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Pure Ideals on Ordered Power Ternary Semigroups on Ternary Semihypergroups Induced by Posets

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Abstract. This article is devoted to the investigation of some algebraic connections between algebraic structures and algebraic hyperstructures. Firstly, we introduce the concept of ordered power ternary semigroups on ternary semihypergroups induced by posets which are generalizations of power ternary semigroups on ternary semihypergroups. Then, the ideas of ideals and pure ideals in ordered power ternary semigroups on ternary semihypergroups induced by posets are introduced. We also study some algebraic properties of pure ideals and weakly pure ideals on the ordered ternary semigroups.

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1. Introduction

A ternary semigroup is a generalization of a semigroup by extending the binary operation to a ternary operation. A ternary semigroup consists of a set with a ternary operation satisfying associativity. The concept of a ternary semigroup was introduced by Lehmer in 1932 [1], who studied the so-called a triplex, which emerges as a commutative ternary group. Banach gave an example to show that a ternary semigroup is not necessarily reduced to an ordinary semigroup. The notion of ternary semigroups was introduced by Los in 1955 [2], who studied some properties of ternary semigroups. Sioson introduced the ideal theory in ternary semigroups [3]. For more information, see [4–7].

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A ternary semihypergroup is an algebraic structure with an associative ternary hyperoperation. Davvaz, Dudek, and Vougiouklis introduced the concept of an n-ary hypergroup, which generalizes a hypergroup in accordance with Marty's definition [8]. A ternary semihypergroup is a specific instance of an n-ary semihypergroup with $n = 3$. Recently, Nongmanee and Leeratanavalee extended Cayley's theorem to study on ternary semihypergroups. Moreover, they discovered several conditions of ternary semihypergroups which can be reduced to semihypergroups [9]. In 2024, they presented some algebraic connections between ternary semigroups and ternary semihypergroups via regularity [10].

The concept of a power semigroup was first explicitly studied by Dubreil [11]. Subsequent studies on power semigroups were conducted by many authors such as Tamura and Shafer [12]. They introduced the notion of a power semigroup $P(S)$ and explored isomorphism problems related to a power semigroup [13]. Jeenkeaw and Leeratanavalee [14] introduced the concept of power semigroups on semihypergroups by using the concept of power groups on hypergroups introduced by Ma, Mi, and Huo [15]. By extending the concept of power semigroup on semihypergroups, the concept of ternary semigroups on ternary semihypergroups was introduced by Nongmanee et al. in 2025 [16].

The notion of power ordered sets was presented by Szymanska and Schweigert [17]. This concept stands between power sets and power relations. They proved that this power relation is antisymmetric by using the theorem of Cantor-Bernstein. Jeenkeaw and Leeratanavalee combined this power relation and the concept of power semigroups on semihypergroups to construct the algebraic structure called ordered power semigroups on semihypergroups induced by posets and study its algebraic properties [18]. Ahsan and Takahashi studied pure ideals in semigroups [19]. Bashir and Shabir studied pure ideals in ternary semigroups [20]. After that, Changphas and Sanborisoot extended their results to study on ordered semigroups and ordered ternary semigroups [21, 22].

In this paper, we establish the notion of ordered power ternary semigroups on ternary semihypergroups induced by posets. Then, we study pure ideals in ordered power ternary semigroups on ternary semihypergroups induced by posets and examine their algebraic properties. Finally, we investigate some interesting results of weakly pure ideals in ordered power ternary semigroups on ternary semihypergroups induced by posets.

2. Preliminaries

First, we recall the definition of ternary semigroups by Lehmer [1] as follows: Let T be a nonempty set and $* : T \times T \times T \to T$ be a mapping which satisfies the *ternary associative law*, i.e., $*(*(a, b, c), d, e) = *(a, *(b, c, d), e) = *(a, b, *(c, d, e))$ for all $a, b, c, d, e \in T$. Then we call $(T, *)$, a *ternary semigroup*. Let H be a nonempty subset of a ternary semigroup $(T, *)$. If H is closed under $*$, i.e., $*(x, y, z) \in H$ for all $x, y, z \in H$ then $(H, *)$ is called a *ternary subsemigroup*.

The algebraic structures of ternary semigroups can be considered as the special case of ternary semihypergroups. We now recall some definitions of an algebraic hyperstructure of ternary semihypergroups [9] as follows: Let S be a nonempty set and $\diamond : S \times S \times S \to \mathcal{P}^*(S)$ be a mapping which is called a *ternary hyperoperation* where $\mathcal{P}^*(S)$

is the family of nonempty subsets of S. An algebraic hyperstructure $(S, \diamond)$ is called a *ternary hypergroupoid*. Additionally, for any $A, B, C \in \mathcal{P}^*(S)$, we can define $\diamond(A, B, C) = \bigcup_{a\in A, b\in B, c\in C} \diamond(a, b, c)$. If a ternary hyperoperation $\diamond$ satisfies the ternary associative law then $(S, \diamond)$ is called a *ternary semihypergroup.*

Definition 1. [17] *Let $(E, \leq)$ be a poset. The relation $\leq_p$ is defined on $P^*(E)$ as follows. For any $\{a_i \mid i \in I\}$, $\{b_j \mid j \in J\} \in P^*(E)$, we have $\{a_i \mid i \in I\} \leq_p \{b_j \mid j \in J\}$ if and only if there exists an injective mapping $\pi : \{a_i \mid i \in I\} \to \{b_j \mid j \in J\}$ such that $a_i \leq \pi(a_i)$ for $i \in I$ and $\{\pi(a_i) \mid i \in I\} \subseteq \{b_j \mid j \in J\}$.*

Theorem 1. [17] *The relation $\leq_p$ is antisymmetric.*

We can see that the relation $\leq_p$ is also reflexive and transitive. That means, it is a partial order. Then $(P^*(E), \leq_p)$ is a partially ordered set which is called an **ordered power set** [17]. We now recall the definition of a power ternary semigroup on a ternary semihypergroup.

Definition 2. [16] *Let $(S, \diamond)$ be a ternary semihypergroup and $\emptyset \neq \mathcal{T} \subseteq \mathcal{P}^*(S)$. Define a ternary operation $\bullet$ on $\mathcal{T}$ by $\bullet(X, Y, Z) = \cup\{\diamond(x, y, z) \mid x \in X, y \in Y, z \in Z\}$ for all $X, Y, Z \in \mathcal{T}$.*

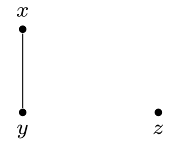
Note here that the ternary semigroup $(\mathcal{T}, \bullet)$ is called a *power ternary semigroup on a ternary semihypergroup* $(S, \diamond)$. From now on, we will combine the concept of an ordered power set together with a power ternary semigroup on a ternary semihypergroup and construct a new algebraic structure called an ordered power ternary semigroup on a ternary semihypergroup induced by a poset.

3. Ordered Power ternary semigroups on ternary semihypergroups induced by Posets

Definition 3. *Let $(S, \leq)$ be a poset and $(\mathcal{T}, \bullet)$ be a power ternary semigroup on ternary semihypergroup $(S, \diamond)$. If the relation $\leq_p$, which is defined as in Definition 1, is compatible with the operation $\bullet$ restricted to $\mathcal{T}$, i.e. for all $X, Y \in \mathcal{T}$, $X \leq_p Y$ implies $\bullet(Z_1, Z_2, X) \leq_p \bullet(Z_1, Z_2, Y)$, $\bullet(Z_1, X, Z_2) \leq_p \bullet(Z_1, Y, Z_2)$ and $\bullet(X, Z_1, Z_2) \leq_p \bullet(Y, Z_1, Z_2)$ for all $Z_1, Z_2 \in \mathcal{T}$, then we call $(\mathcal{T}, \bullet, \leq_p)$ is an* ***ordered power ternary semigroup on a ternary semihypergroup*** *$(S, \diamond)$* ***induced by a poset*** *$(S, \leq)$.*

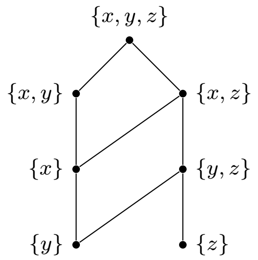
Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ and $\emptyset \neq \mathcal{A} \subseteq \mathcal{T}$. We denote $(\mathcal{A}]_p = \{X \in \mathcal{T} \mid X \leq_p Y$ for some $Y \in \mathcal{A}\}$. If $\mathcal{A} = \{X\}$ then we denote $(\mathcal{A}]_p$ by $(X]_p$.

Example 1. *Let $S = \{x, y, z\}$ and $\leq = \{(x, x), (y, y), (z, z), (y, x)\}$ be a binary relation on S. It is easily seen that $(S, \leq)$ is a poset as Hasse's diagram and we define the ternary hyperoperation $\diamond$ on S as follows.*



$\diamond$	x	y	z
x,x	$\{x\}$	$\{y\}$	$\{z\}$
x,y	$\{y\}$	$\{y\}$	$\{y,z\}$
x,z	$\{z\}$	$\{y,z\}$	$\{z\}$
y,x	$\{y\}$	$\{y\}$	$\{y,z\}$
y,y	$\{y\}$	$\{y\}$	$\{y,z\}$
y,z	$\{y,z\}$	$\{y,z\}$	$\{y,z\}$
z,x	$\{z\}$	$\{y,z\}$	$\{z\}$
z,y	$\{y,z\}$	$\{y,z\}$	$\{y,z\}$
z,z	$\{z\}$	$\{y,z\}$	$\{z\}$

We can see that $(S,\diamond)$ is a ternary semihypergroup. We defined $\leq_p$ on $P^*(S)$. Then we have the following Hasse's diagram. Let $\mathcal{T} = \{\{x\},\{x,y\}\}$ be a subset of $P^*(S)$. We have the Cayley's table of the operation $\bullet$ on $\mathcal{T}$ as follows.



$\bullet$	$\{x\}$	$\{x,y\}$
$\{x\},\{x\}$	$\{x\}$	$\{x,y\}$
$\{x\},\{x,y\}$	$\{x,y\}$	$\{x,y\}$
$\{x,y\},\{x\}$	$\{x,y\}$	$\{x,y\}$
$\{x,y\},\{x,y\}$	$\{x,y\}$	$\{x,y\}$

Then $(\mathcal{T},\bullet)$ is a power ternary semigroup on a ternary semihypergroup $(S,\diamond)$. We have $\{x\} \leq_p \{x\}$, $\{x,y\} \leq_p \{x,y\}$ and $\{x\} \leq_p \{x,y\}$. We can see that $\leq_p$ is compatible with the operation $\bullet$ restricted to $\mathcal{T}$. Therefore $(\mathcal{T},\bullet,\leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S,\diamond)$ induced by a poset $(S,\leq)$.

Definition 4. *Let $(\mathcal{T},\bullet)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S,\diamond)$ induced by a poset $(S,\leq)$. Let $\mathcal{W}$ be a nonempty subset of $\mathcal{T}$. If $\bullet(A,B,C) \in \mathcal{W}$ for any $A,B,C \in \mathcal{W}$ then $\mathcal{W}$ is a* ***power subternary semigroup*** *of $\mathcal{S}$ and every power subternary semigroup of $\mathcal{T}$ is an ordered power ternary semigroup on a ternary semihypergroup $(S,\diamond)$ induced by a poset $(S,\leq)$.*

For nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{T}$, let $\bullet(\mathcal{A},\mathcal{B},\mathcal{C}) = \{\bullet(A,B,C) \mid A \in \mathcal{A}, B \in \mathcal{B}$ and $C \in \mathcal{C}\}$. A nonempty subset $\mathcal{A}$ of $\mathcal{T}$ is a power subternary semigroup of $\mathcal{T}$ if and only if $\bullet(\mathcal{A},\mathcal{A},\mathcal{A}) \subseteq \mathcal{A}$. We denote $\bullet(\mathcal{A},\mathcal{B},\{X\}) = \bullet(\mathcal{A},\mathcal{B},X)$, $\bullet(\mathcal{A},\{X\},\mathcal{B}) = \bullet(\mathcal{A},X,\mathcal{B})$ and $\bullet(\{X\},\mathcal{A},\mathcal{B}) = \bullet(X,\mathcal{A},\mathcal{B})$ for any $\mathcal{A},\mathcal{B} \subseteq \mathcal{T}$ and $X \in \mathcal{T}$.

Corollary 1. *Let $(\mathcal{T},\bullet)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S,\diamond)$ induced by a poset $(S,\leq)$. Let $\{\mathcal{W}_i \mid i \in I\}$ be a nonempty family of power subternary semigroups of $\mathcal{T}$ and $\bigcap_{i\in I}\mathcal{W}_i$ is a nonempty set. Then $\bigcap_{i\in I}\mathcal{W}_i$ is a power subsemigroup of $\mathcal{T}$. Moreover, $\bigcap_{i\in I}\mathcal{W}_i$ is an ordered power ternary semigroup on a ternary semihypergroup $(S,\diamond)$ induced by a poset $(S,\leq)$ and $\bigcup_{i\in I}\mathcal{W}_i$ is not necessary be a power subternary semigroup of $\mathcal{S}$.*

Let $(\mathcal{T}, \bullet)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ and $\mathcal{A} \subseteq \mathcal{T}$. We denoted $(\mathcal{A}]_p := \{X \in \mathcal{T} \mid X \leq_p Y$ for some $Y \in \mathcal{A}\}$.

Lemma 1. *Let $(\mathcal{T}, \bullet)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ and $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{T}$. Then the following statements hold.*

(i) $\mathcal{A} \subseteq (\mathcal{A}]_p$.

(ii) *If* $\mathcal{A} \subseteq \mathcal{B}$ *then* $(\mathcal{A}]_p \subseteq (\mathcal{B}]_p$.

(iii) $\bullet((\mathcal{A}]_p, (\mathcal{B}]_p, (\mathcal{C}]_p) \subseteq (\bullet(\mathcal{A}, \mathcal{B}, \mathcal{C})]_p$.

(iv) $((\mathcal{A}]_p]_p = (\mathcal{A}]_p$.

(v) $(\mathcal{A} \cup \mathcal{B}]_p = (\mathcal{A}]_p \cup (\mathcal{B}]_p$.

(vi) $(\mathcal{A} \cap \mathcal{B}]_p \subseteq (\mathcal{A}]_p \cap (\mathcal{B}]_p$.

(vii) $(\bullet((\mathcal{A}]_p, (\mathcal{B}]_p, (\mathcal{C}]_p)]_p = (\bullet(\mathcal{A}, \mathcal{B}, \mathcal{C})]_p$.

Proof. The proof is straightforward.

Definition 5. *Let $(\mathcal{T}, \bullet)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$. A nonempty subset $\mathcal{A}$ of $\mathcal{T}$ is called a **left (resp. right and lateral) ideal of** $\mathcal{T}$ if and only if (i) $\bullet(\mathcal{T}, \mathcal{T}, \mathcal{A}) \subseteq \mathcal{A}$ (resp. $\bullet(\mathcal{A}, \mathcal{T}, \mathcal{T}) \subseteq \mathcal{A}$ and $\bullet(\mathcal{T}, \mathcal{A}, \mathcal{T}) \subseteq \mathcal{A}$); (ii) $\mathcal{A} = (\mathcal{A}]_p$.*

A nonempty subset $\mathcal{A}$ of $\mathcal{T}$ is called an **ideal of** $\mathcal{T}$ if $\mathcal{A}$ is a left , right and lateral ideal of $\mathcal{T}$. An ideal $\mathcal{A}$ of $\mathcal{T}$ is called a **proper ideal if** $\mathcal{A} \neq \mathcal{T}$. A proper ideal $\mathcal{A}$ of $\mathcal{T}$ is called **the greatest ideal** if every proper ideal is contained in $\mathcal{A}$. A proper ideal $\mathcal{A}$ of $\mathcal{T}$ is called a **maximal ideal** if whenever there exists an ideal $\mathcal{B}$ of $\mathcal{T}$ such that $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{B} = \mathcal{T}$. If $\mathcal{T}$ contains no proper ideals then $\mathcal{T}$ is called **simple**.

Proposition 1. *Let $(\mathcal{T}, \bullet)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$. Then the following statements hold.*

(i) *If $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are ideals of $\mathcal{T}$ then $(\bullet(\mathcal{A}, \mathcal{B}, \mathcal{C})]_p$ is an ideal of $\mathcal{T}$.*

(ii) *If $\mathcal{A}_1, ..., \mathcal{A}_n$ are ideals of $\mathcal{T}$ for any $n = 2k + 1$ and $k \in \mathcal{N}$ then $\bullet(\mathcal{A}_1, \mathcal{A}_2, \bullet ... \bullet (\mathcal{A}_{n-2}, \mathcal{A}_{n-1}, \mathcal{A}_n)) \subseteq \mathcal{A}_1 \cap ... \cap \mathcal{A}_n$.*

(iii) *The union of ideals of $\mathcal{T}$ is an ideal of $\mathcal{T}$.*

(iv) *The finite intersection of ideals of $\mathcal{T}$ is an ideal of $\mathcal{T}$.*

(v) *If $\mathcal{A} \subseteq \mathcal{T}$ then $(\bullet(\mathcal{T}, \mathcal{A}, \mathcal{T})]_p$ is an ideal of $\mathcal{T}$.*

Proof. The proof is straightforward.

Proposition 2. *Let $(\mathcal{T}, \bullet)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ and $\emptyset \neq \mathcal{A} \subseteq \mathcal{T}$. Then $(\mathcal{A} \cup \bullet(\mathcal{T}, \mathcal{T}, \mathcal{A})]_p$ (resp. $(\mathcal{A} \cup \bullet(\mathcal{A}, \mathcal{T}, \mathcal{T})]_p$, $(\mathcal{A} \cup \bullet(\mathcal{T}, \mathcal{A}, \mathcal{T})]_p$ and $(\mathcal{A} \cup \bullet(\mathcal{T}, \mathcal{T}, \mathcal{A}) \cup \bullet(\mathcal{A}, \mathcal{T}, \mathcal{T}) \cup \bullet(\mathcal{T}, \mathcal{A}, \mathcal{T})]_p$) is the smallest left ideal (resp. right ideal, lateral ideal and ideal) of $\mathcal{T}$ containing $\mathcal{A}$.*

Proof. The proof is straightforward.

4. Pure Ideals and Weakly Pure Ideals in Ordered Power Ternary Semigroups on Ternary Semihypergroups Induced by Posets

Definition 6. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$. A* ***left*** *(resp.* ***right, lateral****) ideal $\mathcal{A}$ of $\mathcal{T}$ is called a* ***one-sided left*** *(resp.* ***right, lateral****)* ***pure ideal*** *if for $X \in \mathcal{A}$, there exists $Y, Z \in \mathcal{A}$ such that $X \leq_p \bullet(Y, Z, X)$ (resp. $X \leq_p \bullet(X, Y, Z), X \leq_p \bullet(Y, X, Z)$).*

Definition 7. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$. An ideal $\mathcal{A}$ of $\mathcal{T}$ is called a* ***left (resp. right, lateral) pure ideal*** *if for $X \in \mathcal{A}$ there exists $Y, Z \in \mathcal{A}$ such that $X \leq_p \bullet(Y, Z, X)$ (resp. $X \leq_p \bullet(X, Y, Z), X \leq_p \bullet(Y, X, Z)$).*

We can say that an ideal $\mathcal{A}$ of $\mathcal{T}$ is called a left (resp. right, lateral) pure ideal if $X \in (\bullet(\mathcal{A}, \mathcal{A}, X)]_p$ (resp. $X \in (\bullet(X, \mathcal{A}, \mathcal{A})]_p, X \in (\bullet(\mathcal{A}, X, \mathcal{A})]_p$).

Example 2. *From Example 1, let $\mathcal{H} = \{\{x\}, \{y\}, \{x, y\}\}$. We have the Cayley's table of the operation $\bullet$ on $\mathcal{H}$ as follows. We can see that $(\mathcal{H}, \bullet, \leq_p)$ is an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$.*

$\bullet$	$\{x\}$	$\{y\}$	$\{x,y\}$
$\{x\},\{x\}$	$\{x\}$	$\{y\}$	$\{x,y\}$
$\{x\},\{y\}$	$\{y\}$	$\{y\}$	$\{x,y\}$
$\{x\},\{x,y\}$	$\{x,y\}$	$\{y\}$	$\{x,y\}$
$\{y\},\{x\}$	$\{y\}$	$\{y\}$	$\{y\}$
$\{y\},\{y\}$	$\{y\}$	$\{y\}$	$\{y\}$
$\{y\},\{x,y\}$	$\{y\}$	$\{y\}$	$\{y\}$
$\{x,y\},\{x\}$	$\{x,y\}$	$\{y\}$	$\{x,y\}$
$\{x,y\},\{y\}$	$\{y\}$	$\{y\}$	$\{y\}$
$\{x,y\},\{x,y\}$	$\{x,y\}$	$\{y\}$	$\{x,y\}$

Let $\mathcal{A} = \{\{y\}\}$ be a subset of $\mathcal{H}$. We can see that $\bullet(\mathcal{H}, \mathcal{H}, \mathcal{A}) \subseteq \mathcal{A}$ and $\mathcal{A} = (\mathcal{A}]_p$. Then $\mathcal{A}$ is a left ideal of $\mathcal{H}$. There exists $\{y\} \in \mathcal{A}$ such that $\{y\} \leq_p \bullet(\{y\}, \{y\}, \{y\}) = \{y\}$. Then $\mathcal{A}$ is a one-sided left pure ideal of $\mathcal{H}$. Moreover, we have $\bullet(\mathcal{H}, \mathcal{A}, \mathcal{H}) \subseteq \mathcal{A}$ and $\bullet(\mathcal{A}, \mathcal{H}, \mathcal{H}) \subseteq \mathcal{A}$. So, $\mathcal{A}$ is an ideal of $\mathcal{H}$. Therefore, $\mathcal{A}$ is a left pure ideal of $\mathcal{H}$.

Theorem 2. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ and $\mathcal{A}$ be an ideal of $\mathcal{T}$. Then $\mathcal{A}$ is a left (resp. right) pure ideal if and only if $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = (\bullet(\mathcal{A}, \mathcal{B}, \mathcal{C})]_p$ for all left ideals $\mathcal{C}$, all lateral ideal $\mathcal{B}$ of $\mathcal{T}$ (resp. $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = (\bullet(\mathcal{B}, \mathcal{C}, \mathcal{A})]_p$ for all right ideals $\mathcal{B}$, all lateral ideal $\mathcal{C}$ of $\mathcal{T}$ and $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = (\bullet(\mathcal{B}, \mathcal{A}, \mathcal{C})]_p$ for all left ideals $\mathcal{B}$, all right ideal $\mathcal{C}$ of $\mathcal{T}$).*

Proof. ($\Rightarrow$) Let $\mathcal{A}$ be a left pure ideal and $\mathcal{B}, \mathcal{C}$ be a left ideal of $\mathcal{T}$. Then $\bullet(\mathcal{A}, \mathcal{B}, \mathcal{C}) \subseteq \bullet(\mathcal{A}, \mathcal{T}, \mathcal{T}) \subseteq \mathcal{A}$. Let $X \in (\bullet(\mathcal{A}, \mathcal{B}, \mathcal{C})]_p$. Then there exists $Y \in \bullet(\mathcal{A}, \mathcal{B}, \mathcal{C})$ such that $X \leq_p Y$. That is, $Y \in \bullet(\mathcal{A}, \mathcal{B}, \mathcal{C}) \subseteq \mathcal{A}$. Since $\mathcal{A}$ is an ideal of $\mathcal{T}$, $X \leq_p Y$ and $Y \in \mathcal{A}$, we have $X \in \mathcal{A}$. Also, $Y \in \bullet(\mathcal{A}, \mathcal{B}, \mathcal{C}) \subseteq \bullet(\mathcal{T}, \mathcal{T}, \mathcal{C}) \subseteq \mathcal{C}$. Since $\mathcal{C}$ is a left ideal of $\mathcal{T}$, $X \leq_p Y$ and $Y \in \mathcal{C}$, we have $X \in \mathcal{C}$. Also, $Y \in \bullet(\mathcal{A}, \mathcal{B}, \mathcal{C}) \subseteq \bullet(\mathcal{T}, \mathcal{B}, \mathcal{T}) \subseteq \mathcal{B}$. Since $\mathcal{B}$

is a lateral ideal of $\mathcal{T}$, $X \leq_p Y$ and $Y \in \mathcal{B}$, we have $X \in \mathcal{B}$. Hence $X \in \mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$. Then $(\bullet(\mathcal{A},\mathcal{B},\mathcal{C})]_p \subseteq \mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$. Now, let $X \in \mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$. Since $\mathcal{A}$ is left pure ideal, there exists $Y, Z \in \mathcal{A}$ such that $X \leq_p \bullet(Y,Z,X)$. Since $\bullet(Y,Z,X) \in \bullet(\mathcal{A},\mathcal{B},\mathcal{C})$ and $X \leq_p \bullet(Y,Z,X)$, we have $X \in (\bullet(\mathcal{A},\mathcal{B},\mathcal{C})]_p$. Hence $\mathcal{A} \cap \mathcal{B} \cap \mathcal{B} \subseteq (\bullet(\mathcal{A},\mathcal{B},\mathcal{C})]_p$. Therefore $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = (\bullet(\mathcal{A},\mathcal{B},\mathcal{C})]_p$ for all left ideals $\mathcal{C}$, all lateral ideals $\mathcal{B}$ of $\mathcal{T}$.
($\Leftarrow$) Let $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = (\bullet(\mathcal{A},\mathcal{B},\mathcal{C})]_p$ for all left ideals $\mathcal{C}$, all lateral ideals $\mathcal{B}$ of $\mathcal{T}$ and $X \in \mathcal{A}$. We will show that $X \in (\bullet(\mathcal{A},\mathcal{A},X)]_p$. Consider $X \in \mathcal{A} \cap \mathcal{A} \cap \{X\} = (\bullet(\mathcal{A},\mathcal{A},X)]_p$. Therefore $\mathcal{A}$ is a left pure ideal. Similarly, we can proof that $\mathcal{A}$ is a right (resp. lateral) pure ideal if and only if $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = (\bullet(\mathcal{B},\mathcal{C},\mathcal{A})]_p$ for all right ideals $\mathcal{B}$, all lateral ideal $\mathcal{C}$ of $\mathcal{T}$ and $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = (\bullet(\mathcal{B},\mathcal{A},\mathcal{C})]_p$ for all left ideals $\mathcal{B}$, all right ideal $\mathcal{C}$ of $\mathcal{T}$).

Definition 8. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ and $X \in \mathcal{T}$. If $\bullet(X,Y,Z) = \bullet(Y,X,Z) = \bullet(Z,Y,X) = X$ for all $Y, Z \in \mathcal{T}$ then $\{X\}$ is called a **zero element** of $\mathcal{T}$, denoted by 0.*

Theorem 3. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ with a zero element 0. Then*

(i) *$\{0\}$ is both a left and right pure ideal of $\mathcal{T}$.*

(ii) *The union of left (resp. right) pure ideals of $\mathcal{T}$ is a left (resp. right) pure ideal of $\mathcal{T}$.*

(iii) *The finite intersection of left (resp. right) pure ideals of $\mathcal{T}$ is a left (resp. right) pure ideal of $\mathcal{T}$.*

Proof.

(i) Since $\{0\}$ is an ideal of $\mathcal{T}$ and $0 \leq_p 0 = \bullet(0,0,0)$, we have $\{0\}$ is both a left and right pure ideal of $\mathcal{T}$.

(ii) Let $\{\mathcal{A}_i \mid i \in I\}$ be a family of left (resp. right) pure ideals of $\mathcal{T}$. Then $\bigcup_{i \in I} \mathcal{A}_i$ is an ideal of $\mathcal{T}$. Let $X \in \bigcup_{i \in I} \mathcal{A}_i$. Then $X \in \mathcal{A}_j$ for some $j \in I$. Since $\mathcal{A}_j$ is a left (resp. right) pure ideal, there exists $Y, Z \in \mathcal{A}_j$ such that $X \leq_p \bullet(Y,Z,X)$ (resp. $X \leq_p \bullet(X,Y,Z)$). We have $Y, Z \in \mathcal{A}_j \subseteq \bigcup_{i \in I} \mathcal{A}_i$. Then $\bigcup_{i \in I} \mathcal{A}_i$ is a left (resp. right) pure ideal of $\mathcal{T}$.

(iii) Let $\{\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n\}$ be a finite family of left (resp. right) pure ideals of $\mathcal{T}$. Then $\bigcap_{i=1}^{n} \mathcal{A}_i$ is an ideal of $\mathcal{T}$. Let $X = \bigcap_{i=1}^{n} \mathcal{A}_i$. Then $X \in (\bullet(\mathcal{A}_j, \mathcal{A}_k, X)]_p$ (resp. $X \in (\bullet(X, \mathcal{A}_j, \mathcal{A}_k)]_p$), for some $j, k \in \{1, ..., n\}$. That is, $\bigcap_{i=1}^{n} \mathcal{A}_i$ is a left (resp. right) pure ideal of $\mathcal{T}$.

Theorem 4. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ with a zero element 0 and $\mathcal{A}$ be an ideal of $\mathcal{T}$. Then $\mathcal{A}$ contains the largest left and right pure ideals of $\mathcal{T}$, denoted by $\mathcal{T}(\mathcal{A})$.*

Proof. By Theorem 3(i), we have $\{0\}$ is both a left and right pure ideal contained in $\mathcal{A}$. Then there exists the union of all left and right pure ideals of $\mathcal{T}$ contained in $\mathcal{A}$. That is, the largest left and right pure ideals of $\mathcal{T}$ contained in $\mathcal{A}$.

Theorem 5. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ with a zero element 0 and $\mathcal{A}, \mathcal{B}, \mathcal{A}_i$ be ideals of $\mathcal{T}$ for all $i \in I$. Then the following statements hold.*

(i) $\mathcal{T}(\mathcal{A} \cap \mathcal{B}) = \mathcal{T}(\mathcal{A}) \cap \mathcal{S}(\mathcal{B})$.

(ii) $\bigcup_{i\in I} \mathcal{T}(\mathcal{A}_i) \subseteq \mathcal{T}(\bigcup_{i\in I} \mathcal{A}_i)$.

Proof.

(i) Since $\mathcal{T}(\mathcal{A})$ is the largest left and right pure ideals of $\mathcal{T}$ contained in $\mathcal{A}$, $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{A}$. Likewise, $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{B}$. Then $\mathcal{T}(\mathcal{A}) \cap \mathcal{T}(\mathcal{B}) \subseteq \mathcal{A} \cap \mathcal{B}$. By Theorem 3.11.(iii), we have $\mathcal{T}(\mathcal{A}) \cap \mathcal{T}(\mathcal{B})$ is a pure ideal of $\mathcal{T}$ contained in $\mathcal{A} \cap \mathcal{B}$. Since $\mathcal{T}(\mathcal{A} \cap \mathcal{B})$ is the largest left and right pure ideals of $\mathcal{T}$ contained in $\mathcal{A} \cap \mathcal{B}$, so $\mathcal{T}(\mathcal{A}) \cap \mathcal{T}(\mathcal{B}) \subseteq \mathcal{T}(\mathcal{A} \cap \mathcal{B})$. Since $\mathcal{T}(\mathcal{A} \cap \mathcal{B}) \subseteq \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$, $\mathcal{T}(\mathcal{A} \cap \mathcal{B}) \subseteq \mathcal{T}(\mathcal{A})$. Similarly, $\mathcal{T}(\mathcal{A} \cap \mathcal{B}) \subseteq \mathcal{T}(\mathcal{B})$. Then $\mathcal{T}(\mathcal{A} \cap \mathcal{B}) \subseteq \mathcal{T}(\mathcal{A}) \cap \mathcal{T}(\mathcal{B})$. Therefore $\mathcal{T}(\mathcal{A} \cap \mathcal{B}) = \mathcal{T}(\mathcal{A}) \cap \mathcal{S}(\mathcal{B})$.

(ii) Since $\mathcal{T}(\mathcal{A}_i)$ is the largest left and right pure ideals of $\mathcal{T}$ contained in $\mathcal{A}_i$, $\mathcal{T}(\mathcal{A}_i) \subseteq \mathcal{A}_i$ for all $i \in I$. Then $\bigcup_{i\in I} \mathcal{T}(\mathcal{A}_i) \subseteq \bigcup_{i\in I} \mathcal{A}_i$. By Theorem 3(ii), we have $\bigcup_{i\in I} \mathcal{T}(\mathcal{A}_i)$ is both a left and right pure ideals of $\mathcal{S}$ contained in $\bigcup_{i\in I} \mathcal{A}_i$ for all $i \in I$. Therefore $\bigcup_{i\in I} \mathcal{T}(\mathcal{A}_i) \subseteq \mathcal{T}(\bigcup_{i\in I} \mathcal{A}_i)$.

Definition 9. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$. A left (resp. right) pure ideal $\mathcal{A}$ of $\mathcal{T}$ is said to be* ***left (resp. right) purely maximal*** *if for any proper left (resp. right) pure ideal $\mathcal{B}$ of $\mathcal{T}$ then $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{A} = \mathcal{B}$.*

Definition 10. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$. Let $\mathcal{A}$ be a proper left (resp. right) pure ideal of $\mathcal{T}$. Then $\mathcal{A}$ is called* ***left (resp. right) purely prime*** *if $\mathcal{B}_1 \cap \mathcal{B}_2 \subseteq \mathcal{A}$ implies $\mathcal{B}_1 \subseteq \mathcal{A}$ or $\mathcal{B}_2 \subseteq \mathcal{A}$ for any left (resp. right) pure ideals $\mathcal{B}_1, \mathcal{B}_2$ of $\mathcal{T}$.*

Theorem 6. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$. Then every left (resp. right) purely maximal ideal of $\mathcal{T}$ is purely prime.*

Proof. Let $\mathcal{A}$ be a left (resp. right) purely maximal ideal of $\mathcal{T}$ and $\mathcal{B}, \mathcal{C}$ be left (resp. right) pure ideals of $\mathcal{T}$ such that $\mathcal{B} \cap \mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{B} \nsubseteq \mathcal{A}$. Since $\mathcal{A}, \mathcal{B}$ are left (resp. right) pure ideals of $\mathcal{T}$ and $\mathcal{B} \nsubseteq \mathcal{A}$, $\mathcal{A} \cup \mathcal{B}$ is a left (resp. right) pure ideal of $\mathcal{S}$ such that $\mathcal{A} \subset \mathcal{A} \cup \mathcal{B}$. Since $\mathcal{A}$ is a left (resp. right) purely maximal ideal of $\mathcal{T}$, $\mathcal{T} = \mathcal{A} \cup \mathcal{B}$. Then $\mathcal{C} = \mathcal{S} \cap \mathcal{C} = (\mathcal{A} \cup \mathcal{B}) \cap \mathcal{C} = (\mathcal{A} \cap \mathcal{C}) \cup (\mathcal{B} \cap \mathcal{C}) \subseteq \mathcal{A}$. So $\mathcal{A}$ is left (resp. right) purely prime.

Theorem 7. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ with a zero element 0. Then $\mathcal{T}(\mathcal{A})$ is a left (resp. right) purely prime for any left (resp. right) maximal ideal $\mathcal{A}$ of $\mathcal{T}$.*

Proof. Let $\mathcal{A}$ be a left (resp. right) maximal ideal of $\mathcal{T}$. We will show that $\mathcal{T}(\mathcal{A})$ is a left (resp. right)purely prime. Let $\mathcal{B}, \mathcal{C}$ be left (resp. right) pure ideals of $\mathcal{T}$ such that $\mathcal{B} \cap \mathcal{C} \subseteq \mathcal{S}(\mathcal{A})$. If $\mathcal{B} \subseteq \mathcal{A}$ then $\mathcal{B} \subseteq \mathcal{T}(\mathcal{A})$. Suppose $\mathcal{B} \nsubseteq \mathcal{A}$. Since $\mathcal{A}, \mathcal{B}$ is left (resp. right) pure ideals of $\mathcal{T}$ and $\mathcal{B} \nsubseteq \mathcal{A}$, $\mathcal{A} \cup \mathcal{B}$ is a left (resp. right) pure ideal of $\mathcal{T}$ such that $\mathcal{A} \subset \mathcal{A} \cup \mathcal{B}$. Since $\mathcal{A}$ is a left (resp. right) maximal ideal of $\mathcal{T}$, $\mathcal{T} = \mathcal{A} \cup \mathcal{B}$. Then $\mathcal{C} = \mathcal{T} \cap \mathcal{C} = (\mathcal{A} \cup \mathcal{B}) \cap \mathcal{C} = (\mathcal{A} \cap \mathcal{C}) \cup (\mathcal{B} \cap \mathcal{C}) \subseteq \mathcal{A}$. That is, $\mathcal{C} \subseteq \mathcal{T}(\mathcal{A})$. Therefore $\mathcal{T}(\mathcal{A})$ is a left (resp. right) purely prime.

Theorem 8. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ and $\mathcal{A}$ be a left (resp. right) pure ideal of $\mathcal{T}$. If $X \in \mathcal{T} \backslash \mathcal{A}$ then there exists a left (resp. right) purely prime ideal $\mathcal{B}$ of $\mathcal{T}$ such that $\mathcal{A} \subseteq \mathcal{B}$ and $X \notin \mathcal{B}$.*

Proof. Let $X \in \mathcal{T} \backslash \mathcal{A}$. We set $P = \{\mathcal{B} \mid \mathcal{B}$ is a left (resp. right) pure ideal of $\mathcal{T}$, $\mathcal{A} \subseteq \mathcal{B}$ and $X \notin \mathcal{B}\}$. Since $\mathcal{A} \in P$, $P \neq \emptyset$. We have P is a partially ordered set under the usual inclusion. Let $\{\mathcal{B}_i \mid i \in I\}$ be any totally ordered subset of P. By Theorem 3.11., $\bigcup_{i \in I} \mathcal{B}_i$ is a left (resp. right) pure ideal. Since $\mathcal{A} \subseteq \bigcup_{i \in I} \mathcal{B}_i$ and $X \notin \bigcup_{i \in I} \mathcal{B}_i$, $\bigcup_{i \in I} \mathcal{B}_i \in P$. By Zorn's lemma, P has a maximal element. Let $\mathcal{M}$ be a maximal element of P. We will show that $\mathcal{M}$ is left (resp. right) purely prime ideal. Suppose that $\mathcal{B}_1, \mathcal{B}_2$ are left (resp. right) pure ideals of $\mathcal{T}$ such that $\mathcal{B}_1 \nsubseteq \mathcal{M}$ and $\mathcal{B}_2 \nsubseteq \mathcal{M}$. Since $\mathcal{B}_1, \mathcal{B}_2, \mathcal{M}$ are left (resp. right) pure ideals of $\mathcal{S}$, $\mathcal{B}_1 \cup \mathcal{M}$, $\mathcal{B}_2 \cup \mathcal{M}$ are left (resp. right) pure ideals of $\mathcal{T}$. Since $\mathcal{B}_1 \cup \mathcal{M}$, $\mathcal{B}_2 \cup \mathcal{M}$ are left (resp. right) pure, $\mathcal{A} \subseteq \mathcal{B}_1 \cup \mathcal{M}$, $\mathcal{A} \subseteq \mathcal{B}_2 \cup \mathcal{M}$ and $\mathcal{M}$ is a maximal element of P, then $X \in \mathcal{B}_1 \cup \mathcal{M}, \mathcal{B}_2 \cup \mathcal{M}$. Since $X \in \mathcal{B}_1 \cup \mathcal{M}$ and $X \notin \mathcal{M}$, $X \in \mathcal{B}_1$. Similarly, $X \in \mathcal{B}_2$. Then $X \in \mathcal{B}_1 \cap \mathcal{B}_2$. That is, $\mathcal{B}_1 \cap \mathcal{B}_2 \nsubseteq \mathcal{M}$. Therefore $\mathcal{M}$ is a left (resp. right) purely prime ideal.

Theorem 9. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ and $\mathcal{A}$ be a proper left (resp. right) pure ideal of $\mathcal{T}$. Then $\mathcal{A}$ is an intersection of all the left (resp. right) purely prime ideals of $\mathcal{T}$ containing $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be a proper left (resp. right) pure ideal of $\mathcal{T}$. By Theorem 8, there exists a left (resp. right) purely prime ideal of $\mathcal{T}$ containing $\mathcal{A}$. Let $\{\mathcal{B}_i \mid i \in I\}$ be a family of all purely prime ideals of $\mathcal{T}$ containing $\mathcal{A}$. Then $\mathcal{A} \subseteq \bigcap_{i \in I} \mathcal{B}_i$. Let $X \in \bigcap_{i \in I} \mathcal{B}_i$. Then $X \in \mathcal{B}_i$ for all $i \in I$. Suppose $X \notin \mathcal{A}$. We have $X \in \mathcal{T} \backslash \mathcal{A}$ and $\mathcal{A}$ is a left (resp. right) pure ideal of $\mathcal{T}$. By Theorem 8, there exists a left (resp. right) purely prime ideal $\mathcal{B}$ of $\mathcal{T}$ containing $\mathcal{A}$ and $X \notin \mathcal{B}$. Contradiction with $X \in \mathcal{B}_i$ for all $i \in I$. Then $X \in \mathcal{A}$. Hence $\bigcap_{i \in I} \mathcal{B}_i \subseteq \mathcal{A}$. Therefore $\mathcal{A} = \bigcap_{i \in I} \mathcal{B}_i$.

Next, we introduce the concept of weakly pure ideal in ordered power ternary semigroups on ternary semihypergroups induced by posets.

Definition 11. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$. An ideal $\mathcal{A}$ of $\mathcal{T}$ is said to be **left** (resp. **right**) **weakly pure** if $\mathcal{A} \cap \mathcal{B} = (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{B})]_p$ (resp. $\mathcal{A} \cap \mathcal{B} = (\bullet(\mathcal{B}, \mathcal{A}, \mathcal{A})]_p$) for all two-sided ideals $\mathcal{B}$ of $\mathcal{S}$.*

Lemma 2. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$. Every left (resp. right) pure ideal of $\mathcal{T}$ is left (resp. right) weakly pure.*

Proof. Let $\mathcal{A}$ be a left pure ideal of $\mathcal{T}$ and $X \in \mathcal{A}$. Let $X \in \mathcal{A} \cap \mathcal{B}$ for all two-sided ideals $\mathcal{B}$ of $\mathcal{T}$. Then $X \in \mathcal{A}$ and $X \in \mathcal{B}$. Since $\mathcal{A}$ be a left pure ideal of $\mathcal{T}$, there exists $Y, Z \in \mathcal{A}$ such that $X \leq_p \bullet(Y, Z, X)$. Since $Y, Z \in \mathcal{A}$ and $X \in \mathcal{B}$, $\bullet(Y, Z, X) \in \bullet(\mathcal{A}, \mathcal{A}, \mathcal{B})$. Then $X \in (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{B})]_p$. Hence $\mathcal{A} \cap \mathcal{B} \subseteq (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{B})]_p$. Let $X \in (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{B})]_p$. There exists $Y = \bullet(Y_1, Y_2, Y_3) \in \bullet(\mathcal{A}, \mathcal{A}, \mathcal{B})$ such that $X \leq_p Y$. Since $Y_1, Y_2 \in \mathcal{A}$, $Y_3 \in \mathcal{B}$ and $\mathcal{B}$ is a two-sided ideal of $\mathcal{T}$, we have $Y = \bullet(Y_1, Y_2, Y_3) \in \mathcal{B}$. Since $\mathcal{B}$ is a two-sided ideal of $\mathcal{T}$, $X \in \mathcal{A}$, $Y \in \mathcal{B}$ and $X \leq_p Y$, we have $X \in \mathcal{B}$. Hence $X \in \mathcal{A} \cap \mathcal{B}$. Therefore $\mathcal{A} \cap \mathcal{B} = (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{B})]_p$. Similarly, we can proof that if $\mathcal{A}$ is a right pure ideal of $\mathcal{T}$ then $\mathcal{A} \cap \mathcal{B} = (\bullet(\mathcal{B}, \mathcal{A}, \mathcal{A})]_p$ for all two-sided ideals $\mathcal{B}$ of $\mathcal{T}$.

Theorem 10. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ with a zero element 0 and $\mathcal{A}, \mathcal{B}$ be two-sided ideals of $\mathcal{T}$. Then $\mathcal{B}\mathcal{A}^{-1} = \{T \in \mathcal{T} \mid \text{for all } X, Y \in \mathcal{A}, \bullet(X, Y, T) \in \mathcal{B}\}$ and $\mathcal{A}^{-1}\mathcal{B} = \{T \in \mathcal{T} \mid \text{for all } X, Y \in \mathcal{A}, \bullet(T, X, Y) \in \mathcal{B}\}$ are two-sided ideals of $\mathcal{T}$.*

Proof. Since $0 \in \mathcal{T}$ and $\mathcal{B}$ is a two-sided ideal, then $\bullet(X, Y, 0) = 0 \in \mathcal{B}$ for all $X, Y \in \mathcal{A}$. We have $0 \in \mathcal{B}\mathcal{A}^{-1}$. That is, $\mathcal{B}\mathcal{A}^{-1} \neq \emptyset$. Let $U, V \in \mathcal{T}$ and $T \in \mathcal{B}\mathcal{A}^{-1}$. We will show that $\bullet(U, V, T) \in \mathcal{B}\mathcal{A}^{-1}$. Let $X, Y \in \mathcal{A}$. Since $\bullet(Y, U, V) \in \mathcal{A}$, we have $\bullet(X, Y, \bullet(U, V, T)) = \bullet(X, \bullet(Y, U, V), T) \in \mathcal{B}$. Then $\bullet(U, V, T) \in \mathcal{B}\mathcal{A}^{-1}$. Let $X \in \mathcal{B}\mathcal{A}^{-1} and Y \in \mathcal{T}$ such that $Y \leq_p X$. Let $W, Z \in \mathcal{A}$. Since $\bullet(W, Z, Y) \leq_p \bullet(W, Z, X)$ and $\bullet(W, Z, X) \in \mathcal{B}$, we have $\bullet(W, Z, Y) \in \mathcal{B}$. Hence $Y \in \mathcal{B}\mathcal{A}^{-1}$.Therefore $\mathcal{B}\mathcal{A}^{-1}$ is a two-sided ideal of $\mathcal{T}$. Similarly, we can proof that $\mathcal{A}^{-1}\mathcal{B}$ is a two-sided ideal of $\mathcal{T}$.

Theorem 11. *Let $(\mathcal{T}, \bullet, \leq_p)$ be an ordered power ternary semigroup on a ternary semihypergroup $(S, \diamond)$ induced by a poset $(S, \leq)$ and $\mathcal{A}$ be a two-sided ideal of $\mathcal{T}$. Then $\mathcal{A}$ is a left (resp. right) weakly pure if and only if $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap (\mathcal{B}\mathcal{A}^{-1})$ (resp. $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap (\mathcal{A}^{-1}\mathcal{B})$) for all ideals $\mathcal{B}$ of $\mathcal{T}$.*

Proof. $(\Rightarrow)$ Let $\mathcal{A}$ be a left weakly pure and $\mathcal{B}$ be an ideal of $\mathcal{T}$. By Theorem 10, $\mathcal{B}\mathcal{A}^{-1}$ is an ideal of $\mathcal{T}$. Since $\mathcal{A}$ is a left weakly pure and $\mathcal{B}\mathcal{A}^{-1}$ is an ideal of $\mathcal{T}$, we have $\mathcal{A} \cap (\mathcal{B}\mathcal{A}^{-1}) = (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{B}\mathcal{A}^{-1})]_p$. Since $\bullet(\mathcal{A}, \mathcal{A}, \mathcal{B}\mathcal{A}^{-1}) \subseteq \bullet(\mathcal{A}, \mathcal{A}, \mathcal{T}) \subseteq \mathcal{A}$, we have $(\bullet(\mathcal{A}, \mathcal{A}, \mathcal{B}\mathcal{A}^{-1})]_p \subseteq (\mathcal{A}]_p = A$. Let $Z \in (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{B}\mathcal{A}^{-1})]_p$. There exists $\bullet(X, Y, Z) \in \bullet(\mathcal{A}, \mathcal{A}, \mathcal{B}\mathcal{A}^{-1})$ such that $Z \leq_p \bullet(X, Y, Z)$. By the definition of $\mathcal{B}\mathcal{A}^{-1}$, we have $Z \in \mathcal{B}$. Hence $\mathcal{A} \cap (\mathcal{B}\mathcal{A}^{-1}) \subseteq \mathcal{A} \cap \mathcal{B}$. Let $X \in \mathcal{A} \cap \mathcal{B}$. Since $\bullet(U, V, X) \in \mathcal{B}$ for any $U, V \in \mathcal{A}$, we have $X \in \mathcal{B}\mathcal{A}^{-1}$. That is, $X \in \mathcal{A} \cap (\mathcal{B}\mathcal{A}^{-1})$. Then $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \cap (\mathcal{B}\mathcal{A}^{-1})$. Therefore $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap (\mathcal{B}\mathcal{A}^{-1})$ for all ideals $\mathcal{B}$ of $\mathcal{T}$. $(\Leftarrow)$ Let $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap (\mathcal{B}\mathcal{A}^{-1})$ for all ideals $\mathcal{B}$

of $\mathcal{T}$. We will show that $\mathcal{A} \cap \mathcal{C} = (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p$ for all ideals $\mathcal{C}$ of $\mathcal{S}$. Let $\mathcal{C}$ be an ideal of $\mathcal{T}$. By assumption, $\mathcal{A} \cap \mathcal{C} = \mathcal{A} \cap (\mathcal{C}\mathcal{A}^{-1})$. Since $\mathcal{C} \subseteq \mathcal{S}$ and $\mathcal{A}$ is an ideal of $\mathcal{S}$, we have $\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C}) \subseteq \bullet(\mathcal{A}, \mathcal{A}, \mathcal{T}) \subseteq \mathcal{A}$. Then $(\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p \subseteq (\mathcal{A}]_p$. Since $\mathcal{A}$ is an ideal of $\mathcal{T}$, we have $(\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p \subseteq \mathcal{A}$. Let $X \in (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p$. There exists $Y \in \bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})$ such that $X \leq_p Y$. Then $Y = \bullet(Y_1, Y_2, Y_3)$ for some $Y_1, Y_2 \in \mathcal{A}$, $Y_3 \in \mathcal{C}$. Then $X \leq_p \bullet(Y_1, Y_2, Y_3)$. Let $U, V \in \mathcal{A}$. Since $\bullet(U, \bullet(V, Y_1, Y_2), Y_3) = \bullet(U, V, \bullet(Y_1, Y_2, Y_3)) \in \mathcal{C}$, we have $\bullet(Y_1, Y_2, Y_3) \in \mathcal{C}\mathcal{A}^{-1}$. That is, $X \in \mathcal{C}\mathcal{A}^{-1}$. Then $(\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p \subseteq \mathcal{C}\mathcal{A}^{-1}$. Therefore $(\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p \subseteq \mathcal{A} \cap \mathcal{C}$. For the reverse inclusion, let $C \in \mathcal{C}, A, B \in \mathcal{A}$, we have $\bullet(A, B, C) \in \bullet(\mathcal{A}, \mathcal{A}, \mathcal{C}) \subseteq (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p$. Then $\mathcal{C} \subseteq (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p\mathcal{A}^{-1}$. Hence $\mathcal{A} \cap \mathcal{C} \subseteq \mathcal{A} \cap (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p\mathcal{A}^{-1} = \mathcal{A} \cap (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p \subseteq (\bullet(\mathcal{A}, \mathcal{A}, \mathcal{C})]_p$. Therefore $\mathcal{A}$ is a left weakly pure if and only if $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap (\mathcal{B}\mathcal{A}^{-1})$ for all ideals $\mathcal{B}$ of $\mathcal{S}$. Similarly, we can proof that $\mathcal{A}$ is a right weakly pure if and only if $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap (\mathcal{A}^{-1}\mathcal{B})$ for all ideals $\mathcal{B}$ of $\mathcal{S}$.

5. Conclusion

In this article, we introduced the notion of ordered power ternary semigroups on ternary semihypergroups induced by posets which is an extension of power ternary semigroup on ternary semihypergroups. Particularly, we investigated their algebraic properties including pure ideals and weakly pure ideals.

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