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Supra Regularity and Supra Normality Inspired by Supra-ϵ-Open Sets

M. Aldawood[1], Alaa M. Abd El-latif[2], Khaled A. Aldwoah[3], A. A. Azzam[1,4], Abdelhalim Hasnaoui[2,*], M. I. Elashiry[2], Enas H. Elkordy[1,5], Husham M. Attaalfadeel[2]

[1] *Department of Mathematics, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia*
[2] *Department of Mathematics, College of Science, Northern Border University, Arar 91431, Saudi Arabia*
[3] *Department of Mathematics, Faculty of Science, Islamic University of Madinah, Medinah, Saudi Arabia*
[4] *Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt*
[5] *Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni Suef, Egypt*

Abstract. In this article, as an extension of the concepts of supra-ϵ-T_2-space, supra-ϵ-T_1-space, and supra-ϵ-T_0-space, we present the notion of supra-ϵ-completely space. Furthermore, we demonstrate that for any STS (γ, θ), the concepts of supra-ϵ-T_2-space and supra-ϵ-completely-space are the same if $|\gamma| \leqslant 4$. We also explore the behavior of this notion with respect to specific forms of supra functions. We demonstrate that, under a bijective supra ϵ^*-open function, the image of any supra-ϵ-$T_{2\frac{1}{2}}$-space is a supra-ϵ-$T_{2\frac{1}{2}}$. Additionally, we demonstrate that every supra subspace of supra-ϵ-$T_{2\frac{1}{2}}$-space is supra-ϵ-$T_{2\frac{1}{2}}$. Moreover, four new versions of separation axioms that utilize supra ϵ-open sets are introduced namely: supra-ϵ-regular-space, supra-ϵ-normal-space, supra-ϵ-T_3-space, and supra-ϵ-T_4-space. We also give a general illustration of their key traits and look at the prerequisites for a number of similar links between them. We also propose a figure 1 graphic that shows these linkages. Furthermore, we demonstrate that every supra-ϵ-R-space (γ, θ) is supra-ϵ-N-space if $|\gamma| \leqslant 4$. This implies that the approaches of supra-ϵ-T_3-space and supra-ϵ-T_4-space are the same in this case. The necessary counterexamples that validate our findings are finally presented.

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Key Words and Phrases: Supra-ϵ-Completely Space, Supra-ϵ-Regularity, Supra-ϵ-Normality, Supra Hereditary Property

*Corresponding author.
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Email addresses: m.aldawood@psau.edu.sa (M. Aldawood), alaa.ali@nbu.edu.sa, alaa_8560@yahoo.com (A. M. Abd El-latif), aldwoah@yahoo.com (K. A. Aldwoah), aa.azzam@psau.edu.sa (A. A. Azzam), abdllhalim.hasanawa@nbu.edu.sa (A. Hasnaoui), mustafa.elashiry@nbu.edu.sa (M. I. Elashiry), e.elkordy@psau.edu.sa (E. H. Elkordy), husham.alhassan@nbu.edu.sa (H. M. Attaalfadeel)

1. Introduction

Over the past few decades, supra topologies, topologies, fuzzy topologies, and soft topologies research has been heavily influenced by the study of many types of generalized open, supra open, and soft open sets as well as their basic characteristics. Semi-open sets (continuous maps) were initially proposed by Levine [1] in 1963. Njasta [2] introduced the idea of α-open sets two years later. Mashhour et al. introduced the idea of pre open sets (pre continuous maps) in 1982 [3]. The notion of β-open sets (β-continuous maps) was first presented by Abd-El-Monsef et al. [4] in 1983. In 1966, the definition of b-open sets was investigated in detail [5, 6]. Piotrowski [7] presented the notion of partially open sets (continuous maps) based on [8]. In [9, 10], the approach of somewhere dense sets (also known as sd-sets) was proposed. Other aspects of this concept were explored in [11]. F-open was first introduced by Alqahtani [12]. In 2024, Alqahtani et al. [13] presented the $\mathcal{N}$-open sets approach, which generalizes almost all of the earlier concepts. New kinds of operators were presented by Alghamdi et al. in the context of primal topological spaces [14].

The concept of supra open sets was established by Mashhour et al. [15] and takes into account the fundamental components of supra topology (or STS). They created fundamental topological concepts like as continuity, interior (closure) operators, and separation axioms. The concepts of supra semi- [16] (R- [17], β- [18], b- [19], pre- [20], and α- [21]) open sets have been presented as well as their key characteristics.

A wide range of soft continuity and soft open sets have been produced by the fields of broadly applicable soft open sets [22, 23], soft semi-open sets [24, 25], soft sd-sets [26], and nearly soft β-open sets [27]. Later research was done on soft continuity [28, 29]. In [30], the concept of the soft ideal initially appeared. This concept was then generalized by Fatouh et al. [31] using soft semi-open sets. This approcach is then used to generalize a number of topological features, including soft-I-open sets [32, 33], soft open sets via soft ideals [34], soft compactness [35], soft ideals for congruence representations [36], generalized soft rough sets [37, 38], soft separation axioms [39], and soft connectedness [40]. Recently, several lower soft separation axioms [41] and certain applications of soft δ-closed sets [42] were presented.

El-Sheikh et al. [43] established the definition of supra soft topological space. A variety of supra soft operators have been explored in subsequent studies in terms of supra soft-b-open sets [44], supra (strongly) generalized closed soft sets via soft ideals [45, 46], supra soft sw-open sets [47], supra soft δ_i-open sets [48, 49], and soft separation axioms [50]. The concept of supra soft sd-sets [51, 52] was recently exploited by Abd El-latif et al. to introduce new forms of soft connectedness [53] and several forms of compactness [54, 55]. Soft nodec spaces were given by Alqahtani et al. [56, 57].

In STSs, Abd El-latif et al. established the concept of supra ϵ-open sets [58]. They also provided many kinds of operators, named supra ϵ-closure (boundary, exterior, accumulation, and interior, respectively). Using this notion, he and his colleagues [59] explored novel types of supra maps, named supra ϵ (ϵ^*)-continuous maps, supra ϵ-irresolute maps, supra ϵ-open (closed) maps, and supra ϵ-homeomorphism maps. In [60], in the framework of STSs, the authors introduced new weaker version of the supra septarian axioms based on supra ϵ-open sets, along with its key characteristics, which are referred to as supra-ϵ-T_j-space, $j = 0, 1, 2$.

This manuscript is structured as follows: We give the definitions and findings that are required for the sequel in Preliminaries.
In section 3, we present the notion of supra-ϵ-completely space as an extension of the notions of supra-ϵ-T_2-space, supra-ϵ-T_1-space, and supra-ϵ-T_0-space. We also investigate how this concept behaves in relation to particular supra function forms.
In section 4, four new categories of separation axioms are presented that utilize the employing of supra ϵ-open sets named: supra-ϵ-regular-space, supra-ϵ-normal-space, supra-ϵ-T_3-space, and supra-ϵ-T_4-space. We also present an overview of their key characteristics and look at the prerequisites for a number of similar relationships between them. We also propose a figure 1 that shows these linkages.

$$\begin{array}{ccccccc} \text{supra-}\epsilon\text{-}T_4\text{-space} & \Longrightarrow & \text{supra-}\epsilon\text{-}T_3\text{-space} & \Longrightarrow & \text{supra-}\epsilon\text{-}T_{2\frac{1}{2}}\text{-space} & \Longrightarrow & \text{supra-}\epsilon\text{-}T_2\text{-space} \\ & & & & & & \Downarrow \\ & & & & \text{supra-}\epsilon\text{-}T_0\text{-space} & \Longleftarrow & \text{supra-}\epsilon\text{-}T_1\text{-space.} \end{array}$$

Figure 1. The relationships between different kinds of separation axioms in the context of STSs which are motivated by supra ϵ-open sets

Furthermore, the necessary counterexamples that validate our findings are finally presented.
In Conclusion and future works section, we present an analytical explanation of the concepts and conclusions discussed in this paper, along with a plan for next research based on this study.

2. Preliminaries

Definition 1. *[15] Supra topology (or STS) on γ is the family $\theta \subseteq 2^{\gamma}$ which contains γ and $\emptyset$ and closed under arbitrary union. Additionally, H and H^c are referred to as supra open and supra closed sets, respectively, if $H \in \theta$. Moreover, all supra open sets will additionally have their class indicated by $SO(\gamma)$. Additionally, θ is referred to be an associated STS with σ for a particular topology σ if $\sigma \subset \theta$.*

Definition 2. *[15] The $int^s(W)$ ($cl^s(W)$, $fr^s(W)$) will indicate the supra interior (closure, boundary) for a subset W of an STS (γ, θ), where*

$$int^s(W) = \cup\{Q : Q \in \theta \text{ and } Q \subseteq W\},\ cl^s(W) = \cap\{P : P \in \theta^c \text{ and } W \subseteq P\} \text{ and}$$
$$fr^s(W) = cl^s(W) \backslash int^s(W).$$

Definition 3. *[17] Let P be a subset of an STS (γ, θ). If $int^s(cl^s(P)) \neq \emptyset$, then $P \in SRO(\gamma)$. Also, if $int^s(cl^s(P)) = \emptyset$, then $P \in SND(\gamma)$.*

Definition 4. *[58] Regarding the subset Z of an STS (γ, θ), the class*

$$\theta_Z = \{Z \cap G : G \in \theta\}$$

defines an STS on Z, also known as a supra subspace of (γ, θ).

Definition 5. *[58] A subset W of an STS (γ, θ) is referred to as supra ϵ-open set if either $W = \emptyset$ or*

$$W \subseteq \begin{cases} fr^s(W) \cup int^s(cl^s(W)), & W \in SRO(\gamma), \\ fr^s(W), & W \in SND(\gamma) \text{ and } fr^s(W) \text{ is infinite}. \end{cases}$$

Furthermore, W^c is called supra ϵ-closed-set. Additionally, $SO_\epsilon(\gamma)$ ($SC_\epsilon(\gamma)$) will be used to classify all supra ϵ-open (supra ϵ-closed) sets.

Definition 6. *[58] For a subset W of an STS (γ, θ), the supra ϵ-interior (closure) of W will be indicated by the $int^s_\epsilon(W)$ ($cl^s_\epsilon(W)$), where*

$$int^s_\epsilon(W) = \cup\{Q : Q \in SO_\epsilon(\gamma) \text{ and } Q \subseteq W\} \text{ and}$$
$$cl^s_\epsilon(W) = \cap\{R : R \in SC_\epsilon(\gamma) \text{ and } W \subseteq R\}$$

Theorem 1. *[58] If we consider a subset S of an STS (γ, θ) with $\sigma \subset \theta$, then we obtain that*

(1) $cl^s_\epsilon(S^c) = [int^s_\epsilon(S)]^c$.

(2) $int^s_\epsilon(S^c) = [cl^s_\epsilon(S)]^c$.

(3) $int(S) \subseteq int^s(S) \subseteq int^s_\epsilon(S)$, *where $int(S)$ denotes the interior of S with respect to σ.*

(4) $cl^s_\epsilon(S) \subseteq cl^s(S) \subseteq cl(S)$, *where $cl(S)$ denotes the closure of S with respect to σ.*

Definition 7. *[58] Let W be a subset of an STS (γ, θ) with an arbitrary point $s \in \gamma$. If each supra ϵ-open set G_s containing s, we obtain that*

$$[W \backslash \{s\}] \cap G_s \neq \emptyset,$$

Consequently, s is called a supra ϵ-accumulation point of W. Also, the set of all supra ϵ-accumulation points of W shall be represented by the notation $acc_\epsilon(W)$.

Definition 8. *[60] An STS (γ, θ) is said to be*

(1) *Supra-ϵ-T_0-space if for each distinct points there is a supra-ϵ-open set including one but excluding the other.*

(2) *Supra-ϵ-T_1-space if for each distinct points $\vartheta_1, \vartheta_2 \in \gamma$, then there are two supra-ϵ-open subsets ν_1 and ν_2 of γ, such that $\vartheta_1 \in \nu_1$, $\vartheta_2 \notin \nu_1$, and $\vartheta_1 \notin \nu_2$, $\vartheta_2 \in \nu_2$.*

(3) *Supra-ϵ-T_2-space "supra-ϵ-Hausdorff space" if for each distinct points $\vartheta_1, \vartheta_2 \in \gamma$, then there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, such that $\vartheta_1 \in \nu_1$ and $\vartheta_2 \in \nu_2$.*

Definition 9. *[60] For the subset L of an STS (γ, ϑ), the class*

$$\vartheta_L = \{L \cap O : O \in SO_\epsilon(\gamma)\}$$

defines an STS on L, and it is called an supra ϵ-subspace of (γ, ϑ).

Definition 10. *[60] A function $\gamma_\epsilon : (\gamma_1, \vartheta_1) \rightarrow (\gamma_2, \vartheta_2)$ with θ_1, θ_2 associated STSs with ϑ_1, ϑ_2, respectively, is said to be:*

(1) *Supra ϵ-continuous (abbreviate: supra ϵ-cts) if $\gamma_\epsilon^{-1}(G) \in SO_\epsilon(\gamma_1)$ for each $G \in \vartheta_2$.*

(2) *Supra ϵ-irresolute if $\gamma_\epsilon^{-1}(D) \in SO_\epsilon(\gamma_1)$ for each $D \in SO_\epsilon(\gamma_2)$.*

(3) *Supra ϵ (ϵ^*)-open if $\gamma_\epsilon(U) \in SO_\epsilon(\gamma_2)$ for each $U \in \theta_1$ ($U \in SO_\epsilon(\gamma_1)$).*

(4) *supra-ϵ^*-homeomorphism if it is bijective supra ϵ^*-open and supra ϵ^*-cts.*

3. Supra ϵ-completely spaces

In this section, we provide the notion of supra-ϵ-completely space as a generalization to the approaches of supra-ϵ-T_2-space, supra-ϵ-T_1-space, and supra-ϵ-T_0-space. Additionally, we prove that the notions of supra-ϵ-T_2-space supra-ϵ-completely-space are identical for any STS (γ, θ), if $|\gamma| \leqslant 4$. Furthermore, we investigate this concept's behavior in relation to particular supra function types. In special, we show that the image of each supra-ϵ-$T_{2\frac{1}{2}}$-space is a supra-ϵ-$T_{2\frac{1}{2}}$ under a bijective supra ϵ^*-open function. Finally, we prove that every supra subspace of supra-ϵ-$T_{2\frac{1}{2}}$-space is supra-ϵ-$T_{2\frac{1}{2}}$.

Definition 11. *An STS (γ, θ) is said to be supra-ϵ-$T_{2\frac{1}{2}}$-space "supra-ϵ-completely space" if for each distinct points $\vartheta_1, \vartheta_2 \in \gamma$, then there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, such that $\vartheta_1 \in \nu_1$, $\vartheta_2 \in \nu_2$ and $cl_\epsilon^s(\nu_1) \cap cl_\epsilon^s(\nu_2) = \emptyset$.*

Theorem 2. *Any supra-ϵ-$T_{2\frac{1}{2}}$-space is supra-ϵ-T_2.*

Proof. Follows from Definition 11.

Remark 1. *The converse of Theorem 2 is not hold as the upcoming example will demonstrate.*

Example 1. *Let $\theta = \{\gamma, \emptyset, \{2,4\}, \{1,3\}, \{2,3,4\}, \{1,2,4\}, \{1,2,3\}, \{3,4\}, \{1,3,4\}, \{1,4\}, \{1,2,5\}, \{3,4,5\}, \{2,3,4,5\}, \{1,3,4,5\}, \{1,2,4,5\}, \{1,2\}, \{1,2,3,5\}, \{1,2,3,4\}, \{2,3\}\}$ be an STS on $\gamma = \{1,2,3,4,5\}$. Regarding $1 \neq 2 \in \gamma$, then there are not two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, such that $1 \in \nu_1$, $2 \in \nu_2$ and $cl_\epsilon^s(\nu_1) \cap cl_\epsilon^s(\nu_2) = \emptyset$. Hence, γ is not supra-ϵ-$T_{2\frac{1}{2}}$. Also, it is easy to check that γ is supra-ϵ-T_2.*

Theorem 3. *For any STS (γ, θ), if $|\gamma| \leqslant 4$, then the approaches of supra-ϵ-$T_{2\frac{1}{2}}$-space and supra-ϵ-T_2-space are identical.*

Proof. Given Theorem 2, we have that every supra-$T_{2\frac{1}{2}}$-space is supra-ϵ-T_2. Now, let (γ, θ) be a supra-ϵ-T_2-space and $\vartheta_1 \neq \vartheta_2 \in \gamma$. Then, there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ containing ϑ_1 and ϑ_2, respectively. This implies that,

$$cl^s_\epsilon(\nu_1) \subseteq \nu_2^c \; and \; cl^s_\epsilon(\nu_2) \subseteq \nu_1^c \tag{1}$$

Now, we have two cases:
Case (1), $|\nu_1| = 1$ or $|\nu_1| = 3$, then ν_1 is both supra-ϵ-open and supra-ϵ-closed, which implies that $cl^s_\epsilon(\nu_1) \cap cl^s_\epsilon(\nu_2) = \emptyset$, given Equation 1.
Case (2), $|\nu_1| = 2$, then either $|\nu_2| = 2$ or $|\nu_2| = 1$. If $|\nu_2| = 2$, then both ν_1 and ν_2 are both supra-ϵ-open and supra-ϵ-closed, and hence $cl^s_\epsilon(\nu_1) \cap cl^s_\epsilon(\nu_2) = \emptyset$.
If $|\nu_2| = 1$, then ν_2 is both supra-ϵ-open and supra-ϵ-closed, and thus $cl^s_\epsilon(\nu_1) \cap cl^s_\epsilon(\nu_2) = \emptyset$.
Therefore, γ is supra-ϵ-$T_{2\frac{1}{2}}$.

Definition 12. *A function $\gamma_\epsilon : (\gamma_1, \sigma_1) \rightarrow (\gamma_2, \sigma_2)$ with θ_1, θ_2 associated STSs with σ_1, σ_2 respectively, is said to be supra ϵ (ϵ^*)-closed if $\gamma_\epsilon(U) \in SC_\epsilon(\gamma_2)$ for each $U \in \theta_1^c$ ($U \in SC_\epsilon(\gamma_1)$).*

Theorem 4. *Let $\gamma_\epsilon : (\gamma_1, \sigma_1) \rightarrow (\gamma_2, \sigma_2)$ be a function with θ_1, θ_2 associated STSs with σ_1, σ_2 respectively, and $\beta \subseteq \gamma_1$, then*

$$\gamma_\epsilon \text{ is supra } \epsilon\text{-closed if and only if } cl^s_\epsilon[\gamma_\epsilon(\beta)] \subseteq \gamma_\epsilon(cl^s(\beta)).$$

Proof. " $\Rightarrow$ " Let us suppose that γ_ϵ be a supra ϵ-closed function and $\beta \subseteq \gamma_1$. Since $\gamma_\epsilon(\beta) \subseteq \gamma_\epsilon(cl^s(\beta))$, $cl^s_\epsilon[\gamma_\epsilon(\beta)] \subseteq cl^s_\epsilon[\gamma_\epsilon(cl^s(\beta))] = \gamma_\epsilon(cl^s(\beta))$, given γ_ϵ is supra ϵ-closed function.
" $\Leftarrow$ " Let $\beta \in \theta_1^c$. Considering the assumption,

$$\gamma_\epsilon(\beta) \subseteq cl^s_\epsilon[\gamma_\epsilon(\beta)] \subseteq \gamma_\epsilon(cl^s(\beta)) = \gamma_\epsilon(\beta).$$

Hence,

$$cl^s_\epsilon[\gamma_\epsilon(\beta)] = \gamma_\epsilon(\beta).$$

Therefore,

$$\gamma_\epsilon(\beta) \in SC_\epsilon(\gamma_2), \text{ and hence } \gamma_\epsilon \text{ is a supra } \epsilon\text{-closed function.}$$

Proposition 1. *Let $\gamma_\epsilon : (\gamma_1, \sigma_1) \rightarrow (\gamma_2, \sigma_2)$ be a function with θ_1, θ_2 associated STSs with σ_1, σ_2 respectively, and $\beta \subseteq \gamma_1$, then*

$$\gamma_\epsilon \text{ is supra } \epsilon^*\text{-closed if and only if } cl^s_\epsilon[\gamma_\epsilon(\beta)] \subseteq \gamma_\epsilon(cl^s_\epsilon(\beta)).$$

Proof. By a similar manner to the proof of Theorem 4.

Theorem 5. *Let $\gamma_\epsilon : (\gamma_1, \sigma_1) \rightarrow (\gamma_2, \sigma_2)$ be a bijective function with θ_1, θ_2 associated STSs with σ_1, σ_2 respectively, then γ_ϵ is supra ϵ-open function if and only if it is supra ϵ-closed.*

Proof. " $\Rightarrow$ " Let $W \in \theta_1^c$, then $W^c \in \theta_1$. Since γ_ϵ is supra bijective ϵ-open function,

$$[\gamma_\epsilon(W)]^c = \gamma_\epsilon(W^c) \in SO_\epsilon(\gamma_1).$$

This implies that,

$$\gamma_\epsilon(W) \in SO_\epsilon(\gamma_2).$$

Thus, γ_ϵ is a supra ϵ-closed function.
" $\Leftarrow$ " It is preceded by a similar assertion.

Corollary 1. *Let $\gamma_\epsilon : (\gamma_1, \sigma_1) \rightarrow (\gamma_2, \sigma_2)$ be a bijective function with θ_1, θ_2 associated STSs with σ_1, σ_2 respectively, then γ_ϵ is supra ϵ^*-open function if and only if it is supra ϵ^*-closed.*

Proof. Direct from Theorem 5.

Theorem 6. *The image of each supra-ϵ-$T_{2\frac{1}{2}}$-space is a supra-ϵ-$T_{2\frac{1}{2}}$ under a bijective supra ϵ^*-open function.*

Proof. Let $\gamma_\epsilon : (\gamma_1, \sigma_1) \rightarrow (\gamma_2, \sigma_2)$ with θ_1, θ_2 associated STSs with σ_1, σ_2 respectively, be a bijective supra ϵ^*-open function such that (γ_1, θ_1) is supra-ϵ-$T_{2\frac{1}{2}}$. Let $\alpha_1 \neq \alpha_2 \in \gamma_2$. Since γ_ϵ is bijective, there are $\xi_1 \neq \xi_2 \in \gamma_1$ such that $\gamma_\epsilon^{-1}(\alpha_1) = \xi_1$ and $\gamma_\epsilon^{-1}(\alpha_2) = \xi_2$. Since (γ_1, σ_1) is supra-ϵ-$T_{2\frac{1}{2}}$, there are two disjoint supra-ϵ-open subsets ρ_1 and ρ_2 of γ_1, such that $\xi_1 \in \rho_1$, $\xi_2 \in \rho_2$ and $cl_\epsilon^s(\rho_1) \cap cl_\epsilon^s(\rho_2) = \emptyset$. Since γ_ϵ is supra ϵ^*-open, $\gamma_\epsilon(\rho_1)$ and $\gamma_\epsilon(\rho_2)$ are two disjoint supra-ϵ-open subsets of γ_2 containing α_1, α_2, respectively, such that $\gamma_\epsilon(cl_\epsilon^s(\rho_1)) \cap \gamma_\epsilon(cl_\epsilon^s(\rho_2)) = \emptyset$. Given Corollary 1, γ_ϵ is supra ϵ^*-closed. According to Proposition 1, $cl_\epsilon^s[\gamma_\epsilon(\rho_1)] \subseteq \gamma_\epsilon(cl_\epsilon^s(\rho_1))$ and $cl_\epsilon^s[\gamma_\epsilon(\rho_2)] \subseteq \gamma_\epsilon(cl_\epsilon^s(\rho_2))$ and consequently $cl_\epsilon^s(\gamma_\epsilon(\rho_1)) \cap cl_\epsilon^s(\gamma_\epsilon(\rho_2)) = \emptyset$. Therefore, (γ_2, θ_2) is supra-ϵ-$T_{2\frac{1}{2}}$.

The proof of the following corollary is obvious from Theorem 6.

Corollary 2. *The property of being a supra-ϵ-$T_{2\frac{1}{2}}$-space is a supra hereditary property.*

Proposition 2. *Let (U, ϑ_U) be an supra ϵ-subspace of an STS (γ, ϑ) and V be a subset of γ. Then, $(cl_\epsilon(V))_{\vartheta_U} = U \cap cl_\epsilon(V)$*

Theorem 7. *Every supra subspace of supra-ϵ-$T_{2\frac{1}{2}}$-space is supra-ϵ-$T_{2\frac{1}{2}}$.*

Proof. Suppose that (χ, θ_χ) is a supra subspace of supra-ϵ-$T_{2\frac{1}{2}}$-space (γ, θ) and $\vartheta_1 \neq \vartheta_2 \in \chi \subseteq \gamma$. Given γ is supra-ϵ-$T_{2\frac{1}{2}}$, then there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, such that $\vartheta_1 \in \nu_1$ and $\vartheta_2 \in \nu_2$ and $cl_\epsilon^s(\nu_1) \cap cl_\epsilon^s(\nu_2) = \emptyset$. Given Proposition 2, $(cl_\epsilon(S))_{\theta_\chi} = \chi \cap (cl_\epsilon(S)) = \chi \cap (cl_\epsilon(\nu_1 \cap \chi)) \subseteq \chi \cap cl_\epsilon^s(\nu_1)$ and $(cl_\epsilon(T))_{\theta_\chi} = \chi \cap (cl_\epsilon(T)) = \chi \cap (cl_\epsilon(\nu_1 \cap \chi)) \subseteq \chi \cap cl_\epsilon^s(\nu_2)$. Hence, $\vartheta_1 \in S = \nu_1 \cap \chi$ and $\vartheta_2 \in T = \nu_2 \cap \chi$ such that $(cl_\epsilon(S))_{\theta_\chi} \cap (cl_\epsilon(T))_{\theta_\chi} = \emptyset$. Therefore, χ is supra-ϵ-$T_{2\frac{1}{2}}$-space.

4. Supra ϵ-regularity and supra ϵ-normality

In this section, we introduce four new kinds of separation axioms based on supra ϵ-open sets named supra-ϵ-regular-space, supra-ϵ-normal-space, supra-ϵ-T_3-space, and supra-ϵ-T_4-space. We provide thorough descriptions of each of them. In particular, we examine necessary conditions for several comparable connections between them and provide a general illustration of their salient features. We also suggest a diagram that outlines these relationships [see figure 1]. Moreover, we show prove that every supra-ϵ-R-space (γ, θ) is supra-ϵ-N-space, if $|\gamma| \leqslant 4$, which implies that the two approaches of supra-ϵ-T_3-space and supra-ϵ-T_4-space are identical. Finally, we provide the required counterexamples which confirm our study.

Definition 13. *An STS (γ, θ) is said to be*

(1) *Supra-ϵ-regular-space (or supra-ϵ-R-space) if for each supra-ϵ-closed set H with $\vartheta \notin H$, there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, such that $\vartheta \in \nu_1$ and $H \subseteq \nu_2$.*

(2) *Supra-ϵ-normal-space (or supra-ϵ-N-space) if for each two disjoint supra-ϵ-closed sets H, K, there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, such that $H \subseteq \nu_1$ and $K \subseteq \nu_2$.*

(3) *Supra-ϵ-T_3-space, if it is both supra-ϵ-R-space and supra-ϵ-T_1.*

(4) *Supra-ϵ-T_4-space, if it is both supra-ϵ-N-space and supra-ϵ-T_1.*

Theorem 8. *The following are equivalent for any STS (γ, θ):*

(1) *γ is supra-ϵ-R-space;*

(2) *For every supra-ϵ-open subset ν of γ continuing ϑ, there is a supra-ϵ-open subset η of γ, such that $\vartheta \in \eta \subseteq cl_\epsilon(\eta) \subseteq \nu$;*

(3) *For every $\nu \in SO_\epsilon(\gamma)$ can be expressed by $\nu = \cup\{\eta : \eta \in SO_\epsilon(\gamma)$ and $cl_\epsilon(\eta) \subseteq \nu\}$.*

Proof.

(1) $\Rightarrow$ (2) Let ν be a supra-ϵ-open subset of γ continuing ϑ, then ν^c is supra-ϵ-closed subset of γ with $\vartheta \notin \nu^c$. Given (1), there are two disjoint supra-ϵ-open subsets η_1 and η_2 of γ, such that $\vartheta \in \eta_1$ and $\nu^c \subseteq \eta_2$. Hence, $\vartheta \in \eta_1 \subseteq \eta_2^c \subseteq \nu$. Therefore, $\vartheta \in \eta_1 \subseteq cl_\epsilon(\eta_1) \subseteq \nu$.

(2) $\Rightarrow$ (3) Let $\nu \in SO_\epsilon(\gamma)$. Applying (2), for each $\vartheta \in \nu$, there is a supra-ϵ-open subset η of γ, such that $\vartheta \in \eta \subseteq cl_\epsilon(\eta) \subseteq \nu$. Hence, $\cup\{\eta : \eta \in SO_\epsilon(\gamma)$ *and* $cl_\epsilon(\eta) \subseteq \nu\} = \nu$.

(3) $\Rightarrow$ (1) Let K be a supra-ϵ-closed set with $\vartheta \notin K$, then $K^c \in SO_\epsilon(\gamma)$ with $\vartheta \in K^c$. Given (3), $K^c = \cup\{\eta : \eta \in SO_\epsilon(\gamma)$ *and* $cl_\epsilon(\eta) \subseteq K^c\}$. Since $\vartheta \in K^c$, there is $G_\vartheta \in SO_\epsilon(\gamma)$ including ϑ such that $cl_\epsilon(G_\vartheta) \subseteq K^c$. Thus, $K \subseteq [cl_\epsilon(G_\vartheta)]^c \in SO_\epsilon(\gamma)$, $\vartheta \in G_\vartheta$ and $[cl_\epsilon(G_\vartheta)]^c \cap G_\vartheta = \emptyset$. Therefore, γ is supra-ϵ-R-space.

Theorem 9. *[60] The following are equivalent for any STS* (γ, θ)*:*

(1) γ *is supra-*ϵ*-*T_0*-space;*

(2) *For each* $\vartheta_1 \neq \vartheta_2 \in \gamma$, $cl^s_\epsilon(\{\vartheta_1\}) \neq cl^s_\epsilon(\{\vartheta_2\})$*;*

(3) *For each* $\vartheta \in \gamma$, $acc_\epsilon(\{\vartheta\}) = \cup\{G : G \in SC_\epsilon(\gamma)\}$.

Theorem 10. *[60]*

(1) *Any supra-*ϵ*-*T_j*-space is supra-*ϵ*-*T_{j-1}, $j = 1, 2$.

(2) *Any supra-*T_j*-space is supra-*ϵ*-*T_j, $j = 1, 2$.

Theorem 11. *[60] For any supra-*ϵ*-R-space* (γ, θ)*, the following are equivalent:*

(1) γ *is supra-*ϵ*-*T_0*-space;*

(2) γ *is supra-*ϵ*-*T_2*-space;*

(3) γ *is supra-*ϵ*-*T_1*-space.*

Proof.

(1) $\Rightarrow$ (2) Let $\vartheta_1 \neq \vartheta_2 \in \gamma$. Since γ is supra-ϵ-R-space, $cl^s_\epsilon(\{\vartheta_1\}) \neq cl^s_\epsilon(\{\vartheta_2\})$ according to Theorem 9. Hence, either $\vartheta_2 \notin cl^s_\epsilon(\{\vartheta_1\})$ or $\vartheta_1 \notin cl^s_\epsilon(\{\vartheta_2\})$. Considering $\vartheta_2 \notin cl^s_\epsilon(\{\vartheta_1\})$ and by supra-ϵ-R-spaceness, there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, containing ϑ_2 and $cl^s_\epsilon(\{\vartheta_1$, repetitively. Therefore, γ is supra-ϵ-T_2-space.

(2) $\Rightarrow$ (3) **and** (3) $\Rightarrow$ (1) Follows from Theorem 10 (1).

Theorem 12. *The following are equivalent for any STS* (γ, θ)*:*

(1) γ *is supra-*ϵ*-*T_1*-space;*

(2) *For each* $\vartheta \in \gamma$, $\{\vartheta\} \in SC_\epsilon(\gamma)$*;*

(3) $\cap\{H : H \in SO_\epsilon(\gamma)$ *and* $C \subseteq H\} = C$*;*

(4) *For each* $\vartheta \in \gamma$, $acc_\epsilon(\{\vartheta\}) = \emptyset$.

Theorem 13. *Any supra-*ϵ*-*T_3*-space is supra-*ϵ*-*$T_{2\frac{1}{2}}$.

Proof. Let (γ, θ) be a supra-ϵ-T_3-space and $\vartheta_1 \neq \vartheta_2 \in \gamma$. Given Theorem 12, $\{\vartheta_1\}, \{\vartheta_2\} \in SC_\epsilon(\gamma)$ with $\vartheta_2 \notin \{\vartheta_1\}$ and $\vartheta_1 \notin \{\vartheta_2\}$. Since γ is supra-ϵ-R-space, there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ containing ϑ_1 and ϑ_2, respectively. Hence, there are supra-ϵ-open subsets η_1, η_2 of γ, such that $\vartheta_1 \in \eta_1 \subseteq cl_\epsilon(\eta_1) \subseteq \nu_1$ and $\vartheta_2 \in \eta_2 \subseteq cl_\epsilon(\eta_2) \subseteq \nu_2$ according to Theorem 8. Since ν_1 and ν_2 are disjoint, $cl_\epsilon(\eta_1)$ and $cl_\epsilon(\eta_2)$ are disjoint. Therefore, γ is supra-ϵ-$T_{2\frac{1}{2}}$.

Remark 2. *The converse of Theorem 13 is not hold as the upcoming example will demonstrate.*

Example 2. *Let* $\theta = \{\gamma, \emptyset, \{5,6\}, \{7,8\}, \{5,7\}, \{6,8\}, \{6,7\}, \{5,6,7\}, \{5,6,8\}, \{5,7,8\}, \{6,7,8\},\}$ *be an STS on* $\gamma = \{5,6,7,8\}$*. Regarding* $\{5,8\} \in SC_\epsilon(\gamma)$ *with* $6 \notin \{5,8\}$*, however here are not two disjoint supra-*ϵ*-open subsets* γ *separate* 6 *and* $\{5,8\}$*. Hence,* γ *is not supra-*ϵ*-R-space, which implies that it is not supra-*ϵ*-*T_3*. Also, it is easy to check that* γ *is supra-*ϵ*-*$T_{2\frac{1}{2}}$*.*

Theorem 14. *The following are equivalent for any STS* (γ, θ)*:*

(1) γ *is supra-*ϵ*-*N*-space;*

(2) *For every supra-*ϵ*-closed subset* ν *of* γ *and for every supra-*ϵ*-open superset* ω *of* ν*, there is a supra-*ϵ*-open subset* ρ_1 *of* γ*, such that* $\nu \subseteq \rho_1 \subseteq cl_\epsilon(\rho_1) \subseteq \omega$*;*

(3) *For every* ω_1 *and* $\omega_2 \in SO_\epsilon(\gamma)$ *such that* $\gamma = \omega_1 \cup \omega_2$ *, there are two supra-*ϵ*-closed subsets* ν_1 *and* ν_2 *of* γ*, such that* $\omega_1 \subseteq \nu_1$*,* $\omega_2 \subseteq \nu_2$ *and* $\gamma = \nu_1 \cup \nu_2$*.*

Proof.

(1) $\Rightarrow$ (2) Let ν be a supra-ϵ-closed subset of γ and $\omega \in SO_\epsilon(\gamma)$ such that $\nu \subseteq \omega$, then ν *and* ω^c are two disjoint supra-ϵ-closed sets. Given (1), there are two disjoint supra-ϵ-open subsets ρ_1 and ρ_2 of γ, such that $\nu \subseteq \rho_1$ and $\omega^c \subseteq \rho_2$. Therefore, $\nu \subseteq \rho_1 \subseteq cl_\epsilon(\rho_2^c) = \rho_2^c \subseteq \omega$. Thus, $\nu \subseteq \rho_1 \subseteq cl_\epsilon(\rho_1) \subseteq \omega$.

(2) $\Rightarrow$ (3) Let ω_1 *and* $\omega_2 \in SO_\epsilon(\gamma)$ such that $\gamma = \omega_1 \cup \omega_2$. Then, ν_1^c is a supra-ϵ-closed subset of ν_2. Given (2), there is a supra-ϵ-open subset ρ_1 of γ, such that $\nu_1^c \subseteq \rho_1 \subseteq cl_\epsilon(\rho_1) \subseteq \nu_2$. Therefore, $\rho_1^c \subseteq \nu_1$ and $cl_\epsilon(\rho_1) \subseteq \nu_2$ in which ρ_1^c *and* $cl_\epsilon(\rho_1) \in SC_\epsilon(\gamma)$ with $\rho_1^c \cup cl_\epsilon(\rho_1) = \gamma$.

(3) $\Rightarrow$ (1) Let ν_1 and ν_2 are disjoint supra-ϵ-closed sets, then ν_1^c and ν_2^c are supra-ϵ-open sets in which $\gamma = \nu_1^c \cup \nu_2^c$. Given (3), there are two supra-ϵ-closed subsets δ_1 and δ_2 of γ, such that $\delta_1 \subseteq \nu_1^c$, $\delta_2 \subseteq \nu_2^c$ and $\gamma = \delta_1 \cup \delta_2$. Thus, δ_1^c *and* $\delta_2^c \in SO_\epsilon(\gamma)$ containing ν_1, ν_2, respectively in which $\delta_1^c \cap \delta_2^c = \emptyset$. Therefore, γ is supra-ϵ-N-space.

Theorem 15. *Any supra-*ϵ*-*T_j*-space is supra-*ϵ*-*T_{j-1}*,* $j = 3, 4$*.*

Proof. We prove the case when $j = 4$, the other case by a similar manner. Let (γ, θ) be a supra-ϵ-T_4-space, then it is both supra-ϵ-T_1-space and supra-ϵ-N-space. Now, we want to prove that γ is supra-ϵ-R-space. So, let H be a supra-ϵ-closed set with $\vartheta \notin H$. Given Theorem 12, $\{\vartheta\}$ is supra-ϵ-closed. Hence, H and $\{\vartheta\}$ are disjoint supra-ϵ-closed sets. By supra-ϵ-T_4-spaceness, there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, such that $\{\vartheta\} \subseteq \nu_1$ and $H \subseteq \nu_2$. Therefore, $\vartheta \in \nu_1$ and $H \subseteq \nu_2$. Thus, γ is supra-ϵ-R-space, and so it is (γ, θ) be a supra-ϵ-T_3-space.

Remark 3. *The converse of Theorem 15 is not hold as the upcoming examples will demonstrate.*

Examples 1. **(1)** *Let* $\theta = \{\gamma, \emptyset, \{e,r\}, \{t,y\}, \{e,t\}, \{r,y\}, \{r,t\}, \{e,r,t\}, \{e,r,y\}, \{e,t,y\}, \{r,t,y\}\}$ *be an STS on* $\gamma = \{e,r,t,y\}$. *Then we have that* $SO_\epsilon(\gamma) = \theta$. *It follows that,* γ *is supra-*ϵ*-*T_2*-space, and so it is both supra-*ϵ*-*T_1*-space and supra-*ϵ*-*T_0*-space. Moreover, regarding* $\{e,y\} \in SC_\epsilon(\gamma)$ *with* $r \notin \{e,y\}$, *however there are not two disjoint supra-*ϵ*-open subsets* γ *separate* r *and* $\{e,y\}$. *Hence,* γ *is not supra-*ϵ*-*R*-space, and so it is not supra-*ϵ*-*T_3*-space.*

(2) *Let* $\theta = \{\emptyset, W \subseteq \mathbb{N} : 1 \in W \text{ or } 1 \notin W \text{ and } W^c \text{ is finite}\}$ *be a supra topology in* $\mathbb{N} = \{1,2,3,4,5,.......\}$. *It i clear that* γ *is both supra-*ϵ*-*T_1*-space and supra-*ϵ*-*R*-space, and then it is supra-*ϵ*-*T_3*-space. However, regarding the sets* $A = \{n \in \mathbb{N} : n \text{ is even}\}$ *and* $B = \{n \in \mathbb{N} : n \geqslant 5 \text{ and } n \text{ is odd}\}$. *We have* A *and* B *are disjoint supra-*ϵ*-closed subsets of* $\mathbb{N}$, *however there are not two disjoint supra-*ϵ*-open subsets of* γ *containing them. Therefore,* γ *is not supra-*ϵ*-*N*-space and thus* γ *is not supra-*ϵ*-*T_4*-space.*

Proposition 3. *Based on the aforementioned conclusions, the following irreversible implications are held for an STS* (γ, θ).

$$\begin{array}{ccccccc} \text{supra-}\epsilon\text{-}T_4\text{-space} & \Longrightarrow & \text{supra-}\epsilon\text{-}T_3\text{-space} & \Longrightarrow & \text{supra-}\epsilon\text{-}T_{2\frac{1}{2}}\text{-space} & \Longrightarrow & \text{supra-}\epsilon\text{-}T_2\text{-space} \\ & & & & & & \Downarrow \\ & & & & \text{supra-}\epsilon\text{-}T_0\text{-space} & \Longleftarrow & \text{supra-}\epsilon\text{-}T_1\text{-space} \end{array}$$

Figure 1. *The relationships between different kinds of separation axioms in the context of STSs which are motivated by supra* ϵ*-open sets*

Proof. It is follow from [[60], Proposition 1] and Theorems 13 and 15.

Proposition 4. *[58] Let* (Y, θ_Y) *be an supra* ϵ*-subspace of an STS* (γ, θ) *and* W *be a subset of* γ. *Then,* $W \in SC_\epsilon(Y)$ *if and only if there is* $G \in SC_\epsilon(\gamma)$ *such that* $W = Y \cap G$.

Theorem 16. *Every supra subspace of supra-*ϵ*-*R*-space is supra-*ϵ*-*R.

Proof. Assume that (χ, θ_χ) is a supra subspace of supra-ϵ-R-space (γ, θ) and H is a supra-ϵ-closed subset of χ with $\vartheta \notin H$. Given Proposition 4, there is $N \in SC_\epsilon(\gamma)$ such that $H = \chi \cap N$, and then $\vartheta \notin N$. Since (γ, θ) is supra-ϵ-R-space, there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, such that $\vartheta \in \nu_1$ and $N \subseteq \nu_2$. Hence, $\chi \cap \nu_1$ and $\chi \cap \nu_2$ are two disjoint supra-ϵ-open subsets of χ which containing ϑ and H, respectively. Therefore, (χ, θ_χ) is supra-ϵ-R-space.

Corollary 3. *Every supra subspace of supra-*ϵ*-*T_3*-space is supra-*ϵ*-*T_3.

Proof. It is derived from Theorem 10 and Theorem 16.

Theorem 17. *For any STS* (γ, θ), *if* $|\gamma| \leqslant 4$, *then every supra-*ϵ*-*R*-space is supra-*ϵ*-*N*-space.*

Proof. Let H, K be two disjoint supra-ϵ-closed subsets of a supra-ϵ-R-space (γ, θ). Then, we have five cases:
Case (1), $|K| = 4$, then $K = \gamma$ and $H = \emptyset$. Hence, we get our result.
Case (2), $|K| = 3$, then $|H| = 1$, and so $K^c = H$ and $H^c = K$. Hence, we get our result.
Case (3), $|K| = 2$, then either $|H| = 1$ or $|H| = 2$. If $|H| = 2$, then both H, K are disjoint supra-ϵ-open sets. Hence, we get our result.
If $|H| = 1$, then H is singleton say $H = \{a\}$ and so $a \notin K$. Since γ is supra-ϵ-R-space, there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, containing H, K, respectively. Hence, we get our result.
Case (4), $|K| = 1$, then K is singleton say $K = \{s\}$ and so $s \notin H$. Since γ is supra-ϵ-R-space, there are two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, containing H, K, respectively. Hence, we get our result.
Case (5), $K = \emptyset$, then $H = \gamma$. Hence, we get our result.
Consequently, γ is supra-ϵ-N-space.

Corollary 4. *For any STS (γ, θ), if $|\gamma| \leqslant 4$, then then the two approaches of supra-ϵ-T_3-space and supra-ϵ-T_4-space are identical.*

Proof. Follows from Theorem 17.

Remark 4. *If $|\gamma| > 4$ in Theorem 17, then the result will not be achieved, as demonstrated in the following example.*

Example 3. *Let $\theta = 2^\gamma \backslash \{\{3,4\},\{1,2\},\{1\},\{3\}\}$ be an STS on $\gamma = \{1,2,3,4,5\}$. Regarding the two disjoint supra-ϵ-closed subsets $\{1,2\}$ and $\{3,4\}$ of γ, then there are not two disjoint supra-ϵ-open subsets ν_1 and ν_2 of γ, containing them, respectively. Hence, γ is not supra-ϵ-N-space. Also, it is easy to check that γ is supra-ϵ-R.*

5. Conclusion and future works

In this article, we provide new versions of supra septation axioms inspired by supra-ϵ-open sets. To name a few: supra-ϵ-$T_{2\frac{1}{2}}$-space, supra-ϵ-regular-space, supra-ϵ-normal-space, supra-ϵ-T_3-space, and supra-ϵ-T_4-space. The behavior of these concepts with regard to particular types of supra functions is also examined. We also give a general illustration of their key traits and look at the prerequisites for a number of similar links between them. Finally, the required counterexamples that support our conclusions are provided.
The following topics could be examined in further research on the theoretical aspects of these generalized notions based on the particular methodologies discussed in this paper: Introducing more investigation of septarian axioms term in supra ϵ-open sets by using the ideal notions. Also, presenting these notion to soft topological spaces [22]. Moreover, using fuzzy supra soft topological spaces to generalize these notions [61, 62].

Conflicts of interest

The authors of this work state that they have no conflicting interests with regard to its publication.

Authors contributions

All authors made equal contributions.

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