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# L-Hop Independent Sequences in Graphs

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**Abstract.** Let $G$ be a graph. A sequence of distinct vertices $Q = (a_1, a_2, \dots, a_n)$ of $G$ is called an L-hop independent sequence if $n = 1$ or if $d_G(a_i, a_j) \neq 2$ for each $i \neq j$, where $i, j \in \{1, 2, \dots, n\}$ and $N_G[a_s] \setminus \bigcup_{t=1}^{s-1} N_G(a_t) \neq \varnothing$ for each $s \in \{2, \dots, n\}$. The L-hop independence number of $G$, denoted by $\alpha_{Lh}(G)$, is the maximum length among all L-hop independent sequences in $G$. This study explores and characterizes the L-hop independent sequences in some graphs, and in the join of two graphs. Some formulas and bounds of L-hop independence number with respect to the order of a graph and other parameters in graph theory are derived. Moreover, some relationships of L-hop independence with hop independence and legal hop independence are established.

**2020 Mathematics Subject Classifications**: 05C69

**Key Words and Phrases**: L-sequence, clique L-sequence, clique L-Grundy dominating sequence, L-hop independent sequence, L-independence number

## 1. Introduction

Graph Theory is relatively new area of mathematics, first studied by the super famous mathematician Leonhard Euler in 1735. Since then it has blossomed into a powerful tool used in nearly every branch of science and is currently an active area of mathematics research.

One of the hottest topics in Graph Theory is the concept of independent sets in graphs. A set $S \subseteq V(G)$ is called an independent set of $G$ if no two pair of distinct vertices of $S$ are adjacent. The maximum cardinality of an independent set of $G$, denoted by $\alpha(G)$, is called the independence number of $G$ [1]. The concept of an independent set is often studied in the context of maximum independent sets, which refers to the largest possible

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independent set within a graph. This set has applications in areas like scheduling, resource allocation, and even social network analysis, where finding independent sets can represent groups of individuals or resources that do not interfere with one another.

In 2022, hop independent set in a graph and its parameter was introduced by J. Hassan et al. [2]. A set $S \subseteq V(G)$ is called a hop independent set of $G$ if $d_G(u, w) \neq 2$ for any distinct vertices $u, w \in S$. The maximum cardinality of a hop independent set of $G$, is called the hop independence number of $G$, and is denoted by $\alpha_h(G)$. They have shown that the hop independence number of a graph is always greater than or equal to the hop domination number. Moreover, they derived some bounds and formulas of hop independence numbers of some special graphs and graphs under some binary operations. Some studies related to independent sets, its variations, and other hop-related concepts can be found in [3–11].

In this paper, new variant of hop independence called L-hop independence sequence in a graph is introduced. The authors add some properties to hop independence wherein the order of choosing vertices and its neighborhoods are important. This parameter is investigated on some special graphs, and on the join of any two graphs. Some bounds and exact values are determined. Moreover, some characterizations of this newly defined sequence are presented, and used to solve the said bounds and exact values. The authors are confident that this study would lead to another interesting studies and application in the future.

## 2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. The *distance* $d_G(u, v)$ in $G$ of two vertices $u, v$ is the length of a shortest *u-v path* in $G$.

A subset $I$ of $V(G)$ is called an *independent* if for every pair of distinct vertices $x, y \in I$, $d_G(x, y) \neq 1$. The maximum cardinality of an independent set in $G$, denoted by $\alpha(G)$, is called the *independence* number of $G$. Any independent set $I$ with cardinality equal to $\alpha(G)$ is called an $\alpha$-set of $G$.

A subset $S$ of $V(G)$ is called a *hop independent* set of $G$ if $d_G(u, v) \neq 2$ for any two distinct vertices $u, v \in S$. The *hop independence number* of $G$, denoted by $\alpha_h(G)$, is the maximum cardinality of a hop independent set of $G$.

Given a graph $G$ and a sequence $S = (v_1, \ldots, v_k)$ of distinct vertices of $G$, for every $i \in \{2, 3, \cdots, k\}$ we define the set $\phi_s$ by $\phi_s(Vi) = N[V_i] \setminus \bigcup_{j=1}^{i-1} N(V_j)$. The sequence is called L-sequence if $\phi_s(V_i) \neq \varnothing$ for every $i \in \{2, 3, \cdots, k\}$.

Let $S_1 = (v_1, \ldots, v_n)$ and $S_2 = (u_1, \ldots, u_m)$ be two sequences of distinct vertices of $G$. The *concatenation* of $S_1$ and $S_2$, denoted by $S_1 \oplus S_2$, is the sequence given by $S_1 \oplus S_2 = (v_1, \ldots, v_n, u_1, \ldots, u_m)$.

A sequence $L = (w_1, \ldots, w_k)$ of distinct vertices of $G$ is called a legal hop independent sequence if $k = 1$ or $L$ is a hop independent and $N_G[w_i] \setminus \bigcup_{j=1}^{i-1} N_G[w_j] \neq \varnothing$ for every $i \in \{2, \cdots, k\}$. The maximum length of a legal hop independent sequence in $G$, denoted by $\alpha_{\ell h}(G)$, is called the legal hop independence number of $G$.

A graph is *complete* if every pair of distinct vertices are adjacent. A *complete graph* of order $n$ is denoted by $K_n$. A set $S \subseteq V(G)$ is called a *clique* in $G$ if the subgraph $\langle S \rangle$ induced by $S$ is a complete graph. The maximum size or cardinality of a clique of $G$, denoted by $\omega(G)$, is called the *clique number* of $G$.

Let $G$ and $H$ be any two graphs. The *join* $G+H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

## 3. Results

We shall now define the L-hop independent sequence and L-hop independence number of a graph as follows:

**Definition 1.** Let $G$ be a graph. A sequence of distinct vertices $Q = (a_1, a_2, \ldots, a_n)$ of $G$ is called an L-hop independent sequence if $n = 1$ or if $d_G(a_i, a_j) \neq 2$ for each $i \neq j$, where $i, j \in \{1, 2, \ldots, n\}$ and $N_G[a_s] \backslash \bigcup_{t=1}^{s-1} N_G(a_t) \neq \varnothing$ for each $s \in \{2, \ldots, n\}$. The L-hop independence number of $G$, denoted by $\alpha_{Lh}(G)$, is the maximum length among all L-hop independent sequences in $G$. Moreover, we call $\hat{Q} = \{a_1, a_2, \cdots, a_k\}$ an $L-$hop independent set of $G$.

**Example 1.** Consider the graph in Figure 1. Let $L = (a_1, a_2)$. Then $d_{P_4}(a_1, a_2) = 1$. Observe that $N_{P_4}(a_1) = \{a_2\}$ and $N_{P_4}[a_2] = \{a_1, a_2, a_3\}$. Thus,

$$N_{P_4}[a_2] \backslash N_{P_4}(a_1) = \{a_1, a_2, a_3\} \backslash \{a_2\} = \{a_1, a_3\} \neq \varnothing.$$

Therefore, $L = (a_1, a_2)$ is an L-hop independent sequence of $P_4$, and so $\alpha_{Lh}(P_4) \geq 2$. Now, since $d_{P_4}(a_1, a_3) = 2$, it follows that $\alpha_{Lh}(P_4) \neq 4$. Since

$$d_{P_4}(a_1, a_3) = 2 = d_{P_4}(a_2, a_4),$$

it follows that $L$ is a maximum L-hop independent sequence of $P_4$. Therefore, $\alpha_{Lh}(P_4) = 2$.

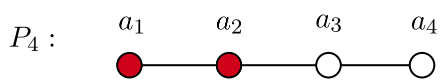



Figure 1: A path graph of order 4.

**Theorem 1.** Let $G$ be a graph. Then

i. $\alpha_{Lh}(G) \leq \alpha_h(G)$;

ii. $1 \leq \alpha_{Lh}(G) \leq |V(G)|$; and

iii. $\alpha_{Lh}(G) \geq \alpha_{\ell h}(G)$.

*Proof.* [i.] Let $G$ be a graph and let $D$ be a maximum L-hop independent sequence of $G$. Then $\alpha_{Lh}(G) = |\hat{D}|$ and $\hat{D}$ is a hop independent set of $G$, where $\hat{D}$ is a corresponding set of $D$. Since, $\alpha_h(G)$ is the maximum cardinality among all hop independent sets in $G$, it follows that $\alpha_h(G) \geq |\hat{D} = \alpha_{Lh}(G)$.

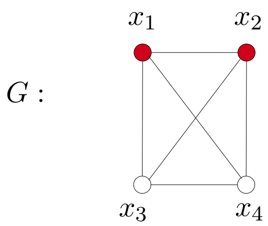
[ii.] Since any sequence $(v)$, where $v \in V(G)$, is an L-hop independent sequence of $G$, we have $\alpha_{Lh}(G) \geq 1$. Since $\alpha_h(G) \leq |V(G)|$, it follows that $\alpha_{Lh}(G) \leq |V(G)|$ by (i). Consequently, $1 \leq \alpha_{Lh}(G) \leq |V(G)|$.

[iii.] Let $S = (v_1, v_2, \ldots, v_n)$ be a maximum legal hop independent sequence in $G$. Then $N_G[v_i]\backslash \bigcup_{j=1}^{i-1} N_G[v_j] \neq \varnothing$ for each $i \in \{2, 3, \ldots, n\}$. Since $N_G(a) \subseteq N_G[a]$ for all $a \in V(G)$, it follows that $\varnothing \neq N_G[v_i]\backslash \bigcup_{j=1}^{i-1} N_G[v_j] \subseteq N_G[v_i]\backslash \bigcup_{j=1}^{i-1} N_G(v_j)$. Hence, $N_G[v_i]\backslash \bigcup_{j=1}^{i-1} N_G(v_j) \neq \varnothing$ for each $i \in \{2, 3, \ldots, n\}$. Therefore, $S$ is an L-sequence in $G$. Since $\hat{S}$ is a hop independent set of $G$, it follows that $S$ is an L-hop independent sequence in $G$. Since $\alpha_{Lh}(G)$ refers to the maximum length of an L-hop independent sequence in $G$, it follows that $\alpha_{Lh}(G) \geq |\hat{S}| = \alpha_{\ell h}(G)$. □

**Remark 1.** The strict inequality of Theorem 1(i) is attainable. Moreover, the inequality is also attainable.

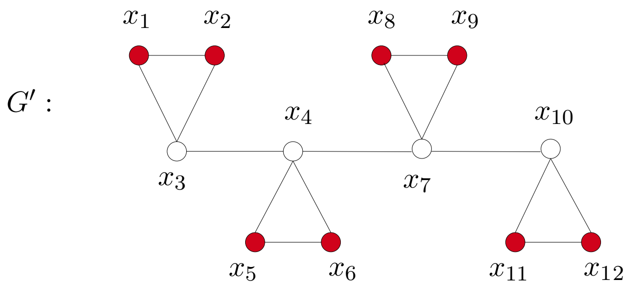
Consider the following two examples below:

**Example 2.** Consider the graph $G$ below.




Let $S = \{x_1, x_2, x_3, x_4\}$. Then $S$ is a maximum hop independent set of $G$. Thus, $\alpha_h(G) = 4$. Next, let $B = (x_1, x_2)$. Then $N_G[x_2] = V(G)$ and $N_G(x_1) = \{x_2, x_3, x_4\}$. Hence, $N_G[x_2] \setminus N_G(x_1) = \{x_1\} \neq \varnothing$, showing that $B$ is an L-hop independent sequence of $G$. Since $N_G(x_i) \cup N_G(x_j) = V(G)$ for all $i, j \in \{1, 2, 3, 4\}$, where $i \neq j$, it follows that $B$ is a maximum L-hop independent sequence of $G$ . That is, $\alpha_{Lh}(G) = 2$.

**Example 3.** Consider the graph $G'$ below.




Let $P = (x_1, x_2, x_5, x_6, x_8, x_9, x_{11}, x_{12})$ and let $\hat{P} = \{x_1, x_2, x_5, x_6, x_8, x_9, x_{11}, x_{12}\}$. Then $\hat{P}$ is a maximum hop independent set of $G'$ and $P$ is a maximum L-hop independent sequence of $G'$. Therefore, $\alpha_h(G') = 8 = \alpha_{Lh}(G')$.

We shall now state the following remark:

**Remark 2.** Let $G$ be a graph. Then each of the following holds:

(i) Every legal hop independent sequence of $G$ is an L-hop independent sequence.

(ii) Every L-hop independent set is a hop independent set, however, the converse need not be true.

**Proposition 1.** Let $n$ be a positive integer. Then

$$\alpha_{Lh}(K_n) = \begin{cases} 1 \text{ if } n = 1 \\ 2 \text{ if } n \geq 2. \end{cases}$$

*Proof.* Clearly, $\alpha_{Lh}(K_1) = 1$. For $n = 2$, let $V(K_2) = \{v_1, v_2\}$. Then $N_{K_2}[v_2] = \{v_1, v_2\}$ and $N_{K_2}(v_1) = v_2$. Thus, $N_{K_2}[v_2] \setminus N_{K_2}(v_1) = v_1 \neq \varnothing$, showing that $C = (v_1, v_2)$ is an L-sequence of $K_2$. Since $d_{K_2}(v_2, v_1) = 1$, it follows that $C$ is an L-hop independent sequence of $K_2$. Therefore, $\alpha_{Lh}(K_2) = 2$. Next, suppose that $n \geq 3$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Then $C' = (v_1, v_2)$ is an L-hop independent sequence of $K_n$. Hence, $\alpha_{Lh}(K_n) \geq 2$. Suppose that $\alpha_{Lh}(K_n) \geq 3$, say $L = (v_1, v_2, \ldots, v_m)$ is a maximum L-hop independent sequence of $K_n$, where $m \geq 3$. Note that $N_{K_n}(v_1) \cup N_{K_n}(v_2) = V(K_n)$. It follows that $N_{K_n}[v_s] \setminus \bigcup_{i=1}^{s-1} N_{K_n}(v_i) = \varnothing$ for all $3 \leq s \leq m$, a contradiction. Therefore, $\alpha_{Lh}(K_n) = 2$ for all $n \geq 2$. □

**Proposition 2.** Let $G$ be a graph. If $\alpha_{Lh}(G) = |V(G)|$, then $\alpha_h(G) = |V(G)|$. However, the converse is not necessarily true.

*Proof.* Suppose that $\alpha_{Lh}(G) = |V(G)|$. Then $\alpha_h(G) \geq |V(G)|$ by Theorem 1. Since $\alpha_h(G) \leq |V(G)|$, it follows that $\alpha_h(G) = |V(G)|$.

Now, consider $K_5$ and let $V(K_5) = \{x_1, x_2, x_3, x_4, x_5\}$. Observe that $d_{K_5}(x_i, x_j) = 1$ for all $i, j \in \{1, 2, 3, 4, 5\}$, where $i \neq j$. It follows that $V(K_5)$ is a maximum hop independent set of $K_5$. Thus, $\alpha_h(K_5) = 5$. Now, by Proposition 1, $\alpha_{Lh}(K_5) = 2$. Hence, the assertion follows. □

**Theorem 2.** [2] Let $G$ be any graph on $n$ vertices. Then

i. $\alpha_h(G) = n$ if and only if every component of $G$ is complete; and

ii. for $n \geq 3$, $\alpha_h(G) = n - 1$ if and only if all but a single component $C$ of $G$ are complete and $C \backslash v$ is a complete graph for some vertex $v \in V(C)$.

**Theorem 3.** Let $G$ be a graph. Then $\alpha_{Lh}(G) = |V(G)|$ if and only if every component of $G$ is either $K_1$ or $K_2$.

*Proof.* Suppose that $\alpha_{Lh}(G) = |V(G)|$. Then by Proposition 2, $\alpha_h(G) = |V(G)|$. By Theorem 2(i) , every component of $G$ is complete. If it is either $K_1$ or $K_2$, then we are done. Now, suppose that every component of $G$ is $K_n$, where $n \geq 3$. Then by Proposition 1, $\alpha_{Lh}(K_n) = 2$ for all $n \geq 3$. It follows that $\alpha_{Lh}(G) \leq |V(G)| - 1$, a contradiction. Therefore, the assertion is true.

Conversely, suppose that every component of $G$ is either $K_1$ or $K_2$. If every component of $G$ is $K_1$. Then by Theorem 1, $\alpha_{Lh}(K_1) = 1$. Let $m$ be the number of components $K_1$ of $G$. Then $\alpha_{Lh}(G) = \sum_{i=1}^{m} \alpha_{Lh}(K_1) = m = |V(G)|$. Next, assume that every component of $G$ is $K_2$. Then by Proposition 1, $\alpha_{Lh}(K_2) = 2$. Let $s$ be the number of components $K_2$ of $G$. Then $\alpha_{Lh}(G) = \sum_{j=1}^{s} \alpha_{Lh}(K_2) = 2s = |V(G)|$. Now, let $r$ and $q$ be the number of $K_1$ and $K_2$ components of $G$, respectively. Then by Proposition 1, $\alpha_{Lh}(G) = r + q = |V(G)|$. □

**Corollary 1.** (i) Let $n$ be a positive integer. Then $\alpha_{Lh}(\overline{K}_n) = n$ for all $n \geq 1$.

(ii) $\alpha_{Lh}(G) = \alpha_h(G) = |V(G)|$ if and only if every component of $G$ is either $K_1$ or $K_2$.

To characterize the L-hop independent sequences and the join of two graphs, we first define the following concepts:

**Definition 2.** Let $G$ be a graph. A sequence $L = (a_1, \ldots, a_n)$ of distinct vertices of $G$ is called a clique L-sequence if $n = 1$ or if $L$ is an L-sequence and its corresponding set $\hat{L} = \{a_1, a_2, \cdots, a_n\}$ induces a complete graph. The maximum length of a clique L-sequence in $G$, denoted by $\alpha^L_{ch}(G)$, is called the L-clique number of $G$. Moreover, we call $\hat{L}$ a clique L-set of $G$.

**Definition 3.** Let $G$ be any graph. A clique L-sequence $L$ is called a clique L-dominating sequence or a clique L-Grundy dominating sequence if its corresponding set $\hat{L}$ is a dominating set of $G$. The maximum length of a clique L-Grundy dominating sequence in $G$, denoted by $\gamma_{cgr}^{L}(G)$, is called the clique L-Grundy domination number of $G$. Moreover, a clique L-sequence $L$ of $G$ is called a clique non-dominating L-sequence if $\hat{L}$ is not a dominating set of $G$.

**Theorem 4.** *[2]*Let $G$ and $H$ be graphs. Then $S$ is a non-empty hop independent set of $G + H$ if and only if one of the following statement holds:

(i) $S \cap V(H) = \varnothing$ and $S \cap V(G)$ is a clique of $G$.

(ii) $S \cap V(G) = \varnothing$ and $S \cap V(H)$ is a clique of $H$.

(iii) $S \cap V(G)$ and $S \cap V(H)$ are cliques in $G$ and $H$, respectively.

**Theorem 5.** [12] Let $G$ and $H$ be two non-complete graphs. A sequence $D$ of distinct verices of $G + H$ is a Grundy dominating sequence in $G + H$ if and only if one of the following conditions holds:

(i) $D$ is a Grundy dominating sequence of $G$.

(ii) $D$ is a Grundy dominating sequence of $H$.

(iii) $D = D_G \oplus (w)$ for some non-dominating legal closed neighborhood sequence $D_G$ of $G$ and $w \in V(H)$.

(iv) $D = D_H \oplus (v)$ for some non-dominating legal closed neighborhood sequence $D_H$ of $H$ and $v \in V(G)$.

**Theorem 6.** Let $H$ and $K$ be two non-complete graphs. A sequence $A$ of distinct vertices of $H + K$ is an L-hop independent sequence in $H + K$ if and only if one of the following conditions holds:

(i) $A$ is a clique L-sequence in $H$

(ii) $A$ is a clique L-sequence in $K$

(iii) $A = A_H \oplus (a)$, where $A_H$ is a clique non-dominating L-sequence in $H$ and $a \in V(K)$.

(iv) $A = A_K \oplus (b)$, where $A_K$ is a clique non-dominating L-sequence in $K$ and $b \in V(H)$.

(v) $A = (x, y)$ for some $x \in V(H)$ and $y \in V(K)$.

*Proof.* Suppose that $A$ is a L-hop independent sequence of $H + K$. Assume that $\hat{A} \subseteq V(H)$. Then $\hat{A}$ is a clique in $H$ by Theorem 4. By Theorem 5, $A$ is a legal sequence in $H$. Thus, $A$ is an L-sequence in $H$, showing that $A$ is a clique L-sequence in $H$, that is, (i) holds. Similarly, if $\hat{A} \subseteq V(K)$, then $A$ is a clique L-sequence in $K$. That is, (ii) holds.

Now, let $A_H$ and $A_K$ be subsequences of $A$ such that $\hat{A}_H = \hat{A} \cap V(H)$ and $\hat{A}_K = \hat{A} \cap V(K)$, where $\hat{A}_H \neq \varnothing$ and $\hat{A}_K \neq \varnothing$. Then $A = A_H \oplus (a)$ for some non-dominating legal sequence $A_H$ in $H$ and $a \in V(K)$ by Theorem 5. Since every legal sequence is an L-sequence, $A_H$ is a non-dominating L-sequence in $H$. By Theorem 4, $A_H$ is clique in $H$. Thus, $A_H$ is a clique non-dominating L-sequence in $H$, and so (iii) holds. Similarly, by Theorem 4 and Theorem 5, (iv) holds. Now, it is also easy to see that $(v)$ follows whenever $A_H$ or $A_K$ is a clique L-Grundy dominating sequence of $H$ or $K$.

Conversely, assume that (i) holds. Then by Theorem 4, $\hat{A}$ is a hop independent set of $H + K$. Since $A$ is an L-sequence, it follows that $A$ is an L-hop independent sequence of $H + K$. Similarly, the assertion follows whenever (ii) holds. Suppose that (iii) holds. Then the corresponding set of $A_H \oplus (a)$ is a hop independent set of $H + K$. Since $A_H$ is a non-dominating, there exists $x \in V(H) \setminus \hat{A_H}$ such that $x \notin N_{H+K}(\hat{A}_H)$. It follows that $x \in N_{H+K}[a] \setminus N_{H+K}(\hat{A}_H)$. Thus, $A = A_H \oplus (a)$ is an L-hop independent sequence of $H + K$. Similarly, the result follows when (iv) is true. Moreover, it is clear that $A = (x, y)$ for some $x \in V(H)$ and $y \in V(K)$ is an L-hop independent sequence of $H + K$.

□

**Theorem 7.** *[12]*Let $G$ be a complete graph and let $H$ be a non-complete graph. A sequence $D$ of distinct vertices of $G + H$ is a Grundy dominating sequence in $G + H$ if and only if one of the following condition holds:

(i) $D = (v)$ for some $v \in V(G)$.

(ii) $D$ is a Grundy dominating sequence of $H$.

(iii) $D = D_H \oplus (v)$ for some non-dominating legal closed neighborhood sequence $D_H$ of $H$ and $v \in V(G)$.

**Theorem 8.** Let $Q$ and $R$ be complete and non-complete graph, respectively. A sequence $B$ of distinct vertices of $Q + R$ is an L-hop independent sequence if and only if one of the following conditions holds:

(i) $B$ is a clique L-sequence of $Q$.

(ii) $B$ is a clique L-sequence of $R$.

(iii) $B = B_R \oplus (w)$, where $B_R$ is a clique non-dominating L-sequence in $R$ and $w \in V(Q)$.

(iv) $A = (x, y)$ for some $x \in V(H)$ and $y \in V(K)$.

*Proof.* Let $B$ be an L-hop independent sequence of $Q + R$. Assume that $\hat{B} \subseteq V(Q)$. Since $\hat{B}$ is hop independent in $Q + R$, $\hat{B}$ is clique in $R$ by Theorem 4. Hence, (i) follows since $B$ is an L-sequence in $Q + R$. Similarly, $(ii)$ follows whenever $\hat{B} \subseteq V(R)$ .

Now, assume that $\hat{B} = \hat{B}_Q \cup \hat{B}_R$, where $\hat{B}_q = \hat{B} \cap V(Q) \neq \varnothing$ and $\hat{B}_R = \hat{B} \cap V(R) \neq \varnothing$. By Theorem 7, $B = B_R \oplus (w)$ for some non-dominating legal closed neighborhhood sequence $B_R$ of $R$ and $w \in V(Q)$. Since every legal closed neighborhood sequence is an L-sequence and $\hat{B}$ is a hop independent set in $Q+R$, $B_R$ must be a clique non-dominating L-sequence in $R$. Hence, $(iii)$ holds.

The converse can be proved easily. □

The following Theorem can be proved easily.

**Theorem 9.** Let $Q$ and $R$ be complete graphs. A sequence $B$ of distinct vertices of $Q+R$ is an L-hop independent sequence if and only if one of the following conditions holds:

(i) $B$ is a clique L-sequence of $Q$.

(ii) $B$ is a clique L-sequence of $R$.

(iii) $A = (x, y)$ for some $x \in V(H)$ and $y \in V(K)$.

The following result follows from Proposition 1, Theorem 6, Theorem 8, and Theorem 9.

**Corollary 2.** Let $H$ and $K$ be two graphs. Then

(i) $2 \leq \alpha_{Lh}(H + K) \leq 3$; and

(ii) $\alpha_{Lh}(H + K) = 2$ if both $H$ and $K$ are complete graphs.

## 4. Conclusion

The concept of L-hop independent sequence has been introduced and initially investigated in this study. Some characterizations and formulas have been obtained on some special graphs, complementary prism, and on the join of any two graphs. Interested researchers may consider studying the complexity of this newly defined concept, and they may also consider providing real-world applications.

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