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The Conformable Double Sumudu-Shehu Transform and its Properties with Applications

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Abstract. We introduce a new transform called the conformable double Sumudu-Shehu transform. It helps solve fractional partial differential equations that appear in physical and engineering problems. The transform uses the conformable derivative idea. We explain its basic properties and show how it works. Then we apply it to some well-known equations like the wave and Klein-Gordon equations.

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1. Introduction

Fractional partial differential equations are used in many real-life problems in physics, circuits, fluids, optics, and biology. One important idea used to deal with such equations is the conformable derivative, introduced in [1], which keeps most of the key features of classical derivatives.

Several approaches were proposed to solve these equations. The conformable double Laplace transform was discussed in [2], [3], and the conformable double Sumudu transform appeared in [4]. More results about these transforms are found in [5] and [6].

Later, a method called the double Sumudu-Shehu transform was introduced in [7]. It was successfully applied to different equations. More studies on integral transforms appear in [8–15].

In this paper, we define the conformable double Sumudu-Shehu transform (CD-SSH). We explain when the transform exists and how it behaves with derivatives. Then we show how it can be used to solve some well-known conformable equations.

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2. Preliminaries

In this section, we give the main definitions and results related to conformable fractional derivatives.

Definition 1. *[1] Let $0 < \theta \leq 1$ and $\chi : (0,\infty) \to \mathbb{R}$. The conformable fractional derivative of order θ is defined as:*

$$\frac{d^\theta}{d\lambda^\theta}\chi(\lambda) = \lim_{\eta\to 0}\frac{\chi(\lambda+\eta\lambda^{1-\theta})-\chi(\lambda)}{\eta},$$

where $\lambda > 0$, and $\frac{\partial^\theta}{\partial\lambda^\theta}$ is referred to as the fractional derivative of order θ.

Definition 2. *[16] Let $0 < \theta_1, \theta_2 \leq 1$ and $\chi(\lambda,\delta) : (0,\infty)\times(0,\infty) \to \mathbb{R}$. The conformable partial derivatives of orders θ_1 and θ_2 of the function $\chi(\lambda,\delta)$ are defined as:*

$$\frac{\partial^{\theta_1}}{\partial\lambda^{\theta_1}}\chi(\lambda,\delta) = \lim_{\eta\to 0}\frac{\chi(\lambda+\eta\lambda^{1-\theta_1},\delta)-\chi(\lambda,\delta)}{\eta},$$

$$\frac{\partial^{\theta_2}}{\partial\delta^{\theta_2}}\chi(\lambda,\delta) = \lim_{\eta\to 0}\frac{\chi(\lambda,\delta+\eta\delta^{1-\theta_2})-\chi(\lambda,\delta)}{\eta},$$

where $\lambda, \delta > 0$, $\frac{\partial^{\theta_1}}{\partial\lambda^{\theta_1}}$ and $\frac{\partial^{\theta_2}}{\partial\delta^{\theta_2}}$ are referred to as fractional derivatives of orders θ_1 and θ_2, respectively.

Theorem 1. *[17]Suppose that $\chi(\lambda,\delta)$ is differentiable at a point $\lambda, \delta > 0$, $0 < \theta_1, \theta_2 \leq 1$, then:*

$$\frac{\partial^{\theta_1}\chi}{\partial\lambda^{\theta_1}} = \lambda^{1-\theta_1}\frac{\partial\chi}{\partial\lambda},$$

$$\frac{\partial^{\theta_2}\chi}{\partial\delta^{\theta_2}} = \delta^{1-\theta_2}\frac{\partial\chi}{\partial\delta}.$$

3. The Conformable Double Sumudu-Shehu transform

In this section, we introduce the CD-SSH transform and explain its main properties such as linearity. We also present a new result related to partial derivatives. Finally, we show how these ideas help in finding the CD-SSH of some basic functions.

Definition 3. *Let $\chi(\lambda,\delta)$ be a continuous function on $(0,\infty)\times(0,\infty)$. Then*
1- The Conformable Sumudu transformation (C-S) of $\chi(\lambda,\delta)$, denoted by $S_\lambda^\theta[\chi(\lambda,\delta)]$, is defined as:

$$\Phi(\rho) = S_\lambda^\theta(\chi(\lambda,\delta)) = \frac{1}{\rho}\int_0^\infty e^{-\frac{\lambda^\theta}{\rho^\theta}}\chi(\lambda,\delta)\lambda^{\theta-1}d\lambda,\ \rho\in\mathbb{C}.$$

2- The Conformable Shehu transformation (C-SH) of $\chi(\lambda,\delta)$, denoted by $H_\delta^\theta[\chi(\lambda,\delta)]$, is defined as:

$$\Omega(\epsilon,\eta) = H_\delta^\theta(\chi(\lambda,\delta)) = \int_0^\infty e^{-\epsilon\frac{\delta^\theta}{\eta\theta}}\chi(\lambda,\delta)\delta^{\theta-1}d\delta,\ \epsilon,\eta\in\mathbb{C}.$$

3- The Conformable Sumudu-Shehu transformation (CD-SSH) of $\chi(\lambda,\delta)$, denoted by $S_\lambda^{\theta_1}H_\delta^{\theta_2}[\chi(\lambda,\delta)]$, is defined as:

$$\Psi(\rho,\epsilon,\eta) = S_\lambda^{\theta_1}H_\delta^{\theta_2}[\chi(\lambda,\delta)] = \frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)}\chi(\lambda,\delta)\lambda^{\theta_1-1}\delta^{\theta_2-1}d\lambda d\delta.$$

Theorem 2. *Assume that $\chi:(0,\infty)\times(0,\infty)\to\mathbb{R}$ such that $\Psi(\rho,\epsilon,\eta) = S_\lambda^{\theta_1}H_\delta^{\theta_2}[\chi(\frac{\lambda^{\theta_1}}{\theta_1},\frac{\delta^{\theta_2}}{\theta_2})]$ exist, then*

$$S_\lambda^{\theta_1}H_\delta^{\theta_2}[\chi(\frac{\lambda^{\theta_1}}{\theta_1},\frac{\delta^{\theta_2}}{\theta_2})] = S_\lambda H_\delta[\chi(\lambda,\delta)],$$

where

$$S_\lambda H_\delta[\chi(\lambda,\delta)] = \frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda}{\rho}+\frac{\epsilon\delta}{\eta}\right)}\chi(\lambda,\delta)\,d\lambda\,d\delta.$$

Lemma 1. *$S_\lambda^{\theta_1}H_\delta^{\theta_2}(\chi(\lambda,\delta))$ is a linear transformation.*

Proof. for nonzero constants a_1 and a_2, we have

$S_\lambda^{\theta_1}H_\delta^{\theta_2}(a_1\chi_1(\lambda\delta)+a_2\chi_2(\lambda,\delta))$

$$\begin{aligned}
&= \frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)}(a_1\chi_1(\lambda,\delta)+a_2\chi_2(\lambda,\delta))\lambda^{\theta_1-1}\delta^{\theta_2-1}d\lambda d\delta \\
&= \frac{a_1}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)}\chi_1(\lambda,\delta)\lambda^{\theta_1-1}\delta^{\theta_2-1}d\lambda d\delta + \\
&\quad \frac{a_2}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)}\chi_2(\lambda,\delta)\lambda^{\theta_1-1}\delta^{\theta_2-1}d\lambda d\delta \\
&= a_1 S_\lambda^{\theta_1}H_\delta^{\theta_2}(\chi_1(\lambda,\delta)) + a_2 S_\lambda^{\theta_1}H_\delta^{\theta_2}(\chi_2(\lambda,\delta)).
\end{aligned}$$

If $\chi(\lambda,\delta)$ can be written as $\chi(\lambda,\delta)=p(\lambda)q(\delta)$ for some continuous functions p and q, then $S_\lambda^{\theta_1}H_\delta^{\theta_2}(\chi(\lambda,\delta)) = S_\lambda^{\theta_1}(p(\lambda))H_\delta^{\theta_2}(q(\delta))$. In fact

$$S_\lambda^{\theta_1}H_\delta^{\theta_2}(\chi(\lambda,\delta)) = S_\lambda^{\theta_1}H_\delta^{\theta_2}(p(\lambda)q(\delta))$$

$$
\begin{aligned}
&= \frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)} p(\lambda)q(\delta)\lambda^{\theta_1-1}\delta^{\theta_2-1}d\lambda d\delta \\
&= \left(\frac{1}{\rho}\int_0^\infty e^{-\frac{\lambda^{\theta_1}}{\rho\theta_1}} p(\lambda)\lambda^{\theta_1-1}d\lambda\right)\left(\int_0^\infty e^{-\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}} q(\delta)\delta^{\theta_2-1}d\delta\right) \\
&= S_\lambda^{\theta_1}(p(\lambda))H_\delta^{\theta_2}(q(\delta)).
\end{aligned}
$$

3.1. The Conformable Double Sumudu-Shehu transform for some basic functions

(i)

$$S_\lambda^{\theta_1}H_\delta^{\theta_2}[c] = S_\lambda^{\theta_1}H_\delta^{\theta_2}[c] = \frac{c\eta}{\epsilon}, \quad c \in \mathbb{R},$$

(ii)

$$S_\lambda^{\theta_1}H_\delta^{\theta_2}\left[e^{a_1\frac{\lambda^{\theta_1}}{\theta_1}+a_2\frac{\delta^{\theta_2}}{\theta_2}}\right]$$

$$= S_\lambda^{\theta_1}H_\delta^{\theta_2}[e^{a_1\lambda+a_2\delta}] = \frac{\eta}{(1-a_1\rho)(\epsilon-a_2\eta)}, \mathrm{Re}(\frac{1}{\rho}) > \mathrm{Re}(a_1),$$

(iii)

$$S_\lambda^{\theta_1}H_\delta^{\theta_2}\left[\left(\frac{\lambda^{\theta_1}}{\theta_1}\right)^{a_1}\left(\frac{\delta^{\theta_2}}{\theta_2}\right)^{a_2}\right]$$

$$
\begin{aligned}
&= S_\lambda^{\theta_1}H_\delta^{\theta_2}[\lambda^{a_1}\delta^{a_2}] \\
&= \frac{\rho^{a_1}\eta^{a_2+1}}{\epsilon^{a_2+1}}\Gamma(a_1+1)\Gamma(a_2+1),\ \mathrm{Re}(\frac{1}{\rho}) > 0 \text{ and } \mathrm{Re}(a_1) > -1.
\end{aligned}
$$

3.2. Existence condition for the Conformable Double Sumudu-Shehu transform

Definition 4. *Let $0 < \theta_1, \theta_2 \leq 1$. Then a function $\chi(\lambda, \delta)$ is said to be of conformable exponential orders a_1 and a_2 on $0 < \lambda < \infty$ and $0 < \delta < \infty$. If there exist $A, B, C > 0$ such that $|\chi(\lambda,\delta)| \leq Ae^{a_1\frac{\lambda^{\theta_1}}{\theta_1}+a_2\frac{\delta^{\theta_2}}{\theta_2}}$, for all $\frac{\lambda^{\theta_1}}{\theta_1} > B$, $\frac{\delta^{\theta_2}}{\theta_2} > C$.*

Theorem 3. *Let $0 < \theta_1, \theta_2 \leq 1$ and $\chi(\lambda, \delta)$ be a continuous function on the region $(0,\infty)\times(0,\infty)$ of conformable exponential orders a_1 and a_2. Then $\Psi(\rho, \epsilon, \eta) = S_\lambda^{\theta_1}H_\delta^{\theta_2}[\chi(\lambda, \delta)]$ exists for ρ, η whenever $\mathrm{Re}(\frac{1}{\rho}) > a_1$ and $\mathrm{Re}\left(\frac{\epsilon}{\eta}\right) > a_2$.*

Proof. We have

$$
\begin{aligned}
|\Psi(\rho,\epsilon,\eta)| &= \left|\frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)}\chi(\lambda,\delta)\lambda^{\theta_1-1}\delta^{\theta_2-1}\, d\lambda d\delta\right| \\
&\le \frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)}\left|\chi(\lambda,\delta)\right|\lambda^{\theta_1-1}\delta^{\theta_2-1} d\lambda d\delta \\
&\le \frac{A}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)} e^{a_1\frac{\lambda^{\theta_1}}{\theta_1}+a_2\frac{\delta^{\theta_2}}{\theta_2}}\lambda^{\theta_1-1}\delta^{\theta_2-1} d\lambda d\delta \\
&= A\left(\frac{1}{\rho}\int_0^\infty e^{-(\frac{1}{\rho}-a_1)\frac{\lambda^{\theta_1}}{\theta_1}}\lambda^{\theta_1-1}d\lambda\right)\left(\int_0^\infty e^{-(\frac{\epsilon}{\eta}-a_2)\frac{\delta^{\theta_2}}{\theta_2}}\delta^{\theta_2-1}d\delta\right) \\
&= \frac{A\eta}{(1-a_1\rho)\,(\epsilon-a_2\eta)},
\end{aligned}
$$

where $\mathrm{Re}(\frac{1}{\rho}) > a_1$ and $\mathrm{Re}\left(\frac{\epsilon}{\eta}\right) > a_2$.

3.3. Derivatives properties

Now, we present some basic properties of the CD-SSH

Let $\Psi(\rho,\epsilon,\eta) = S_\lambda^{\theta_1}H_\delta^{\theta_2}\left(\chi(\lambda,\delta)\right)$ where $\chi(\lambda,\delta)$ is a continuous function on $(0,\infty)\times(0,\infty)$. Then

(i)

$$S_\lambda^{\theta_1}H_\delta^{\theta_2}\left(\frac{\partial^{\theta_1}\chi(\lambda,\delta)}{\partial\lambda^{\theta_1}}\right) = \frac{1}{\rho}\Psi(\rho,\epsilon,\eta) - \frac{1}{\rho}H_\delta^{\theta_2}(\chi(0,\delta)), \tag{1}$$

(ii)

$$S_\lambda^{\theta_1}H_\delta^{\theta_2}\left(\frac{\partial^{2\theta_1}\chi(\lambda,\delta)}{\partial\lambda^{2\theta_1}}\right) = \frac{1}{\rho^2}\Psi(\rho,\epsilon,\eta) - \frac{1}{\rho^2}H_\delta^{\theta_2}(\chi(0,\delta)) - \frac{1}{\rho}H_\delta^{\theta_2}(\frac{\partial^{\theta_1}\chi(0,\delta)}{\partial\lambda^{\theta_1}}), \tag{2}$$

(iii)

$$S_\lambda^{\theta_1}H_\delta^{\theta_2}\left(\frac{\partial^{\theta_2}\chi(\lambda,\delta)}{\partial\delta^{\theta_2}}\right) = \frac{\epsilon}{\eta}\Psi(\rho,\epsilon,\eta) - S_\lambda^{\theta_1}(\chi(\lambda,0)), \tag{3}$$

(iv)

$$S_\lambda^{\theta_1}H_\delta^{\theta_2}\left(\frac{\partial^{2\theta_2}\chi(\lambda,\delta)}{\partial\delta^{2\theta_2}}\right) = \frac{\epsilon^2}{\eta^2}\Psi(\rho,\epsilon,\eta) - \frac{\epsilon}{\eta}S_\lambda^{\theta_1}(\chi(\lambda,0)) - S_\lambda^{\theta_1}(\frac{\partial^{\theta_2}\chi(\lambda,0)}{\partial\delta^{\theta_2}}). \tag{4}$$

Proof. Proof of Equation 1

$$S_\lambda^{\theta_1} H_\delta^{\theta_2}\left(\frac{\partial^{\theta_1}\chi(\lambda,\delta)}{\partial\lambda^{\theta_1}}\right) = \frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)}\frac{\partial^{\theta_1}\chi(\lambda,\delta)}{\partial\lambda^{\theta_1}}\lambda^{\theta_1-1}\delta^{\theta_2-1}d\lambda d\delta.$$

By Theorem 1, we have $\frac{\partial^{\theta_1}\chi(\lambda,\delta)}{\partial\lambda^{\theta_1}} = \lambda^{1-\theta_1}\frac{\partial\chi(\lambda,\delta)}{\partial\lambda}$. So,

$$S_\lambda^{\theta_1} H_\delta^{\theta_2}\left(\frac{\partial^{\theta_1}\chi(\lambda,\delta)}{\partial\lambda^{\theta_1}}\right) = \frac{1}{\rho}\int_0^\infty e^{-\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}}\delta^{\theta_2-1}\int_0^\infty e^{-\frac{\lambda^{\theta_1}}{\rho\theta_1}}\frac{\partial\chi(\lambda,\delta)}{\partial\lambda}d\lambda d\delta.$$

By integrating by parts, we get

$$S_\lambda^{\theta_1} H_\delta^{\theta_2}\left(\frac{\partial^{\theta_1}\chi(\lambda,\delta)}{\partial\lambda^{\theta_1}}\right)$$

$$= \frac{1}{\rho}\int_0^\infty e^{-\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}}\delta^{\theta_2-1}\left(-\chi(0,\delta)+\frac{1}{\rho}\int_0^\infty e^{-\frac{\lambda^{\theta_1}}{\rho\theta_1}}\chi(\lambda,\delta)\lambda^{\theta_1-1}\ d\lambda\right)d\delta$$

$$= -\frac{1}{\rho}\int_0^\infty e^{-\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}}\chi(0,\delta)\delta^{\theta_2-1}d\delta$$

$$+\frac{1}{\rho^2}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^{\theta_1}}{\rho\theta_1}+\epsilon\frac{\delta^{\theta_2}}{\eta\theta_2}\right)}\chi(\lambda,\delta)\lambda^{\theta_1-1}\delta^{\theta_2-1}d\lambda d\delta$$

$$= \frac{1}{\rho}\Psi(\rho,\epsilon,\eta) - \frac{1}{\rho}H_\delta^{\theta_2}(\chi(0,\delta)).$$

The proof of Equations 2, 3 and 4 follow using the same steps.

In Table 1, we have the CD-SSH of some basic functions.

Table 1: Table of CD-SSH

$\chi(\lambda,\delta)$	$S_\lambda^{\theta_1} H_\delta^{\theta_2}\left(\chi(\lambda,\delta)\right)$
c	$\frac{c\eta}{\epsilon}$, $\mathrm{Re}(\rho)>0$
$\left(\frac{\lambda^{\theta_1}}{\theta_1}\right)^{a_1}\left(\frac{\delta^{\theta_2}}{\theta_2}\right)^{a_2}$	$\frac{\rho^{a_1}\eta^{a_2+1}}{\epsilon^{a_2+1}}\Gamma(a_1+1)\Gamma(a_2+1)$, $\mathrm{Re}(\rho)>0$ and $\mathrm{Re}(a_1)>-1$
$e^{a_1\frac{\lambda^{\theta_1}}{\theta_1}+a_2\frac{\delta^{\theta_2}}{\theta_2}}$	$\frac{\eta}{(1-a_1\rho)(\epsilon-a_2\eta)}$, $\mathrm{Re}(\frac{1}{\rho})>\mathrm{Re}(a_1)$
$e^{i\left(a_1\frac{\lambda^{\theta_1}}{\theta_1}+a_2\frac{\delta^{\theta_2}}{\theta_2}\right)}$	$\frac{i\eta}{(i+a_1\rho)(\epsilon-ia_2\eta)}$, $\mathrm{Im}(a_1)+\mathrm{Re}(\frac{1}{\rho})>0$
$\sin\left(a_1\frac{\lambda^{\theta_1}}{\theta_1}+a_2\frac{\delta^{\theta_2}}{\theta_2}\right)$	$\frac{\eta(\rho\epsilon a_1+\eta a_2)}{(1+a_1^2\rho^2)(\epsilon^2+a_2^2\eta^2)}$, $\lvert\mathrm{Im}(a_1)\rvert<\mathrm{Re}(\frac{1}{\rho})$
$\cos\left(a_1\frac{\lambda^{\theta_1}}{\theta_1}+a_2\frac{\delta^{\theta_2}}{\theta_2}\right)$	$\frac{\eta(\epsilon-\rho\eta a_1a_2)}{(1+a_1^2\rho^2)(\epsilon^2+a_2^2\eta^2)}$, $\lvert\mathrm{Im}(a_1)\rvert<\mathrm{Re}(\frac{1}{\rho})$
$\sinh\left(a_1\frac{\lambda^{\theta_1}}{\theta_1}+a_2\frac{\delta^{\theta_2}}{\theta_2}\right)$	$\frac{\eta(\rho\epsilon a_1+\eta a_2)}{(\rho^2a_1^2-1)(\epsilon^2-a_2^2\eta^2)}$, $\mathrm{Re}(\frac{1}{\rho})>\mathrm{Re}(a_1)$ and $\mathrm{Re}(\frac{1}{\rho})+\mathrm{Re}(a_1)>0$
$\cosh\left(a_1\frac{\lambda^{\theta_1}}{\theta_1}+a_2\frac{\delta^{\theta_2}}{\theta_2}\right)$	$\frac{\eta(\epsilon-\rho\eta a_1a_2)}{(\rho^2a_1^2-1)(\epsilon^2-a_2^2\eta^2)}$, $\mathrm{Re}(\frac{1}{\rho})>\mathrm{Re}(a_1)$ and $\mathrm{Re}(\frac{1}{\rho})+\mathrm{Re}(a_1)>0$
$p(\lambda)q(\delta)$	$S_\lambda^{\theta_1}(p(\lambda))H_\delta^{\theta_2}(q(\delta))$

4. Applications

In this section, we use the CD-SSH for solving conformable partial differential equations

Example 1. *Consider the conformable wave equation*

$$\frac{\partial^{2\theta_1}\chi(\lambda,\delta)}{\partial\lambda^{2\theta_1}}+9\frac{\partial^{2\theta_2}\chi(\lambda,\delta)}{\partial\delta^{2\theta_2}}=18,\ \textit{where}\ \lambda,\delta>0. \tag{5}$$

With initial conditions (ICs)
$\chi(\lambda,0)=\cosh\left(3\frac{\lambda^{\theta_1}}{\theta_1}\right)$, $\frac{\partial^{\theta_2}\chi(\lambda,0)}{\partial\delta^{\theta_2}}=0$,
and boundary conditions (BCs)
$\chi(0,\delta)=\cos\left(\frac{\delta^{\theta_2}}{\theta_2}\right)+\left(\frac{\delta^{\theta_2}}{\theta_2}\right)^2$, $\frac{\partial^{\theta_1}\chi(0,\delta)}{\partial\lambda^{\theta_1}}=0$.

Solution 1. *By applying the C-S to the ICs and the C-SH to the BCs, we get*
$S_\lambda^{\theta_1}\left(\cosh\left(3\frac{\lambda^{\theta_1}}{\theta_1}\right)\right)=\frac{1}{1-9\rho^2}$, $S_\lambda^{\theta_1}(0)=0$, $H_\delta^{\theta_2}\left(\cos\left(\frac{\delta^{\theta_2}}{\theta_2}\right)+\left(\frac{\delta^{\theta_2}}{\theta_2}\right)^2\right)=\frac{\epsilon\eta}{\epsilon^2+\eta^2}+\frac{2\eta^3}{\epsilon^3}$,
$H_\delta^{\theta_2}(0)=0$.
Apply the CD-SSH to Equation 5, we get

$$\frac{1}{\rho^2}\Psi-\frac{\epsilon\eta}{\rho^2\left(\epsilon^2+\eta^2\right)}-\frac{2\eta^3}{\rho^2\epsilon^3}+\frac{9\epsilon^2}{\eta^2}\Psi-\frac{9\epsilon}{\eta\left(1-9\rho^2\right)}=\frac{18\eta}{\epsilon}.$$

So,

$$\Psi(\rho,\epsilon,\eta)\quad=\quad\frac{\frac{\epsilon\eta}{\rho^2(\epsilon^2+\eta^2)}+\frac{2\eta^3}{\rho^2\epsilon^3}+\frac{9\epsilon}{\eta(1-9\rho^2)}+\frac{18\eta}{\epsilon}}{\frac{1}{\rho^2}+\frac{9\epsilon^2}{\eta^2}}$$

$$= \frac{\frac{\epsilon\eta^2-9\rho^2\epsilon\eta^2+9\rho^2\epsilon^3+9\rho^2\epsilon\eta^2}{\rho^2\eta(1-9\rho^2)(\epsilon^2+\eta^2)}+\frac{2\eta^3+18\rho^2\epsilon^2\eta}{\rho^2\epsilon^3}}{\frac{\eta^2+9\rho^2\epsilon^2}{\rho^2\eta^2}}.$$

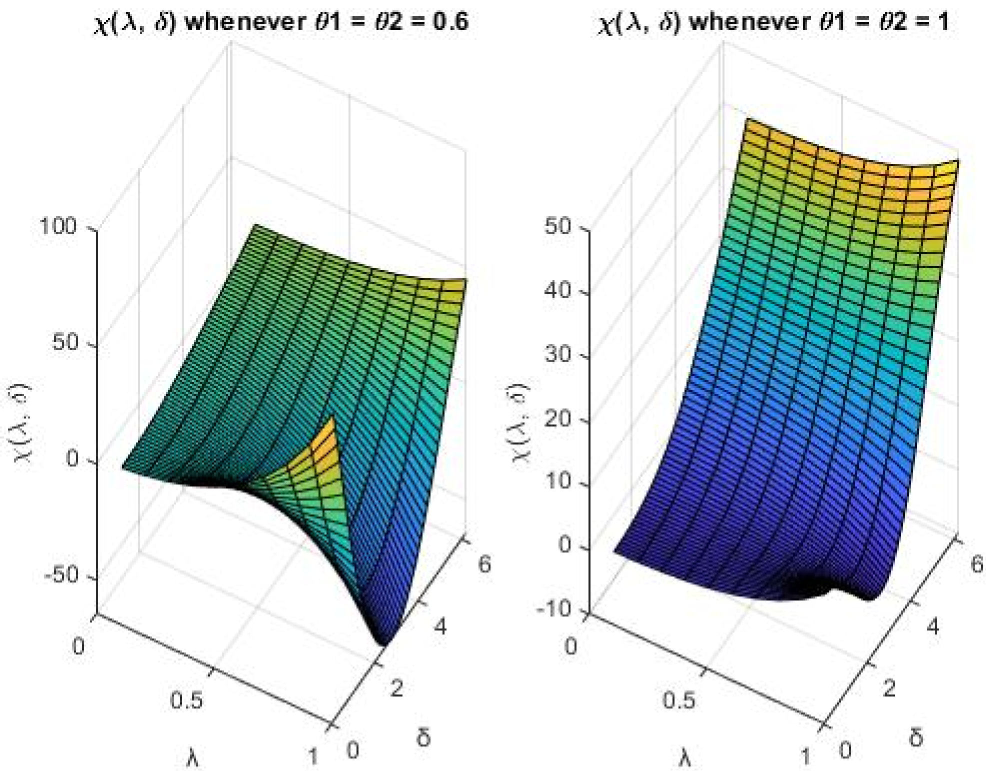
By simplify,

$$\Psi(\rho,\epsilon,\eta)=\frac{\epsilon\eta}{(1-9\rho^2)(\epsilon^2+\eta^2)}+\frac{2\eta^3}{\epsilon^3}.$$

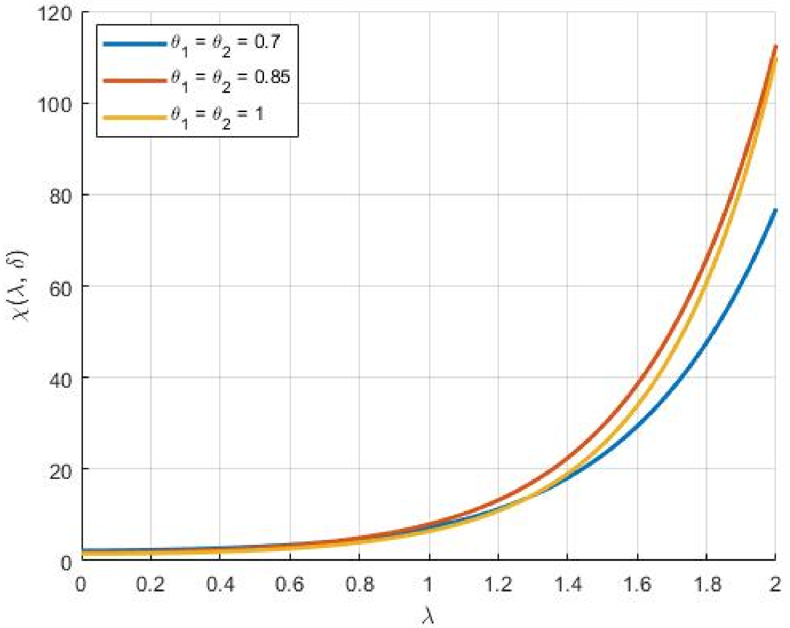
So,

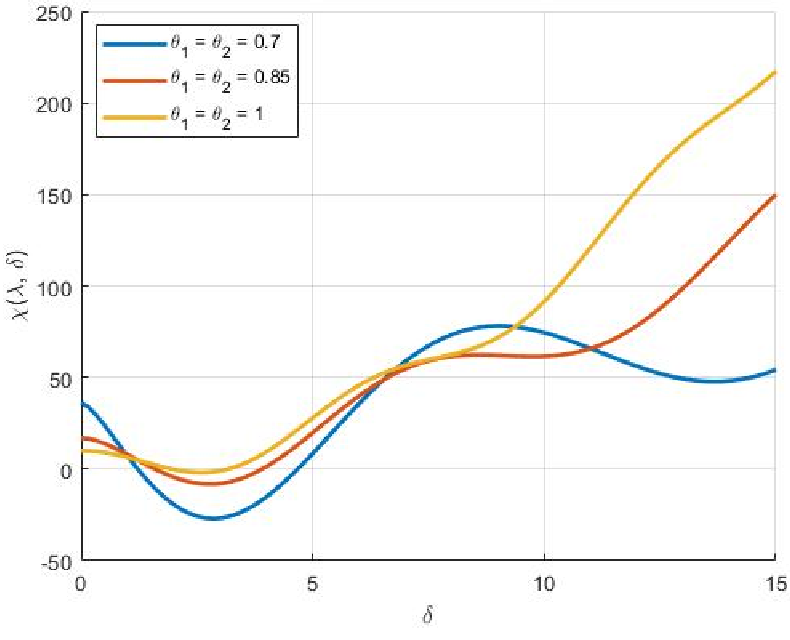
$$\begin{aligned}\chi(\lambda,\delta) &= \left(S_\lambda^{\theta_1}\right)^{-1}\left(H_\delta^{\theta_2}\right)^{-1}\left(\frac{\epsilon\eta}{(1-9\rho^2)(\epsilon^2+\eta^2)}+\frac{2\eta^3}{\epsilon^3}\right)\\ &= \cosh\left(3\frac{\lambda^{\theta_1}}{\theta_1}\right)\cos\left(\frac{\delta^{\theta_2}}{\theta_2}\right)+\left(\frac{\delta^{\theta_2}}{\theta_2}\right)^2.\end{aligned}$$

The following figures show the 3D representation of the solution at $\theta_1=\theta_2=0.6,1$.



The following two figures illustrates the 2D graph of the solution with respect to λ and δ at $\theta_1=\theta_2=0.7,0.85,1$.





Example 2. *Consider the conformable Klein-Gordon equation*

$$2\frac{\partial^{2\theta_1}\chi(\lambda,\delta)}{\partial\lambda^{2\theta_1}} + \frac{\partial^{2\theta_2}\chi(\lambda,\delta)}{\partial\delta^{2\theta_2}} = \chi(\lambda,\delta) - 3\left(\frac{\lambda^{\theta_1}}{\theta_1}\right)\left(\frac{\delta^{\theta_2}}{\theta_2}\right), \text{ where } \lambda,\delta > 0. \tag{6}$$

With ICs
$\chi(\lambda,0)=0,\ \frac{\partial^{\theta_2}\chi(\lambda,0)}{\partial\delta^{\theta_2}}=e^{2-\frac{\lambda^{\theta_1}}{\theta_1}}+3\frac{\lambda^{\theta_1}}{\theta_1},$
and BCs
$\chi(0,\delta)=e^2\sin\left(\frac{\delta^{\theta_2}}{\theta_2}\right),\ \frac{\partial^{\theta_1}\chi(0,\delta)}{\partial\lambda^{\theta_1}}=-e^2\sin\left(\frac{\delta^{\theta_2}}{\theta_2}\right)+3\frac{\delta^{\theta_2}}{\theta_2}.$

Solution 2. *By applying the C-S to the ICs and the C-SH to the BCs, we get*
$S_\lambda^{\theta_1}(0)=0,\ S_\lambda^{\theta_1}\left(e^{2-\frac{\lambda^{\theta_1}}{\theta_1}}+3\frac{\lambda^{\theta_1}}{\theta_1}\right)=\frac{e^2}{1+\rho}+3\rho,\ H_\delta^{\theta_2}\left(e^2\sin\left(\frac{\delta^{\theta_2}}{\theta_2}\right)\right)=\frac{e^2\eta^2}{\epsilon^2+\eta^2},$
$H_\delta^{\theta_2}\left(-e^2\sin\left(\frac{\delta^{\theta_2}}{\theta_2}\right)+3\frac{\delta^{\theta_2}}{\theta_2}\right)=\frac{-e^2\eta^2}{\epsilon^2+\eta^2}+\frac{3\eta^2}{\epsilon^2}.$
Apply the CD-SSH to Equation 6, we get

$$\frac{2}{\rho^2}\Psi-\frac{2e^2\eta^2}{\rho^2\left(\epsilon^2+\eta^2\right)}+\frac{2e^2\eta^2}{\rho\left(\epsilon^2+\eta^2\right)}-\frac{6\eta^2}{\rho\epsilon^2}+\frac{\epsilon^2}{\eta^2}\Psi-\frac{e^2}{1+\rho}-3\rho=\Psi-\frac{3\rho\eta^2}{\epsilon^2}.$$

So,

$$\Psi(\rho,\epsilon,\eta)=\frac{\frac{2e^2\eta^2}{\rho^2(\epsilon^2+\eta^2)}-\frac{2e^2\eta^2}{\rho(\epsilon^2+\eta^2)}+\frac{6\eta^2}{\rho\epsilon^2}+\frac{e^2}{1+\rho}+3\rho-\frac{3\rho\eta^2}{\epsilon^2}}{\frac{2}{\rho^2}+\frac{\epsilon^2}{\eta^2}-1}.$$

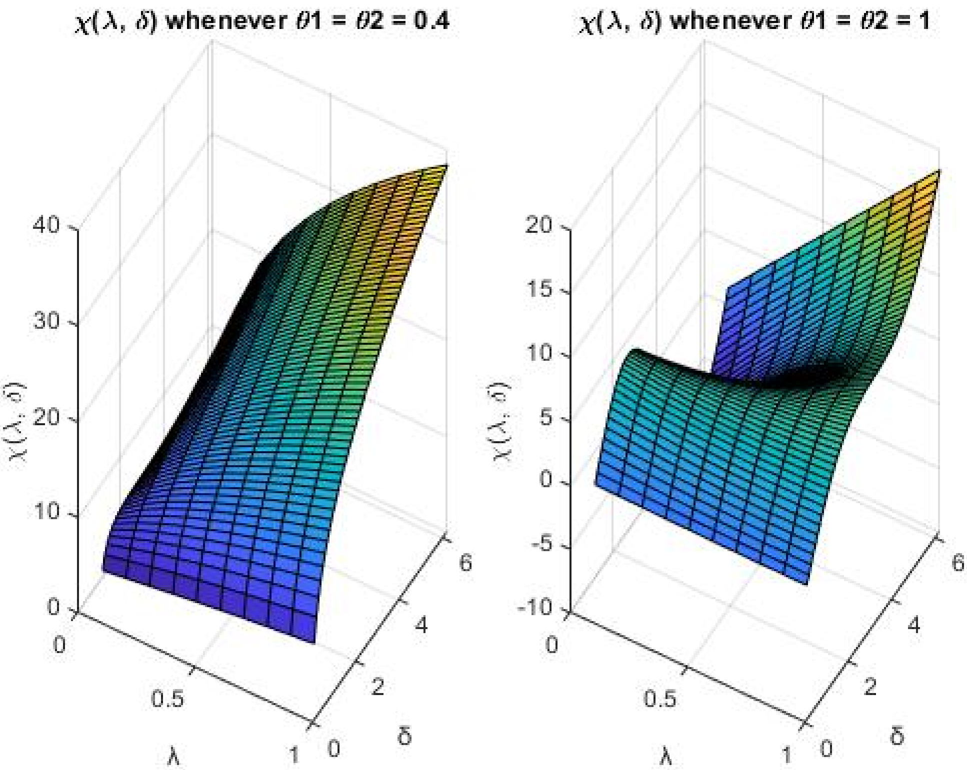
By simplify,

$$\Psi(\rho,\epsilon,\eta)=\frac{e^2\eta^2}{(\rho+1)\left(\epsilon^2+\eta^2\right)}+\frac{3\rho\eta^2}{\epsilon^2}.$$

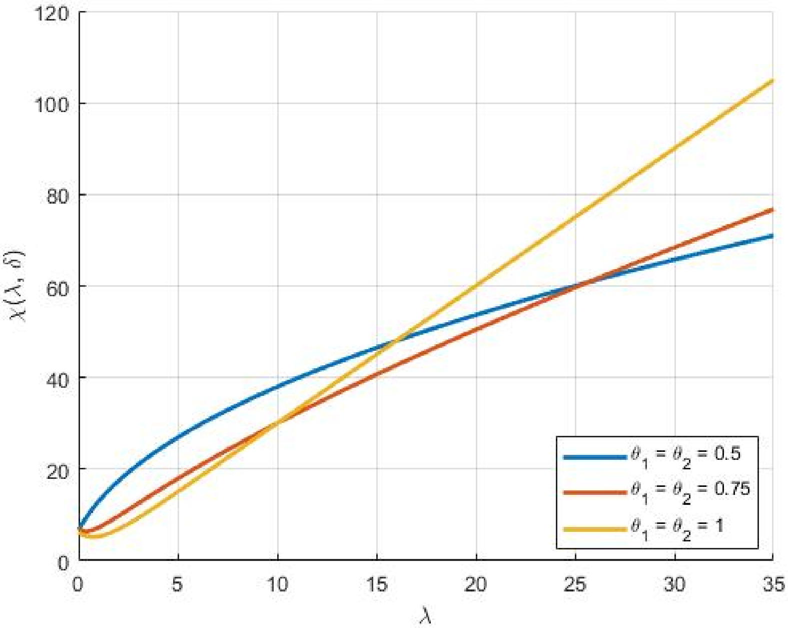
So,

$$\begin{aligned}\chi(\lambda,\delta) &= \left(S_\lambda^{\theta_1}\right)^{-1}\left(H_\delta^{\theta_2}\right)^{-1}\left(\frac{e^2\eta^2}{(\rho+1)\left(\epsilon^2+\eta^2\right)}+\frac{3\rho\eta^2}{\epsilon^2}\right)\\ &= e^{2-\frac{\lambda^{\theta_1}}{\theta_1}}\sin\left(\frac{\delta^{\theta_2}}{\theta_2}\right)+3\left(\frac{\lambda^{\theta_1}}{\theta_1}\right)\left(\frac{\delta^{\theta_2}}{\theta_2}\right).\end{aligned}$$

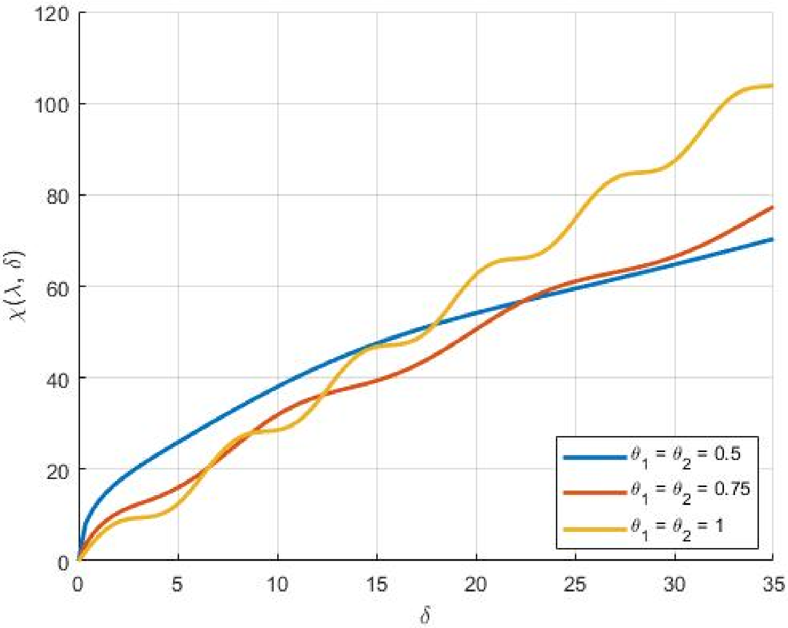
The following figures show the 3D representation of the solution at $\theta_1=\theta_2=0.4,1$.



The following two figures illustrates the 2D graph of the solution with respect to λ *and* δ *at* $\theta_1 = \theta_2 = 0.5, 0.75, 1.$



$\chi(\lambda,\delta)$
λ
$\theta_1 = \theta_2 = 0.5$
$\theta_1 = \theta_2 = 0.75$
$\theta_1 = \theta_2 = 1$



$\chi(\lambda,\delta)$
δ
$\theta_1 = \theta_2 = 0.5$
$\theta_1 = \theta_2 = 0.75$
$\theta_1 = \theta_2 = 1$

5. Conclusion

We introduced the CD-SSH and showed its main properties. We applied it to solve several conformable equations. The results show that the method is useful and works well. It may help with other equations in future work.

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