{k_1\} \subseteq \mathcal{E}$, $\mathfrak{r} \in \mathcal{D}$.*

Theorem 3.20. *Every $\mathcal{G}_{pt}$-regular with $\mathcal{G}_{pt}$-$S_g^*T_c$ space is $\mathcal{G}_{pt}$-S_g^*-regular.*

Proof. Under the condition that $\mathcal{V}$ is $\mathcal{G}_{pt}$-regular and $\mathcal{G}_{pt}$-$S_g^*T_c$. Take a set $\mathcal{F}$ which belongs to $\mathcal{G}_{pt}$-S_g^*C($\mathcal{V}$) along with an element $\mathfrak{r}$ belonging to both $\mathcal{V}$ and non-corresponding to $\mathcal{F}$. Because $\mathcal{V}$ functions as a $\mathcal{G}_{pt}$-$S_g^*T_c$ space, its concluding that $\mathcal{F}$ is closed yet $\mathfrak{r}$ belongs to the exterior of $\mathcal{F}$. Since $\mathcal{V}$ possesses the property of $\mathcal{G}_{pt}$-regularity a pair of open sets named $\mathcal{E}$ and $\mathcal{D}$ exists with the property that $\mathcal{F}$ belongs to $\mathcal{E}$ while $\mathfrak{r}$ belongs to $\mathcal{D}$ and these open sets are disjoint. The space $\mathcal{V}$ meets the criteria for being a $\mathcal{G}_{pt}$-S_g^*-regular. □

Theorem 3.21. *If $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-regular, then it is $\mathcal{G}_{pt}$-regular space.*

Proof. According to the fact, every closed set belongs to $\mathcal{G}_{pt}$-S_g^*C($\mathcal{V}$). □

Theorem 3.22. *The subsequent statements are equivalent:*

(i) *$\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-regular.*

(ii) *For all $\mathfrak{r} \in \mathcal{V}$ and each $\mathcal{G}_{pt}$-S_g^*-open neighbourhood $\mathcal{U}_\alpha$ there exists open neighbourhood $\mathcal{N}_\alpha$ of $\mathcal{V}$ such that Cl° ($\mathcal{N}_\alpha$) $\subseteq \mathcal{U}_\alpha$.*

Proof. (1) implies (2) Assume $\mathcal{U}_\alpha$ is $\mathcal{G}_{pt}$-S_g^*-neighbourhood of $\mathfrak{r}$, there exists $\mathcal{E} \in \mathcal{G}_{pt}$-$S_g^*$O($\mathcal{V}$) such that $\mathfrak{r} \in \mathcal{E} \subseteq \mathcal{U}_\alpha$. Now, $\mathcal{E}^c \in \mathcal{G}_{pt}$-$S_g^*$C($\mathcal{V}$) and $\mathfrak{r} \notin \mathcal{E}^c$. From (1), there exists $\mathcal{R}_\alpha$, $\mathcal{S}_\alpha$ such that $\mathcal{E}^c \subseteq \mathcal{R}_\alpha$, $\mathfrak{r} \in \mathcal{S}_\alpha$, $\mathcal{R}_\alpha \cap \mathcal{S}_\alpha = \emptyset$. Thus $\mathcal{S}_\alpha \subseteq \mathcal{M}_\alpha{}^c$. Now, $\mathrm{Cl}^\circ(\mathcal{S}_\alpha) \subseteq \mathrm{Cl}^\circ(\mathcal{R}_\alpha{}^c) = \mathcal{E}^c$ and $\mathcal{E}^c \subseteq \mathcal{R}_\alpha$ implies $\mathcal{R}_\alpha{}^c \subseteq \mathcal{E} \subseteq \mathcal{U}_\alpha$. Thus $\mathrm{Cl}^\circ(\mathcal{S}_\alpha) \subseteq \mathcal{U}_\alpha$. (2) implies (1) Consider $\mathcal{G}_{pt}$-S_g^*-closed $\mathcal{F}$ in $\mathcal{V}$ and $\mathfrak{r} \notin \mathcal{F}$ or $\mathfrak{r} \in (\mathcal{F})^c$ and $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-open implies $(\mathcal{F})^c$ is

$\mathcal{G}_{pt}$-S_g^*-neighbourhood of $\mathfrak{r}$. By hypothesis, there exists an open neighbourhood $\mathcal{N}_\alpha$ such that $\mathfrak{r} \in \mathcal{N}_\alpha$, Cl° $(\mathcal{N}_\alpha) \subseteq (\mathcal{F})^c$ implies $\mathcal{F} \subseteq \{\ \mathcal{V}$ - Cl° $(\mathcal{N}_\alpha)\ \}$ and $\mathcal{N}_\alpha \cap \{\ \mathcal{V}$ - Cl°$(\mathcal{N}_\alpha)\ \}$ $= \emptyset$. Thus, $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-regular. □

Theorem 3.23. *Assume $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-regular iff for every $\mathcal{E} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$ and point $\mathfrak{p} \in (\mathcal{V}$ - $\mathcal{E})$ then $\mathfrak{r} \in \mathcal{U}_\alpha$, $\mathcal{E} \subseteq \mathcal{N}_\alpha$ and Cl°$(\mathcal{N}_\alpha) \cap$ Cl°$(\mathcal{U}_\alpha) = \emptyset$, where $\mathcal{U}_\alpha$ and $\mathcal{N}_\alpha$ are open sets.*

Proof. Given that $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-regular. Assume $\mathcal{E} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$ and $\mathfrak{r} \notin \mathcal{E}$. Then, $\mathfrak{p} \in \mathcal{M}_\alpha$ and $\mathcal{E} \subseteq \mathcal{N}_\alpha$ and $\mathcal{M}_\alpha \cap \mathcal{N}_\alpha = \emptyset$, where $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ are open sets implies $\mathcal{M}_\alpha \cap$ Cl°$(\mathcal{N}_\alpha) = \emptyset$. As $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-regular, $\mathfrak{p} \in \mathcal{R}_\alpha$ and Cl°$(\mathcal{N}_\alpha) \subseteq \mathcal{S}_\alpha$, $\mathcal{R}_\alpha \cap \mathcal{N}_\alpha = \emptyset$ where $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$ are open. Furthermore, Cl°$(\mathcal{R}_\alpha) \cap \mathcal{S}_\alpha = \emptyset$. Assume $\mathcal{U}_\alpha = \mathcal{M}_\alpha \cap \mathcal{R}_\alpha$ implies $\mathfrak{p} \in \mathcal{U}_\alpha$, $\mathcal{E} \subseteq \mathcal{N}_\alpha$ and Cl°$(\mathcal{N}_\alpha) \cap$ Cl°$(\mathcal{U}_\alpha) = \emptyset$ where $\mathcal{N}_\alpha$ and $\mathcal{U}_\alpha$ are open in $\mathcal{V}$. On the other hand, consider $\mathcal{N}_\alpha$ and $\mathcal{U}_\alpha$ are open sets. $\mathfrak{p} \in \mathcal{U}_\alpha$, $\mathcal{E} \subseteq \mathcal{N}_\alpha$ and Cl°$(\mathcal{N}_\alpha) \cap$ Cl°$(\mathcal{U}_\alpha)$ $= \emptyset$ for all $\mathcal{E} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$ and $\mathfrak{p} \in (\mathcal{V}$ - $\mathcal{E})$ implies $\mathfrak{p} \in \mathcal{U}_\alpha$, $\mathcal{E} \subseteq \mathcal{N}_\alpha$ and $\mathcal{U}_\alpha \cap \mathcal{N}_\alpha = \emptyset$. Thus, $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-regular. □

Theorem 3.24. *A subspace $\mathcal{Z}$ of $\mathcal{G}_{pt}$-S_g^*-regular $(\mathcal{Z}, \mathcal{G}_\tau, \mathcal{P})$ is $\mathcal{G}_{pt}$-S_g^*-regular.*

Proof. Obvious. □

Theorem 3.25. *Assume $\mathfrak{f}$ is bijective, $\mathcal{G}_{pt}$-S_g^*-irresolute and open map from $\mathcal{G}_{pt}$-S_g^*-regular $\mathcal{V}$ into $\mathcal{Z}$, then $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-regular.*

Proof. Let $\mathfrak{r}_\alpha \in \mathcal{Z}$ and $\mathcal{F} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$ and $\mathfrak{r}_\alpha \notin \mathcal{F}$. Furthermore, $\mathfrak{f}$ is $\mathcal{G}_{pt}$-S_g^*-irresolute, then $\mathfrak{f}^{-1}(\mathcal{F}) \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$. Now, assume $\mathfrak{r}_\alpha = \mathfrak{f}(\mathfrak{r})$ then $\mathfrak{f}^{-1}(\mathfrak{r}_\alpha) = \mathfrak{r}$ and $\mathfrak{r} \notin \mathfrak{f}^{-1}(\mathcal{F})$. As $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-regular then there exists $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$ such that $\mathfrak{r} \in \mathcal{R}_\alpha$ and $\mathfrak{f}^{-1}(\mathcal{F}) \subseteq \mathcal{S}_\alpha$, $\mathcal{R}_\alpha$ $\cap\ \mathcal{S}_\alpha = \emptyset$. Since $\mathfrak{f}$ is open and bijective implies $\mathfrak{r}_\alpha \in \mathfrak{f}(\mathcal{R}_\alpha)$, $\mathcal{F} \subseteq \mathfrak{f}(\mathcal{S}_\alpha)$ and $\mathfrak{f}(\mathcal{R}_\alpha \cap \mathcal{S}_\alpha) = \mathfrak{f}(\emptyset) = \emptyset$. Then, $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-regular. □

3.2.3. $\mathcal{G}_{pt}$-S_g^*-Normal Space

Definition 3.23. *Assume $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*normal if for each pair $\mathcal{E}$, $\mathcal{D} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$, there exists open sets $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$ in $\mathcal{V}$ such that $\mathcal{D} \subseteq \mathcal{R}_\alpha$ and $\mathcal{E} \subseteq \mathcal{S}_\alpha$.*

Theorem 3.26. *Every $\mathcal{G}_{pt}$-S_g^*-normal is $\mathcal{G}_{pt}$-normal.*

Proof. As $\mathcal{V}$ is a $\mathcal{G}_{pt}$-S_g^*-normal. Assume disjoint sets $\mathcal{E}$ and $\mathcal{D}$ in $\mathcal{V}$. So $\mathcal{E}$, $\mathcal{D} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$. As $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-normal implies there exist a pair $\mathcal{F}$, $\mathcal{H}_\alpha$ such that $\mathcal{D} \subseteq \mathcal{F}$, $\mathcal{E} \subseteq \mathcal{H}_\alpha$. Thus, $\mathcal{V}$ is $\mathcal{G}_{pt}$-normal. □

Example 3.9. *Consider $\mathcal{V} = \{j_1, k_1, l_1\}$ and $\mathcal{G}_\tau = \{\ \emptyset,\ \mathcal{V},\ \{k_1\},\ \{l_1\},\ \{k_1, l_1\},\ \{j_1, k_1\}\ \}$, $\mathcal{P} = \{\ \emptyset,\ \{j_1\}, \{l_1\}, \{j_1, l_1\}\ \}$. Here, $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{P})$ is $\mathcal{G}_{pt}$-normal but not $\mathcal{G}_{pt}$-S_g^*-normal space. For disjoint sets $\{j_1\}$, $\{k_1, l_1\} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$, there does not exist open sets $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$ in $\mathcal{V}$.*

Theorem 3.27. *If $\mathcal{Z}$ is $\mathcal{G}_{pt}$-normal, $\mathcal{G}_{pt}$-$S_g^*T_c$ space, then $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-normal.*

Proof. Since $\mathcal{Z}$ is $\mathcal{G}_{pt}$-normal. Consider disjoint set $\mathcal{E}$, $\mathcal{D} \in \mathcal{G}_{pt}$-$S_g^*$C($\mathcal{Z}$). As $\mathcal{G}_{pt}$-$S_g^*T_c$ space, then $\mathcal{E}$ and $\mathcal{D}$ are closed. Since $\mathcal{Z}$ is $\mathcal{G}_{pt}$-normal, then there exist disjoint open sets $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$ in $\mathcal{Z}$ such that $\mathcal{E} \subseteq \mathcal{R}_\alpha$ and $\mathcal{D} \subseteq \mathcal{S}_\alpha$. Thus, $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-normal. □

Theorem 3.28. *Every $\mathcal{G}_{pt}$-S_g^*-normal is $\mathcal{G}_{pt}$-g-normal.*

Proof. As $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-normal. Assume disjoint set $\mathcal{E}$, $\mathcal{D} \in \mathcal{G}_{pt}$-$S_g^*$C($\mathcal{Z}$) implies there exist a disjoint $\mathcal{R}_\alpha$, $\mathcal{S}_\alpha$ such that $\mathcal{E} \subseteq \mathcal{R}_\alpha$ and $\mathcal{D} \subseteq \mathcal{S}_\alpha$. Thus, $\mathcal{V}$ is $\mathcal{G}_{pt}$-g-normal. □

Remark 3.5. *Every $\mathcal{G}_{pt}$-g-normal is not $\mathcal{G}_{pt}$-S_g^*-normal.*

Example 3.10. *Consider $\mathcal{V} = \{j_1, k_1, l_1\}$ and $\mathcal{G}_\tau = \{ \emptyset, \mathcal{V}, \{k_1\}, \{l_1\}, \{k_1, l_1\}, \{j_1, l_1\} \}$, $\mathcal{P} = \{ \emptyset, \{j_1\}, \{l_1\}, \{j_1, l_1\} \}$. Here, ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is $\mathcal{G}_{pt}$-g-normal but not $\mathcal{G}_{pt}$-S_g^*-normal space. For disjoint sets $\{j_1\}$, $\{k_1, l_1\} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$, there does not exist open sets $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$ in $\mathcal{V}$.*

Theorem 3.29. *Every $\mathcal{G}_{pt}$-S_g^*-normal is $\mathcal{G}_{pt}$-w-normal.*

Proof. Similar to theorem 3.28. □

Remark 3.6. *An illustration disproves the invalidity of the converse statement derived from above.*

Example 3.11. *Consider $\mathcal{V} = \{j_1, k_1, l_1\}$ and $\mathcal{G}_\tau = \{ \emptyset, \mathcal{V}, \{k_1\}, \{l_1\}, \{k_1, l_1\}, \{j_1, l_1\} \}$, $\mathcal{P} = \{ \emptyset, \{j_1\}, \{l_1\}, \{j_1, l_1\} \}$. Here, ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) is $\mathcal{G}_{pt}$-w-normal but not $\mathcal{G}_{pt}$-S_g^*-normal space. For disjoint sets $\{j_1\}$, $\{k_1, l_1\} \in \mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$, there does not exists open sets $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$ in $\mathcal{V}$.*

Theorem 3.30. *If $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-closed subspace of $\mathcal{G}_{pt}$-S_g^*-normal $\mathcal{V}$, then $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-normal.*

Proof. Assume $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-normal and $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-closed subspace. Consider a pair of disjoint sets $\mathcal{E}$ and $\mathcal{D} \in \mathcal{G}_{pt}$-$S_g^*$C($\mathcal{Z}$) implies there exists $\mathcal{F}$, $\mathcal{H}_\alpha \in \mathcal{V}$ such that $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{D} \subseteq \mathcal{H}_\alpha$ implies $\mathcal{F} \cap \mathcal{Z}$ and $\mathcal{H}_\alpha \cap \mathcal{Z}$ are open in $\mathcal{Z}$. Furthermore, $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{D} \subseteq \mathcal{H}_\alpha$ implies $\mathcal{E} \cap \mathcal{Z} \subseteq \mathcal{Z} \cap \mathcal{F}$ and $\mathcal{Z} \cap \mathcal{D} \subseteq \mathcal{Z} \cap \mathcal{H}_\alpha$ and $(\mathcal{F} \cap \mathcal{Z}) \cap (\mathcal{Z} \cap \mathcal{H}_\alpha) = \mathcal{Z} \cap (\mathcal{F} \cap \mathcal{H}_\alpha)$ $= \emptyset$. Thus, $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-normal. □

Theorem 3.31. *The following conditions in ($\mathcal{V}$, $\mathcal{G}_\tau$, $\mathcal{P}$) are equivalent:*

1) *The space $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-normal.*
2) *For each $\mathcal{E}$ belonging to $\mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$, there exists an open set $\mathcal{T}_1$ such that $\mathcal{E} \subseteq \mathcal{T}_1 \subseteq Cl(\mathcal{T}_1) \subseteq \mathcal{T}_2$ for some $\mathcal{T}_2$ in $\mathcal{G}_{pt}$-$S_g^*O(\mathcal{V})$ with $\mathcal{E} \subseteq \mathcal{T}_2$.*

3) Given two disjoint sets $\mathcal{E}$ and $\mathcal{D}$ in $\mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$, there exists an open set $\mathcal{T}_1$ such that $\mathcal{E} \subseteq \mathcal{T}_1$ and $Cl(\mathcal{T}_1) \cap \mathcal{D} = \emptyset$.

4) For any two disjoint sets $\mathcal{E}$, $\mathcal{D}$ in $\mathcal{G}_{pt}$-$S_g^*C(\mathcal{V})$, there exist open sets $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{E} \subseteq \mathcal{T}_2$, $\mathcal{D} \subseteq \mathcal{T}_1$, and $Cl(\mathcal{T}_2) \cap Cl(\mathcal{T}_1) = \emptyset$.

Proof. (1) implies (2): Suppose $\mathcal{E}$ belongs to $\mathcal{G}_{pt}$-$S_g^*\mathrm{C}(\mathcal{V})$ and $\mathcal{T}_2$ is an element of $\mathcal{G}_{pt}$-$S_g^*\mathrm{C}(\mathcal{V})$ with $\mathcal{E} \subseteq \mathcal{T}_2$. Since $\mathcal{E}$ and $\mathcal{V} - \mathcal{T}_2$ are disjoint, the assumption of $\mathcal{G}_{pt}$-S_g^*-normality guarantees open sets $\mathcal{T}_1$ and $\mathcal{T}_3$ such that $\mathcal{E} \subseteq \mathcal{T}_1$ and $\mathcal{V} - \mathcal{T}_2 \subseteq \mathcal{T}_3$ with $\mathcal{T}_1 \cap \mathcal{T}_3 = \emptyset$. This ensures $\mathcal{T}_1 \subseteq \mathcal{V} - \mathcal{T}_3$ and further $\mathrm{Cl}(\mathcal{T}_1) \subseteq \mathcal{V} - \mathcal{T}_3 \subseteq \mathcal{T}_2$, leading to $\mathrm{Cl}(\mathcal{T}_1) \subseteq \mathcal{T}_2$.

(2) implies (3): Given two disjoint sets $\mathcal{E}$ and $\mathcal{D}$ in $\mathcal{G}_{pt}$-$S_g^*\mathrm{C}(\mathcal{V})$, we note that $\mathcal{E} \subseteq \mathcal{V} - \mathcal{D}$. By (2), there exists an open set $\mathcal{T}_1$ such that $\mathcal{E} \subseteq \mathcal{T}_1$ and $\mathrm{Cl}(\mathcal{T}_1) \subseteq \mathcal{V} - \mathcal{D}$, ensuring that $\mathrm{Cl}(\mathcal{T}_1) \cap \mathcal{D} = \emptyset$.

(3) implies (4): Given two disjoint sets $\mathcal{E}$ and $\mathcal{D}$ in $\mathcal{G}_{pt}$-$S_g^*\mathrm{C}(\mathcal{V})$, (3) guarantees an open set $\mathcal{T}_2$ such that $\mathcal{E} \subseteq \mathcal{T}_2$ and $\mathrm{Cl}(\mathcal{T}_2) \cap \mathcal{D} = \emptyset$. Since $\mathrm{Cl}(\mathcal{T}_2)$ is closed, applying (3) again ensures the existence of an open set $\mathcal{T}_1$ containing $\mathcal{D}$ such that $\mathrm{Cl}(\mathcal{T}_2) \cap \mathrm{Cl}(\mathcal{T}_1) = \emptyset$.

(4) implies (1): Suppose $\mathcal{E}$ and $\mathcal{D}$ are two disjoint sets in $\mathcal{G}_{pt}$-$S_g^*\mathrm{C}(\mathcal{V})$. By (4), there exist disjoint open sets $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{E} \subseteq \mathcal{T}_2$ and $\mathcal{D} \subseteq \mathcal{T}_1$. This confirms that $\mathcal{V}$ satisfies the definition of $\mathcal{G}_{pt}$-S_g^*-normality. □

Theorem 3.32. *Assume a mapping $\mathfrak{f} : \mathcal{V} \to \mathcal{Z}$. If $\mathfrak{f}$ is bijective open, $\mathcal{G}_{pt}$-S_g^*-irresolute from $\mathcal{G}_{pt}$-S_g^*-normal $\mathcal{V}$ onto $\mathcal{Z}$, then $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-normal.*

Proof. Assume disjoint sets $\mathcal{E}$, $\mathcal{D} \in \mathcal{G}_{pt}$-$S_g^*\mathrm{C}(\mathcal{V})$. Since $\mathfrak{f}$ is $\mathcal{G}_{pt}$-S_g^*-irresolute then $\mathfrak{f}^{-1}(\mathcal{E})$ and $\mathfrak{f}^{-1}(\mathcal{D})$ are in $\mathcal{G}_{pt}$-$S_g^*\mathrm{C}(\mathcal{V})$. As $\mathcal{V}$ is $\mathcal{G}_{pt}$-S_g^*-normal implies $\mathfrak{f}^{-1}(\mathcal{E}) \subseteq \mathcal{T}_2$ and $\mathfrak{f}^{-1}(\mathcal{D}) \subseteq \mathcal{T}_1$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are open in $\mathcal{V}$. Also, as $\mathfrak{f}$ is bijective and open, $\mathfrak{f}(\mathcal{T}_2)$ and $\mathfrak{f}(\mathcal{T}_1)$ are open and $\mathcal{E} \subseteq \mathfrak{f}(\mathcal{T}_2)$, $\mathcal{D} \subseteq \mathfrak{f}(\mathcal{T}_1)$. Thus, $\mathcal{Z}$ is $\mathcal{G}_{pt}$-S_g^*-normal. □

Theorem 3.33. *The following statements hold equivalently in $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{P})$:*

1) $\mathcal{V}$ is $\mathcal{G}_{pt}$-g-normal.

2) There exist disjoint open sets $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathcal{G}_{pt}$-$S_g^*O(\mathcal{V})$ such that $\mathcal{E} \subseteq \mathcal{T}_2$ and $\mathcal{D} \subseteq \mathcal{T}_1$ for any disjoint sets $\mathcal{E}$ and $\mathcal{D}$.

Proof. (1) implies (2) Suppose that $\mathcal{V}$ is $\mathcal{G}_{pt}$-g-normal, and let $\mathcal{E}$ and $\mathcal{D}$ be two disjoint subsets of $\mathcal{V}$. By the assumption that $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{P})$ is $\mathcal{G}_{pt}$-g-normal, there exist disjoint $\mathcal{G}_{pt}$-g-open sets $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{E} \subseteq \mathcal{T}_2$ and $\mathcal{D} \subseteq \mathcal{T}_1$. Consequently, $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathcal{G}_{pt}$-$S_g^*\mathrm{O}(\mathcal{V})$, satisfying $\mathcal{E} \subseteq \mathcal{T}_2$ and $\mathcal{D} \subseteq \mathcal{T}_1$ while ensuring $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$.

(2) implies (1) consider that for any two disjoint $\mathcal{G}_{pt}$-S_g^*-closed sets $\mathcal{E}$, $\mathcal{D} \in \mathcal{G}_{pt}$-$S_g^*\mathrm{C}(\mathcal{V})$, there exist disjoint open sets $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{E} \subseteq \mathcal{T}_2$, $\mathcal{D} \subseteq \mathcal{T}_1$, and $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ where $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathcal{G}_{pt}$-$S_g^*\mathrm{O}(\mathcal{V})$. Since $\mathcal{E}$ is contained in $\mathcal{G}_{pt}$-$\mathrm{gInt}(\mathcal{T}_2)$ and $\mathcal{D}$ in $\mathcal{G}_{pt}$-$\mathrm{gInt}(\mathcal{T}_1)$, and their interiors remain disjoint, it follows that $(\mathcal{V}, \mathcal{G}_\tau, \mathcal{P})$ is $\mathcal{G}_{pt}$-g-normal. □

4. Methodology

A theoretical framework demonstrates the examination of compactness as well as connectedness and separation axioms in generalized primal topological spaces. The research method includes the following main stages of development:

- **Review of Relevant Literature:** This research explores the developmental history of compactness and connectedness and separation properties through extensive study of published literature across generalized contexts. The research focuses on reviewing studies that created fundamental educational frameworks to transfer the principles of S_g^*-compactness and S_g^*-connectedness to generalized primal topological spaces.
- **Definition Development:** The article introduces new definitions of S_g^*-compactness and S_g^*-connectedness under generalized primal topology. New definitions for generalized primal structures are developed but go through extensive evaluation to confirm their alignment with core principles of primal topology.
- **Analyzing the Separation Axioms:** Generalized primal topology utilizes S_g^*-open sets to study the classical separation conditions T_0, T_1 and T_2. This study establishes the S_g^*-T_0, S_g^*-T_1 together with the S_g^*-T_2 conditions before performing their respective analyses.
- **Establishing Theoretical Proofs:** The formal verification process proves the various attributes defined in the proposed constructions. Mathematical proofs establish internal validity while ensuring logical consistency of new concepts in order to provide respectable grounds for theoretical future research.
- **Comparative Analysis:** The paper evaluates the new developed theory by showing its distinctions and correspondences to traditional theories alongside demonstrating advantages for accepting generalized primal topological methods.
- **Results Synthesis:** The research findings transform into structural models for describing S_g^*-compact as well as S_g^*-connected spaces through the generalized primal topological framework, together with separation axiom analysis. Furthermore, the work discusses what aspects the advancement means for developing topological theory.

5. Conclusions

Researchers perform an extensive investigation into separation axioms together with connectedness and compactness phenomena in generalized primal topological spaces. This paper extends traditional concepts to the new framework, which increases our understanding of S_g^*-compactness and S_g^*-connectedness at a theoretical level. The new definitions specifically designed for generalized primal spaces combined with their formal support system enhances these concepts for application throughout nonclassical spaces.

The study introduces a new set of classification tools through S_g^*-T_0, S_g^*-T_1, and S_g^*-T_2 separation axiom reinterpretations to analyze spaces based on their topological properties. In their independent studies, S_g^*-compactness and S_g^*-connectedness do not strictly mirror traditional separation axioms, yet their examination broadens the fundamental understanding of generalized primal spaces. This research provides significant value to the expansion of generalized primal topology as a field of exploration.

Author contributions

Conceptualization, M.S., U.I. and I.L.P.; methodology, M.I. and I.L.P.; software, T.K., U.I. and I.L.P.; validation, M.A. and I.L.P.; formal analysis, U.I. and M.A.; investigation, M.A.; resources, T.K.; data curation, U.I.; writing—original draft preparation, M.S., M.I., and U.I.; writing—review and editing, T.K.; visualization, M.A. supervision, T.K. and I.L.P.; project administration, U.I. and I.L.P.; funding acquisition, M.A. All authors have read and agreed to the published version of the manuscript.

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Competing interests

The authors declare that they have no competing interests.

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