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The Effect of Variation in the Order of Zeta Function in Strip $(0,1)$ on the Upper Bound of $\mathcal{N}_p(x)$

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Abstract. During the third decade of the last century, Arne Beurling introduced the generalise primes as any increasing positive real sequence starting with a real number greater than 1 called "Beurling primes". Where the fundamental theorem of arithmetics gives Beurling integers. This work study Beurling's prime systems and concentrates on the upper bound of Beurling zeta function in the region $(0,1)$. This reflects of course on the size of the error term of Beurling counting function of integers $\mathcal{N}_p(x)$.

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Key Words and Phrases: Beurling primes, Beurling integers, Beurling zeta function

1. Introduction

The theory of numbers is one of the important branch in mathematics that deals with properties of counting number involving Riemann zeta function.
Analytic number theory is that branch of number theory which deals with problems of integers in analytic way and some times to find approximate solutions of number theoritical functions where exact solutions are out of reach. Analytic number theory has a well known results on prime number called Prime Number Theorem which states that the number of primes less than x is about $\frac{x}{\log x}$. Since prime number theorem was proved in 1896, independently by Hadamard and de la Vallee Poussin [1] , Mathematitian have wondered which condition on the primes were really necessary to this kind of theorems.
During the 1930's Arne Beurling defined the idea of generalised prime numbers or (Beurling primes): any real sequence $P = \{p_1, p_2, p_3, ...\}$ satisfying $1 < p_1 \leq p_2 \leq p_3 \leq .,.,.,.,.,\leq p_n \leq ....$ and $p_n \longrightarrow \infty$ as $n \longrightarrow \infty$. So P called the generalised primes and also he

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formed the generalised integers $\mathcal{N}$ (Beurling integers) which is the product of the form $\mathcal{N} = \prod_{i=1}^{k} p_i^{a_i}$, where $k \in \mathbb{N}$ and $a_i \in \mathbb{N} \cup \{0\}$. Therefore, the generalised integer or Beurling integers can be formed by the sense of the Fundamental Theorem of Arithmetic. In this sense, Beurling generalises the notion of prime numbers and natural numbers. From the definition realising that the generalised prime need not be atcual prime , nor even integers. Beurling also defined $\pi_p(x)$ to be the counting function of generalised prime and $\mathcal{N}_p(x)$ to be the counting function of generalised integers. Beurling also interested to find condition on $\mathcal{N}$ which let a prime number theorem holds.

In 1937, Beurling proved [2] that if

$$\mathcal{N}_p(x) = ax + O(\frac{x}{(\log x)^{\gamma}}) \text{ for some } a \geq 0 \ and \ \gamma \text{ greater than } 3/2,$$

then $\pi_p(x) \sim \frac{x}{\log x}$, this is called Beurling Prime Number Theorem.

Lator on Diamond [3] modified the definition of counting function and zeta function using Beurling's definitions. Since last century till now many authors have been dealing with Beurling (generalised) prime system such as Bteman and Diamond[4] and so many papers of Diamond[3, 5–7] , Maliavin[8], Nyman[9], Hall[10], Kahane[11], Lagarias[12] and Zhang[13]. The major reason for that is related to the difficulities of prove or disprove of Riemann Hypothesis as a special case of Beurling generalised prime.

This article introduces some concepts of generalised prime counting function $\pi_p(x)$ and concentrates on the behavior of Beurling zeta function $\zeta_p(s)$ in the strip $(0, 1)$ and its effection on the error term of generalised integer counting function $\mathcal{N}_p(x)$.

2. Preliminaries

This section gives some basic concepts and properties that are needed for the aim of this paper.

First of all the Chebyshev function which is equivalent to prime counting function [1] is let P be the set of actual prime, The Chebyshev counting function for any positive real x is defined to be $\psi(x) = \sum\limits_{(p^k \leq x)} \log p$. where $k \in \mathbb{N}$ and $p \in P$ Prime counting function [1, 14] is a number prime less than or equal to x. That is $\pi(x) = \sum\limits_{(p \leq x)} 1$ is a counting function of primes, for a large value x, and also Counting function of integers [1] is $\mathcal{N}(x) = \sum\limits_{(n \leq x)} 1$ for a large value of x, $n \in \mathbb{N}$.

Riemann zeta function has an important role in analytic number theory and distribution of prime which defined by Riemann [1] as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s \in \mathbb{C}$ and $Re(s) \geq 1$.

Beurling, as we mentioned before, generalises the notion of prime number and the natural number. Beurling defined The generalised prime counting function [15] as $\pi_p(x) = \sum\limits_{p \leq x, p \in P} 1$ and generalised integer counting function $\mathcal{N}_p(x) = \sum\limits_{n \leq x, n \in \mathcal{N}} 1$. Beurling also generalised the zeta function [15] as $\zeta_p(s) = \sum\limits_{n \in \mathcal{N}_p} n^{-s}$ when $Re(s) > 1$, $s \in \mathbb{C}$ which is called Beurling zeta function and also has several definitions related to its relation with the counting function of primes and integers:

(i) $\zeta_p(s) = \int_1^\infty x^{-s} d\mathcal{N}(x)$,

(ii) $\frac{-\zeta_p(s)}{\zeta_p(s)} = \int_1^\infty x^{-s} d\psi(x)$,

(iii) $\zeta_p(s) = \exp \int_1^\infty x^{-s} d\pi_p(x)$

The above definitions shows the link between $\zeta_p(s)$, $\pi_p(x)$ or $\psi_p(x)$, and $\mathcal{N}_p(x)$ which indicated that if we know the behavior of each one, say $\psi_p(x)$, then we could know the behavior of $\zeta_p(s)$ and $\mathcal{N}_p(x)$.

The following basic definitions are introduced since this work is focusing on upper bound of generalised counting function $\mathcal{N}_p(x)$.

Definition 1. ***Big-O-Notation****[1]. Let $g(x) \geq 0$ for all $x \geq a$. We write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ to mean that the quotient $\left|\frac{f(x)}{g(x)}\right|$ is bounded for $x \geq a$; that is, there exists a constant $M > 0$ such that $|f(x)| \leq Mg(x)$ for all $x \geq a$. An equation of the form $f(x) = O(g(x)) + h(x)$ means that $f(x) - h(x) = O(g(x))$.*

Definition 2. ***Asymptotic Notation*** *[1]Let $g(x) > 0$ for all $x \geq a$. If $\lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = 1$ we say that $f(x)$ is asymptotic to $g(x)$ as $(x \to \infty)$, and we write $f(x) \sim g(x)$ as $(x \to \infty)$.*

Definition 3. ***Big-Omega Notation*** *[1] Let $g(x) \geq 0$ for all $x \geq a$. We write $f(x) = \Omega(g(x))$ or $f(x) \gg g(x)$ to mean that the quotient $\left|\frac{f(x)}{g(x)}\right|$ is bounded for $x \geq a$; that is, there exists a constant $M > 0$ such that $|f(x)| \geq Mg(x)$ for all $x \geq a$.*

Lemma 1. *[16] Suppose $\psi_p(x) = x + O(x^\alpha)$ for some $\alpha \in [0, 1)$, then $\zeta_p(s)$ has analytic continuation to the half-plane $H_\alpha = \{s \in \mathbb{C} : Re(s) > \alpha\}$ except for a simple pole at $s = 1$ and $\zeta_p(s) \neq 0$ in this region.*

3. Some Known Results

Most of authers on this subject had been working on connecting the asymptotic behavior of the generalised prime systems and generalised integer counting function involving Beurling zeta function $\zeta_p(s)$.

We mean by the word {system} as follows:

As Beurling definition satisfies for any real sequence the condition of Beurlings, this means that there are infinitly many real sequence of Beurling primes which indicate that there are infinitely many explanations of the counting functions and Beurling zeta function related to each other. So, if we assume that B is a set of all arithmetical functions. Let $S_0 = \{h \in B : \ h(1) = 0\}$ and $S_1 = \{g \in B : \ g(1) = 1\}$, then the order pair (h, g) is called an outer generalised prime system.

Inspite of the fact that Beurling answered so many questions about prime number theorem and generalised prime and integer counting functions such as Beurling prime number theorem, there are many more questions regarding these functions.

The generalised Riemann zeta fuction also has an important role in the connection of asymptotic behavior of $\pi_p(x)$ and $\mathcal{N}_p(x)$ since in showing the link between these results the author might observe the Beurlinng zeta function $\zeta_p(s)$ is involved which means that The three functions are related to each others in the sense that an assumption made on $\pi_p(x)$ which is then showing to the property of $\zeta_p(s)$ (this is related to the size of zeta function along the vertical line) and this is then shown to imply a property of $\mathcal{N}_p(x)$ and vice versa.

Moving our attention to list some previous work which are relevant of this work.

(i) In 1977, Diamond [7] showed the converse of Beurling's prime number theorem as he stated: suppose that $\int_2^\infty t^{-2}|\Pi_p(t) - \frac{t}{\log t}|dt < \infty$, then there exist a positive constant c such that $N_P(x) \sim cx$ as $x \to \infty$

(ii) In 1983, Landau [17] proved that if

$$\mathcal{N}_p(x) = ax + O(x^\theta), \quad (\theta < 1) \tag{1}$$

then

$$\pi_p(x) = li(x) + O(xe^{-k\sqrt{\log x}}) \quad \text{for some } k > 0,$$

where $li(x) = \int_2^x \frac{dt}{\log t}$

(iii) In 2006, Diamond, Montogomery and Vorhauer [18] showed that Landau's result is best possiblre. That is they proved that here is a discrete generalised prime system for which equation (1) holds but

$$\pi_p(x) = li(x) + \Omega(xe^{-q\sqrt{logx}})$$

for some $q > 0$

(iv) In 1969, Malliavin [8] showed that for $\alpha \in (0, 1)$ and $a, c > 0$

$$\mathcal{N}_p(x) = ax + O(xe^{-c(\log x)^\beta})$$

implies

$$\Pi_p(x) = li(x) + O(xe^{-k(\log x)^\alpha})$$

for some $k > 0$ where $\beta = 10\alpha$

(v) In 1970, Diamond [3] showed the converse of Malliavin's result, as he proved that if

$$\Pi_p(x) = li(x) + O(xe^{-c(logx)^\alpha})$$

holds for $\alpha \in (0, 1)$ and some $c > 0$, then

$$\mathcal{N}_p(x) = \psi x + O(xe^{-b(\log x \log\log x)^\beta})$$

for some $b > 0$ where $\beta = \frac{\alpha}{1+\alpha}$.

(vi) In 1998, Balanzario [19] showed by example that there exists a continuous generalised prime system for which

$$\Pi_p(x) = li(x) + O(xe^{-(\log x)^{\alpha}})$$

and

$$\mathcal{N}_p(x) = \rho x + \Omega \pm (xe^{-c(logx)^{\beta}})$$

holds for some positive constants ρ and c with $\alpha = \beta = \frac{1}{2}$.

(vii) In 2006, Hilberdink[16] extended Diamond's result in 6 to the case of $\alpha = 1$ as follows: suppose $\psi_p(x) = x + O(x^{\alpha})$ for some $\alpha \in (0, 1)$ then there exists positive constants ρ and c such that

$$\mathcal{N}_p(x) = \rho x + O(xe^{-c\sqrt{\log x \log\log x}}).$$

(viii) In 2014, Al-Maamori [15] showed by example that there exists a continuous generalised prime system for which

$$\Pi_p(x) = li(x) + O(xe^{-(\log x)^{\alpha}})$$

and

$$\mathcal{N}_p(x) = \rho x + \Omega \pm (xe^{-c(\log x)^{\beta}})$$

holds for some positive constants ρ and c with $\alpha = \beta$.

(ix) In 2015, Al-Maamori and Hilberdink [20] showed that

Theorem 1. *Suppose that for some $\alpha \in (0, 1)$, $\zeta_p(s)$ has an analytic continuation to the half plane H_α except for a simple pole at $s = 1$ with residue β. Further assume that for some $c < 1$*

$$\zeta_p(\sigma + it) = O(t^c), \quad for\ some \quad \sigma \geq 1 - \frac{1}{f(\log t)}$$

where f is positive, strictly increasing continuous function, tending to infinity. Then for $\gamma = 1 - c$,

$$\mathcal{N}_p(x) = \beta x + O(xe^{-\frac{\gamma}{2}h^{-1}(\gamma^{-1}\log x)})$$

where $h(u) = uf(u)$.

The next section focus on the theorem (1) above. In particular, the following work concentrating on the effect of the constant c in the order of Beurling zeta function on the upper bound of $\mathcal{N}_p(x)$.

4. The Effect of the Constant c on the Upper Bound of Generalised Integr Counting Function $\mathcal{N}_p(x)$

This work is studying the behavior of $\mathcal{N}_p(x)$ when Beurling zeta function $\zeta_p(s)$ has known asymptotic behavior.

There is no loss of generality if we rewrite the statement of the theorem(1) above in another style:

(i) Suppose for some $\alpha \in (0,1)$.

(ii) $\zeta_p(s)$ has an analytic continuation to the half plane H_α except for a simple pole at $s=1$ with residue β.

(iii) For some $c<1$, we have

$$\zeta_p(\sigma+it)=O(t^c), \text{ for some } \sigma \geq 1-\frac{1}{f(\log t)}$$

with f is positive, strictly increasing continuous function, tending to infinity.

(iv) Then for $\gamma = 1-c$,

$$\mathcal{N}_p(x)=\beta x+O(xe^{-\frac{\gamma}{2}h^{-1}(\gamma^{-1}\log x)})$$

where $h(u)=uf(u)$.

Our aim here is to show the effect of the constant c appearing in the order of Beurling zeta function on the error term of $\mathcal{N}_p(x)$. For instance assume that theorem(1) exist for $c=\frac{1}{2}$. Its worthwhile to mention that we adapted the same strategy of the proof mentioned in [20, theorem 2.1,p 387], for the purpose of this work.

It is more clear to rewrite the theorem (1) with $c=\frac{1}{2}$ and one can see how the error term give different result.

Theorem 2. *Suppose that for some $\alpha \in (0,1)$, $\zeta_p(s)$ has an analytic continuation to the half plane H_α except for a simple pole at $s=1$ with residue β. Further assume that for some $c<1$*

$$\zeta_p(\sigma+it)=O(t^{\frac{1}{2}}), \quad for\ some\ \ \sigma \geq 1-\frac{1}{f(\log t)}$$

where f is positive, strictly increasing continuous function, tending to infinity. Then for $\gamma=\frac{1}{2}$,

$$\mathcal{N}_p(x)=\beta x+O\left(x\ \exp\left(-\frac{1}{4}h^{-1}(2\log x)\right)\right)$$

where $h(u)=uf(u)$.

Proof. By given assume that the upper bound of $\zeta_p(s)=O(t^{\frac{1}{2}})$, and to find approximate formula for $\mathcal{Z}_p(x)$. We know by Perron's formula [21],

$$\mathcal{N}_p(y)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\zeta_p(s)y^s}{s}ds,\ where\ b>1$$

$$\mathcal{Z}_p(x) = \int_0^x \mathcal{N}_p(y)dy = \int_0^x \left(\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\zeta_p(s)y^s}{s}ds\right)dy = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\zeta_p(s)x^{s+1}}{s(s+1)}ds,\ b>1$$

pushing the contour to the left of the line $Re(s) = b$ past the simple pole at $s = 1$, we get for any $T > 0$

$$\mathcal{Z}_p(x) = \frac{\beta}{2}x^2 + \frac{1}{2\pi i}\int\limits_{\lambda T} \zeta_p(s)\frac{x^{s+1}}{s(s+1)}ds + \frac{1}{2\pi i}\int_{1-\frac{1}{f(logt)}+iT}^{b+iT} \zeta_p(s)\frac{x^{s+1}}{s(s+1)}ds$$

$$+\frac{1}{2\pi i}\int_{b-iT}^{1-\frac{1}{f(logt)}-iT} \zeta_p(s)\frac{x^{s+1}}{s(s+1)}ds + \frac{1}{2\pi i}\int_{b+iT}^{b+i\infty} \zeta_p(s)\frac{x^{s+1}}{s(s+1)}ds \qquad (2)$$

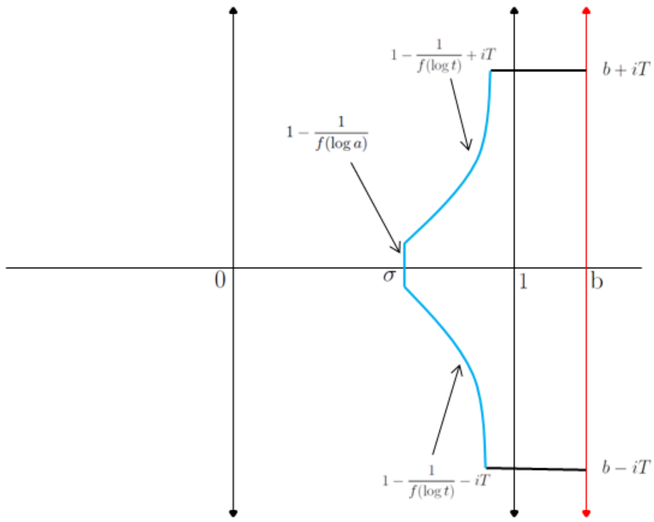

Figure 1: Contour λT

Here λT is the contour $s = 1 - \frac{1}{f(\log t)} + it$ for $a < |t| \leq T$ and $s = 1 - \frac{1}{f(\log a)} + it$ for $|t| \leq a$. The constant a is chosen such that $a > e$ and $1 - \frac{1}{f(\log a)} > \alpha$.

The integration of the third term of the equation (2) on $[1 - \frac{1}{f(\log t)} + iT, b + iT]$ is equal to

$O\left(\frac{x^{b+1}}{T^{3/2}\log x}\right) \longrightarrow 0\ as\ T \longrightarrow \infty.$

Similarily the fourth term of the a equation (2) is also 0 when $T \longrightarrow \infty$.

So, equation (2) becomes

$$\mathcal{Z}_p(x) = \frac{\beta}{2}x^2 + \frac{1}{2\pi i}\int\limits_{\lambda T} \zeta_p(s)\frac{x^{s+1}}{s(s+1)}ds$$

where λT is the contour $s = 1 - \frac{1}{f(\log t)} + it$ for $|t| > a > e$ and $s = 1 - \frac{1}{f(\log a)} + it$ for $|t| < a$.

Therefore,

$$\begin{aligned}
\left|\mathcal{Z}_p(x)-\tfrac{\beta}{2}x^2\right| &= \left|\tfrac{1}{2\pi i}\int\limits_{\lambda T}\zeta_p(s)\tfrac{x^{s+1}}{s(s+1)}ds\right| \\
&= O\left(\int_a^\infty \frac{\left|\zeta_p(1-\frac{1}{f(\log t)}-it)\right|}{t^2}.x^{2-\frac{1}{f(\log t)}}dt\right)+O\left(x^{2-\frac{1}{f(\log a)}}\right) \\
&= O\left(x^2\int_{\log a}^\infty \exp\left[\log\zeta_p(1-\tfrac{1}{f(\log t)}-it)-\log t^2+\log x^{-\frac{1}{f(\log t)}}dt\right]\right)+O\left(x^{2-\frac{1}{f(\log a)}}\right) \\
&= O\left(x^2\int_{\log a}^\infty \exp\left[\log T^{1/2}-2\log T-\tfrac{1}{f(\log t)}.\log x\right]dt\right)+O\left(x^{2-\frac{1}{f(\log a)}}\right).
\end{aligned}$$

By using $u=\log t$, to get:

$$=O\left(x^2\int_{\log a}^\infty exp\left[-\left(\frac{1}{2}u+\frac{\log x}{f(u)}\right)\right]du\right)+O\left(x^{2-\frac{1}{f(\log a)}}\right)$$

By using some manipulations to the above integral to get:

$$\int_{loga}^\infty exp\left[-\left(\frac{1}{2}u+\frac{logx}{f(u)}\right)\right]du=\left(\int_{loga}^A+\int_A^\infty\right)exp\left[-\left(\frac{1}{2}u+\frac{logx}{f(u)}\right)\right]du$$

for some $A>\log a$.

Where the first integral over $(\log a, A)$ is

$$\int_{\log a}^A exp\left[-\left(\frac{1}{2}+\frac{\log x}{f(u)}\right)du\right]\le e^{-\frac{\log x}{f(A)}}\int_{\log a}^A e^{-\frac{1}{2}u}du=O\left(e^{-\frac{\log x}{f(A)}}\right)$$

Whilest the second integeral over the interval (A,∞) is,

$$\int_A^\infty exp\left[-\left(\frac{1}{2}u+\frac{logx}{f(u)}\right)du\right]\le\int_A^\infty e^{-\frac{1}{2}u}du=O\left(e^{-\frac{1}{2}A}\right)$$

Using the optimality for the two parts above to get:

$$O\left(e^{-\frac{\log x}{f(A)}}\right)=O\left(e^{-\frac{1}{2}A}\right)$$

$$-\frac{\log x}{f(A)}=-\frac{1}{2}A$$

which tells as that

$$h(A)=Af(A)=2\log x$$

which means,

$$A=h^{-1}(2\log x).$$

Hence,

$$\left|\mathcal{Z}_p(x) - \frac{\beta}{2}x^2\right| = O\left(x^2 exp\left[-\frac{1}{2}h^{-1}(2\log x)\right]\right) \tag{3}$$

we know from given that the function $\mathcal{N}_p$ is increasing function, so for every $0 < y < x$, we have

$$\int_0^x \mathcal{N}_p(u)du - \int_0^{x-y} \mathcal{N}_p(u)du = \int_{x-y}^x \mathcal{N}_p(u)du \leq y\mathcal{N}_p(x)$$

on the other hand,

$$\int_0^{x+y} \mathcal{N}_p(u)du - \int_0^x \mathcal{N}_p(u)du = \int_x^{x+y} \mathcal{N}_p(u)du \geq y\mathcal{N}_p(x).$$

Therefore,

$$\frac{\mathcal{Z}_p(x) - \mathcal{Z}_p(x-y)}{y} \leq \mathcal{N}_p(x) \leq \frac{\mathcal{Z}_p(x+y) - \mathcal{Z}_p(x)}{y}$$

Using equation (3) the left hand side of the above inequality is

$$= \frac{1}{y}\left(\frac{\beta}{2}(x^2 - (x-y)^2)\right) + O\left(x^2 \exp\left[-\frac{1}{2}h^{-1}(2\log(x-y))\right]\right).$$

That is the left hand side is,

$$= \frac{1}{y}\frac{\beta}{2}\left(x^2 - (x^2 - 2xy + y^2)\right) + O\left(x^2 exp\left[-\frac{1}{2}h^{-1}(2\log(x-y))\right]\right) \tag{4}$$

$$= \frac{1}{y}(\beta xy - \frac{\beta y^2}{2}) + O\left(x^2 \exp\left[-\frac{1}{2}h^{-1}(2\log(x-y))\right]\right)$$

. Similarly, the right hand side is

$$\frac{1}{y}(\beta xy + \frac{\beta y^2}{2}) + O\left(x^2 \exp\left[-\frac{1}{2}h^{-1}(2\log(x))\right]\right)$$

Now for some $\epsilon > 0$ and $d > 0$ we have,

$$\begin{aligned} h(x) - h(x-d) &= xf(x) - (x-d)f(x-d) \ (by\ given) \\ &= x(f(x) - f(x-d)) + df(x-d) \geq \epsilon > 0 \end{aligned}$$

This means that $h(x) - \epsilon \geq h(x-d)$, therefore with $y = o(x)$ (since $0 < y < x$)

$$h^{-1}(2\log(x-y)) \geq h^{-1}(2\log(x-\epsilon)) \geq h^{-1}(2\log(x-d))$$

So replacing $x - y$ by x of equation (4), we have for some $m > 0$

$$\frac{1}{y}\left(\beta xy - \frac{\beta y^2}{2} + M\left(x^2 \exp\left[-\frac{1}{2}h^{-1}(2\log x)\right]\right)\right) \tag{5}$$

Assuming $B = -\frac{1}{2}h^{-1}(2\log x)$, so formula (5) appears as:

$$\frac{1}{y}\left(\beta xy - \frac{\beta y^2}{2} + M\left(x^2 e^{\frac{B}{2}}\right)\right)$$

Take $y = xe^{\frac{B}{2}}$ then we have ,

$$\frac{1}{y}\left(\beta xy - \frac{\beta y^2}{2} + My^2\right) = \beta x - \frac{\beta y}{2} + My = \beta x + y\left(m - \frac{y}{2}\right) = \beta x + O(y)$$

Hence,

$$\mathcal{N}_p(x) = \beta x + O(y)$$

where $y = x\ \exp\left(-\frac{1}{4}h^{-1}(2\log x)\right)$

$$\mathcal{N}_p(x) = \beta x + O\left(x\ \exp\left(-\frac{1}{4}h^{-1}(2\log x)\right)\right).$$

The above work shows that "how the size of error term of $\mathcal{N}_p(x)$ affected by changing the size of error term of $\zeta_p(s)$".

Therefore, by using the same strategies by repeating the above proof, observing the following table:

Table 1: Size of error term of $\mathcal{N}_p(x)$ affected by c between $(0, 1)$

$\zeta_p(s)$	$\mathcal{N}_p(x)$
$O(t^{0.1})$	$\beta x + O\left(x\ \exp\left(-\frac{9}{20}h^{-1}(\frac{10}{9}\log x)\right)\right)$
$O(t^{0.5})$	$\beta x + O\left(x\ \exp\left(-\frac{1}{4}h^{-1}(2\log x)\right)\right)$
$O(t^{0.9})$	$\beta x + O\left(x\ \exp\left(-\frac{1}{20}h^{-1}(10 log x)\right)\right)$

From the above table, one can see that the error terms of $\mathcal{N}_p(x)$ is always negative when $0 < c < 1$ and get smaller as c get closer to 0 and it gets bigger as c closer to 1.

Now the interseting point is that what will happen to the size of error term of $\mathcal{N}_p(x)$ when $c > 1$. First, we will find the error term of $\mathcal{N}_p(x)$ when $c = \frac{3}{2}$.

The proof has the same step untill we get to the following step:

$$\left|\mathcal{Z}_p(x) - \frac{\beta}{2}x^2\right| = O\left(x^2\int_{\log a}^{\infty} \exp\left[-\left((1-c)u + \frac{\log x}{f(u)}\right)\right] du\right) + O\left(x^{2-\frac{1}{f(\log a)}}\right)$$

since $c = \frac{3}{2}$, then $1 - c = \frac{-1}{2}$, so the above equation becomes

$$\left|\mathcal{Z}_p(x) - \frac{\beta}{2}x^2\right| = O\left(x^2\int_{\log a}^{\infty} \exp\left[-\left(\frac{-1}{2}u + \frac{\log x}{f(u)}\right)\right] du\right) + O\left(x^{2-\frac{1}{f(\log a)}}\right)$$

After doing some manipulations one could see that the first integral over $(\log a, A)$ is :

$$\int_{\log a}^{A} \exp\left[-\left(-\frac{1}{2}u + \frac{\log x}{f(u)}\right)\right] du = O\left(e^{-\frac{\log x}{f(A)}}\right)$$

.

$$\int_{A}^{\infty} \exp\left[-\left(-\frac{1}{2}u + \frac{\log x}{f(u)}\right)\right] du = O\left(e^{\frac{1}{2}A}\right)$$

Similarly choosing A opitamally such that $O-$ terms will be:

$$O\left(e^{-\frac{\log x}{f(A)}}\right) = O\left(e^{\frac{1}{2}A}\right)$$

$$A = h^{-1}(-2\log x)$$

Hence

$$\int_{\log a}^{\infty} exp\left[-\left(-\frac{1}{2}u + \frac{\log x}{f(u)}\right)\right] du = O\left(exp\left(\frac{1}{2}h^{-1}(-2\log x)\right)\right)$$

and

$$\mathcal{N}_p(x) = \beta x + O\left(x\exp\left(\frac{1}{4}h^{-1}(-2\log x)\right)\right)$$

From the above result, one can see that the error term of $\mathcal{N}_p(x)$ where $1 < c < 2$ is positive and big.

Its remain here to mention what is the effection of $c = 1$ and $c = 2$ on the error term of $\mathcal{N}_p(x)$ by the following lemmas:

Lemma 2. *Suppose $\zeta_p(s)$ has an analytic continuation to the half plane H_α except for a simple pole at $s = 1$ with residue β, and $\zeta_p(\sigma + it) = O(t^c)$ where $c = 1$ and $\sigma \geq 1 - \frac{1}{f(\log t)}$, then for $\gamma = 1 - c$, $\mathcal{N}_p(x)$ diverges.*

Proof. Since $c = 1$, then $\gamma = 0$. after the same calculation as $c = 1/2$, we get the following equation

$$\left|\mathcal{Z}_p(x) - \frac{\beta}{2}x^2\right| = O\left(x^2\int_{\log a}^{\infty} \exp(-\frac{\log x}{f(u)})du\right)$$

as f is increasing and goes to infinity, we get

$$O\left(x^2\int_{\log a}^{\infty} 1du\right) \longrightarrow \infty$$

Hence $\mathcal{N}_p(x)$ diverges.

Lemma 3. *Suppose $\zeta_p(s)$ has an analytic continuation to the half plane H_α except for a simple pole at $s = 1$ with residue β, and $\zeta_p(\sigma + it) = O(t^c)$ where $c = 2$ and $\sigma \geq 1 - \frac{1}{f(\log t)}$, then $\mathcal{N}_p(x)$ diverges.*

Proof. Since the third term of equation (2) is $O\left(\frac{x^{b+1}}{\log x}\right)$ which is constant and

$$\left|\mathcal{Z}_p(x) - \frac{\beta}{2}x^2\right| = O\left(x^2 \int_{\log a}^{\infty} x^{-\frac{1}{f(u)}} \exp(u)du\right)$$

as f is increasing and goes to infinity, we get

$$O\left(x^2 \int_{\log a}^{\infty} \exp(u)du\right) \longrightarrow \infty$$

Hence $\mathcal{N}_p(x)$ diverges.

Remark 1. *For $c > 2$, $\mathcal{N}_p(x)$ goes to ∞ since the third term of equation (2) goes to ∞ which leads to $\mathcal{N}_p(x)$ to be ∞.*

It's worthwhile to mention again that the effection of c on the error term of $\mathcal{N}_p(x)$ is positive and big when $1 < c < 2$, but by adding the condition for f to be even in theorem (1) the error term will decrease as shown in the following lemma

Lemma 4. *Suppose for some $\alpha \in (0, 1)$, $\zeta_p(s)$ has analytic continuation to the half plane H_α except for a simple pole at $s = 1$ with residue β. Furthermore assume that for $1 < c < 2$, $\zeta_p(s) = O(t^c)$ for $\sigma \geq 1 - \frac{1}{f(\log t)}$, where f is positive, strictly increasing continuous, even function and tends to infinity then*

$$\mathcal{N}_p(x) = \beta x + O\left(x \exp\left(+\frac{\gamma}{2}h^{-1}(k \log x)\right)\right),$$

where $h(u) = uf(u)$, $\gamma = 1 - c < 0$ and $k = -\gamma^{-1}$.

Proof. By theorem (1) $\mathcal{N}_p(x) = \beta x + O\left(x \exp\left(-\frac{\gamma}{2}h^{-1}(\gamma^{-1} \log x)\right)\right)$ for $0 < c < 1$

If $1 < c < 2$, then $\gamma = -(1 - c) > 0$ and $\gamma^{-1} = \frac{1}{1-c} < 0$

Let $-k = \gamma^{-1}$ where $k > 0$ and h^{-1} is odd (since f is even).

Hence

$$\mathcal{N}_p(x) = \beta x + O\left(x \exp\left(-\frac{\gamma}{2}h^{-1}(-k \log x)\right)\right) = \beta x + O\left(x \exp\left(\frac{\gamma}{2}h^{-1}(k \log x)\right)\right)$$

Where $\gamma = 1 - c < 0$ for $1 < c < 2$

The following is the counter example to apply theorem(2).

Example 1. *Let $f(x) = \frac{x^n}{\log x}$. Then $h(x) = \frac{x^{n+1}}{\log x}$, $h^{-1}(x) \sim \sqrt[n+1]{\frac{x^n \log x}{n+1}}$, and*

$h^{-1}(\log x) \sim \sqrt[n+1]{\frac{(\log x)^n \log\log x}{n+1}}$ Now if $c = 1/2$, we have $\zeta_p(\sigma + it) = O(t^{1/2})$ for $\sigma \geq 1 - \frac{\log\log t}{n \log t}$ and

$$\mathcal{N}_p(x) = \rho x + O\left(x \exp\left(-\frac{1}{4}\sqrt[n+1]{\frac{(2\log x)^n \; 2\log\log x}{n+1}}\right)\right).$$

5. The Connection Between the Constant c and the Real Part σ of Beurling Zeta Function $\zeta_p(s)$

This part concentrated on figuring out the connection between constant c in the order of Beurling zeta function $\zeta_p(s)$ and σ(the real part of Beurling zeta function $\zeta_p(s)$). In other meaning the reader can see earlier thatt there is a connection between zeta function and beurling integer counting function, so we are interested to figure out this type of connection. fore more details the reader could see ([20], pp.392).

Our aim is to apply Theorem (2) and have to show for which region of σ,

$$\zeta_0(\sigma + it) = O(t^{1/2}).$$

So in order for $|\zeta_0(\sigma + it)| \ll t^c$ to hold for $c = \frac{1}{2}$, we have the following:

$$|\zeta_0(\sigma + it)| \ll \exp\left(1 + t^{100(1-\sigma)^{\frac{3}{2}}} (\log t)^{\frac{2}{3}}\right)$$

So, we get

$$\exp\left[\left(1 + t^{100(1-\sigma)^{\frac{3}{2}}}\right)(\log t)^{\frac{2}{3}}\right] \leq t^{\frac{1}{2}}$$

$$\left(1 + t^{100(1-\sigma)^{\frac{3}{2}}}\right)(\log t)^{\frac{2}{3}} \leq \log t^{\frac{1}{2}}$$

$$\left(1 + e^{100(1-\sigma)^{\frac{3}{2}}\log t}\right) \leq \frac{1}{2}(\log t)^{\frac{1}{3}}$$

$$e^{100(1-\sigma)^{\frac{3}{2}}\log t} \leq 1 + e^{100(1-\sigma)^{\frac{3}{2}}\log t} \leq \frac{1}{2}(\log t)^{\frac{1}{3}}$$

$$\text{so } e^{100(1-\sigma)^{\frac{3}{2}}\log t} \leq \frac{1}{2}(\log t)^{\frac{1}{3}}$$

$$\log\left(e^{100(1-\sigma)^{\frac{3}{2}}\log t}\right) \leq \log\left(\frac{1}{2}(\log t)^{\frac{1}{3}}\right)$$

$$100(1-\sigma)^{\frac{3}{2}}\log t \leq \log(\frac{1}{2}) + log\left((\log t)^{\frac{1}{3}}\right)$$

$$(1-\sigma)^{\frac{3}{2}} \leq \frac{\log\frac{1}{2} + \frac{1}{3}\log\log t}{100\log t}$$

$$\log\left(e^{100(1-\sigma)^{\frac{3}{2}}\log t}\right) \leq \log\left(\frac{1}{2}(\log t)^{\frac{1}{3}}\right)$$

$$100(1-\sigma)^{\frac{3}{2}}\log t \leq \log(\frac{1}{2}) + log\left((\log t)^{\frac{1}{3}}\right)$$

$$(1-\sigma)^{\frac{3}{2}} \leq \frac{\log\frac{1}{2} + \frac{1}{3}\log\log t}{100\log t}$$

$$(1-\sigma) \leq \left(\frac{\log \frac{1}{2} + \frac{1}{3}\log\log t}{100\log t}\right)^{\frac{2}{3}}$$

$$-\sigma \leq -1 + \left(\frac{\log \frac{1}{2} + \frac{1}{3}\log\log t}{100\log t}\right)^{\frac{2}{3}}$$

$$\sigma \geq 1 - \left(\frac{\log \frac{1}{2} + \frac{1}{3}\log\log t}{100\log t}\right)^{\frac{2}{3}}$$

Hence for the above σ, we have $\zeta_0(\sigma + it) = O(t^{\frac{1}{2}})$.

One can show the connection between σ and $0 < c < 2$ by the following table:

Table 2: The Effect of c on σ

c	$\zeta_0(\sigma+it)$	σ
$\frac{1}{10}$	$O(t^{0.1})$	$\geq 1 - \left(\frac{\log \frac{1}{10} + \frac{1}{3}\log\log t}{100\log t}\right)^{\frac{2}{3}}$
$\frac{1}{2}$	$O(t^{0.5})$	$\geq 1 - \left(\frac{\log \frac{1}{2} + \frac{1}{3}\log\log t}{100\log t}\right)^{\frac{2}{3}}$
$\frac{9}{10}$	$O(t^{0.9})$	$\geq 1 - \left(\frac{\log \frac{9}{10} + \frac{1}{3}\log\log t}{100\log t}\right)^{\frac{2}{3}}$
$\frac{3}{2}$	$O(t^{\frac{3}{2}})$	$\geq 1 - \left(\frac{\log \frac{3}{2} + \frac{1}{3}\log\log t}{100\log t}\right)^{\frac{2}{3}}$

From the above table one can see that when c get closer to 1,the region of σ get smaller and also when $1 < c < 2$, the region of σ get smaller and smaller.

6. Conclusion

In conclusion, the purpose of this work was to concentrate on the impact of the order of Beurling zeta function $\zeta_p(s)$ on the size of error term of Beurling integer counting function $\mathcal{N}_p(x)$.

In particular, we discovered by changing the interval of constant c in Theorem (1), different results has been achieved. First, when constant $0 < c < 1$ the error term of $\mathcal{N}_p(x)$ has different explanation. Second if constant $1 < c < 2$, then the error term of $\mathcal{N}_p(x)$ is big as it shown in Section (4). But then by adding some condition to Theorem (1), the error term of $\mathcal{N}_p(x)$ became smaller as it shown in lemma (4).Further more, when constant $c = 1$ and $c \geq 2$, $\mathcal{N}_p(x)$ diverges.

In Addition, this work also focused on the connection between constant c in the order of $\zeta_p(s)$ and the real part σ of $\zeta_p(s)$.

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