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Analytical and Numerical Investigation of a Fractional-Order 4D Chaotic System via Caputo Fractional Derivative

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Abstract. This work applies two efficient techniques to investigate fractional-order systems: the Residual Power Series Method (RPSM) and the Caputo fractional derivative (CFD). Both are utilized, especially to evaluate chaotic behavior and investigate the complex dynamics associated with a four-dimensional fractional-order chaotic system. Reliable and efficient numerical simulations with the complexity of chaotic behavior are obtained using the CFD approach. Furthermore, derived analytical solutions of the fractional-order 4D system using the RPSM are obtained. This approach is highly preferred as it can handle several beginning conditions, is computationally efficient, and is numerically stable, hence producing rather precise results. Combining RPSM's analytical capacity with CFD's strong numerical accuracy offers a comprehensive understanding of the system dynamics. Although RPSM is best for stability and simplicity of use, the CFD technique is especially helpful because of its great accuracy in predicting chaotic behavior. The suggested methods precisely define their dynamics, provide exact solutions, and clearly identify chaotic attractors. For instance, representative parameter values used in our analysis are ($\nu = 0.95, 0.99, 1$). These results show how well CFD- and RPSM-based methods represent and solve challenging problems in engineering and scientific research.

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Key Words and Phrases: Fractional Derivatives, Caputo fractional derivative, Residual Power Series Method, Chaos, Simulation

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1. Introduction

In recent decades, fractional calculus has been an excellent generalization of traditional integer-order differentiation and integration and has been shown to be a more realistic description of advanced dynamic phenomena than other traditional descriptions [1–6]. It is an extremely flexible and realistic mathematical tool for the representation of many physical, biological, and engineering phenomena [7–10]. Its performance is stronger in long-term memory and dependency systems, where traditional processes fail [11–14].

Various definitions of fractional derivatives exist, such as those proposed by Riemann-Liouville, Caputo, Caputo-Fabrizio, and Atangana-Baleanu, which offer different benefits regarding mathematical precision, ease of calculation, and applicability in general to many physical systems [15–18]. The clever command of the aforementioned formulations is the secret to their use in the fields of physics, engineering, and biology, where the incorporation of fractional calculus is the underlying addition to precision and analytical competence [19–21]. As classical differentiation finds it difficult to deal with the irregular and infinitely detailed characters of fractals, fractal derivatives provide a better mathematical instrument to explain their complicated structures and dynamic behaviors.

The power of fractional-order systems to integrate long-term memory and spatial heterogeneity into mathematical models makes them a crucial tool for studying and managing chaos. This is especially crucial in fields like economics, biology, engineering, and physics, where system behavior frequently demonstrates historical dependence and fractal features. Fractional derivatives allow for richer dynamical behavior, such as delayed reactions, ongoing correlations, and more seamless transitions between periodic and chaotic regimes, by permitting the order of the system to be either constant or variable. Therefore, a key component in improving our comprehension of complicated systems is fractional calculus.

The Caputo derivative offers a realistic representation of physical systems with memory effects and is therefore particularly well-adapted to represent real-world dynamics. Its non-singular kernel can simulate processes with exponential decay behavior efficiently. The Caputo-Fabrizio derivative is renowned for addressing nonlinear differential equations efficiently, reducing the computational complexity and the time taken to find the solution at times. Its solid theoretical foundation and applicability are the keys to popularizing positive applications of fractional calculus and therefore play a more important role in an increasingly broad area of science and engineering [22–25] .

Since Rossler has been trailblazing on hyperchaotic systems, there has been significant advancement to identify multiple applications. Significance includes 2D systems in image encryption, pseudo-random number generation, and secure communications [26], and [27] operated in 3D in secure transmission and multi-image encryption, while [28, 29] took it to the 4D with dynamic analysis and synchronization added. Hyperchaotic systems have been widely applied in information processing, electronics, neuroscience, and secure communications, including the encryption of images, audio files, and videos, and also in the generation of random numbers [30–33].

Fractional-order systems give a better description of complex dynamics, capturing memory effects and long-term dependencies beyond the reach of traditional models. Fractional system solutions improve the simulation accuracy in most fields, and comparative studies [34, 35] establish the advantages and limitations of different approaches to enhance more efficient modeling techniques. Further, chaos analysis of fractional systems is crucial for safe communication, encryption, and control systems. Modern research on hyperchaotic systems, which are sensitive and have complicated behavior, is important for progressing secure data transmission, cryptography, and signal processing. It is also necessary to study these systems to advance science and technology [36–39].

Fu et al. [40] introduced the following 4-dimensional hyperchaotic system:

$$\begin{cases} \dfrac{dx_1}{dt} = a(x_2 - x_1) + x_4, \\ \dfrac{dx_2}{dt} = cx_2 - 10x_1x_3, \\ \dfrac{dx_3}{dt} = -bx_3 + 10x_1x_2, \\ \dfrac{dx_4}{dt} = dx_2 + x_1^2, \end{cases} \tag{1}$$

where a, b, and c are constants, $d > 0$ is a variable parameter, and x_1, x_2, x_3, and x_4 are driving variables.

Fractional hyperchaotic systems utilizing the Caputo derivative are a generalization of conventional hyperchaotic systems, described by incorporating fractional-order dynamics described by:

$$\begin{cases} {}^{C}D^{\nu}_{0,t}x_1 = a(x_2 - x_1) + x_4, \\ {}^{C}D^{\nu}_{0,t}x_2 = cx_2 - 10x_1x_3, \\ {}^{C}D^{\nu}_{0,t}x_3 = -bx_3 + 10x_1x_2, \\ {}^{C}D^{\nu}_{0,t}x_4 = dx_2 + x_1^2, \end{cases} \tag{2}$$

where ${}^{C}D^{\nu}_{0,t}$ is the Caputo fractional derivative of order ν, while a, b, and c are constants, $d > 0$ is a variable parameter, and (x_1), (x_2), (x_3), and (x_4) are driving variables.

Despite extensive studies on chaotic dynamics in systems characterized by fractional orders, there is a lack of appropriate approaches that provide both numerical accuracy and analytical stability. Many existing approaches are either just numerical simulations lacking theoretical insight or exclusively analytical approximations that may not sufficiently represent chaotic behavior. This work fills this gap by integrating the merits of both RPSM and CFD to provide a well-balanced technique for the analysis of fractional-order chaotic systems. Moreover, previous research has concentrated much on low-dimensional systems, while the present study investigates higher-dimensional studies of the 4D fractional-order system to provide a more precise and realistic approximation of chaotic dynamics. The results demonstrate that CFD- and RPSM-based methods have a significant potential for enhancing modeling and solving complicated dynamical systems, indicating their applicability in various scientific and engineering fields [41, 42].

This article presents a novel approach, which combines the Caputo fractional derivative (CFD) technique with the residual power series method (RPSM), to investigate the complex dynamics within a fractional-order 4D system. Fractional-order chaotic systems have been studied individually through numerical or analytical methods in the literature before; however, combined usage of CFD and RPSM in this article presents a more profound and precise insight into chaotic dynamics. The CFD approach guarantees numerical simulations of high accuracy exactly with numerical certainty, elegantly approximating complicated chaotic dynamics, while the RPSM guarantees computationally efficient and stable means of determining analytical solutions. The combination enhances the reliability and accuracy of fractional-order chaotic systems, marking a significant advancement in the field. The paper demonstrates how the two approaches complement each other in their ways to close the gap between numerical and analytical methods to fractional-order chaos.

2. Mathematical Preliminaries

Definition 1[43]: The Riemann–Liouville (R-L) fractional integral of a function $Q : \mathbb{R}^+ \to \mathbb{R}$ of order $\nu > 0$ is characterized as follows:

$$ {}_0J_t^{\nu}(Q(t)) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\mu)^{\nu-1} Q(\mu) d\mu, \quad t > 0, $$

where, $\Gamma(\nu)$ represents the Euler gamma function.

Definition 2[43]: The R–L fractional differential operator applied to $Q(t)$ of order $\nu > 0$ is written as:

$$ {}^{RL}D_{0,t}^{\nu}(Q(t)) = \frac{1}{\Gamma(m-\nu)} \frac{d^m}{dt^m} \int_0^t (t-\mu)^{m-\nu-1} Q(\mu) d\mu, \quad t > 0, $$

where, $m-1 < \nu < m,\ and\, m \in \mathbb{N}$. In this definition, the associated kernel may exhibit a singularity at one endpoint of the integration interval.

Definition 3[43]: The fractional derivative of a function $Q(t)$ is defined according to the Caputo approach of a specific order $\nu > 0$ is defined as:

$$ {}^{C}D_{0,t}^{\nu}(Q(t)) = \frac{1}{\Gamma(m-\nu)} \int_0^t (t-\mu)^{m-\nu-1} Q^{(m)}(\mu) d\mu, \quad t > 0, $$

where, $m-1 < \nu \leq m,\ and\, m \in \mathbb{N}$.

This Caputo definition is tailored to tackle the difficulties associated with substituting fractional-order initial conditions using the R-L fractional differential operator in practical applications. This is accomplished by implementing standard integer-order initial conditions, thus removing the singular kernel found in the R-L definition.

Definition 4: [44] A power series representation of the type

$$ \sum_{m=0}^{\infty} d_m (t-t_0)^{m\nu} = d_0 + d_1 (t-t_0)^{\nu} + d_2 (t-t_0)^{2\nu} + ..., \tag{3} $$

where, $m-1<\nu\leq m$ and $t\geq t_0$ is called fractional PS about t_0, where t denotes a variable and d_m represents constants known as the coefficients of the series.

Theorem 1: [44] Assume that the function Q possesses a fractional PS representation at $t=t_0$ characterized by the following form:

$$Q(t)=\sum_{m=0}^{\infty} d_m(t-t_0)^{m\nu}, \tag{4}$$

where, $m-1<\nu\leq m$ and $t_0\leq t<t_0+R$. If the derivatives $D^{m\nu}Q(t)$ are continuous in the interval (t_0,t_0+R), where $m=0,1,2,...$, then the coefficients d_m of Eq. (4) can be determined using the following formula:

$$d_m=\frac{D^{m\nu}Q(t_0)}{\Gamma(m\nu+1)}, \tag{5}$$

where, $D^{m\nu}=D^{\nu}D^{\nu}\cdots D^{\nu}$ (m-times) and R denotes the radius of convergence.

Definition 5: [44] For $m-1<\nu\leq m$, a power series representation of the type

$$\sum_{m=0}^{\infty} Q_m(x)(t-t_0)^{m\nu}=Q_0(x)+Q_1(x)(t-t_0)^{\nu}+Q_2(x)(t-t_0)^{2\nu}+..., \tag{6}$$

this is referred to as a multiple fractional PS centered at $t=t_0$, where t represents a variable and $Q_m(x)$ denotes the functions of x which are known as the coefficients of the series.

3. Numerical Scheme for the Fractional-Order 4D System with Caputo Derivative

In this part, we present an approximate solution for the model under the fractional Caputo–Fabrizio operator employing a highly effective numerical approach [45] . The governing equations are given as:

$$^{*}D^{\nu}_{0,t}Q(t)=F(Q(t)),\quad t\in[0,a],\quad Q(0)=Q_0, \tag{7}$$

where, $Q=(x_1,y,z)\in\mathbb{R}^3_+$, and the function $F(Q)$ meets the Lipschitz condition:

$$\|F(Q_1(t))-F(Q_2(t))\|\leq K\|Q_1(t)-Q_2(t)\|, \tag{8}$$

with $K>0$ as a Lipschitz constant. From Equation (7), we can express the solution as:

$$Q(t)=Q_0+{}_0J^{\nu}_tF(Q(t)),\quad t\in[0,a], \tag{9}$$

where, ${}_0J^{\nu}_t$ is the fractional integral operator derived from the Caputo operators.

Consider the interval $[0,a]$ divided into n equal parts with step length $\Delta t=\frac{a}{n}$ (e.g., $\Delta t=0.05$ in simulations). Let x_{1_r} be the approximation of $x_1(t)$ in $t=t_r$, where $r=$

$0, 1, \ldots, n$. Using finite differences, the numerical scheme for Equation (7) is formulated as:

$$c_{x_{1_{r+1}}} = x_{1_0} + \frac{(\Delta t)^{\omega}}{\Gamma(\omega+1)} \sum_{k=0}^{r} \left[(1+r-k)^{\omega} - (r-k)^{\omega}\right] F(x_{1_k}) + \mathcal{O}(\Delta t^2), \tag{10}$$

Where, $\mathcal{O}(\Delta t^2)$ is the order of approximation accuracy.

4. Impact of Fractional Order on the Dynamics of Chaotic Systems via CFD

In this part, we analyze how the parameter ν influences the behavior of the system, as illustrated in the figures 1, 2, and 3. Using the Caputo fractional derivative, plots show how the system changes as the fractional order parameter α takes on different values, showing that it does have an effect on how the system behaves. From Figure 1, for $\nu = 1$, the system provides classical chaotic dynamics for integer-order derivatives with densely structured trajectories in the phase space and the states (x_1, x_2, x_3) exhibiting strong oscillation activity but no fading out in the time series. Lowering ν to 0.99 in Figure 2; adds a fractional-order component, hence lowering the system's chaos with more organized phase space trajectories. The time series plots have little damping of the oscillations, showing early energy dissipation. An even smaller attractor is formed by further lowering alpha to 0.95 in Figure 3; this shows a significant reduction in chaotic behavior. Where the oscillations of x_1, x_2, x_3 decrease with time significantly, showing a trend towards stability, the damping is evident even in the time-series plots. Overall, the plots show that lowering ν shifts the system from a state characterized by classical chaos to a less chaotic and stable one. This shows that fractional derivatives can be used to stabilize a system and its behavior. This analysis is important for explaining flow behavior from idealized stability to actual turbulence in the real world and for controlling the management of systems in engineering applications.

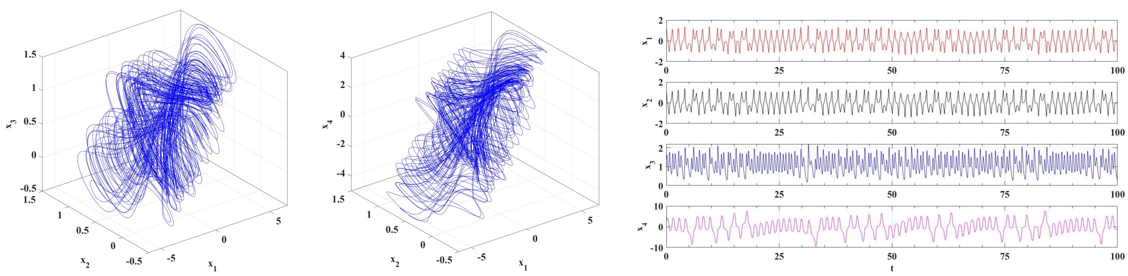

Figure 1: System dynamics for $\nu = 1$ employing the Caputo fractional derivative.

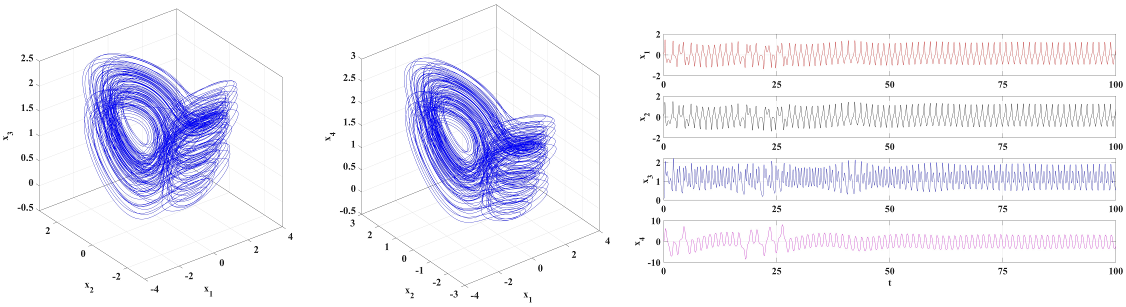

Figure 2: System dynamics for $\nu = 0.99$ employing the Caputo fractional derivative.

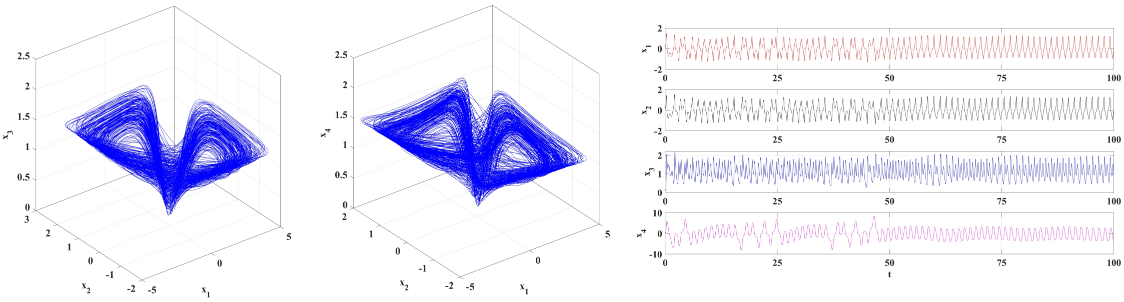

Figure 3: System dynamics for $\nu = 0.95$ using the Caputo fractional derivative.

The numerical solutions displayed in Tables 1, 2, and 3 demonstrate the behavior of the system 2 for various fractional orders ($\nu = 1,\ 0.99,\ 0.95$) in $t = 2$. When $\nu = 1$, the system functions as an integer-order differential equation, and the CFD method produces solutions that converge to the 4th-order Runge-Kutta (RK4) reference values with decreasing step size h. As ν decreases to 0.99 and 0.95, the influence of fractional order effects results in memory dependence, causing a minor reduction in the values of x_1, x_2, x_3, x_4. The CFD method exhibits consistent convergence in all cases, with solutions aligning closely with the Adams-Bashforth-Moulton method (ABM) reference values for fractional orders. The findings show that a decrease in h is directly proportional to increased accuracy, which supports the consistency of CFD in solving fractional order systems. The nature of solutions with decreasing ν represents the effect of fractional dynamics, which retards the evolution of the system relative to the classical case. The study demonstrates the efficiency of CFD in the management of systems of integer and fractional order with high precision.

Table 1: Solutions of system 2 by using CFD for $(x_{10}, x_{20}, x_{30}, x_{40}) = (1, 1, 1, 1)$ where $\nu = 1$, and $t = 2$

h	x_1	x_2	x_3	x_4
1/320	0.973059863422542	0.800578111799992	1.591562295686195	2.536897475278398
1/640	0.970285557191089	0.796036503750927	1.593731307460380	2.546833743002026
1/1280	0.969355303143915	0.794508406142886	1.594465238932689	2.550156263728269
1/2560	0.968888986811479	0.793741513533585	1.594834214830244	2.551819454601515
1/5120	0.968608795575532	0.793280445317936	1.595056238342953	2.552817983348334
1/10240	0.967953794109018	0.792201844098075	1.595576140721881	2.555149666256854
RK4	0.967484855614437	0.791428991073163	1.595949065222047	2.556816674404389

Table 2: Solutions of system 2 by using CFD for $(x_{10}, x_{20}, x_{30}, x_{40}) = (1, 1, 1, 1)$ where $\nu = 0.99$, and $t = 2$

h	x_1	x_2	x_3	x_4
1/320	0.958724904468643	0.784005868539537	1.586882323107994	2.552704617046767
1/640	0.957783855967246	0.782526656938586	1.587511271988065	2.555919465717640
1/1280	0.957312261656128	0.781784430216698	1.587827659031779	2.557528691550405
1/2560	0.0.957028942786839	0.781338234845598	1.588018096550532	2.558494798445208
1/5120	0.956839907974789	0.781040408427304	1.588145305517077	2.559139106587635
1/10240	0.955883492764598	0.779514856699298	1.588816353323760	2.562396671913593
ABM	0.955344465148585	0.779339094329801	1.589131949997077	2.567697787348436

Table 3: Solutions of system 2 by using CFD for $(x_{10}, x_{20}, x_{30}, x_{40}) = (1, 1, 1, 1)$ where $\nu = 0.95$, and $t = 2$

h	x_1	x_2	x_3	x_4
1/320	0.915581297407851	0.741241155418083	1.557139066272577	2.575478404944152
1/640	0.915134211844442	0.740621399693080	1.557299568921548	2.576919332084313
1/1280	0.914865705487542	0.740248850488766	1.557396449608853	2.577784401113457
1/2560	0.914686591456969	0.740000191391083	1.557461275589489	2.578361325295418
1/5120	0.913702159352431	0.738585624017066	1.557873715167465	2.581343575041200
1/10240	0.914387867668505	0.739585234231689	1.557569739937361	2.579323238417623
ABM	0.914496444818986	0.739349393578095	1.557768565532089	2.579413747913888

5. Residual power series approach for hyperchaotic system (RPSM)

The fractional hyperchaotic system is recognized as a significant universal nonlinear model that appears in various physical systems. This article aims to derive the solution for this equation employing the advantageous RPS method.

$$\begin{cases} {}^{C}D_{0,t}^{\nu}x_1 = a(x_2 - x_1) + x_4, \\ {}^{C}D_{0,t}^{\nu}x_2 = cx_2 - 10x_1x_3, \\ {}^{C}D_{0,t}^{\nu}x_3 = -bx_3 + 10x_1x_2, \\ {}^{C}D_{0,t}^{\nu}x_4 = dx_2 + x_1^2, \end{cases} \tag{11}$$

subject to the following constraints initial conditions:

$$x_1(0) = x_{10},\ x_2(0) = x_{20},\ x_3(0) = x_{30},\ x_4(0) = x_{40}. \tag{12}$$

To numerically address the fractional order 4D hyperchaotic system using the Caputo derivative, we adhere to the following procedure outlined below.
The RPS method involves representing the solutions of the Eq. (11), given the i.c outlined in Eq. (12), as a series of various fractional PS expansions centered around the starting point $t = 0$. Assume a solution expressed as a power series;

$$\begin{cases} x_1(t) = \sum_{k=0}^{\infty} a_{1k} \frac{t^{k\nu}}{\Gamma(1+k\nu)}, \\ x_2(t) = \sum_{k=0}^{\infty} a_{2k} \frac{t^{k\nu}}{\Gamma(1+k\nu)}, \\ x_3(t) = \sum_{k=0}^{\infty} a_{3k} \frac{t^{k\nu}}{\Gamma(1+k\nu)}, \\ x_4(t) = \sum_{k=0}^{\infty} a_{4k} \frac{t^{k\nu}}{\Gamma(k\nu+1)}. \end{cases} \tag{13}$$

The RPS offers analytical approximate solutions expressed as an infinite multiple fractional PS. To derive the numerical representation of the values depicted in this series, it is necessary to truncate the resulting series and follow a practical procedure to achieve this objective. So; let $x_{1n}(t),\ x_{2n}(t),\ x_{3n}(t),\ x_{4n}(t)$ to denot the n-th truncated series solution of $x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)$, in that order. That is :

$$\begin{cases} x_{1n}(t) = x_{10} + \sum_{k=0}^{\infty} a_{1k} \frac{t^{k\nu}}{\Gamma(1+k\nu)}, \\ x_{2n}(t) = x_{20} + \sum_{k=0}^{\infty} a_{2k} \frac{t^{k\nu}}{\Gamma(1+k\nu)}, \\ x_{3n}(t) = x_{30} + \sum_{k=0}^{\infty} a_{3k} \frac{t^{k\nu}}{\Gamma(1+k\nu)}, \\ x_{4n}(t) = x_{40} + \sum_{k=0}^{\infty} a_{4k} \frac{t^{k\nu}}{\Gamma(1+k\nu)}. \end{cases} \tag{14}$$

Define the residual functions for the system :

$$\begin{cases} Res\ x_1(t) =^C D^{\nu}_{0,t} x_1 - a(x_2 - x_1) - x_4, \\ Res\ x_2(t) = {}^C D^{\nu}_{0,t} x_2 - c x_2 + 10 x_1 x_3, \\ Res\ x_3(t) = {}^C D^{\nu}_{0,t} x_3 + b x_3 - 10 x_1 x_2, \\ Res\ x_4(t) = {}^C D^{\nu}_{0,t} x_4 - d x_2 - x_1^2. \end{cases} \tag{15}$$

Hence, the n-th residual functions of $x_1,\ x_2,\ x_3,\ x_4$ are :

$$\begin{cases} Res\ x_{1n}(t) =^C D^{\nu}_{0,t} x_{1n} - a(x_{2n} - x_{1n}) - x_{4n}, \\ Res\ x_{2n}(t) = {}^C D^{\nu}_{0,t} x_{2n} - c x_{2n} + 10 x_{1n} x_{3n}, \\ Res\ x_{3n}(t) = {}^C D^{\nu}_{0,t} x_{3n} + b x_{3n} - 10 x_{1n} x_{2n}, \\ Res\ x_{4n}(t) = {}^C D^{\nu}_{0,t} x_{4n} - d x_{2n} - x_{1n}^2. \end{cases} \tag{16}$$

Obviously; $Res\ x_1(t) = Res\ x_2(t) = Res\ x_3(t) = Res\ x_4(t) = 0, \quad \forall t \geq 0$. So, $\lim_{n\to\infty} Res\ x_{1n}(t) = Res\ x_1(t)$; $\lim_{n\to\infty} Res\ x_{2n}(t) = Res\ x_2(t)$; $\lim_{n\to\infty} Res\ x_{3n}(t) = Res\ x_3(t)$; $\lim_{n\to\infty} Res\ x_{4n}(t) = Res\ x_4(t)$. As, the Caputo derivative of any constant is zero, then :

$$\begin{cases} {}^C D^{(k-1)\nu}_{0,t} Res\ x_1(0) =^C D^{(k-1)\nu}_{0,t} Res\ x_{1k}(0), \\ {}^C D^{(k-1)\nu}_{0,t} Res\ x_2(0) =^C D^{(k-1)\nu}_{0,t} Res\ x_{2k}(0), \\ {}^C D^{(k-1)\nu}_{0,t} Res\ x_3(0) =^C D^{(k-1)\nu}_{0,t} Res\ x_{3k}(0), \\ {}^C D^{(k-1)\nu}_{0,t} Res\ x_4(0) =^C D^{(k-1)\nu}_{0,t} Res\ x_{4k}(0), \end{cases} \tag{17}$$

for $k = 1, ..., n$.

Now, to obtain the coefficients $a_{1k},\ a_{2k},\ a_{3k},\ a_{4k},\ k = 1, 2, 3, ..., n$, Substituting the n-truncated series of $x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)$ into (13), and then apply the Caputo operators ${}^C D^{(n-1)\nu}_{0,t}$ on $Res\ x_{1n}(t),\ Res\ x_{2n}(t),\ Res\ x_{3n}(t),\ Res\ x_{4n}(t)$ resp., we get :

$$\begin{cases} {}^C D^{(n-1)\nu}_{0,t} Res\ x_{1n}(0) = 0, \\ {}^C D^{(n-1)\nu}_{0,t} Res\ x_{2n}(0) = 0, \\ {}^C D^{(n-1)\nu}_{0,t} Res\ x_{3n}(0) = 0, \\ {}^C D^{(n-1)\nu}_{0,t} Res\ x_{4n}(0) = 0, \end{cases} \tag{18}$$

for $n = 1, 2, 3, ...$

Let us find the first few coefficients :

- For $n = 1$:

$$\begin{cases} x_1(t) = x_{10} + a_{11}\dfrac{t^\nu}{\Gamma(\nu+1)} = 1 + \dfrac{a_{11}}{\Gamma(\nu+1)}t^\nu, \\ x_2(t) = x_{20} + a_{21}\dfrac{t^\nu}{\Gamma(\nu+1)} = 1 + \dfrac{a_{21}}{\Gamma(\nu+1)}t^\nu, \\ x_3(t) = x_{30} + a_{31}\dfrac{t^\nu}{\Gamma(\nu+1)} = 1 + \dfrac{a_{31}}{\Gamma(\nu+1)}t^\nu, \\ x_4(t) = x_{40} + a_{41}\dfrac{t^\nu}{\Gamma(\nu+1)} = 1 + \dfrac{a_{41}}{\Gamma(\nu+1)}t^\nu. \end{cases} \tag{19}$$

According to (16), the first residual functions of $x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)$ are :

$$\begin{cases} Res\ x_{11}(t) = ^C D^\nu_{0,t}x_{11} - a(x_{21} - x_{11}) - x_{41}, \\ \qquad = a_{11} - 1 - \dfrac{a_{41}t^\nu}{\Gamma(\nu+1)} - a\dfrac{a_{21}t^\nu}{\Gamma(\nu+1)} + a\dfrac{a_{11}t^\nu}{\Gamma(\nu+1)}, \end{cases} \tag{20}$$

and

$$\begin{cases} Res\ x_{21}(t) = ^C D^\nu_{0,t}x_{21} - cx_{2n} + 10x_{1n}x_{3n} \\ \qquad = a_{21} - 2 + d\dfrac{a_{11}t^\nu}{\Gamma(\nu+1)} - c\dfrac{a_{21}t^\nu}{\Gamma(\nu+1)} + d\dfrac{a_{31}t^\nu}{\Gamma(\nu+1)} + d\dfrac{a_{11}a_{31}t^{2\nu}}{(\Gamma(\nu+1))^2}, \end{cases} \tag{21}$$

also

$$\begin{cases} Res\ x_{31}(t) = ^C D^\nu_{0,t}x_{31} + bx_{31} - 10x_{11}x_{21}, \\ \qquad = a_{31} - 7 + b\dfrac{a_{31}t^\nu}{\Gamma(\nu+1)} - d\dfrac{a_{21}t^\nu}{\Gamma(\nu+1)} - d\dfrac{a_{11}t^\nu}{\Gamma(\nu+1)} - d\dfrac{a_{11}a_{21}t^{2\nu}}{(\Gamma(\nu+1))^2}, \end{cases} \tag{22}$$

and

$$\begin{cases} Res\ x_{41}(t) = ^C D^\nu_{0,t}x_{41} - dx_{21} - x_{11}^2, \\ \qquad = a_{41} - 11 - d\dfrac{a_{21}t^\nu}{\Gamma(\nu+1)} - 2\dfrac{a_{11}t^\nu}{\Gamma(\nu+1)} - \dfrac{a_{11}^2t^{2\nu}}{(\Gamma(\nu+1))^2}. \end{cases} \tag{23}$$

Using (18), we get $a_{11},\ a_{21},\ a_{31},\ a_{41}$, so that the first RPS solution of Eqs. (11-12) can be articulated as :

$$\begin{cases} x_{11}(t) = 1 + \dfrac{t^\nu}{\Gamma(\nu+1)}, \\ x_{21}(t) = 1 + \dfrac{2t^\nu}{\Gamma(\nu+1)}, \\ x_{31}(t) = 1 + \dfrac{7t^\nu}{\Gamma(\nu+1)}, \\ x_{41}(t) = 1 + \dfrac{11t^\nu}{\Gamma(\nu+1)}. \end{cases} \tag{24}$$

- For $n = 2$:

$$\begin{cases} x_{12}(t) = 1 + \dfrac{t^{\nu}}{\Gamma(\nu+1)} + \dfrac{a_{12}t^{2\nu}}{\Gamma(2\nu+1)}, \\ x_{22}(t) = 1 + \dfrac{2t^{\nu}}{\Gamma(\nu+1)} + \dfrac{a_{22}t^{2\nu}}{\Gamma(2\nu+1)}, \\ x_{32}(t) = 1 + \dfrac{7t^{\nu}}{\Gamma(\nu+1)} + \dfrac{a_{32}t^{2\nu}}{\Gamma(2\nu+1)}, \\ x_{42}(t) = 1 + \dfrac{11t^{\nu}}{\Gamma(\nu+1)} + \dfrac{a_{42}t^{2\nu}}{\Gamma(2\nu+1)}, \end{cases} \tag{25}$$

and the second residual functions are :

$$\begin{cases} Res\ x_{12}(t) =^{C} D^{\nu}_{0,t}x_{12} - a(x_{22} - x_{12}) - x_{42}, \\ \qquad = (a_{22-46})\dfrac{t^{\nu}}{\Gamma(\nu+1)} - (a_{42} + a(a_{22} - a_{12})))\dfrac{t^{2\nu}}{\Gamma(2\nu+1)}, \end{cases} \tag{26}$$

and

$$\begin{cases} Res\ x_{22}(t) =^{C} D^{\nu}_{0,t}x_{22} - cx_{22} + dx_{12}x_{32} \\ \qquad = (a_{22} + 56)\dfrac{t^{\nu}}{\Gamma(\nu+1)} + (d(a_{32} + a_{12} - 12a_{22}))\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} + 70\dfrac{t^{2\nu}}{(\Gamma(\nu+1))^2} \\ \qquad + (da_{32} + 70a_{12})\dfrac{t^{3\nu}}{\Gamma(\nu+1)\Gamma(2\nu+1)} + \dfrac{da_{12}a_{32}t^{4\nu}}{(\Gamma(2\nu+1))^2}, \end{cases} \tag{27}$$

also

$$\begin{cases} Res\ x_{32}(t) =^{C} D^{\nu}_{0,t}x_{32} + bx_{32} - 10x_{12}x_{22}, \\ \qquad = (a_{32} - 9)\dfrac{t^{\nu}}{\Gamma(\nu+1)} + (ba_{32} - d(a_{22} + a_{12}))\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} - 20\dfrac{t^{2\nu}}{(\Gamma(\nu+1))^2} \\ \qquad - (da_{22} + 20a_{12})\dfrac{t^{3\nu}}{\Gamma(\nu+1)\Gamma(2\nu+1)} - da_{12}a_{22}\dfrac{t^{4\nu}}{(\Gamma(2\nu+1))^2}, \end{cases} \tag{28}$$

and

$$\begin{cases} Res\ x_{42}(t) =^{C} D^{\nu}_{0,t}x_{42} - dx_{22} - x_{12}^2, \\ \qquad = (a_{42} - 22)\dfrac{t^{\nu}}{\Gamma(\nu+1)} - (d + 2a_{12}\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} - \dfrac{t^{2\nu}}{(\Gamma(\nu+1))^2} - 2a_{12}\dfrac{t^{3\nu}}{\Gamma(\nu+1)\Gamma(2\nu+1)} \\ \qquad - \dfrac{a_{12}^2t^{4\nu}}{(\Gamma(2\nu+1))^2}. \end{cases} \tag{29}$$

Using (18), we get $a_{12},\ a_{22},\ a_{32},\ a_{42}$, therefore, the second RPS solution of Eqs. (11-12) are given by the following form :

$$\begin{cases} x_{12}(t) = 1 + \dfrac{t^{\nu}}{\Gamma(\nu+1)} + 46\dfrac{t^{2\nu}}{\Gamma(2\nu+1)}, \\ x_{22}(t) = 1 + \dfrac{2t^{\nu}}{\Gamma(\nu+1)} - 56\dfrac{t^{2\nu}}{\Gamma(2\nu+1)}, \\ x_{32}(t) = 1 + \dfrac{7t^{\nu}}{\Gamma(\nu+1)} + 9\dfrac{t^{2\nu}}{\Gamma(2\nu+1)}, \\ x_{42}(t) = 1 + \dfrac{11t^{\nu}}{\Gamma(\nu+1)} + 22\dfrac{t^{2\nu}}{\Gamma(2\nu+1)}. \end{cases} \tag{30}$$

- In the same manner, implementing the identical procedures for $n = 3$, leads to :

$$\begin{cases} x_{13}(t) = 1 + \dfrac{t^{\nu}}{\Gamma(\nu+1)} + 46\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} + a_{13}\dfrac{t^{3\nu}}{\Gamma(3\nu+1)}, \\ x_{23}(t) = 1 + \dfrac{2t^{\nu}}{\Gamma(\nu+1)} - 56\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} + a_{23}\dfrac{t^{3\nu}}{\Gamma(3\nu+1)}, \\ x_{33}(t) = 1 + \dfrac{7t^{\nu}}{\Gamma(\nu+1)} + 9\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} + a_{33}\dfrac{t^{3\nu}}{\Gamma(3\nu+1)}, \\ x_{43}(t) = 1 + \dfrac{11t^{\nu}}{\Gamma(\nu+1)} + 22\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} + a_{43}\dfrac{t^{3\nu}}{\Gamma(3\nu+1)}. \end{cases} \tag{31}$$

By the same process, we get :

$$a_{13} = 3548,\ a_{23} = -1222 - \tfrac{70\Gamma(2\nu+1)}{(\Gamma(\nu+1))^2},\ a_{33} = -127 + \tfrac{20\Gamma(2\nu+1)}{(\Gamma(\nu+1))^2},\ a_{43} = -514 + \tfrac{\Gamma(2\nu+1)}{(\Gamma(\nu+1))^2}.$$

Hence,

$$\begin{cases} x_{13}(t) = 1 + \dfrac{t^{\nu}}{\Gamma(\nu+1)} + 46\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} + 3548\dfrac{t^{3\nu}}{\Gamma(3\nu+1)}, \\ x_{23}(t) = 1 + \dfrac{2t^{\nu}}{\Gamma(\nu+1)} - 56\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} - (1222 + \dfrac{70\Gamma(2\nu+1)}{(\Gamma(\nu+1))^2})\dfrac{t^{3\nu}}{\Gamma(3\nu+1)}, \\ x_{33}(t) = 1 + \dfrac{7t^{\nu}}{\Gamma(\nu+1)} + 9\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} + (-127 + \dfrac{20\Gamma(2\nu+1)}{(\Gamma(\nu+1))^2})\dfrac{t^{3\nu}}{\Gamma(3\nu+1)}, \\ x_{43}(t) = 1 + \dfrac{11t^{\nu}}{\Gamma(\nu+1)} + 22\dfrac{t^{2\nu}}{\Gamma(2\nu+1)} + (-514 + \dfrac{\Gamma(2\nu+1)}{(\Gamma(\nu+1))^2})\dfrac{t^{3\nu}}{\Gamma(3\nu+1)}. \end{cases} \tag{32}$$

This process can be reiterated until the coefficients associated with the multiple fractional PS solutions of equations (11-12) are acquired in the desired order.

Here, the parameter values are taken as $a = 35,\ b = 3,\ c = 12$, and $d = 10$.

6. Application of the CFD and RPSM to Chaotic Dynamical System

In this section, we present numerical solutions acquired with the Residual Power Series Method (RPSM) to solve the system 2. The solutions are computed for different values of fractional order (0.95, 0.99, and 1) with homogeneous initial conditions. The objective is to investigate the effect of fractional order on the dynamics of the system and to analyze the accuracy and convergence of the RPSM method. Tables 4, 5, and 6 exhibit numerical data for the first 4 terms of the solutions, showing solutions of the system 2 employing the Residual Power Series Method (RPSM) for various fractional order values of ν (0.95, 0.99, and 1). The data show a decrease with decreasing t values and eventually converge, hence verifying stability and the correctness of the approach. Different ν values compared reveal little variance in the solutions, which result from the fractional-order parameter on system behavior. The proximity of values with diminishing step sizes further attests to the effectiveness of RPSM. These results confirm the method's efficacy in fractional-order differential systems, hence proving its relevance in mathematical modeling.

Table 4: Solutions of system (2) using RPSM for $(x_{10}, x_{20}, x_{30}, x_{40}) = (1, 1, 1, 1)$ where $\nu = 1$.

t	x_1	x_2	x_3	x_4
0.5	0.501189960570766	1.010000344410005	3.540065400660771	5.564300000813071
0.6	0.600000699400806	1.210000155410008	4.200061400760091	6.663400200643092
0.7	0.700460199106844	1.441241155418083	4.966107145270071	7.764400204643071
0.8	0.821342118443442	1.630061378692061	5.665176578821620	8.885010322064221
0.9	0.933755705487631	1.850027650460755	6.369196439800855	9.976584402223558
1.0	1.009700731466999	2.000800191391083	7.000761275589489	11.00897725296401

Table 5: Solutions of system (2) using RPSM for $(x_{10}, x_{20}, x_{30}, x_{40}) = (1, 1, 1, 1)$ where $\nu = 0.99$.

t	x_1	x_2	x_3	x_4
0.5	0.505595555300738	1.011193002380041	3.539167400662136	5.561546530022142
0.6	0.605608366300064	1.211224063300052	4.239265004152555	6.661698300312111
0.7	0.705455377406621	1.410910065307072	4.938186043162656	7.761698403922141
0.8	0.805157220074431	1.809471378646070	6.333169556721431	9.952118122074211
0.9	0.904737604366541	1.809476670465756	6.333165237606752	9.952162304313457
1.0	1.004275561446867	2.008410162371001	7.029432265578476	11.04625221527541

Table 6: Solutions of system (2) using RPSM for $(x_{10}, x_{20}, x_{30}, x_{40}) = (1, 1, 1, 1)$ where $\nu = 0.95$.

t	x_1	x_2	x_3	x_4
0.5	0.528261457200942	1.056520060720154	3.697832000373467	5.810878940022073
0.6	0.628160365000042	1.256321072320061	4.397125003153668	6.909767440722251
0.7	0.727227431023701	1.454451329793310	5.090599567821537	7.999519332062421
0.8	0.825854310237016	1.651171388672142	5.779199568921548	9.081457211094202
0.9	0.923218096487542	1.454457750267755	5.090595337506652	7.999574401223463
1.0	1.020536481457856	2.041060182371174	7.143732264476477	11.22595132227552

7. Conclusion

To investigate the intricate dynamics within the four-dimensional fractional order system, this study shows the efficiency of the residual power series method (RPSM) and the Caputo fractional derivative (CFD). The RPSM is a reliable and simple way to get analytical answers; the CFD approach is seen to provide extremely accurate numerical simulations and effectively capture chaotic behavior, and by means of these two approaches, one gains additional insight into the complex dynamics of the system. In addition to providing useful tools for modeling and solving fractional-order systems, the findings reveal that both techniques are successful in discovering chaotic attractors and closely approximating the behavior of these systems. Due to the accuracy and consistency of these methodologies, they offer tremendous potential for future usage in engineering and scientific research, particularly in areas of study where fractional-order models play a key role. In the future, we intend to solve some new fractional models, such as in [46–48] and make comparisons with other numerical methods [49–51].

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