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Analytical Investigation for Some Problems Under Different Fractional Differential and Integral Operators

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Abstract. This article presents the three well-known fractional order differential operators including the Caputo, Caputo-Fabrizio, and Atangana-Baleanu to describe the solutions of a class of non-linear fractional partial differential equation (FPDE). A method called the Adomian decomposition method (ADM) is used to find series solutions in a semi-analytical way, using different transforms like Laplace, Elzaki, Sumudu, Aboodh, Mohand, Yang, Natural, and Shehu. The solutions obtained by the proposed method have precision and a high rate of convergence. We then verify the derived solutions numerically and graphically for both fractional and integer orders. Furthermore, the solutions under these transformations are the same. The proposed simulations show that as the number of iterations increases, the corresponding absolute error reduces. Moreover, fractional order solutions are converging to integer order solutions.

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Key Words and Phrases: Adomian decomposition method, integral transforms, Caputo derivative operator, Caputo-Fabrizio derivative operator, Atangana-Baleanu derivative operator

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1. Introduction

In the past few decades, mathematicians have established many mathematical models utilizing the idea of fractional order derivatives. These fractional-order mathematical models provided an extremely accurate match to the data from experiments of the associated problems as opposed to traditional models. The scientists and scholars have initiated unique mathematical models employing fractional differential/integral equations. Due to

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the numerous uses of fractional differential equations (FDEs) in the fields of science and engineering, mathematicians are increasingly focusing on this area of study. Fractional calculus is widely used in mathematical biology [1, 2], quantum field theory [3], fluid mechanics [4], and plasma physics [5]. In control engineering, it is essential for fractional order systems [6] and PID controllers [7]. Applications also encompass neural networks [8], optical communications [9], and electrochemical processes [10]. Moreover, fractional calculus is employed in the modeling of time-space fractional diffusion equations [11], groundwater contamination [12], heat conduction in materials [13], and acoustic wave propagation [14]. Emerging applications encompass fractional dynamical systems in economics [15], fractional modeling in image processing [16], and machine learning algorithms [17].

Nonlinear partial differential equations (PDEs) can be tricky to solve precisely. Consequently, in recent decades, mathematicians and scholars have investigated and introduced several numerical methods and methodologies for addressing nonlinear PDEs. Given the absence of universal methods applicable to all equation types and the necessity to treat each equation as an individual problem, solving nonlinear PDEs presents significant challenges. We can assess and understand the numerical methods and the physical phenomena clarified through analytical solutions to FPDEs. Numerical and novel analytical methodologies that exist for solving FPDEs are Numerical methods include the Grünwald-Letnikov method [18], finite difference methods [19], fractional Adams-Bashforth and Adams-Moulton methods [20], spectral methods [21], fractional order Runge-Kutta methods [22], wavelet methods [23], and fractional discrete Fourier transform [24]. Analytical methods include the Laplace transform [25], series solutions [26], Fourier series technique [27], the homotopy perturbation method [28], variational methods [29], and the method of successive approximations [30]. Besides these, other methods are [31–35].

The primary objective is to illustrate the solutions of non-linear FPDEs with Initial Conditions (ICs) by employing the Caputo fractional derivative (CFD) operator, the Caputo-Fabrizio fractional (CFFD) operator, and the Atangana-Baleanu fractional (ABFD) operator in a mathematical model. Fractional derivatives of the proposed operators will be utilized to illustrate the positive effects of employing these operators to solve a mathematical problem. Outcomes from these operators are analyzed to demonstrate their efficacy and precision. In 1966, Caputo [36] proposed the Caputo Derivative, that includes a singular kernel (power law kernel). In 2015, Caputo and Fabrizio [37] introduced the Caputo-Fabrizio Fractional Derivative, characterized by a non-singular kernel featuring an exponential function. In 2016, Atangana and Baleanu [38] presented the derivative with a non-singular kernel utilizing the Mittag-Leffler function.

The ADM, developed by George Adomian between the 1970s and 1990s, is a semi-analytical technique that decomposes the unknown term of equations both linear and nonlinear differential and integral equations into components of varying degrees. These components are subsequently summed, and the method seeks to determine the solutions for each order, which constitutes the fundamental principle of ADM. The method's essential element is the utilization of "Adomian Polynomials" which facilitate the convergence of solutions for the nonlinear portion of the equation by simple linearization of the problem. This study integrates ADM with various transforms and subsequently analyzes the results using tables and graphs.

Analytical methods have a significant role in fractional calculus because solutions obtained by analytical methods are exact and closed-form. This ensures the accuracy of the resulting quantitative predictions. On the other side, the numerical methods generate errors that may lead to results significantly diverging from the exact solutions. ADM is a semi-analytical method that has precision and a high rate of convergence to closed-form solutions.

The Laplace [39], Elzaki [40], Aboodh [41], Sumudu[42],Yang[43], Mohand [44], Natural [45], and Shehu [46] transforms are integral transforms employed to solve FPDEs across different fields, including engineering and mathematics. These transforms, essential instruments, convert time-domain functions into the complex frequency domain.

There are some basic definitions along with some mathematical calculations for different transformations and derivative operators in section (2). A simple, straightforward, and general methodology in section (3). In section (4), we solve a non-linear FPDE under different derivative operators and transformations.

2. Preliminaries

Definition 1. *The Laplace transform [39] of $g(\tau)$ is a function $G(s)$ of another variable s defined by*

$$G(s) = \mathcal{L}[g(\tau)] = \int_0^\infty \mathrm{e}^{-s\,\tau} g(\tau) d\tau, \quad \textit{where} \quad \tau \geq 0. \tag{1}$$

Where $g(\tau)$ be a continuous or sectionally continuous function of τ.

Definition 2. *The Elzaki transform [40] of $g(\tau)$ is defined by*

$$G(s) = \mathfrak{E}[g(\tau)] = s \int_0^\infty \mathrm{e}^{-\frac{\tau}{s}} g(\tau) d\tau, \qquad \tau \geq 0. \tag{2}$$

Definition 3. *The Aboodh transform [41] of $g(\tau)$ is defined by*

$$G(s) = \mathcal{A}[g(\tau)] = \frac{1}{s} \int_0^\infty \mathrm{e}^{-s\,\tau} g(\tau) d\tau, \qquad \tau \geq 0. \tag{3}$$

Definition 4. *The Sumudu transform [42] of $g(\tau)$ is defined by*

$$G(s) = \mathcal{S}[g(\tau)] = \frac{1}{s} \int_0^\infty \mathrm{e}^{-\frac{\tau}{s}} g(\tau) d\tau, \qquad \tau \geq 0. \tag{4}$$

Definition 5. *The Yang transform [43] of $g(\tau)$ is defined by*

$$G(s) = \mathcal{Y}[g(\tau)] = \int_0^\infty \mathrm{e}^{-\frac{\tau}{s}} g(\tau) d\tau, \qquad \tau \geq 0. \tag{5}$$

Definition 6. *The Mohand transform [44] of $g(\tau)$ is defined by*

$$G(s) = \mathcal{M}[g(\tau)] = s^2 \int_0^\infty \mathrm{e}^{-s\tau} g(\tau) d\tau, \qquad \tau \geq 0. \tag{6}$$

Definition 7. *The Natural transform [45] of $g(\tau)$ is defined by*

$$F(s,\theta) = \mathcal{N}[g(\tau)] = \frac{1}{\theta}\int_0^\infty \mathrm{e}^{-\frac{s\tau}{\theta}} g(\tau)d\tau, \qquad \tau \geq 0. \tag{7}$$

Definition 8. *The Shehu transform [46] of $g(\tau)$ is defined by*

$$F(s,\theta) = \mathfrak{S}[g(\tau)] = \int_0^\infty \mathrm{e}^{-\frac{s\tau}{\theta}} g(\tau)d\tau, \qquad \tau \geq 0. \tag{8}$$

Definition 9. *The kernel of the integral equation*

$$w(\theta) = \int_{p(\theta)}^{q(\theta)} \kappa(\chi,\theta)w(\theta)d\theta. \tag{9}$$

is $\kappa(\chi,\theta)$. If the kernel is singular, then equation (9) becomes a singular integral equation. The Kernel is singular if one or both limits of integration become infinite, or if it becomes infinite at one point or more points in the interval of integration, mathematically

$$\kappa(\chi,\theta) \to \infty \qquad \textit{with} \qquad \theta \to \chi.$$

Definition 10. *Consider, g_1 and g_2 are two piecewise continuous functions of exponential order on the interval $[0,\infty)$, then the convolution is defined on g_1 and g_2 as*

$$g_1 * g_2 = \int_0^\tau g_1(\tau-\tau)g_2(\tau)d\tau = \int_0^\tau g_2(\tau)g_1(\tau-\tau)d\tau = g_2 * g_1, \tag{10}$$

Showing, $g_1 * g_2 = g_2 * g_1$.

Further, let take LT

$$\mathcal{L}(g_1 * g_2) = \mathcal{L}\left[\int_0^\tau g_1(\tau-\tau)g_2(\tau)d\tau\right] = \mathcal{L}[g_1(\tau)]\mathcal{L}[g_2(\tau)] = G_1(s)G_2(s). \tag{11}$$

Similarly, Convolution under different transformations; given in the Equation (12).

$$\begin{cases} \mathcal{L}[g_1 * g_2] = \mathcal{L}[g_1(\tau)]\mathcal{L}[g_2(\tau)] = G_1(s)G_2(s). \\ \mathfrak{E}[g_1 * g_2] = \frac{\mathfrak{E}[g_1(\tau)]\mathfrak{E}[g_2(\tau)]}{s} = \frac{G_1(s)G_2(s)}{s}. \\ \mathcal{A}[g_1 * g_2] = s\,\mathcal{A}[g_1(\tau)]\mathcal{A}[g_2(\tau)] = s\,G_1(s)G_2(s). \\ \mathcal{S}[g_1 * g_2] = s\,\mathcal{S}[g_1(\tau)]\mathcal{S}[g_2(\tau)] = s\,G_1(s)G_2(s). \\ \mathcal{Y}[g_1 * g_2] = \mathcal{Y}[g_1(\tau)]\mathcal{Y}[g_2(\tau)] = G_1(s)G_2(s). \\ \mathcal{M}[g_1 * g_2] = \frac{\mathcal{M}[g_1(\tau)]\mathcal{M}[g_2(\tau)]}{s^2} = \frac{G_1(s)G_2(s)}{s^2}. \\ \mathcal{N}[g_1 * g_2] = \theta\,\mathcal{N}[g_1(\tau)]\mathcal{N}[g_2(\tau)] = \theta\,G_1(s)G_2(s). \\ \mathfrak{S}[g_1 * g_2] = \mathfrak{S}[g_1(\tau)]\mathfrak{S}[g_2(\tau)] = G_1(s)G_2(s). \end{cases} \tag{12}$$

Now,first order derivative under different transform are

$$\begin{cases} \mathcal{L}[g'(\tau)] = s\,G(s) - g(0). \\ \mathfrak{E}[g'(\tau)] = \frac{G(s)}{s} - s\,g(0). \\ \mathcal{A}[g'(\tau)] = s\,G(s) - \frac{g(0)}{s}. \\ \mathcal{S}[g'(\tau)] = \frac{G(s)}{s} - \frac{g(0)}{s}. \\ \mathcal{Y}[g'(\tau)] = \frac{G(s)}{s} - g(0). \\ \mathcal{M}[g'(\tau)] = s\,G(s) - s^2\,g(0). \\ \mathcal{N}[g'(\tau)] = \frac{s\,G(s,\theta)}{\theta} - \frac{g(0)}{\theta}. \\ \mathfrak{S}[g'(\tau)] = \frac{s\,G(s,\theta)}{\theta} - g(0). \end{cases} \tag{13}$$

Definition 11. *The CFD [36] is defined as*

$$^{C}D^{\alpha}_{\theta}\chi(\theta) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\theta} (\theta-\tau)^{-\alpha} \frac{\partial}{\partial \tau}\chi(\tau)\,\mathrm{d}\tau. \tag{14}$$

Taking LT, implies

$$\begin{aligned} \mathcal{L}[^{C}D^{\alpha}_{\theta}\chi(\theta)] &= \mathcal{L}\left[\frac{1}{\Gamma(1-\alpha)}\left(\theta^{-\alpha} * \frac{\partial}{\partial\theta}\chi(\theta)\right)\right] \\ &= \frac{1}{\Gamma(1-\alpha)}\left(\mathcal{L}[\theta^{-\alpha}]\,\mathcal{L}\left[\frac{\partial}{\partial\theta}\chi(\theta)\right]\right) \\ &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}(sX(s)-\chi(0))\right) \\ &= s^{\alpha-1}(sX(s)-\chi(0)) \\ &= s^{\alpha}X(s) - s^{\alpha-1}\chi(0). \end{aligned} \tag{15}$$

Implies

$$\mathcal{L}[\,^{C}D^{\alpha}_{\theta}\chi(\theta)\,] = s^{\alpha}X(s) - s^{\alpha-1}\chi(0), \quad \text{where} \quad 0 < \alpha \leq 1. \tag{16}$$

Similarly, under different transformations are given in the equation (17).

$$\begin{cases} \mathcal{L}[\,^{C}D^{\alpha}_{\theta}\chi(\theta)\,] = s^{\alpha}X(s) - s^{\alpha-1}\chi(0). \\ \mathfrak{E}[\,^{C}D^{\alpha}_{\theta}\chi(\theta)\,] = s^{-\alpha}X(s) - s^{2-\alpha}\chi(0). \\ \mathcal{A}[\,^{C}D^{\alpha}_{\theta}\chi(\theta)\,] = s^{\alpha}X(s) - s^{\alpha-2}\chi(0). \\ \mathcal{S}[\,^{C}D^{\alpha}_{\theta}\chi(\theta)\,] = s^{-\alpha}X(s) - s^{-\alpha}\chi(0). \\ \mathcal{X}[\,^{C}D^{\alpha}_{\theta}\chi(\theta)\,] = s^{-\alpha}X(s) - s^{1-\alpha}\chi(0). \\ \mathcal{M}[\,^{C}D^{\alpha}_{\theta}\chi(\theta)\,] = s^{\alpha}X(s) - s^{\alpha+1}\chi(0). \\ \mathcal{N}[\,^{C}D^{\alpha}_{\theta}\chi(\theta)\,] = \left(\frac{s}{\varsigma}\right)^{\alpha}X(s,\varsigma) - \left(\frac{s}{\varsigma}\right)^{\alpha}\frac{1}{s}\,\chi(0). \\ \mathfrak{S}[\,^{C}D^{\alpha}_{\theta}\chi(\theta)\,] = \left(\frac{s}{\varsigma}\right)^{\alpha}X(s,\varsigma) - \left(\frac{s}{\varsigma}\right)^{\alpha-1}\chi(0). \end{cases} \tag{17}$$

Definition 12. *The CFFD [37] is defined as:*

$$ {}^{CF}D_{\theta}^{\alpha}\chi(\theta) = \frac{\mathbf{N}(\alpha)}{1-\alpha}\int_{0}^{\theta} \mathrm{e}^{-\frac{\alpha}{1-\alpha}(\theta-\tau)}\frac{\partial}{\partial\tau}\chi(\tau)\,\mathrm{d}\tau, \tag{18} $$

where $\mathbf{N}(0) = \mathbf{N}(1) = 1$.

Taking LT

$$ \begin{aligned} \mathcal{L}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] &= \mathcal{L}\left[\frac{\mathbf{N}(\alpha)}{1-\alpha}\left(\mathrm{e}^{-\frac{\alpha}{1-\alpha}\theta} * \frac{\partial}{\partial\theta}\chi(\theta)\right)\right] \\ &= \frac{\mathbf{N}(\alpha)}{1-\alpha}\left(\mathcal{L}\left[\mathrm{e}^{-\frac{\alpha}{1-\alpha}\theta}\right]\,\mathcal{L}\left[\frac{\partial}{\partial\theta}\chi(\theta)\right]\right) \\ &= \frac{\mathbf{N}(\alpha)}{1-\alpha}\,\frac{1}{s-\left(-\frac{\alpha}{1-\alpha}\right)}(sX(s)-\chi(0)) \\ &= \frac{\mathbf{N}(\alpha)}{(1-\alpha)(s+\frac{\alpha}{1-\alpha})}(sX(s)-\chi(0)), \end{aligned} \tag{19} $$

implies

$$ \mathcal{L}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] = \frac{\mathbf{N}(\alpha)}{(1-\alpha)s+\alpha}(sX(s)-\chi(0)). \tag{20} $$

Similarly, under different transformations in equation (21)

$$ \begin{cases} \mathcal{L}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] = \frac{\mathbf{N}(\alpha)}{s(1-\alpha)+\alpha}(sX(s)-\chi(0)). \\ \mathfrak{E}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] = \frac{\mathbf{N}(\alpha)}{1+\alpha(s-1)}\left(X(s)-s^{2}\,\chi(0)\right). \\ \mathcal{A}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] = \frac{\mathbf{N}(\alpha)}{s^{2}(1-\alpha)+\alpha s}\left(s^{2}X(s)-\chi(0)\right). \\ \mathcal{S}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] = \frac{\mathbf{N}(\alpha)}{1+\alpha(s-1)}\left(X(s)-\ \chi(0)\right). \\ \mathcal{X}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] = \frac{\mathbf{N}(\alpha)}{1+\alpha(s-1)}(X(s)-s\,\chi(0)). \\ \mathcal{M}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] = \frac{\mathbf{N}(\alpha)}{s(1-\alpha)+\alpha}(sX(s)-s^{2}\,\chi(0)). \\ \mathcal{N}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] = \frac{\mathbf{N}(\alpha)}{\frac{s}{\varsigma}(1-\alpha)+\alpha}\left(\frac{s}{\varsigma}X(s,\varsigma)-\frac{\chi(0)}{\varsigma}\right). \\ \mathfrak{S}[{}^{CF}D_{\theta}^{\alpha}\chi(\theta)] = \frac{\mathbf{N}(\alpha)}{\frac{s}{\varsigma}(1-\alpha)+\alpha}\left(\frac{s}{\varsigma}X(s,\varsigma)-\chi(0)\right). \end{cases} \tag{21} $$

Definition 13. *The ABFD [38] defined as:*

$$ {}^{AB}D_{\theta}^{\alpha}\chi(\theta) = \frac{\mathbf{AB}(\alpha)}{1-\alpha}\int_{0}^{\theta} E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(\theta-\tau)^{\alpha}\right)\frac{\partial}{\partial\tau}\chi(\tau)\,\mathrm{d}\tau, \tag{22} $$

where $\mathbf{AB}(0) = \mathbf{AB}(1) = 1$ *and* $E_{\alpha}(\cdot)$ *denoted the Mittag-Leffler function [47] i.e.*

$$ E_{\alpha}(\theta) = E_{\alpha,1}(\theta) = \sum_{i=0}^{\infty}\frac{\theta^{i}}{\Gamma(i\alpha+\beta)}\bigg|_{\beta=1} \qquad \alpha > 0. \tag{23} $$

Taking LT

$$\begin{aligned}\mathcal{L}[^{AB}D_\theta^\alpha \chi(\theta)] &= \mathcal{L}\left[\frac{\mathbf{AB}(\alpha)}{1-\alpha}\left(E_\alpha\left(-\frac{\alpha}{1-\alpha}\,\theta^\alpha\right) * \frac{\partial}{\partial\theta}\chi\left(\theta\right)\right)\right]\\ &= \frac{\mathbf{AB}(\alpha)}{1-\alpha}\left(\mathcal{L}\left[E_\alpha\left(-\frac{\alpha}{1-\alpha}\,\theta^\alpha\right)\right]\,\mathcal{L}\left[\frac{\partial}{\partial\theta}\chi\left(\theta\right)\right]\right)\\ &= \frac{\mathbf{AB}(\alpha)}{1-\alpha}\left(\frac{s^{\alpha-1}}{s^\alpha-\left(-\frac{\alpha}{1-\alpha}\right)}(sX(s)-\chi(0))\right)\\ &= \frac{s^{\alpha-1}\mathbf{AB}(\alpha)}{s^\alpha(1-\alpha)+\alpha}(sX(s)-\chi(0)),\end{aligned} \tag{24}$$

implies

$$\mathcal{L}[\,^{AB}D_\theta^\alpha \chi(\theta)\,] = \frac{s^{\alpha-1}\mathbf{AB}(\alpha)}{s^\alpha(1-\alpha)+\alpha}(sX(s)-\chi(0)). \tag{25}$$

Similarly, the equation (26) for different transformations.

$$\begin{cases}\mathcal{L}[\,^{AB}D_\theta^\alpha \chi(\theta)\,] = \frac{s^{\alpha-1}\mathbf{AB}(\alpha)}{(1-\alpha)\left(\frac{\alpha}{1-\alpha}+s^\alpha\right)}\left(sX\left(s\right)-\chi\left(0\right)\right).\\ \mathfrak{E}[\,^{AB}D_\theta^\alpha \chi(\theta)\,] = \frac{s\ \mathbf{AB}(\alpha)}{(1-\alpha)\left(\frac{\alpha}{1-\alpha}s^\alpha+1\right)}\left(\frac{X(s)}{s}-s\ \chi\left(0\right)\right).\\ \mathcal{A}[\,^{AB}D_\theta^\alpha \chi(\theta)\,] = \frac{s^{\alpha-1}\mathbf{AB}(\alpha)}{(1-\alpha)\left(\frac{\alpha}{1-\alpha}+s^\alpha\right)}\left(sX\left(s\right)-\frac{\chi(0)}{s}\right).\\ \mathcal{S}[\,^{AB}D_\theta^\alpha \chi(\theta)\,] = \frac{s\ \mathbf{AB}(\alpha)}{(1-\alpha)\left(\frac{\alpha}{1-\alpha}s^\alpha+1\right)}\left(\frac{X(s)}{s}-\frac{\chi(0)}{s}\right).\\ \mathcal{X}[\,^{AB}D_\theta^\alpha \chi(\theta)\,] = \frac{s\ \mathbf{AB}(\alpha)}{(1-\alpha)\left(\frac{\alpha}{1-\alpha}s^\alpha+1\right)}\left(\frac{X(s)}{s}-\chi\left(0\right)\right).\\ \mathcal{M}[\,^{AB}D_\theta^\alpha \chi(\theta)\,] = \frac{s^{\alpha-1}\mathbf{AB}(\alpha)}{(1-\alpha)\left(\frac{\alpha}{1-\alpha}+s^\alpha\right)}\left(sX\left(s\right)-s^2\chi\left(0\right)\right).\\ \mathcal{N}[\,^{AB}D_\theta^\alpha \chi(\theta)\,] = \frac{\varsigma s^{\alpha-1}\mathbf{AB}(\alpha)}{(1-\alpha)\left(\frac{\alpha}{1-\alpha}\varsigma^\alpha+s^\alpha\right)}\left(\frac{s}{\varsigma}X\left(s\right)-\frac{\chi(0)}{\varsigma}\right).\\ \mathfrak{S}[\,^{AB}D_\theta^\alpha \chi(\theta)\,] = \frac{\varsigma s^{\alpha-1}\mathbf{AB}(\alpha)}{(1-\alpha)\left(\frac{\alpha}{1-\alpha}\varsigma^\alpha+s^\alpha\right)}\left(\frac{s}{\varsigma}X\left(s\right)-\chi\left(0\right)\right).\end{cases} \tag{26}$$

At last, we have calculated the transformations of some basic functions, which are given in the table (1).

Table 1: Basic functions under different transformations.

Functions	**Laplace**	**Elzaki**	**Aboodh**	**Sumudu**
1	$\frac{1}{s}$	s^2	$\frac{1}{s^2}$	1
μ	$\frac{\mu}{s}$	$s^2\mu$	$\frac{\mu}{s^2}$	μ
θ	$\frac{1}{s^2}$	s^3	$\frac{1}{s^3}$	s
θ^2	$\frac{2}{s^3}$	$2\,s^4$	$\frac{2}{s^4}$	$2\,s^2$
θ^n	$\frac{\Gamma(n+1)}{s^{n+1}}$	$s^{2+n}\Gamma\left(n+1\right)$	$\frac{\Gamma(n+1)}{s^{n+2}}$	$\Gamma\left(n+1\right)s^n$
$\sin\left(\mu\,\theta\right)$	$\frac{\mu}{\mu^2+s^2}$	$\frac{s^3\mu}{\mu^2s^2+1}$	$\frac{\mu}{(\mu^2+s^2)s}$	$\frac{\mu\,s}{\mu^2s^2+1}$
$\cos\left(\mu\,\theta\right)$	$\frac{s}{\mu^2+s^2}$	$\frac{s^2}{\mu^2s^2+1}$	$\frac{1}{\mu^2+s^2}$	$\frac{1}{\mu^2s^2+1}$
$\sinh\left(\mu\,\theta\right)$	$\frac{\mu}{-\mu^2+s^2}$	$-\frac{s^3\mu}{\mu^2s^2-1}$	$\frac{\mu}{-\mu^2s+s^3}$	$-\frac{\mu\,s}{\mu^2s^2-1}$
$\cosh\left(\mu\,\theta\right)$	$\frac{s}{-\mu^2+s^2}$	$-\frac{s^2}{\mu^2s^2-1}$	$\frac{1}{s^2-\mu^2}$	$\frac{1}{1-\mu^2s^2}$
$\mathrm{e}^{\mu\,\theta}$	$\frac{1}{s-\mu}$	$\frac{s^2}{1-\mu\,s}$	$\frac{1}{(s-\mu)s}$	$\frac{1}{1-\mu\,s}$
$\frac{\mathrm{e}^{p\theta}-\mathrm{e}^{q\theta}}{p-q}$	$\frac{1}{(s-p)(s-q)}$	$\frac{s^2}{\left(p-\frac{1}{s}\right)\left(q-\frac{1}{s}\right)}$	$\frac{\frac{1}{s}}{(p-s)(q-s)}$	$\frac{1}{\left(p-\frac{1}{s}\right)\left(q-\frac{1}{s}\right)}$
$E_\alpha\left(-\mu\,\theta^\alpha\right)$	$\frac{s^\alpha}{s(\mu+s^\alpha)}$	$\frac{s^2}{\mu\,s^\alpha+1}$	$\frac{s^\alpha}{s^2(\mu+s^\alpha)}$	$\frac{1}{(\mu\,s^\alpha+1)}$
$1-E_\alpha\left(-\mu\,\theta^\alpha\right)$	$\frac{\mu}{s(\mu+s^\alpha)}$	$\frac{s^{\alpha+2}\mu}{w\,s^\alpha+1}$	$\frac{\mu}{s^2(\mu+s^\alpha)}$	$\frac{\mu\,s^\alpha}{w\,s^\alpha+1}$
Functions	**Yang**	**Mohand**	**Natural**	**Shehu**
1	s	s	$\frac{1}{s}$	$\frac{\varsigma}{s}$
μ	$\mu\,s$	$\mu\,s$	$\frac{\mu}{s}$	$\frac{\mu\,\varsigma}{s}$
θ	s^2	1	$\frac{\varsigma}{s^2}$	$\frac{\varsigma^2}{s^2}$
θ^2	$2\,s^3$	$\frac{2}{s}$	$\frac{2\,\varsigma^2}{s^3}$	$\frac{2\,\varsigma^3}{s^3}$
θ^n	$\Gamma\left(n+1\right)s^{n+1}$	$\frac{\Gamma(n+1)}{s^{n-1}}$	$\frac{\Gamma(n+1)\varsigma^n}{s^{n+1}}$	$\frac{\Gamma(n+1)\varsigma^{n+1}}{s^{n+1}}$
$\sin\left(\mu\,\theta\right)$	$\frac{s^2\mu}{\mu^2s^2+1}$	$\frac{s^2\mu}{\mu^2+s^2}$	$\frac{\mu\,\varsigma}{\mu^2\varsigma^2+s^2}$	$\frac{\mu\,\varsigma^2}{\mu^2\varsigma^2+s^2}$
$\cos\left(\mu\,\theta\right)$	$\frac{s}{\mu^2s^2+1}$	$\frac{s^3}{\mu^2+s^2}$	$\frac{s}{\mu^2\varsigma^2+s^2}$	$\frac{s\,\varsigma}{\mu^2\varsigma^2+s^2}$
$\sinh\left(\mu\,\theta\right)$	$-\frac{s^2\mu}{\mu^2s^2-1}$	$\frac{s^2\mu}{-\mu^2+s^2}$	$\frac{\mu\,\varsigma}{-\mu^2\varsigma^2+s^2}$	$\frac{\mu\,\varsigma^2}{-\mu^2\varsigma^2+s^2}$
$\cosh\left(\mu\,\theta\right)$	$-\frac{s}{\mu^2s^2-1}$	$\frac{s^3}{-\mu^2+s^2}$	$\frac{s}{-\mu^2\varsigma^2+s^2}$	$\frac{s\,\varsigma}{-\mu^2\varsigma^2+s^2}$
$\mathrm{e}^{\mu\,\theta}$	$\frac{s^2}{s-\mu}$	$-\frac{s^3}{\mu\,s-1}$	$\frac{1}{s-\mu\,\varsigma}$	$\frac{\varsigma}{-\mu\,\varsigma+s}$
$\frac{\mathrm{e}^{p\theta}-\mathrm{e}^{q\theta}}{p-q}$	$\frac{s}{\left(p-\frac{1}{s}\right)\left(q-\frac{1}{s}\right)}$	$\frac{s^2}{(p-s)(q-s)}$	$\frac{1}{\left(p-\frac{s}{\varsigma}\right)\left(q-\frac{s}{\varsigma}\right)}$	$\frac{\varsigma}{\left(p-\frac{s}{\varsigma}\right)\left(q-\frac{s}{\varsigma}\right)}$
$E_\alpha\left(-\mu\,\theta^\alpha\right)$	$\frac{s}{\mu\,s^\alpha+1}$	$\frac{s^{\alpha+1}}{\mu+s^\alpha}$	$\frac{s^\alpha}{s(\mu\,\varsigma^\alpha+s^\alpha)}$	$\frac{s^\alpha\varsigma}{s(\mu\,\varsigma^\alpha+s^\alpha)}$
$1-E_\alpha\left(-\mu\,\theta^\alpha\right)$	$\frac{\mu\,s^{\alpha+1}}{\mu\,s^\alpha+1}$	$\frac{s\mu}{\mu+s^\alpha}$	$\frac{\mu\,\varsigma^\alpha}{s(\mu\,\varsigma^\alpha+s^\alpha)}$	$\frac{\mu\,\varsigma^{\alpha+1}}{s(\mu\,\varsigma^\alpha+s^\alpha)}$

The table 1 demonstrates some convolutions results against different fractional order operators.

3. Suggested ADM Method by Using Different Fractional Differential Operators

Let us have a general nonlinear FPDE in the form of

$$^{C}D_{\theta}^{\alpha}\nu\,(\varkappa,\theta)+F\nu(\varkappa,\theta)+N\nu(\varkappa,\theta)=f(\varkappa,\theta),\qquad u(\varkappa,0)=f(\varkappa). \tag{27}$$

Where $F\nu(\varkappa,\theta)$ and $N\nu(\varkappa,\theta)$ are linear and nonlinear parts, respectively. Taking LT (27) implies

$$\mathcal{L}[^{C}D_{\theta}^{\alpha}\nu\,(\varkappa,\theta)]=-\mathcal{L}\Big[F\nu(\varkappa,\theta)+N\nu(\varkappa,\theta)-f(\varkappa,\theta)\Big]. \tag{28}$$

By using equation (16), implies

$$U(\varkappa,s)=\frac{\nu(\varkappa,0)}{s}-\frac{1}{s^{\alpha}}\,\mathcal{L}\Big[F\nu(\varkappa,\theta)+N\nu(\varkappa,\theta)-f(\varkappa,\theta)\Big].$$

Now, taking the inverse LT

$$\begin{aligned}\mathcal{L}^{-1}[U(\varkappa,s)]&=\nu(\varkappa,\theta)\\&=\mathcal{L}^{-1}\left[\frac{\nu(\varkappa,0)}{s}-\frac{1}{s^{\alpha}}\,\mathcal{L}\Big[F\nu(\varkappa,\theta)+N\nu(\varkappa,\theta)-f(\varkappa,\theta)\Big]\right].\end{aligned} \tag{29}$$

$N\nu(\varkappa,\theta)$ will be decomposed by Adomian Polynomial, defined as

$$N\nu(\varkappa,\theta)=\sum_{n=0}^{\infty}P_n,$$

where

$$P_n=\frac{1}{n!}\,\frac{\partial^n}{\partial\gamma^n}\left(N(\sum_{i=0}^{\infty}\gamma^i\,\nu_i)\right)\Bigg|_{\gamma=0}\quad n=0,1,2,3,\ldots\quad .$$

Thus, LADM decompose equation (29) as

$$\nu_{k+1}(\varkappa,\theta)=\mathcal{L}^{-1}\left[-\frac{1}{s^{\alpha}}\,\mathcal{L}\Big[F\nu_k(\varkappa,\theta)+P_k\Big]\right]\quad,\qquad k\geq 0. \tag{30}$$

Where

$$\nu_0(\varkappa,\theta)=\mathcal{L}^{-1}\left[\frac{\nu(\varkappa,0)}{s}+\frac{1}{s^{\alpha}}\,\mathcal{L}\Big[f(\varkappa,\theta)\Big]\right]. \tag{31}$$

Similarly, ADM iterative terms under different transformations for CFD in equation (32).

$$\begin{cases} \nu_{k+1}(\varkappa,\theta) = \mathcal{L}^{-1}\left[-\frac{1}{s^{\alpha}}\,\mathcal{L}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{A}^{-1}\left[-\frac{1}{s^{\alpha}}\,\mathcal{A}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{M}^{-1}\left[-\frac{1}{s^{\alpha}}\,\mathcal{M}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{Y}^{-1}\left[-s^{\alpha}\,\mathcal{Y}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathfrak{E}^{-1}\left[-s^{\alpha}\,\mathfrak{E}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{S}^{-1}\left[-s^{\alpha}\,\mathcal{S}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathfrak{G}^{-1}\left[-\frac{\varsigma^{\alpha}}{s^{\alpha}}\,\mathfrak{G}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{N}^{-1}\left[-\frac{\varsigma^{\alpha}}{s^{\alpha}}\,\mathcal{N}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \end{cases} \tag{32}$$

Similarly, using CFFD, iterative terms for ADM under different transformations, are given in equation (35)

$$\nu_{k+1}(\varkappa,\theta) = \mathcal{L}^{-1}\left[-\frac{1-\alpha}{\mathbf{N}(\alpha)s}\left(s+\frac{\alpha}{1-\alpha}\right)\mathcal{L}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right] \quad , \qquad k \geq 0, \tag{33}$$

and

$$\nu_{k+1}(\varkappa,\theta) = \mathcal{L}^{-1}\left[-\frac{s^{\alpha}(1-\alpha)+\alpha}{\mathbf{AB}(\alpha)s^{\alpha}}\,\mathcal{L}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right] \quad , \qquad k \geq 0. \tag{34}$$

$$\begin{cases} \nu_{k+1}(\varkappa,\theta) = \mathcal{L}^{-1}\left[-\frac{1-\alpha}{\mathbf{N}(\alpha)s}\left(s+\frac{\alpha}{1-\alpha}\right)\mathcal{L}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{A}^{-1}\left[-\frac{1-\alpha}{\mathbf{N}(\alpha)s}\left(s+\frac{\alpha}{1-\alpha}\right)\mathcal{A}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{M}^{-1}\left[-\frac{1-\alpha}{\mathbf{N}(\alpha)s}\left(s+\frac{\alpha}{1-\alpha}\right)\mathcal{M}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{Y}^{-1}\left[-\frac{1+\alpha(s-1)}{\mathbf{N}(\alpha)}\,\mathcal{Y}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathfrak{E}^{-1}\left[-\frac{1+\alpha(s-1)}{\mathbf{N}(\alpha)}\,\mathfrak{E}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{S}^{-1}\left[-\frac{1+\alpha(s-1)}{\mathbf{N}(\alpha)}\,\mathcal{S}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathfrak{G}^{-1}\left[-\frac{(1-\alpha)\varsigma}{s\mathbf{N}(\alpha)}\left(\frac{s}{\varsigma}+\frac{\alpha}{1-\alpha}\right)\mathfrak{G}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{N}^{-1}\left[-\frac{(1-\alpha)\varsigma}{s\mathbf{N}(\alpha)}\left(\frac{s}{\varsigma}+\frac{\alpha}{1-\alpha}\right)\mathcal{N}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \end{cases} \tag{35}$$

Similarly, using ABFD, iterative terms for ADM under different transformations, are given in equations (36)

$$\begin{cases} \nu_{k+1}(\varkappa,\theta) = \mathcal{L}^{-1}\left[-\frac{s^\alpha(1-\alpha)+\alpha}{\mathbf{AB}(\alpha)s^\alpha}\,\mathcal{L}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{A}^{-1}\left[-\frac{s^\alpha(1-\alpha)+\alpha}{\mathbf{AB}(\alpha)s^\alpha}\,\mathcal{A}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{M}^{-1}\left[-\frac{s^\alpha(1-\alpha)+\alpha}{\mathbf{AB}(\alpha)s^\alpha}\,\mathcal{M}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{Y}^{-1}\left[-\frac{\alpha(s^\alpha-1)+1}{\mathbf{AB}(\alpha)}\,\mathcal{Y}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathfrak{E}^{-1}\left[-\frac{\alpha(s^\alpha-1)+1}{\mathbf{AB}(\alpha)}\,\mathfrak{E}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{S}^{-1}\left[-\frac{\alpha(s^\alpha-1)+1}{\mathbf{AB}(\alpha)}\,\mathcal{S}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathfrak{S}^{-1}\left[-\frac{s^\alpha(1-\alpha)+\frac{\alpha}{\varsigma^\alpha}}{\mathbf{AB}(\alpha)s^\alpha}\,\mathfrak{S}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \\ \nu_{k+1}(\varkappa,\theta) = \mathcal{N}^{-1}\left[-\frac{s^\alpha(1-\alpha)+\frac{\alpha}{\varsigma^\alpha}}{\mathbf{AB}(\alpha)s^\alpha}\,\mathcal{N}\left[F\nu_k(\varkappa,\theta)+P_k\right]\right]. \end{cases} \tag{36}$$

If $f(\varkappa,\theta)=0$ then $\nu_0(\varkappa,\theta)$ is calculated as

$$\nu_0(\varkappa,\theta) = \mathcal{L}^{-1}\left[\frac{\nu(\varkappa,0)}{s}\right]. \tag{37}$$

Similarly, equation (38) , for $\nu_0(\varkappa,\theta)$ under different transforms.

$$\begin{cases} \nu_0(\varkappa,\theta) = \mathcal{L}^{-1}\left[\frac{\nu(\varkappa,0)}{s}\right]. \\ \nu_0(\varkappa,\theta) = \mathfrak{E}^{-1}\left[s^2\nu(\varkappa,0)\ \right]. \\ \nu_0(\varkappa,\theta) = \mathcal{A}^{-1}\left[\frac{\nu(\varkappa,0)}{s^2}\right]. \\ \nu_0(\varkappa,\theta) = \mathcal{S}^{-1}\left[\ \nu(\varkappa,0)\ \right]. \\ \nu_0(\varkappa,\theta) = \mathcal{Y}^{-1}\left[s\,\nu(\varkappa,0)\ \right]. \\ \nu_0(\varkappa,\theta) = \mathcal{M}^{-1}\left[\ s\,\nu(\varkappa,0)\ \right]. \\ \nu_0(\varkappa,\theta) = \mathcal{N}^{-1}\left[\frac{\nu(\varkappa,0)}{s}\right]. \\ \nu_0(\varkappa,\theta) = \mathfrak{S}^{-1}\left[\nu(\varkappa,0)\,\frac{\varsigma}{s}\right]. \end{cases} \tag{38}$$

The term-wise iterative solution of ADM with different transforms for the equations (32, 35, and 36) are represented as

$$\nu(\varkappa,\theta) = \sum_{n=0}^{\infty}\nu_n(\varkappa,\theta). \tag{39}$$

4. Problem Solution

$$D_\tau^\alpha \mu(\chi,\tau) - \frac{d^2\mu}{d\chi^2} + \frac{d\mu}{d\chi} - \mu - \mu\left(\frac{d^2\mu}{d\chi^2}\right) + \mu^2 = 0; \qquad \mu(\chi,0) = \mathrm{e}^\chi. \tag{40}$$

Having an exact solution

$$\mu(\chi,\tau) = \mathrm{e}^{\chi+\tau}. \tag{41}$$

4.1. Solving under Caputo Fractional Differential Operator

$$\mu_0(\chi,\tau) = \mathrm{e}^\chi. \tag{42}$$

$$\mu_1(\chi,\tau) = \frac{\mathrm{e}^\chi \tau^\alpha}{\Gamma(1+\alpha)}. \tag{43}$$

$$\mu_2(\chi,\tau) = \frac{\mathrm{e}^\chi (\tau^\alpha)^2}{\Gamma(1+2\alpha)}. \tag{44}$$

$$\mu_3(\chi,\tau) = \frac{\mathrm{e}^\chi (\tau^\alpha)^3}{\Gamma(1+3\alpha)}. \tag{45}$$

$$\mu_4(\chi,\tau) = \frac{\mathrm{e}^\chi (\tau^\alpha)^4}{\Gamma(1+4\alpha)}. \tag{46}$$

$$\mu(\chi,\tau) = \mathrm{e}^\chi + \frac{\mathrm{e}^\chi \tau^\alpha}{\Gamma(1+\alpha)} + \frac{\mathrm{e}^\chi (\tau^\alpha)^2}{\Gamma(1+2\alpha)} + \frac{\mathrm{e}^\chi (\tau^\alpha)^3}{\Gamma(1+3\alpha)} + \frac{\mathrm{e}^\chi (\tau^\alpha)^4}{\Gamma(1+4\alpha)} + \cdots. \tag{47}$$

Taking $\alpha = 1$, implies

$$\mu(\chi,\tau) = \mathrm{e}^\chi + \mathrm{e}^\chi\tau + \frac{\mathrm{e}^\chi\tau^2}{2} + \frac{\mathrm{e}^\chi\tau^3}{6} + \frac{\mathrm{e}^\chi\tau^4}{24} + \frac{\mathrm{e}^\chi\tau^5}{120} + \cdots. \tag{48}$$

Clearly Equation (48) is converging to the exact solution.

4.2. Solving under CF Fractional Differential Operator

$$\mu_0(\chi,\tau) = \mathrm{e}^\chi. \tag{49}$$

$$\mu_1(\chi,\tau) = \frac{\mathrm{e}^\chi (\alpha(-1+\tau)+1)}{M(\alpha)}. \tag{50}$$

$$\mu_2(\chi,\tau) = \frac{\left(2 + \alpha^2\left(\tau^2 - 4\tau + 2\right) + (4\tau - 4)\alpha\right)\mathrm{e}^\chi}{2M(\alpha)^2}. \tag{51}$$

$$\mu_3(\chi,\tau) = \frac{e^{\chi}}{6M(\alpha)^3}\left(\alpha^3\left(\tau^3 - 9\tau^2 + 18\tau - 6\right) + 9\alpha^2\left(\tau^2 - 4\tau + 2\right) + 18\alpha(-1+\tau) + 6\right). \tag{52}$$

$$\mu_4(\chi,\tau) = \frac{e^{\chi}}{24M(\alpha)^4}\left(\alpha^4\left(\tau^4 - 16\tau^3 + 72\tau^2 - 96\tau + 24\right) + 16\alpha^3\left(\tau^3 - 9\tau^2 + 18\tau - 6\right) + 72\alpha^2\left(\tau^2 - 4\tau + 2\right) + 96\alpha(-1+\tau) + 24\right). \tag{53}$$

$$\mu(\chi,\tau) = e^{\chi} + \frac{e^{\chi}\left(\alpha(-1+\tau)+1\right)}{M(\alpha)} + \frac{\left(2 + \alpha^2\left(\tau^2 - 4\tau + 2\right) + (4\tau - 4)\alpha\right)e^{\chi}}{2M(\alpha)^2} + \cdots. \tag{54}$$

By putting $\alpha = 1$ Equation(54) implies

$$\mu(\chi,\tau) = e^{\chi} + e^{\chi}\tau + \frac{\tau^2 e^{\chi}}{2} + \frac{e^{\chi}\tau^3}{6} + \frac{e^{\chi}\tau^4}{24} + \frac{e^{\chi}\tau^5}{120} + \cdots. \tag{55}$$

Clearly Equation (55) is converging to the exact solution.

4.3. Solving under ABC Fractional Differential Operator

$$\mu_0(\chi,\tau) = e^{\chi}. \tag{56}$$

$$\mu_1(\chi,\tau) = -\frac{e^{\chi}\left(-1 + \alpha - \frac{\alpha\tau^{\alpha}}{\Gamma(1+\alpha)}\right)}{AB(\alpha)}. \tag{57}$$

$$\mu_2(\chi,\tau) = -\frac{e^{\chi}\left(-\frac{\alpha^2\tau^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{2\tau^{\alpha}\alpha(-1+\alpha)}{\Gamma(1+\alpha)} - 1 + 2\alpha - \alpha^2\right)}{AB(\alpha)^2}. \tag{58}$$

$$\mu_3(\chi,\tau) = -\frac{e^{\chi}\left(-\frac{\alpha^3\tau^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{3\tau^{\alpha}\alpha(-1+\alpha)^2}{\Gamma(1+\alpha)} + \frac{3\tau^{2\alpha}\alpha^2(-1+\alpha)}{\Gamma(1+2\alpha)} - 1 - 3\alpha^2 + 3\alpha + \alpha^3\right)}{AB(\alpha)^3}. \tag{59}$$

$$\mu_4(\chi,\tau) = -\frac{e^{\chi}\left(-1 + 4\alpha^3 - \alpha^4 - \frac{\alpha^4\tau^{4\alpha}}{\Gamma(1+4\alpha)} + 4\alpha - 6\alpha^2 + \frac{4\tau^{\alpha}\alpha(-1+\alpha)^3}{\Gamma(1+\alpha)} - \frac{6\tau^{2\alpha}\alpha^2(-1+\alpha)^2}{\Gamma(1+2\alpha)} + \frac{4\tau^{3\alpha}\alpha^3(-1+\alpha)}{\Gamma(1+3\alpha)}\right)}{AB(\alpha)^4}. \tag{60}$$

$$\mu(\chi,\tau) = e^{\chi} - \frac{e^{\chi}\left(-1 + \alpha - \frac{\alpha\tau^{\alpha}}{\Gamma(1+\alpha)}\right)}{AB(\alpha)} - \frac{e^{\chi}\left(-1 + 2\alpha - \alpha^2 - \frac{\alpha^2\tau^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{2\tau^{\alpha}\alpha(-1+\alpha)}{\Gamma(1+\alpha)}\right)}{AB(\alpha)^2} + \cdots. \tag{61}$$

By putting $\alpha = 1$ Equation(61) implies

$$\mu(\chi,\tau) = \mathrm{e}^{\chi} + \mathrm{e}^{\chi}\tau + \frac{\mathrm{e}^{\chi}\tau^2}{2} + \frac{\mathrm{e}^{\chi}\tau^3}{6} + \frac{\mathrm{e}^{\chi}\tau^4}{24} + \frac{\mathrm{e}^{\chi}\tau^5}{120} + \cdots . \quad (62)$$

Clearly Equation (62) is converging to the exact solution.

Here, in figure 1, we present 2D graphs of our solution.

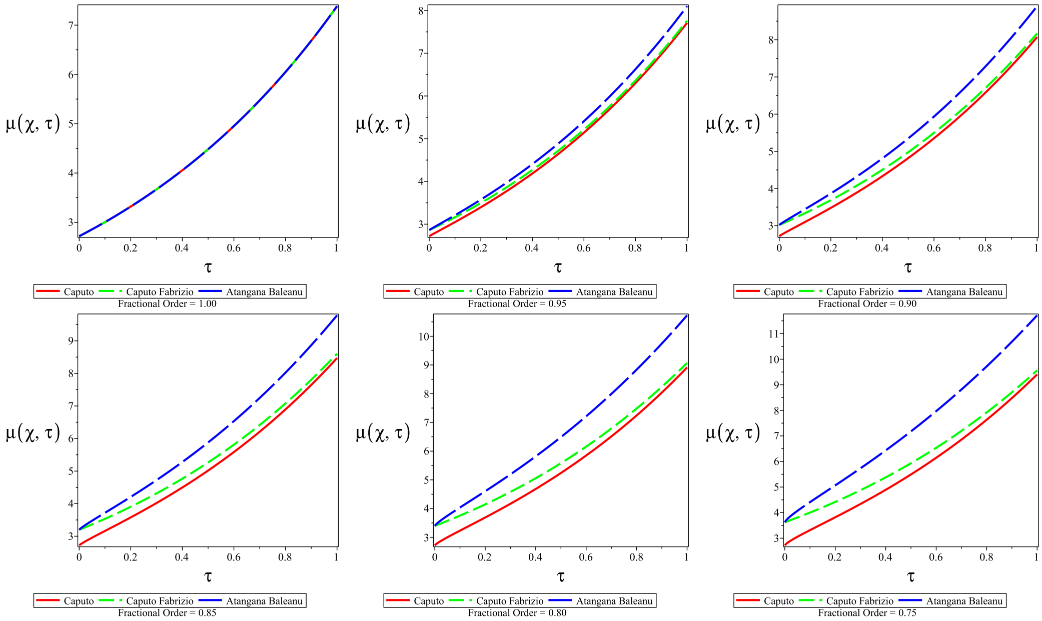


Figure 1: 2D graphs of solutions at different fractional orders.

The error minimization procedure has presented in figure 2.

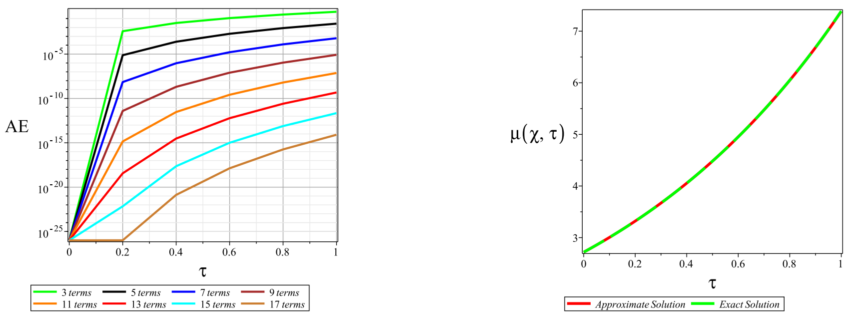
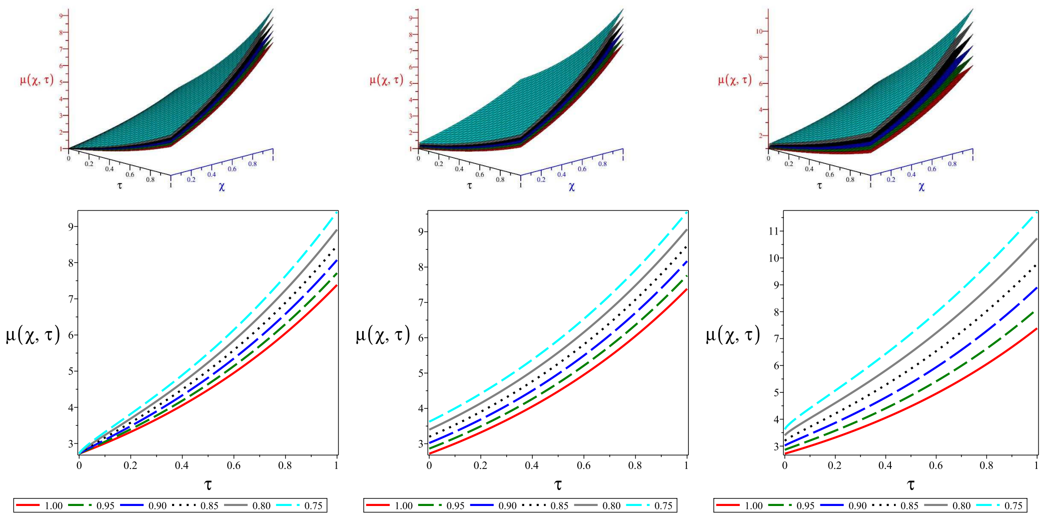


Figure 2: The error is minimized as the number of solution terms increases.

We have presented graphical illustrations of solutions for various operators in figure 3.



(a) **(b)** **(c)**

Figure 3: Solution graphs under different differential operators: **(a)**: CFD **(b)**: CFFD **(c)**: ABFD .

Fractional Order	τ	CFD Operator	CFFD Operator	ABFD Operator
1.00	0.10	3.004166021	3.004166021	3.004166021
	0.20	3.320116674	3.320116674	3.320116674
	0.30	3.669293792	3.669293792	3.669293792
	0.40	4.055183573	4.055183573	4.055183573
	0.50	4.481625588	4.481625588	4.481625588
0.95	0.10	3.049062427	3.162276707	3.209534249
	0.20	3.395391468	3.494836859	3.574059501
	0.30	3.772474744	3.862326090	3.970883434
	0.40	4.185951335	4.268359634	4.405859090
	0.50	4.640664007	4.716892562	4.883957994
0.90	0.10	3.101148306	3.337913899	3.445629081
	0.20	3.480848024	3.688814956	3.867192329
	0.30	3.888474328	4.076407078	4.319300743
	0.40	4.332160489	4.504383671	4.810635889
	0.50	4.817860678	4.976753819	5.347337632
0.85	0.10	3.161737693	3.534026565	3.719068724
	0.20	3.578250433	3.905096972	4.207468721
	0.30	4.019492351	4.314587569	4.723204464
	0.40	4.496450273	4.766204929	5.278274854
	0.50	5.016292557	5.263940139	5.879974675
0.80	0.10	3.232462725	3.754097180	4.037868933
	0.20	3.689797588	4.147155788	4.603938816
	0.30	4.168268841	4.580217667	5.191702098
	0.40	4.682091666	5.056930340	5.817070731
	0.50	5.239738787	5.581190749	6.488496318
0.75	0.10	3.315379658	4.002197630	4.411458682
	0.20	3.818266181	4.418910210	5.066371531
	0.30	4.338252697	4.876937276	5.733580680
	0.40	4.893167590	5.379793868	6.433753129
	0.50	5.492854752	5.931207911	7.176619194

Table 2: Approximate solutions under different differential operators at $\chi = 1$.

In table 2, we have displayed the approximate solutions corresponding to various differential operators of fractional orders.

5. Conclusions

This study conducts an analytic investigation of a non-linear FPDE utilizing various derivative operators and transformations. A clearly stated general methodology is developed for fractional operators under transformations in the context of solving FPDEs. The study concludes that the ADM, when applied with various transforms, is an effective tool for solving both linear and non-linear FPDEs, demonstrating high accuracy and reduced labor effort. Furthermore, the data presented in the tables and plots indicate that the solutions to the FPDEs using the Caputo, Caputo-Fabrizio, and Atangana-Baleanu fractional derivatives at an integer level are equivalent and converge to the exact solution. Furthermore, Plots indicate that absolute errors diminish as the iterations of ADM with various transformations increase. 2D & 3D-Plots show that fractional order solutions under different operators are converging to integer order. The ADM series solutions are the same under different transforms. Our research indicates that ADM with various transformations offers a straightforward and precise solution to FPDEs involving different operators.

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